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Sharp asymptotics of metastable transition times for one dimensional SPDEs[1]

Florent Barret[a,b]

[a] *CMAP UMR 7641, École Polytechnique CNRS, Route de Saclay, 91128 Palaiseau Cedex, France*
[b] *Current address: Max-Planck-Institute für Mathematik in den Naturwissenschaften, Inselstr. 22, 04103 Leipzig, Germany.*
E-mail: barret@mis.mpg.de

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Abstract. We consider a class of parabolic semi-linear stochastic partial differential equations driven by space–time white noise on a compact space interval. Our aim is to obtain precise asymptotics of the transition times between metastable states. A version of the so-called Eyring–Kramers formula is proven in an infinite dimensional setting. The proof is based on a spatial finite difference discretization of the stochastic partial differential equation. The expected transition time is computed for the finite dimensional approximation and controlled uniformly in the dimension.

Résumé. Nous nous intéressons à une famille d'équations aux dérivées partielles stochastiques paraboliques et semi-linéaires, perturbées par un bruit blanc en espace-temps, définies sur un intervalle réel compact. Nous cherchons à calculer les asymptotiques précises des espérances des temps de transitions entre les états métastables. Nous démontrons dans ce cadre une version en dimension infinie de la formule dite d'Eyring–Kramers. La preuve repose sur l'approximation par un schéma aux différences finies de l'équation aux dérivées partielles stochastique. L'espérance du temps de transition est calculée pour l'approximation puis contrôlée uniformément quelque soit la dimension.

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1. Introduction

Metastability is a phenomenon which concerns systems with several stable states. Due to perturbations (either deterministic or stochastic) the system undergoes a shift of regime and reaches a new stable state (see, e.g., [16] by Cassandro, Galves, Olivieri and Vares, [27] by Galves, Olivieri and Vares, the book [37] by Olivieri and Vares and the lecture notes [7] by Bovier). Typical examples of metastable behavior can be found in chemistry, physics (for models of phase transition) and ecology.

In this article, our aim is to understand metastability for a class of stochastic partial differential equations. We consider the Allen–Cahn (or Ginzburg–Landau) model which represents the behavior of an elastic string in a viscous stochastic environment submitted to a potential (see, e.g., Funaki [26]). This model has other interpretations in quantum field theory (see [17,22] and the references therein) and in statistical mechanics as a reaction diffusion equation modeling phase transitions and evolution of interfaces (see Brassesco and Buttà [13,14]).

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More precisely, we deal with the following equation, for $(x,t) \in [0,1] \times \mathbb{R}^+$

$$\partial_t u(x,t) = \gamma \partial_{xx} u(x,t) - V'\big(u(x,t)\big) + \sqrt{2\varepsilon} W, \tag{1.1}$$

where $\gamma > 0$. W is a space–time white noise on $[0,1] \times \mathbb{R}^+$ in the sense of Walsh [39] and $\varepsilon > 0$ is the intensity of the noise. V is a smooth real valued function on $\mathbb{R}$ called a local potential. We consider two boundary conditions: Dirichlet boundary conditions (for all $t \in \mathbb{R}^+$, $u(0,t) = u(1,t) = 0$) and Neumann boundary conditions ($\partial_x u(0,t) = \partial_x u(1,t) = 0$). The initial condition is given by a continuous function u_0 which satisfies the given boundary conditions. Existence and uniqueness of a Hölder-continuous solution in the mild sense have been proved by Gyöngy and Pardoux in [30].

Faris and Jona-Lasinio in [22] were among the first ones to analyze Eq. (1.1) for a double well potential

$$V(x) = \frac{x^4}{4} - \frac{x^2}{2}. \tag{1.2}$$

In this case, V has only two minima which are $+1$ and -1. One expects that the model (1.1) has several stable states and that a metastable behavior occurs. The authors introduced a functional potential S and interpreted (1.1) as the stochastic perturbation of an infinite dimensional gradient system:

$$\partial_t u = -\frac{\delta S}{\delta \phi} + \sqrt{2\varepsilon} W, \tag{1.3}$$

where for ϕ a differentiable function,

$$S(\phi) = \int_0^1 \frac{\gamma}{2} \big|\phi'(x)\big|^2 + V\big(\phi(x)\big)\,\mathrm{d}x. \tag{1.4}$$

S represents the free energy. $\frac{\delta S}{\delta \phi}$ is the Fréchet derivative of S, i.e., the infinite dimensional gradient of S.

For more general functions V (real valued C^3 functions), we can define a similar potential S as in (1.4) which determines a potential landscape. Under the stochastic perturbation, this potential landscape is explored by the process u defined in (1.1). While the system without noise (i.e., $\varepsilon = 0$) has several stable fixed points (which are the minima of S), for $\varepsilon > 0$ transitions between these fixed points will occur at a suitable timescale. The transition paths go through the lowest saddle points. Thus, minima and saddle points of S have a key role to understand metastability but it is often a hard task, given a potential V (and thus S), to completely compute and comprehend the geometrical structure of the energy landscape. However, some elegant methods exist (see, e.g., [23,40]).

The model (1.3) is an infinite dimensional generalization of the finite dimensional systems investigated by Freidlin and Wentzell [25] and by Bovier, Eckhoff, Gayrard and Klein in [10,11]. Moreover, we will see that (1.1) is rigorously the limit of a gradient finite dimensional system (via a spatial finite difference approximation).

