

A uniform dimension result for two-dimensional fractional multiplicative processes

Xiong Jin¹

Mathematical Institute, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, Scotland. E-mail: xj3@st-andrews.ac.uk

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Abstract. Given a two-dimensional fractional multiplicative process $(F_t)_{t \in [0,1]}$ determined by two Hurst exponents H_1 and H_2 , we show that there is an associated uniform Hausdorff dimension result for the images of subsets of $[0, 1]$ by F if and only if $H_1 = H_2$.

Résumé. Etant donné un processus multiplicatif fractionnaire bi-dimensionnel $(F_t)_{t \in [0,1]}$ déterminé par deux exposants de Hurst H_1 et H_2 , nous montrons l'existence d'un résultat uniforme pour la dimension de Hausdorff des images des sous-ensembles de $[0, 1]$ par F si et seulement si $H_1 = H_2$.

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1. Introduction

It is well-known that planar Brownian motion doubles the Hausdorff dimension, in the sense that for any Borel set $E \subset \mathbb{R}_+$,

$$\mathbb{P}\{\dim_H B(E) = 2 \dim_H E\} = 1, \tag{1}$$

where $B : \mathbb{R}_+ \mapsto \mathbb{R}^2$ is a planar Brownian motion and $B(E) = \{B(t) : t \in E\}$ is the image of E through B . This result was first proved by McKean [18] in 1955, following the works of Lévy [17] and Taylor [21] regarding the Hausdorff measure of $B(\mathbb{R}_+)$, and was extended to α -stable processes by Blumenthal and Gettoor [5]. The result cannot be extended to more general Lévy processes, but one can obtain control such as

$$\mathbb{P}\{\beta' \dim_H E \leq \dim_H X(E) \leq \beta \dim_H E\} = 1 \tag{2}$$

for certain parameters β and β' depending on the process X (see [6,19]). In [6] Blumenthal and Gettoor also conjectured that given any Borel set E , there exists a constant $\lambda(X, E)$ such that

$$\mathbb{P}\{\dim_H X(E) = \lambda(X, E)\} = 1.$$

This conjecture is proved by Khoshnevisan and Xiao [16] in 2005, in terms of Lévy exponents.

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The relation (1) involves an exceptional null set $N_E \subset \Omega$ for each fixed E , and it is natural to ask whether there is a null set N such that $N_E \subset N$ holds for uncountably many E . In other words, we would hope for a result like

$$\mathbb{P}\{\dim_H B(E) = 2 \dim_H E \text{ for all } E \in \mathcal{O}\} = 1 \tag{3}$$

for \mathcal{O} as large as possible. In the literature results like (3) are termed as uniform dimension result, and this was first proved by Kaufman [15] for planar Brownian motion when \mathcal{O} is the set of all Borel sets in \mathbb{R}_+ . This result was extended to strictly stable Lévy processes by Hawkes and Pruitt [10]. For general Lévy processes the corresponding uniform dimension result may not hold, but for Lévy subordinators one can obtain either a uniform result as (3) for smaller family \mathcal{O} (collection of Borel sets whose Hausdorff dimension and packing dimension coincide) or a looser dimension result as (2) uniformly for all Borel sets (see [10]). For further information regarding the dimension results of stochastic processes, we refer to the survey papers [22,25].

In this paper we prove a uniform dimension result for two-dimensional fractional multiplicative processes, a class of random continuous functions recently constructed by Barral and Mandelbrot [3]. These processes and their generalisation, multiplicative cascade processes [2], are considered as natural extensions of the classical Mandelbrot measures [14] to functions, or in probabilistic terms, subordinators to general processes. In [12] the author proved a dimension result for two-dimensional multiplicative cascade processes, motivated by the recent works in [4,7,20] that proved the Knizhnik–Polyakov–Zamolodchikov formula from quantum gravity for Gaussian multiplicative chaos and Mandelbrot measures. The KPZ formula is a quadratic (thus nonlinear) relation between dimensions of a given Borel set with respect to the Euclidean metric and the random metric obtained from multiplicative chaos. It is natural to ask whether or not this type of dimension formula holds uniformly for all Borel sets. With the help of multifractal analysis of multiplicative cascades and their graph and range singularity spectra (see [1,11] for example), it can be shown that for the dimension result in [12], as long as the formula is nonlinear, there will be some random sets that break the formula. Thus the only candidate for which the uniform dimension result could possibly hold is the multi-dimensional fractional multiplicative processes (in one-dimensional case there is the level set of the process that breaks the formula), which leads to the present study.