Our aim is to derive precise asymptotics of the expected transition time, i.e., the time needed, starting from a minimum ϕ_0 of S, to hit a set of lower minima. We define the hitting time $\tau_\varepsilon(B)$ by $\tau_\varepsilon(B) = \inf\{t > 0, u(t) \in B\}$ where B is a disjoint union of small ball around some minima of S lower than ϕ_0. We prove that the expected time, $\mathbb{E}_{\phi_0}[\tau_\varepsilon(B)]$, has a very distinctive form known as the Arrhenius equation (Theorem 2.6). This expectation reads

$$\mathbb{E}_{\phi_0}\big[\tau_\varepsilon(B)\big] = A\mathrm{e}^{E/\varepsilon}\big(1 + \mathrm{O}\big(\sqrt{\varepsilon}\big|\ln(\varepsilon)\big|^{3/2}\big)\big) \quad (\varepsilon \to 0), \tag{1.5}$$

where E is the activation energy and A is the prefactor. E was computed by Faris and Jona-Lasinio for the double well potential (1.2) using a large deviation approach (Theorem 1.1 [22]). E is exactly the minimum height of potential that a pathway has to overcome to reach B starting from ϕ_0. The prefactor A is a constant (for our set of hypotheses) and depends only on the local geometry of the potential S near the minimum ϕ_0 and near the passes (or saddle points) from ϕ_0 to the set B. The order $\mathrm{O}(\sqrt{\varepsilon}|\ln(\varepsilon)|^{3/2})$ of the error term comes directly from the local approximation of the potential S by its quadratic part.

For the double well potential (1.2) with Dirichlet boundary conditions, Faris and Jona-Lasinio proved that S has only two global minima, denoted m and $-m$ (which away from the boundary, correspond roughly to the constant functions 1 and -1 resp.). For some γ, this model has a unique saddle point $\sigma = 0$ (the constant function 0). We deduce from Theorem 2.6 that $\mathbb{E}_{-m}[\tau_\varepsilon(B^+)]$, for a small ball B^+ in the suitable norm around m, takes the form (1.5) with $E = S(\sigma) - S(-m)$ and

$$A = \frac{2\pi}{|\lambda^-(\sigma)|}\sqrt{\prod_{k=1}^{+\infty}\frac{|\lambda_k(\sigma)|}{|\lambda_k(-m)|}}, \tag{1.6}$$

where $(\lambda_k(\phi))_{k\geq 1}$ are the eigenvalues of the second Fréchet derivative of the potential S at a point ϕ and $\lambda^-(\sigma)$ is the unique negative eigenvalue at the saddle point σ. Using asymptotic expansion of the eigenvalues, we prove that the infinite product converges. This quantity is similar to the ratio of the determinants of the Hessian matrices obtained in the finite dimensional case (see [10]). We also mention the fact that this infinite product has a nice expression in terms of solutions of linear differential equations (see, e.g., Levit and Smilansky [33]).

Kramers in [32] investigated the case of a one dimensional diffusion as a model for chemical reactions and expresses rates instead of expectations. Previous computations leading to similar rates were made by Eyring in [21]. Their formula is known as the Eyring–Kramers formula. It takes the form (1.5) with the prefactor given by a formula similar to (1.6) but with a single factor in the product (there is only one eigenvalue).

Similar Eyring–Kramers formulas exist through a wide range of reversible Markovian models from Markov chains, stochastic differential equations. For finite dimensional diffusions, Freidlin and Wentzell in [25], proving that these systems obey a large deviation principle, obtained the activation energy in terms of the rate function. In recent years, the potential theory approach initiated by Bovier, Eckhoff, Gayrard and Klein in [10,11] allowed to give very precise results and led to a proof of the Eyring–Kramers formula for gradient drift diffusions in finite dimension. Moreover, the potential approach was originated from Markov chains (see [7–9]) and have been refined to obtain metastable transition times for specific models (see, e.g., [6,12]).

Formula (1.6) is then the extension of the Eyring–Kramers formula to a class of one-dimensional SPDEs (1.1). Maier and Stein in [34] obtained heuristically this formula and Vanden-Eijnden and Westdickenberg in [38] conducted similar computations.

Specifically, the system (1.1) and its metastable behavior have been studied for at least thirty years using mainly large deviation principle and comparison estimates between the deterministic process ((1.1) with $\varepsilon = 0$) and the stochastic process defined by (1.1). Cassandro, Olivieri, Picco [17] obtained asymptotics similar to those obtained by Faris and Jona-Lasinio [22] when the size of the space interval is not fixed and goes to infinity as ε goes to 0 sufficiently slowly. These results first prove the existence of a suitable exponential timescale in which the process undergoes a transition.

In the same case as (1.2), Martinelli, Olivieri and Scoppola [35] obtained the asymptotic exponentiality of the transition times (Theorem 4.1 [35]). Also, Brassesco [13] proved that the trajectories of this system exhibit characteristics of a metastable behavior: the escape from the basin of attraction of the minimum $-m$ occurs through the lowest saddle points (Theorem 2.1 [13]) and the process starting from $-m$ spends most of its time before the transition near $-m$ (Theorem 2.2 [13]).

In this paper, we consider a local potential V (satisfying Assumptions 2.1 and 2.4) and we rigorously prove an infinite dimensional version of the Eyring–Kramers formula. Our method relies on a spatial finite difference approximation of Eq. (1.1) introduced by Berglund, Fernandez and Gentz in [4,5] as a model of coupled particles submitted to a potential. The computation of the expected transition time for the approximated system gives us the prefactor, the activation energy and some error terms. We need to control the behavior of these error terms as the step of discretization goes to 0 (or equivalently as the dimension N of the approximated system goes to $+\infty$). To this aim, we adapt results from [3] by Bovier, Méléard and the author.

As proved by Funaki [26] and Gyöngy [29], the solution of the approximated system converges to the solution of the SPDE. By combining different results from SPDE theory, large deviation theory (from Chenal and Millet [18]) and Sturm–Liouville theory we are able to take the limit of the finite dimensional model in order to retrieve the SPDE (1.1). We also need to adapt estimates on the loss of the memory of the initial condition (from Martinelli, Scoppola and Sbano [35,36]) uniformly in the dimension.

The use of spatial finite difference approximation is quite natural since we consider our SPDEs in the sense of Walsh [39], limited to the case of space–time white noise. Other approximations could be possible, notably the Galerkin approximation should lead to similar results for a different class of SPDEs in the framework of Da Prato and Zabczyk (see the book [20]).

The article is organized as follows. In Section 2, we present the equation, the assumptions, the main theorem (Theorem 2.6) and a sketch of its proof. Then in Section 3, we adapt the convergence of the approximations and prove convergence of the approximated transition times. In Section 4, we state large deviations estimates by Chenal and Millet [18], contraction results by Martinelli, Olivieri, Scoppola and Sbano [35,36] and prove a uniform control in the initial condition uniformly in the dimension. In Section 5, we recall results about eigenvalues and eigenvectors of Sturm–Liouville problems and prove the convergence of the prefactor. In the last section, we compute the expected transition times uniformly in the dimension.

We will use the following notations henceforth. For a functional space $\mathcal{C}$, equipped with a norm $\|\cdot\|_{\mathcal{C}}$, we denote by $\mathcal{C}_{bc}$ the closed subspace in the $\mathcal{C}$ topology of the functions in $\mathcal{C}$ satisfying the suitable boundary conditions (Dirichlet or Neumann). For $f \in L^\infty([0,1]\times[0,T])$ we set the norm of this space $\|f\|_{\infty,T}$ or simply $\|f\|_\infty$ when $T=+\infty$.

2. Results

2.1. *The equation*

The assumptions are of two kinds: some on the local potential V, others on the functional potential S. We first start with the hypotheses on V.

Assumption 2.1. *We suppose that*:

- *V is C^3 on $\mathbb{R}$.*
- *V is convex at infinity: there exist $R, c > 0$ such that for $|u| > R$*

$$V''(u) > c > 0. \tag{2.1}$$

- *V grows at infinity at most polynomially: there exist $p, C > 0$ such that*

$$V(u) < C\bigl(1+|u|^p\bigr). \tag{2.2}$$

These hypotheses are made to avoid complications for the definition of the solution u of (1.1) and to allow the computations of the derivatives of S. Note that Eq. (2.1) implies in particular that the drift $-V'$ satisfies a one sided linear growth condition: there exists $C > 0$ such that for all $u \in \mathbb{R}$

$$-uV'(u) < C\bigl(1+u^2\bigr). \tag{2.3}$$

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space on which we define a space–time white noise W as defined in [39] equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ with the usual properties. The integrable processes for the white noise are the predictable measurable processes in $L^2(\Omega \times \mathbb{R}_+ \times [0,1])$. We denote by $g_t(x,y)$ the density of the semi-group generated by $\gamma\partial_{xx}$ on $[0,1]$ with the suitable boundary conditions.

Let us recall that a random field u is a mild solution of (1.1) if

(1) u is almost surely continuous on $[0,1]\times\mathbb{R}^+$ and predictable
(2) for all $(x,t) \in [0,1]\times\mathbb{R}^+$

$$\begin{aligned} u(x,t) = {} & \int_0^1 g_t(x,y)u_0(y)\,\mathrm{d}y - \int_0^t\int_0^1 g_{t-s}(x,y)V'\bigl(u(y,s)\bigr)\,\mathrm{d}y\,\mathrm{d}s \\ & + \sqrt{2\varepsilon}\int_0^t\int_0^1 g_{t-s}(x,y)W(\mathrm{d}y,\mathrm{d}s). \end{aligned} \tag{2.4}$$

We state from [30] the following result on the existence, uniqueness and regularity of the solution (valid under the one sided linear growth condition (2.3)) for Dirichlet and Neumann boundary conditions.

Proposition 2.2 ([30]). *For every initial condition $u_0 \in C_{bc}([0,1])$, the stochastic partial differential Eq. (1.1) has a unique mild solution. Moreover for all $T > 0$ and $p \geq 1$,*

$$\mathbb{E}\Big[\sup_{[0,T]\times[0,1]} \big|u(x,t)\big|^p\Big] \leq C(T,p). \tag{2.5}$$

The random field u is essentially $\frac{1}{2}$-Hölder in space and $\frac{1}{4}$-Hölder in time.