Recall that as a special case of [12], we have the following dimension result for the two-dimensional fractional multiplicative process $F = (F_1, F_2)$ with parameters $1/2 < H_1 \leq H_2 < 1$ (see Section 2 for precise definition): for every Borel set $E \subset [0, 1]$,

$$\mathbb{P}\left\{\dim_H F(E) = \frac{\dim_H E}{H_1} \wedge \left(1 + \frac{\dim_H E - H_1}{H_2}\right)\right\} = 1. \tag{4}$$

In particular when $H_1 = H_2 = H \in (1/2, 1)$, we have

$$\mathbb{P}\left\{\dim_H F(E) = \frac{1}{H} \dim_H E\right\} = 1. \tag{5}$$

The result (4) has exactly the same form as in [24] for Gaussian vector fields, which is shown in [23] to be uniform if and only if the parameters of Gaussian vector fields coincide. In this paper we show that the same phenomenon occurs for two-dimensional fractional multiplicative processes:

Theorem 1.1. *If $H_1 = H_2 = H \in (1/2, 1)$, then (5) is uniform, that is*

$$\mathbb{P}\left\{\dim_H F(E) = \frac{1}{H} \dim_H E \text{ for all sets } E \subset [0, 1]\right\} = 1. \tag{6}$$

If $H_1 < H_2$, then the result (4) cannot hold almost surely for all Borel sets.

The proof of (6) relies on a stopping time technique used in [9] for stable Markov processes, with certain non-trivial modifications due to the fact that fractional multiplicative processes are neither stable nor Markovian. To show that result (4) cannot be uniform when $H_1 \neq H_2$, we use the same trick as in [23] to show that the level set of F_1 breaks the formula.

2. Two-dimensional fractional multiplicative processes

2.1. 2-d fractional multiplicative processes

Fix two parameters $1/2 < H_1 \leq H_2 < 1$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2)$ be a random vector such that for $j = 1, 2$,

$$\varepsilon_j = \begin{cases} +1, & \text{with probability } (1 + 2^{H_j-1})/2; \\ -1, & \text{with probability } (1 - 2^{H_j-1})/2. \end{cases}$$

Denote by $\{0, 1\}^* = \bigcup_{n \geq 1} \{0, 1\}^n$ the set of finite dyadic words. Let

$$\{\varepsilon(w) = (\varepsilon_1(w), \varepsilon_2(w)): w \in \{0, 1\}^*\}$$

be a sequence of independent copies of ε encoded by $\{0, 1\}^*$.

Let $j \in \{1, 2\}$. For each $w = w_1 \cdots w_n \in \{0, 1\}^*$ let

$$t_w = \sum_{m=1}^n w_m 2^{-m}$$

be the corresponding dyadic point in $[0, 1)$, and let

$$I_w = [t_w, t_w + 2^{-n})$$

be the corresponding dyadic interval. Then let

$$\bar{\varepsilon}_j(w) = \prod_{m=1}^n \varepsilon_j(w_1 \cdots w_m)$$

be the random weight on I_w .

For $x \in [0, 1)$ and $n \geq 1$ let $x|_n = x_1 \cdots x_n \in \{0, 1\}^n$ be the unique word such that $x \in I_{x|_n}$. For $n \geq 1$ define the piecewise linear function

$$F_{j,n} : t \in [0, 1] \mapsto 2^{n(1-H_j)} \int_0^t \bar{\varepsilon}_j(x|_n) dx.$$

From Theorem 1.1 in [3] one has that almost surely $\{F_{j,n}\}_{n \geq 1}$ converges uniformly to a limit F_j , and F_j is α -Hölder continuous for any $\alpha \in (0, H_j)$. Then the two dimensional fractional multiplicative process considered in this paper is the mapping

$$F = (F_1, F_2) : t \in [0, 1] \mapsto (F_1(t), F_2(t)) \in \mathbb{R}^2.$$

We shall always assume that $\mathbb{P}\{\varepsilon_1 = \varepsilon_2\} < 1$, to ensure that the process F does not degenerate to one-dimensional case.

Remark 2.1. If we take the parameter $H_j \in (-\infty, 1/2]$, then the corresponding sequence $\{F_{j,n}\}_{n \geq 1}$ is not bounded in L^2 -norm. Moreover it is shown in [3] that the normalised sequence

$$X_{j,n} = \begin{cases} F_{j,n}/(2^{n(1/2-H_j)} \sqrt{1 + (2^{2-2H_j} - 2)^{-1}}), & \text{if } H < 1/2, \\ F_{j,n}/\sqrt{n/2}, & \text{if } H = 1/2 \end{cases}$$

converges, as $n \rightarrow \infty$, in law to standard Brownian motion restricted on $[0, 1]$.

2.2. Statistical self-similarity

For any $w \in \{0, 1\}^*$ we can similarly define

$$F_{j,n}^{[w]} : t \in [0, 1] \mapsto 2^{n(1-H_j)} \int_0^t \prod_{m=1}^n \varepsilon_j(w \cdot x|_m) dx.$$

Denote by $|w|$ the length of w and

$$g_w : t \mapsto 2^{|w|}(t - t_w)$$

the canonical mapping from I_w to $[0, 1]$. Then by definition for any $s, t \in I_w$ and $n \geq |w|$ one has

$$F_{j,n}(t) - F_{j,n}(s) = 2^{-|w|H_j} \cdot \bar{\varepsilon}_j(w) \cdot [F_{j,n-|w|}^{[w]}(g_w(t)) - F_{j,n-|w|}^{[w]}(g_w(s))]. \tag{7}$$