The proof of Proposition 2.2 is standard and uses mainly estimates on the density $g_t(x,y)$. The estimate (2.5) is straightforward if the drift is globally Lipschitz and bounded. For our case, it holds via a localization argument and the use of a comparison theorem from [30]. Details can be found in the Ph.D. thesis of the author [2] (Lemma A.3.9).

Remark 1. *The definition of the stochastic convolution (the last expression of the right-hand side of (2.4)) requires the density of the semi-group to be in $L^2([0,1]\times[0,T])$ for every $T > 0$. Unfortunately, that is only true in dimension one. For higher dimensions, the stochastic convolution does not define a classical function but a distribution in a Sobolev space of negative index [39].*

2.2. *Stationary points*

As for the finite dimensional case, the minima and saddle points of S play a crucial role. To this end, we first specify what is the "gradient" (or the Fréchet derivative) of the functional S. Let us recall that S is defined, for $\phi \in H^1_{bc}$, by

$$S(\phi) = \int_0^1 \frac{\gamma}{2}\big|\phi'(x)\big|^2 + V\big(\phi(x)\big)\,\mathrm{d}x. \tag{2.6}$$

For ϕ, h in $C^2_{bc}([0,1])$ we have a Taylor expansion of S at the second order in h

$$S(\phi + h) = S(\phi) + D_\phi S(h) + \frac{1}{2}D^2_\phi S(h,h) + \mathrm{O}\big(\|h\|^2_{C^2}\big), \tag{2.7}$$

where $\|h\|_{C^2} = \|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty$. By integration by parts we compute the differentials $D_\phi S$ and $D^2_\phi S$. The first order differential is a linear functional which takes the form

$$D_\phi S(h) = \int_0^1 \big[-\gamma\phi''(x) + V'\big(\phi(x)\big)\big]h(x)\,\mathrm{d}x. \tag{2.8}$$

The Fréchet derivative is $\frac{\delta S}{\delta\phi} = -\gamma\phi''(x) + V'(\phi(x))$. The second order derivative (the Hessian operator) takes the form

$$D^2_\phi S(h,h) = \int_0^1 h(x)\big[-\gamma h''(x) + V''\big(\phi(x)\big)h(x)\big]\,\mathrm{d}x. \tag{2.9}$$

We denote by $\mathcal{H}_\phi S$ the Hessian operator at ϕ:

$$\mathcal{H}_\phi S h(x) = -\gamma h''(x) + V''\big(\phi(x)\big)h(x). \tag{2.10}$$

The Hessian operator is a Sturm–Liouville operator.

We say that ϕ is a *stationary point* of S if ϕ is solution of the non-linear differential equation

$$\frac{\delta S}{\delta\phi} = -\gamma\phi'' + V'(\phi) = 0. \tag{2.11}$$

Let us now fix two points $\phi, \psi \in C_{bc}([0,1])$ and define some quantities.

$$\Gamma(\phi \to \psi) = \{f, f(0) = \phi, f(1) = \psi, f \in C([0,1], C_{bc}([0,1]))\} \tag{2.12}$$

is the set of continuous paths from ϕ to ψ. For $f \in \Gamma(\phi \to \psi)$, $\widehat{f}$ denotes the set of maxima of the path f,

$$\widehat{f} = \Big\{ f(t_0), t_0 \in \operatorname*{arg\,max}_{t\in[0,1]} S(f(t)) \Big\}. \tag{2.13}$$

The saddle points are passes from a valley to another one. The definition uses this idea.

Definition 2.3 (Saddles). *For any $\phi, \psi \in C_{bc}([0,1])$, we define $\widehat{S}(\phi,\psi)$, the minimum height needed to go from ϕ to ψ*

$$\widehat{S}(\phi,\psi) = \widehat{S}(\psi,\phi) = \inf\{S(x), x \in \widehat{f}, f \in \Gamma(\phi \to \psi)\}. \tag{2.14}$$

For a finite subset $\mathcal{A}$ of $C_{bc}([0,1])$, we also define $\widehat{S}(\phi,\mathcal{A}) = \min_{\psi\in A} \widehat{S}(\phi,\psi)$. It is the minimum height attained to reach a point in $\mathcal{A}$ from ϕ.

For ϕ, ψ such that $\widehat{S}(\phi,\psi) < \infty$, we denote $\mathscr{S}(\phi,\psi)$ the set of admissible saddles: the points which realize the maximum along a minimal pathway

$$\mathscr{S}(\phi,\psi) = \{\sigma \in C_{bc}([0,1]), S(\sigma) = \widehat{S}(\phi,\psi), \exists f \in \Gamma(\phi \to \psi), \sigma \in \widehat{f}\}. \tag{2.15}$$

Accordingly, for a finite subset $\mathcal{A}$ of $C_{bc}([0,1])$, we define $\mathscr{S}(\phi,\mathcal{A})$

$$\mathscr{S}(\phi,\mathcal{A}) = \{\sigma \in C_{bc}([0,1]), S(\sigma) = \widehat{S}(\phi,\mathcal{A}), \exists \psi \in \mathcal{A}, \exists f \in \Gamma(\phi \to \psi), \sigma \in \widehat{f}\}. \tag{2.16}$$

The set of admissible saddle points is very important to compute the prefactor of the mean transition times. Near these points the process spends the most crucial time as it passes from a basin of attraction to another one.