Let $F_j^{[w]}$ be the limit of $\{F_{j,n}^{[w]}\}_{n \geq 1}$. It certainly has the same law as F_j , and it is independent of $\bar{\varepsilon}_j(w)$. Moreover for any $s, t \in I_w$ one gets from (7) that

$$F_j(t) - F_j(s) = 2^{-|w|H_j} \cdot \bar{\varepsilon}_j(w) \cdot [F_j^{[w]}(g_w(t)) - F_j^{[w]}(g_w(s))]. \tag{8}$$

2.3. Boundary values and oscillations

Let $Z_j = F_j(1)$ and $Z_j(w) = F_j^{[w]}(1)$ for $w \in \{0, 1\}^*$. Clearly they have the same law. Also from (8) one has

$$F_j(t_w + 2^{-|w|}) - F_j(t_w) = 2^{-|w|H_j} \cdot \bar{\varepsilon}_j(w) \cdot Z_j(w),$$

where $\bar{\varepsilon}_j(w)$ and $Z_j(w)$ are independent. Let

$$\varphi(u, v) = \mathbb{E}(e^{i(uZ_1+vZ_2)}) \quad \text{and} \quad \varphi_j(u) = \mathbb{E}(e^{iuZ_j})$$

be the characteristic functions of (Z_1, Z_2) and Z_j respectively. The following lemma (see Lemma 2 in [12]) is essential to our proof:

Lemma 2.1. *One has $\varphi \in L^1(\mathbb{R}^2)$ if $\mathbb{P}\{\varepsilon_1 = \varepsilon_2\} < 1$ and $\varphi_j \in L^1(\mathbb{R})$. Consequently, (Z_1, Z_2) has a bounded joint density function f with*

$$\|f\|_\infty \leq \|\varphi\|_1 = \iint_{\mathbb{R}^2} |\varphi(u, v)| du dv < \infty,$$

provided $\mathbb{P}\{\varepsilon_1 = \varepsilon_2\} < 1$, and Z_j has a bounded density function f_1 with

$$\|f_j\|_\infty \leq \|\varphi_j\|_1 = \int_{\mathbb{R}} |\varphi_j(u)| du < \infty.$$

Let $X_j = \sup_{s,t \in [0,1]} |F_j(t) - F_j(s)|$ and $X_j(w) = \sup_{s,t \in [0,1]} |F_j^{[w]}(t) - F_j^{[w]}(s)|$ for $w \in \{0, 1\}^*$. They have the same law, and from (8) one has

$$\sup_{s,t \in I_w} |F_j(t) - F_j(s)| = 2^{-|w|H_j} \cdot X_j(w).$$

Moreover, from Lemma 3.1 in [2] one has that for all $q \in \mathbb{R}$,

$$\mathbb{E}(X_j^q) < \infty. \tag{9}$$

3. Proof of Theorem 1.1

3.1. Proof of (6)

The idea of the proof is to show that for all n large enough, the number of $w \in \{0, 1\}^{\lfloor n/H \rfloor}$ that the image $F(I_w)$ intersects with a given dyadic square of side length 2^{-n} is bounded above uniformly for all dyadic squares. This will imply the uniform dimension result (6). In order to do so we need to control the probability of $F(t_w) \in S$, given that $F(t_{w_1}), \dots, F(t_{w_k}) \in S$ for $w_j \neq w$, then pass this information to $F(I_w) \cap S \neq \emptyset$ by using the control of the oscillation of F on I_w .

Preliminaries

Let $H_1 = H_2 = H \in (1/2, 1)$ and fix an integer $p > 1/(2H - 1)$ so that $(1 + 1/p)/H < 2$.

For $n \geq 1$ denote by \mathcal{T}_n the set of dyadic numbers of generation n , that is

$$\mathcal{T}_n = \left\{ t_w = \sum_{j=1}^n w_j 2^{-j} : w = w_1 \cdots w_n \in \{0, 1\}^n \right\} \cup \{1\}.$$

Let \mathcal{S}_n be the collection of all dyadic squares in \mathbb{R}^2 with side length 2^{-n} .

Fix $n \geq 1$ and $S \in \mathcal{S}_n$. Let $m = \lfloor n(1 - 1/p)/H \rfloor$. Define

$$N(S) := \#\{w \in \{0, 1\}^m : F(t_w) \in S\}$$

and

$$\tilde{N}(S) := \#\{w \in \{0, 1\}^m : F(I_w) \cap S \neq \emptyset\}.$$

In the following we shall estimate a uniform control of $N(S)$ for all $S \in \mathcal{S}_n$, and use it to get a uniform control of $\tilde{N}(S)$, then show that this uniform control gives the uniform dimension result.

Uniform control of $N(S)$

First notice that

$$\mathcal{T}_m = \{j \cdot 2^{-m} : j = 0, \dots, 2^m\}.$$

For $k = 1, \dots, 2^m$ one can easily get

$$\mathbb{P}(N(S) \geq k) \leq \mathbb{E}(H_k(S)), \tag{10}$$

where

$$H_k(S) := \sum_{s_1 < s_2 < \dots < s_k \in \mathcal{T}_m} \prod_{l=1}^k \mathbf{1}_{\{F(s_l) \in S\}}.$$

We shall control $\mathbb{E}(H_k(S))$ by iterating.

Denote by $|S|$ the diameter of S . One has

$$\begin{aligned} & H_{k+1}(S) \\ &= \sum_{s_1 < s_2 < \dots < s_k \in \mathcal{T}_m, s_k \neq 1} \left(\prod_{l=1}^k \mathbf{1}_{\{F(s_l) \in S\}} \right) \cdot \sum_{s_k < t \in \mathcal{T}_m} \mathbf{1}_{\{F(t) \in S\}} \\ &\leq \sum_{s_1 < s_2 < \dots < s_k \in \mathcal{T}_m, s_k \neq 1} \left(\prod_{l=1}^k \mathbf{1}_{\{F(s_l) \in S\}} \right) \cdot \sum_{s_k < t \in \mathcal{T}_m} \mathbf{1}_{\{|F(t) - F(s_k)| \leq |S|\}}. \end{aligned} \tag{11}$$

Fix $s_1 < \dots < s_k < t \in \mathcal{T}_m$, let N be the smallest integer such that there exists a dyadic word $w = w_1 \dots w_N$ such that $I_w \subset (s_k, t)$. This gives

$$2^{-N} \leq t - s_k \leq 4 \cdot 2^{-N}.$$

From (8) we may write

$$|F(t) - F(s_k)| = \left(\sum_{j=1,2} |A_j(w)Z_j(w) + B_j(w)|^2 \right)^{1/2}, \quad (12)$$

where $A_j(w) = 2^{-NH} \bar{\varepsilon}_j(w)$ and $B_j(w) = F_j(t) - F_j(t_w + 2^{-N}) + F_j(t_w) - F_j(s_k)$. Denote by

$$\mathcal{F}(w) = \sigma(\varepsilon(u): u \in \{0, 1\}^* \setminus \{w \cdot u: u \in \{0, 1\}^*\}). \quad (13)$$

It is easy to check that the random variables $A_1(w)$, $A_2(w)$, $B_1(w)$, $B_2(w)$ and $\prod_{l=1}^k \mathbf{1}_{\{F(s_l) \in S\}}$ are measurable with respect to $\mathcal{F}(w)$. Also notice that

$$\mathbf{1}_{\{|F(t) - F(s_k)| \leq |S|\}} \leq |F(t) - F(s_k)|^{-(1+1/p)/H} \cdot |S|^{(1+1/p)/H}.$$

Recall in Lemma 2.1 the joint density function f of (Z_1, Z_2) , which is bounded by $\|\varphi\|_1 < \infty$. One gets

$$\begin{aligned} & \mathbb{E}(|F(t) - F(s_k)|^{-(1+1/p)/H} | \mathcal{F}(w)) \\ &= \iint_{\mathbb{R}^2} \frac{f(x, y) \, dx \, dy}{(|A_1(w)x + B_1(w)|^2 + |A_2(w)y + B_2(w)|^2)^{(1+1/p)/2H}} \\ &\leq 2^{N(1+1/p)} \iint_{\mathbb{R}^2} \frac{f((u - B_1(w)2^{NH})/\bar{\varepsilon}_1(w), (u - B_1(w)2^{NH})/\bar{\varepsilon}_1(w))}{(u^2 + v^2)^{(1+1/p)/2H}} \, du \, dv \\ &\leq 2^{N(1+1/p)} \cdot \left(1 + \|\varphi\|_1 \cdot \iint_{u^2+v^2 < 1} \frac{du \, dv}{(u^2 + v^2)^{(1+1/p)/2H}} \right) \\ &:= 2^{N(1+1/p)} \cdot C_{(1+1/p)/H}. \end{aligned}$$

The finiteness of $C_{(1+1/p)/H}$ comes from the fact that we already choose p large enough such that $(1 + 1/p)/H < 2$. This gives

$$\begin{aligned} & \mathbb{E} \left(\prod_{l=1}^k \mathbf{1}_{\{F(s_l) \in S\}} \cdot \mathbf{1}_{\{|F(t) - F(s_k)| \leq |S|\}} | \mathcal{F}(w) \right) \\ &\leq |S|^{(1+1/p)/H} \cdot \prod_{l=1}^k \mathbf{1}_{\{F(s_l) \in S\}} \cdot \mathbb{E}(|F(t) - F(s_k)|^{-(1+1/p)/H} | \mathcal{F}(w)) \\ &\leq C_{(1+1/p)/H} |S|^{(1+1/p)/H} \cdot \prod_{l=1}^k \mathbf{1}_{\{F(s_l) \in S\}} \cdot 2^{N(1+1/p)} \\ &\leq 4^{1+1/p} C_{(1+1/p)/H} |S|^{(1+1/p)/H} \cdot \prod_{l=1}^k \mathbf{1}_{\{F(s_l) \in S\}} \cdot |t - s_k|^{-(1+1/p)}. \quad (14) \end{aligned}$$