We now present the assumptions on S.

Assumption 2.4. *We suppose that*:

- *S has a finite number of minima and saddle points.*
- *All the minima and saddle points of S are non-degenerate* (*i.e.*, *hyperbolic*): *at each point*, *the Hessian operator has non-zero eigenvalues.*

Assumption 2.4 is structural. The finite number of stationary points provides a simple generalization of the case where there is only one saddle point. It also implies that the stationary points are isolated, e.g., we do not consider the case of periodic stable orbits. The non-degeneracy condition is necessary in order to approximate locally at the minima and saddle points the potential by its quadratic part. If this is not the case the prefactor in (1.5) is not a constant but should have a dependence in ϵ.

Connections between Assumptions 2.1 and 2.4 are not straightforward. Proving that a given potential S satisfies Assumption 2.4 is not easy, a precise analysis is often needed. Moreover if we want to investigate the dependence of the potential S on the parameter γ, bifurcations can occur and the landscape do not satisfy Assumption 2.4 for some critical values of γ. See Berglund, Fernandez and Gentz [4,5] for the finite and infinite dimensional cases for the double well potential. However, results exist (see [1] and references therein) on the generality of Assumption 2.4.

In addition, under Assumptions 2.4 and 2.1, the deterministic dynamical system (i.e., (1.1) without the white noise) satisfies a Morse–Smale structure (see [15,23] and the references therein). This means that the attractor of the dynamical system consists of equilibria and heteroclinic orbits connecting these equilibria. Methods were developed by Fiedler and Rocha in [23], by Wolfrum in [40] to compute the global attractor of the deterministic system.

Remark 2. *H^1 is the convenient functional space for the process since $S(\phi) < +\infty$ if and only if ϕ is in $H^1([0,1])$. In fact from the upper bound* (2.2) *and lower bound* (2.1) *on V we get*

$$C_1(\|\phi\|_{H^1}^2 - 1) \le S(\phi) \le C_1'(\|\phi\|_{H^1}^2 + \|\phi\|_{H^1}^p + 1). \tag{2.17}$$

Each function in $H^1([0,1])$ *is continuous and even* α*-Hölder continuous* (*for* $0 < \alpha < \frac{1}{2}$).

For each $\phi \in C([0,1])$, we define the quantity $\mathrm{Det}(\mathcal{H}_\phi S)$:

- for Dirichlet boundary conditions, let f be the solution on $[0,1]$ of

$$\mathcal{H}_\phi S f = 0, \qquad f(0) = 1, \qquad f'(0) = 0, \tag{2.18}$$

then $\mathrm{Det}(\mathcal{H}_\phi S) = f(1)$,
- for Neumann boundary conditions, let f be the solution on $[0,1]$ of

$$\mathcal{H}_\phi S f = 0, \qquad f(0) = 0, \qquad f'(0) = 1, \tag{2.19}$$

then $\mathrm{Det}(\mathcal{H}_\phi S) = f'(1)$.

Let us recall that, as a regular Sturm–Liouville operator, $\mathcal{H}_\phi S$ has a countable number of eigenvalues, all of them real. We denote by $(\lambda_k(\phi))_{k\geq 1}$ the sequence of these eigenvalues in the increasing order. The definition of $\mathrm{Det}(\mathcal{H}_\phi S)$ is justified by the following lemma.

Lemma 2.5 ([33]). *For any* ϕ *and* ψ *with non-degenerate Hessian operator, the infinite product* $\prod_{k=1}^{\infty} \frac{\lambda_k(\phi)}{\lambda_k(\psi)}$ *is convergent and we have*

$$\prod_{k=1}^{\infty} \frac{\lambda_k(\phi)}{\lambda_k(\psi)} = \frac{\mathrm{Det}(\mathcal{H}_\phi S)}{\mathrm{Det}(\mathcal{H}_\psi S)}. \tag{2.20}$$

This lemma relates the infinite product of the ratio of eigenvalues to a ratio of terminal values of solutions. We find an elementary proof in [33] by Levit and Smilansky which relies on two different expressions of the Green function associated to the problem $\mathcal{H}_\phi S f = 0$ satisfying the boundary conditions. In fact, the Green function could either be expressed using the spectral decomposition of $\mathcal{H}_\phi S$ or expressed as a linear combination of two well-chosen fundamental solutions (of the second order linear differential equation).

2.3. *Main results*

Before stating the main result, we describe the set of minima and saddle points. In fact, the prefactor depends greatly on the geometry of a graph connecting the minima to each other through the saddle points (so-called the 1-skeleton connection graph by Fiedler and Rocha in [24]). We define this graph and express the prefactor partly as an equivalent conductance on this graph.

We denote by $\mathcal{M}$ the set of minima of S. Since by Assumption 2.4, there is a finite number of stationary points, we order the minima by increasing energy. We denote by $\phi_1, \phi_2, \dots, \phi_m$, $m = |\mathcal{M}|$, the different minima indexed by increasing energy

$$S(\phi_1) \leq S(\phi_2) \leq \cdots \leq S(\phi_m). \tag{2.21}$$

We denote by $\mathcal{M}_l$, the subset of minima $\mathcal{M}_l = \{\phi_1, \phi_2, \dots, \phi_l\}$ for $1 \leq l \leq m$.