For any $s_k \in \mathcal{T}_m$ we have

$$\sum_{s_k < t \in \mathcal{T}_m} |t - s_k|^{-(1+1/p)} \leq 2^{m(1+1/p)} \sum_{l=1}^{\infty} l^{-(1+1/p)} \leq 2^{n(1-1/p)(1+1/p)/H} \sum_{l=1}^{\infty} l^{-(1+1/p)}. \quad (15)$$

Thus by combining (11), (14) and (15) we get

$$\mathbb{E}(H_{k+1}(S)) \leq C \cdot 2^{-n(1/p-1/p^2)/H} \cdot \mathbb{E}(H'_k(S)), \quad (16)$$

where $C = 4^{1+1/p} C_{(1+1/p)/H} 2^{(1+1/p)/2H} \sum_{l=1}^{\infty} l^{-(1+1/p)} < \infty$ and

$$H'_k(S) = \sum_{s_1 < s_2 < \dots < s_k \in \mathcal{T}_m, s_k \neq 1} \prod_{l=1}^k \mathbf{1}_{\{F(s_l) \in S\}}.$$

Obviously $H'_k(S) \leq H_k(S)$, thus from (16) we get

$$\begin{aligned} \mathbb{E}(H_{k+1}(S)) &\leq C \cdot 2^{-n(1/p-1/p^2)/H} \cdot \mathbb{E}(H'_k(S)) \\ &\leq C \cdot 2^{-n(1/p-1/p^2)/H} \cdot \mathbb{E}(H_k(S)) \\ &\leq C^k \cdot 2^{-nk(1/p-1/p^2)/H} \cdot \mathbb{E}(H_1(S)). \end{aligned}$$

Together with (10) this gives

$$\begin{aligned} \mathbb{P}\left(\sup_{S \in \mathcal{S}_n} N(S) \geq k+1\right) &\leq C^k \cdot 2^{-nk(1/p-1/p^2)/H} \cdot \sum_{S \in \mathcal{S}_n} \mathbb{E}\left(\sum_{j=0}^{2^m} \mathbf{1}_{\{F(t_j) \in S\}}\right) \\ &= C^k \cdot 2^{-nk(1/p-1/p^2)/H} \cdot \mathbb{E}\left(\sum_{j=0}^{2^m} \sum_{S \in \mathcal{S}_n} \mathbf{1}_{\{F(t_j) \in S\}}\right) \\ &= C^k \cdot 2^{-nk(1/p-1/p^2)/H} \cdot (2^m + 1) \\ &\leq 2C^k \cdot 2^{-nk(1/p-1/p^2)/H} \cdot 2^{n(1-1/p)/H} \\ &= 2C^k \cdot 2^{-n(1-1/p)(k/p-1)/H}. \end{aligned}$$

We may choose $k = p + 1$. Then by the Borel–Cantelli lemma one gets for \mathbb{P} -almost every $\omega \in \Omega$ that there exists a integer $n_p(\omega)$ such that for any $n \geq n_p(\omega)$,

$$\sup_{S \in \mathcal{S}_n} N(S) \leq p + 2. \quad (17)$$

Uniform control of $\tilde{N}(S)$

Now we want to pass the information of $F(t_w) \in S$ to that of $F(I_w) \cap S \neq \emptyset$. We start with the oscillation

$$X(w) = \sup_{s, t \in [0, 1]} \left(\sum_{j=1, 2} |F_j^{[w]}(s) - F_j^{[w]}(t)|^2 \right)^{1/2}, \quad w \in \{0, 1\}^*.$$

Clearly $X(w)$ has the same law as $X = \sup_{s, t \in [0, 1]} |F(s) - F(t)|$. Then for any $n \geq 1$,

$$\begin{aligned} \mathbb{P}\left(\sup_{w \in \{0, 1\}^n} X(w) \geq 2^{n/p}\right) &\leq \sum_{w \in \{0, 1\}^n} \mathbb{P}(X(w) \geq 2^{n/p}) \\ &\leq 2^n \cdot 2^{-n(1+1/p)} \cdot \mathbb{E}(X^{p+1}). \end{aligned}$$

From (9) it is easy to deduce that $\mathbb{E}(X^{p+1}) < \infty$. Then by the Borel–Cantelli lemma one gets for \mathbb{P} -almost every $\omega \in \Omega$ that there exists a integer $n'_p(\omega)$ such that for any $n \geq n'_p(\omega)$ and $w \in \{0, 1\}^n$,

$$X(w) \leq 2^{n/p}. \quad (18)$$

To combine (17) and (18), let

$$\Omega' = \bigcap_{p>1/(2H-1)} \{\omega \in \Omega: n_p(\omega) \vee n'_p(\omega) < \infty\}.$$

So $\mathbb{P}(\Omega') = 1$.

Fix $\omega \in \Omega'$, $p > 1/(2H - 1)$ and $n \geq n_p(\omega) \vee n'_p(\omega)$ large enough such that

$$\lfloor n(1 - 1/p)/H \rfloor > n(1 - 2/p)/H.$$

We are going to control the number

$$\tilde{N}(S) := \{w \in \{0, 1\}^{\lfloor n(1-1/p)/H \rfloor}: F(I_w) \cap S \neq \emptyset\}.$$

First for any $w \in \{0, 1\}^{\lfloor n(1-1/p)/H \rfloor}$ one has

$$\sup_{s,t \in I_w} |F(s) - F(t)| = 2^{-|w|H} \cdot X(w) \leq 2^{-\lfloor n(1-1/p)/H \rfloor(H-1/p)}.$$

For $S \in \mathcal{S}_n$ let

$$U(S) = \{z \in \mathbb{R}^2: \text{dist}(z, S) \leq 2^{-\lfloor n(1-1/p)/H \rfloor(H-1/p)}\}.$$

Then for any $w \in \{0, 1\}^{\lfloor n(1-1/p)/H \rfloor}$ we have

$$F(I_w) \cap S \neq \emptyset \Rightarrow F(I_w) \subset U(S) \Rightarrow F(t_w) \in U(S). \quad (19)$$

On the other hand, there are at most

$$\begin{aligned} & \text{area}\{z \in \mathbb{R}^2: \text{dist}(z, S) \leq 2^{-\lfloor n(1-1/p)/H \rfloor(H-1/p)} + \sqrt{2} \cdot 2^{-n}\} / 2^{-2n} \\ & \leq \pi \cdot (2^{-\lfloor n(1-1/p)/H \rfloor(H-1/p)} + 2\sqrt{2} \cdot 2^{-n})^2 \cdot 2^{2n} \\ & \leq \pi \cdot (1 + 2\sqrt{2})^2 \cdot (2^{-\lfloor n(1-1/p)/H \rfloor})^{2(H-1/p) - (2n/\lfloor n(1-1/p)/H \rfloor)} \\ & \leq \pi \cdot (1 + 2\sqrt{2})^2 \cdot (2^{-\lfloor n(1-1/p)/H \rfloor})^{2(H-1/p) - 2H/(1-2/p)} \\ & = \pi \cdot (1 + 2\sqrt{2})^2 \cdot (2^{-\lfloor n(1-1/p)/H \rfloor})^{-2(2H/(1-2/p)+1)/p} \end{aligned}$$

many $S \in \mathcal{S}_n$ that intersect $U(S)$, and for each $S \in \mathcal{S}_n$ there are at most $p + 2$ words $w \in \{0, 1\}^{\lfloor n(1-1/p)/H \rfloor}$ such that $F(t_w) \in S$. Together with (19), this implies that for n large enough,

$$\sup_{S \in \mathcal{S}_n} \tilde{N}(S) \leq C' \cdot (2^{-\lfloor n(1-1/p)/H \rfloor})^{-2(2H/(1-2/p)+1)/p}, \quad (20)$$

where $C' = (p + 2) \cdot \pi \cdot (1 + 2\sqrt{2})^2$.

Lower bound of $\dim_H F(E)$

For any set $E \subset [0, 1]$, let \mathcal{C}_N be any dyadic square covering of $F(E)$ such that

$$\sum_{S \in \mathcal{C}_N} |S|^{\dim_H F(E)+1/p} \leq 2^{-N}.$$

Denote by $n(S) = \lfloor -(\log_2 |S|) \cdot (1 - 1/p)/H \rfloor$. For N large enough one has that

$$\bigcup_{S \in \mathcal{C}_N} \{I_w: w \in \{0, 1\}^{n(S)}, F(I_w) \cap S \neq \emptyset\}$$

forms a covering of E . Moreover, due to (20), for

$$s = (\dim_H F(E) + 1/p) \cdot \frac{H}{1 - 1/p} + 2 \left(\frac{2H}{1 - 2/p} + 1 \right) / p$$

one has

$$\sum_{S \in \mathcal{C}_N} \sum_{w \in \{0,1\}^{n(S)}, F(I_w) \cap S \neq \emptyset} |I_w|^s \leq C' \sum_{S \in \mathcal{C}_N} |S|^{\dim_H F(E) + 1/p} \leq C' \cdot 2^{-N},$$

which implies

$$\dim_H E \leq (\dim_H F(E) + 1/p) \cdot \frac{H}{1 - 1/p} + 2 \left(\frac{2H}{1 - 2/p} + 1 \right) / p.$$

Since this holds for all $p > 1/(2H - 1)$, we get $\dim_H E \leq H \dim_H F(E)$.