Recall that from the Freidlin–Wentzell theory (see [25], Theorem 3.1, Chapter 4 and Theorem 5.3, Chapter 6), since the systems considered are gradient with additive noise, the quasi-potential defined by Freidlin–Wentzell is the potential. The variation of the rate function takes a very simple form and depends on the variation of the potential S (and the noise parameter ε). This is only due to the simple form of the rate function for the large deviation of the infinite dimensional process (see Chenal and Millet [18], and Section 4).

We consider the transitions from a minimum ϕ_{l_0} to $\mathcal{M}_l$ for $l < l_0$. These are the only visible metastable transitions. We will see from large deviations estimates, that to go from a minimum ϕ to another ψ, it requires a time of order $\exp([\widehat{S}(\phi,\psi) - S(\phi)]/\varepsilon)$. The time required to make the reverse transition is also of order $\exp([\widehat{S}(\psi,\phi) - S(\psi)]/\varepsilon)$. Therefore if $S(\psi) > S(\phi)$, we get

$$\widehat{S}(\phi,\psi) - S(\phi) > \widehat{S}(\psi,\phi) - S(\psi) \tag{2.22}$$

and the time required to go from ϕ to ψ is much larger than for the reverse transition. So we cannot see the reverse transitions since there are absorbed by the direct ones. If some minima have the same potential, we can suitably order them to consider a transition from one minimum to another one at a same height.

Note that the simple rates we obtain come from two important considerations: the system we investigate is gradient (the rate function for the large deviation and then the quasi-potential is expressed as a variation of the potential S), and we consider only the "relevant" transitions that allow this simple expression of the transition rates (i.e., the ordering of the minima and the hitting set $\mathcal{M}_l$ containing any minima below a fixed level). This last ordering prevents any trajectories to get stuck at an intermediate minimum at a potential below our targets because we do not allow such traps. We stop the process when it reaches (the neighborhood of) a minimum below a defined potential. This could also be seen in Freidlin and Wentzell [25] when we apply Theorem 5.3 of Chapter 6 to our carefully chosen transitions.

Let us now construct the weighted graph of paths from ϕ_{l_0} to $\mathcal{M}_l$. We denote $\widehat{S} = \widehat{S}(\phi_{l_0}, \mathcal{M}_l)$ the common potential of the saddles (Definition 2.3). For two minima ϕ_i, ϕ_j in $\mathcal{M}$, we define the following equivalence relation: we say that ϕ_i and ϕ_j are equivalent if $\widehat{S} > \widehat{S}(\phi_i, \phi_j)$ (note that $\widehat{S}$ is symmetric by Definition 2.3 thus our relation is symmetric). ϕ_i and ϕ_j are equivalent if there is a pathway from one to the other which stays below the value $\widehat{S}$. Then the vertices of the graph are the equivalence classes (K_i) of $\mathcal{M}$. Note that, by definition, ϕ_{l_0} and $\mathcal{M}_l$ are in different equivalence classes.

The saddle points in $\mathscr{S}(\phi_{l_0}, \mathcal{M}_l)$ are the edges. We connect an edge $\widehat{\sigma}$ between two vertices K, J if the saddle $\widehat{\sigma}$ is a pass between the valleys of K and J: there exists $\phi \in K$, $\psi \in J$ and $f \in \Gamma(\phi \to \psi)$ such that $\widehat{f}$ has a unique element and $\widehat{f} = \widehat{\sigma}$. Note that by connecting the different paths, the definition of the edges does not depend on the specific minima considered in an equivalence class. Let us remark also that by definition, the class of ϕ_{l_0} is connected to at least one class containing a point of $\mathcal{M}_l$ but the graph can have disconnected components.

Existence of this graph is ensured by Assumption 2.4 (see [24] and references therein).

Each saddle point in $\mathscr{S}(\phi_{l_0}, \mathcal{M}_l)$ has a unique negative eigenvalue from the Morse–Smale property and the hyperbolicity of the stationary points. The weight associated to an edge $\widehat{\sigma}$ is defined as

$$w(\widehat{\sigma}) = \frac{|\lambda^-(\widehat{\sigma})|}{\sqrt{|\mathrm{Det}\,\mathcal{H}_{\widehat{\sigma}} S|}}, \tag{2.23}$$

where $\lambda^-(\widehat{\sigma})$ is the unique negative eigenvalue of $\mathcal{H}_{\widehat{\sigma}} S$.