Upper bound of $\dim_H F(E)$

Now consider any dyadic interval covering \mathcal{I}_N of E such that

$$\sum_{I \in \mathcal{I}_N} |I|^{\dim_H E + 1/p} \leq 2^{-N}.$$

For N large enough one gets from (18) that

$$\sup_{s,t \in I_w} |F(s) - F(t)| = |I_w|^H \cdot X(w) \leq |I_w|^{H-1/p}$$

for any $I_w \in \mathcal{I}_N$, thus for each I one can use a square of side length $2|I|^{H-1/p}$ to cover $F(I)$. We have

$$\sum_{I \in \mathcal{I}_N} (2|I|^{H-1/p})^{(\dim_H E + 1/p)/(H-1/p)} = C'' \sum_{I \in \mathcal{I}_N} |I|^{\dim_H E + 1/p} \leq C'' 2^{-N},$$

where $C'' = 2^{(\dim_H E + 1/p)/(H-1/p)}$. This gives

$$\dim_H F(E) \leq \frac{\dim_H E + 1/p}{H - 1/p}.$$

Since this holds for all $p > 1/(2H - 1)$, we get $\dim_H E \geq H \dim_H F(E)$.

3.2. Proof of the fact that a uniform result cannot hold when $H_1 < H_2$

We shall use the following result.

Proposition 3.1. *For $j = 1, 2$ almost surely there exists a Borel set $R \subset F_j([0, 1])$ with positive Lebesgue measure such that for each $y \in R$,*

$$\dim_H L_j(y) = 1 - H_j,$$

where $L_j(y) = \{x \in [0, 1]: F_j(x) = y\}$ is the level set of F_j at level y .

From Proposition 3.1 one has that almost surely

$$\dim_H L_1(y) = 1 - H_1 > 0$$

for some $y \in F_1([0, 1])$. On the other hand, since F_2 is α -Hölder continuous for all $\alpha \in (0, H_2)$, and $F(L_1(y)) = \{y\} \times F_2(L_1(y))$, so

$$\dim_H F(L_1(y)) = \dim_H F_2(L_1(y)) \leq \frac{\dim_H L_1(y)}{H_2} < \frac{\dim_H L_1(y)}{H_1}.$$

This shows that the relation in (4) cannot hold almost surely for all Borel sets.

Proof of Proposition 3.1. It is enough to prove the result for F_1 . We will use the same method as used for constructing the local time of fractional Brownian motion to compute the Hausdorff dimension of its level sets, see [13] for example.

Let ν be the occupation measure of F_1 with respect to the Lebesgue measure on $[0, 1]$, that is the Borel measure defined as

$$\nu(B) = \int_0^1 \mathbf{1}_{\{F_1(t) \in B\}} dt \quad \text{for any Borel set } B \subset \mathbb{R}.$$

First we show that almost surely ν is absolutely continuous with respect to the Lebesgue measure. We consider the Fourier transform of ν :

$$\hat{\nu}(u) = \int_0^1 e^{iuF_1(t)} dt.$$

We will show that

$$\mathbb{E} \left(\int_{\mathbb{R}} |\hat{\nu}(u)|^2 du \right) < \infty.$$

This will imply that almost surely $\hat{\nu}$ is in $L^2(\mathbb{R})$. Therefore almost surely ν is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} and its density function belongs to $L^2(\mathbb{R})$.

By using Fubini's theorem one has

$$\mathbb{E} \left(\int_{\mathbb{R}} |\hat{\nu}(u)|^2 du \right) = \mathbb{E} \left(\iint_{s,t \in [0,1]} \int_{\mathbb{R}} e^{iu \cdot (F_1(t) - F_1(s))} du ds dt \right).$$

Fix $0 \leq s < t \leq 1$. Let $N \geq 1$ be the smallest integer such that there exists a dyadic word $w = w_1 \cdots w_N$ such that $I_w \subset (s, t)$, thus

$$2^{-N} \leq |t - s| \leq 4 \cdot 2^{-N}.$$

From (8) we may write

$$F_1(t) - F_1(s) = A_1(w) \cdot Z_1(w) + B_1(w), \tag{21}$$

where $A_1(w) = 2^{-NH_1} \bar{\varepsilon}_1(w)$ and $B_1(w) = F_1(t) - F_1(t_w + 2^{-N}) + F_1(t_w) - F_1(s)$. Recall that $Z_1(w)$ is independent of $A_1(w)$ and $B_1(w)$.

Recall (13) and $\varphi_1(u) = \mathbb{E}(e^{iuZ_1})$ the characteristic function of Z_1 . One has

$$\mathbb{E} \left(\int_{\mathbb{R}} e^{iu \cdot (F_1(t) - F_1(s))} du \middle| \mathcal{F}(w) \right) = \int_{\mathbb{R}} e^{iuB_1(w)} \cdot \varphi_1(A_1(w) \cdot u) du.$$

Thus

$$\begin{aligned} \left| \mathbb{E} \left(\int_{\mathbb{R}} e^{iu \cdot (F_1(t) - F_1(s))} du \right) \right| &\leq \mathbb{E} \left(\int_{\mathbb{R}} |\varphi_1(A_1(w) \cdot u)| du \right) \\ &= \mathbb{E}(|A_1(w)|^{-1}) \cdot \int_{\mathbb{R}} |\varphi_1(u)| du \end{aligned} \tag{22}$$

$$\begin{aligned} &= 2^{NH_1} \cdot \|\varphi_1\|_1 \\ &\leq 4^{H_1} \cdot |s - t|^{-H_1} \cdot \|\varphi_1\|_1. \end{aligned}$$