$K^+(\widehat{\sigma})$ and $K^-(\widehat{\sigma})$ denote the two equivalence classes connected by a given edge $\widehat{\sigma}$ (which could be identical). For a real valued vector a indexed by the equivalence classes, we consider the following quadratic form

$$Q(a) = \sum_{\widehat{\sigma} \in \mathscr{S}(\phi_{l_0}, \mathcal{M}_l)} w(\widehat{\sigma}) \big(a\big(K^+(\widehat{\sigma})\big) - a\big(K^-(\widehat{\sigma})\big)\big)^2. \tag{2.24}$$

Let us distinguish K_0 the class containing ϕ_{l_0}. We define $\mathcal{C}^*(\phi_{l_0}, \mathcal{M}_l)$ the equivalent conductance of the graph between the classes containing ϕ_{l_0} and $\mathcal{M}_l$ as

$$\mathcal{C}^*(\phi_{l_0}, \mathcal{M}_l) = \inf\big\{Q(a), a(K_0) = 1, a(J) = 0, \text{ for all } J \text{ such that } J \cap \mathcal{M}_l \neq \varnothing\big\}. \tag{2.25}$$

Remark 3. *This conductance is an approximation of the capacity between neighborhoods of ϕ_{l_0} and $\mathcal{M}_l$. In some sense, we replace the continuous landscape defined by S by a graph containing the relevant geometric structure of the landscape. Note that the disconnected components from K_0 of the graph (if there exist) do not play any role in the Eq. (2.25) since we only impose the condition that a must be 0 on some of their vertices or no condition is imposed. On both cases, the minimum is 0 and attained for $a = 0$ on the disconnected components from K_0.*

Let us denote by $\mathcal{B}_\rho(\phi)$, for $\phi \in H^1_{bc}[0,1]$, the ball of center ϕ and radius ρ in H^1_{bc}

$$\mathcal{B}_\rho(\phi) = \{\sigma \in H^1_{bc}, \|\sigma - \phi\|_{L^2} \leq \rho, \|\sigma\|_{H^1} < A_1\}, \tag{2.26}$$

where A_1 is a sufficiently large constant. We also define $\mathcal{B}_\rho(\mathcal{M}_l) = \bigcup_{\phi \in \mathcal{M}_l} \mathcal{B}_\rho(\phi)$. We choose this kind of neighborhood because in the following we need to control the norm in the uniform norm and in the α-Hölder norm (for $\alpha < \frac{1}{2}$).

We now state our main result describing the dependence in ε of the mean of the hitting time of a union of balls around the points of $\mathcal{M}_l$ starting from ϕ_{l_0}.

Theorem 2.6. *Under the Assumptions 2.1, 2.4, for any minimum ϕ_{l_0}, and a set of minima $\mathcal{M}_l$ with $l_0 > l$, there exists ρ_0 such that for any $\rho_0 > \rho > 0$*

$$\mathbb{E}_{\phi_{l_0}}\left[\tau_\varepsilon\left(\mathcal{B}_\rho(\mathcal{M}_l)\right)\right] = \frac{2\pi \mathrm{e}^{(\widehat{S}(\phi_{l_0}, \mathcal{M}_l) - S(\phi_{l_0}))/\varepsilon}}{\mathcal{C}^*(\phi_{l_0}, \mathcal{M}_l)\sqrt{\operatorname{Det} \mathcal{H}_{\phi_{l_0}} S}}\left(1 + \Psi(\varepsilon)\right), \tag{2.27}$$

where the error term satisfies $\Psi(\varepsilon) = \mathrm{O}(\sqrt{\varepsilon}|\ln(\varepsilon)|^{3/2})$.

For the simple case where we have only three stationary points, two minima and one saddle we have the following corollary.

Corollary 2.7. *Let ϕ^+ and ϕ^- be the two minima with $S(\phi^-) \geq S(\phi^+)$ and $\widehat{\sigma}$ the unique saddle point. There exists ρ_0 such that for any $\rho_0 > \rho > 0$*

$$\mathbb{E}_{\phi^-}\left[\tau_\varepsilon\left(\mathcal{B}_\rho(\phi^+)\right)\right] = \frac{2\pi}{|\lambda^-(\widehat{\sigma})|}\sqrt{\frac{|\operatorname{Det} \mathcal{H}_{\widehat{\sigma}} S|}{\operatorname{Det} \mathcal{H}_{\phi^-} S}}\mathrm{e}^{(S(\widehat{\sigma}) - S(\phi^-))/\varepsilon}\left(1 + \Psi(\varepsilon)\right), \tag{2.28}$$

where the error term is $\Psi(\varepsilon) = \mathrm{O}(\sqrt{\varepsilon}|\ln(\varepsilon)|^{3/2})$.

Remark 4. *The double well potential (1.2) ($V(x) = \frac{x^4}{4} - \frac{x^2}{2}$) with Neumann boundary conditions and for $\gamma > 1/\pi^2$ satisfies the hypotheses of the Corollary 2.7. In fact, the stationary points are the functions ϕ defined on $[0,1]$ such that $\phi'(0) = \phi'(1) = 0$ and which are solutions of*

$$\gamma\phi''(x) - \phi^3(x) + \phi(x) = 0, \quad \forall x \in [0,1]. \tag{2.29}$$

Then, the three constant solutions ($\phi(x) = 1$, $\phi(x) = -1$ or $\phi(x) = 0$) are stationary points. There are the only stationary points. To prove this, let us define ϕ_α for $\alpha \in \mathbb{R}$, the unique solution of the differential Eq. (2.29) such that $\phi'_\alpha(0) = 0$ and $\phi_\alpha(0) = \alpha$. ϕ_α is a stationary point if and only if $\phi'_\alpha(1) = 0$. By symmetry, we can restrict our study to $\alpha > 0$. By multiplying (2.29) by ϕ'_α and by integration we obtain, for all $x \in [0,1]$

$$\frac{\gamma}{2}\left[\phi'_\alpha(x)\right]^2 - V\left(\phi_\alpha(x)\right) = V(\alpha). \tag{2.30}$$