From Lemma 2.1 we get $\|\varphi_1\|_1 < \infty$, thus

$$\mathbb{E}\left(\int_{\mathbb{R}} |\hat{v}(u)|^2 du\right) \leq 4^{H_1} \cdot \|\varphi_1\|_1 \cdot \iint_{s,t \in [0,1]} |s - t|^{-H_1} ds dt < \infty.$$

We have proved that almost surely ν is absolutely continuous with respect to the Lebesgue measure. This implies that almost surely for ν -almost every $y \in F_1([0, 1])$ the following limit

$$\lim_{r \rightarrow 0} \frac{1}{r} \int_0^1 \mathbf{1}_{\{|F_1(t)-y| \leq r\}} dt$$

exists and belongs to $(0, \infty)$, thus yielding a positive finite Borel measure ν_y carried by $L_1(y) = \{t \in [0, 1] : F_1(t) = y\}$, defined as

$$\int_0^1 g(t) d\nu_y(t) = \lim_{r \rightarrow 0+} \frac{1}{r} \int_0^1 \mathbf{1}_{\{|F_1(t)-y| \leq r\}} g(t) dt, \quad \forall g \in \mathcal{C}([0, 1]).$$

Moreover, for any Borel measurable function $G : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}_+$ one has

$$\int_{y \in F_1([0,1])} \int_{[0,1]} G(t, y) d\nu_y(t) dy = \int_0^1 G(t, F_1(t)) dt.$$

Let $\gamma > 0$. By Fatou's lemma and Fubini's theorem we have

$$\begin{aligned} &\int_{y \in F_1([0,1])} \int_0^1 \int_0^1 |s - t|^{-\gamma} d\nu_y(s) d\nu_y(t) dy \\ &= \int_{y \in F_1([0,1])} \int_0^1 \left[\lim_{r \rightarrow 0} \frac{1}{r} \int_0^1 \mathbf{1}_{\{|F_1(s)-y| \leq r\}} |s - t|^{-\gamma} ds \right] d\nu_y(t) dy \\ &\leq \liminf_{r \rightarrow 0} \frac{1}{r} \int_0^1 \int_{y \in F_1([0,1])} \int_0^1 \mathbf{1}_{\{|F_1(s)-y| \leq r\}} |s - t|^{-\gamma} d\nu_y(t) dy ds \\ &= \liminf_{r \rightarrow 0} \frac{1}{r} \int_0^1 \int_0^1 \mathbf{1}_{\{|F_1(s)-F_1(t)| \leq r\}} |s - t|^{-\gamma} dt ds. \end{aligned} \tag{23}$$

Fix $0 \leq s < t \leq 1$. Recall (21) and the fact that $Z_1(w)$ has a bounded density function f_1 with $\|f_1\|_\infty \leq \|\varphi_1\|_1$, so

$$\begin{aligned} \mathbb{E}(\mathbf{1}_{\{|F_1(s)-F_1(t)| \leq r\}} | \mathcal{F}(w)) &= \int_{\mathbb{R}} \mathbf{1}_{\{|x+B_1(w)/A_1(w)| \leq r/|A_1(w)|\}} f_1(x) dx \\ &= \int_{\mathbb{R}} \mathbf{1}_{\{|z| \leq r/|A_1(w)|\}} f_1\left(z - \frac{B_1(w)}{A_1(w)}\right) dz \\ &\leq \|\varphi_1\|_1 \cdot \frac{2r}{|A_1(w)|} \\ &= 2\|\varphi_1\|_1 \cdot r \cdot 2^{NH_1}. \end{aligned} \tag{24}$$

Again using Fatou's lemma and Fubini's theorem we get from (23) and (24) that

$$\mathbb{E}\left(\int_{y \in F_1([0,1])} \int_0^1 \int_0^1 |s - t|^{-\gamma} d\nu_y(s) d\nu_y(t) dy\right) \leq C \int_0^1 \int_0^1 |s - t|^{-(\gamma+H_1)} ds dt,$$

where $C = 2\|\varphi_1\|_1 4^{H_1}$. Due to the mass distribution principle we get that for any $\gamma < 1 - H_1$, almost surely for ν -almost every $y \in F_1([0, 1])$,

$$\dim_H L_1(y) \geq \gamma.$$

This gives us the desired lower bound.

For the upper bound, we use the fact that almost surely the Hausdorff dimension of the graph of F_1 , defined as $\{(t, F_1(t)) : t \in [0, 1]\}$, is equal to $2 - H_1$ (Theorem 1.1 in [3]). Then from Corollary 7.12 in [8] we know that there cannot exist a subset $R \subset F_1([0, 1])$ with positive Lebesgue measure such that for every $y \in R$, $\dim_H L_1(y) > 1 - H_1$. \square

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