Therefore, the trajectory $(\phi_\alpha(x), \phi'_\alpha(x))_{x\geq 0}$ describes the level set $\{(y,z), \frac{\gamma z^2}{2} - V(y) = V(\alpha)\}$ in $\mathbb{R}^2$. Using this, we obtain that for $\alpha > 1$, ϕ'_α is decreasing and thus $\phi'_\alpha(1)$ can not be zero. For $\alpha \in]0,1[$, $\phi'_\alpha(x) = 0$ has a unique positive minimal solution $x(\alpha)$. This first positive solution can be expressed as

$$x(\alpha) = \sqrt{2\gamma}\alpha \int_0^1 \frac{\mathrm{d}u}{\sqrt{V(\alpha u) - V(\alpha)}} \tag{2.31}$$

$$= 2\sqrt{2\gamma} \int_0^1 \frac{\mathrm{d}u}{\sqrt{(1-u^2)(2-\alpha^2(1+u^2))}}. \tag{2.32}$$

A simple computation shows that $x(\alpha)$ is increasing for $\alpha \in]0,1[$ and that $x(0^+) = \lim_{\alpha\to 0} x(\alpha) = \sqrt{\gamma}\pi$. We have $\phi'_\alpha(1) = 0$ if and only if there exists an integer $k > 0$ such that $kx(\alpha) = 1$. Therefore, if $x(\alpha) > 1$, ϕ_α is not a stationary point. At last, for $\gamma > 1/\pi^2$, since $x(\alpha)$ is increasing, we get that $x(\alpha) \geq x(0^+) > 1$ for all $\alpha \in]0,1[$, which proves that the only possible stationary points are the three constant solutions.

2.4. *Example*

Let us briefly present, for an illustration, the case of a 3-well potential V (see Fig. 1).

Computing the stationary points of S from V is not straightforward. However several articles give some method to obtain them at least geometrically. We refer the reader to [1,15,23,40].

For the fictional V given, for sufficiently small $\gamma > 0$, we can have the situation described by Fig. 2 for Neummann boundary conditions.

In the case of Fig. 2 the "complete" graph (where we draw all the saddles and minima and their heteroclinic

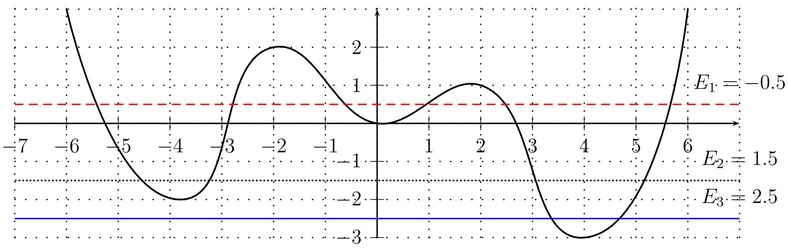


Fig. 1. 3 well potential V.

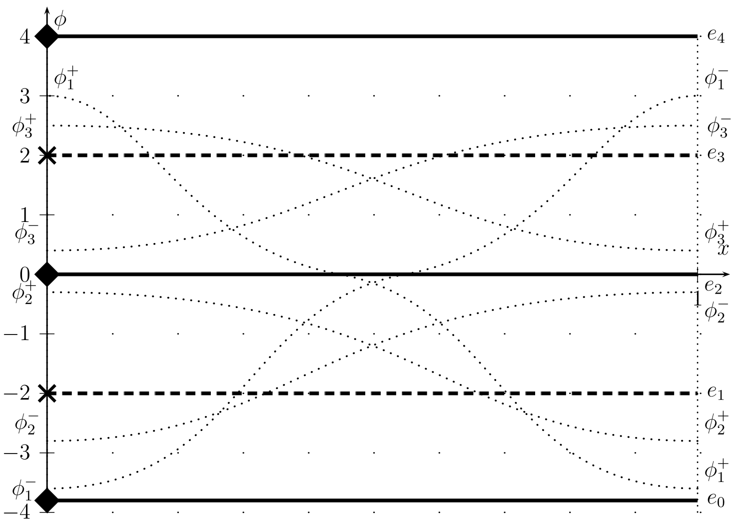
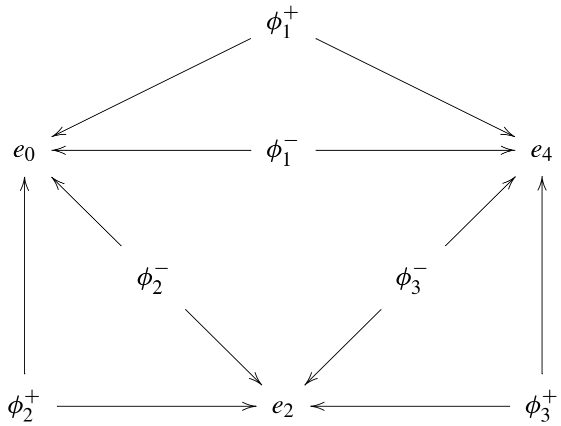


Fig. 2. Example of saddle points (dotted lines), and minima (solid lines), the dashed lines represent the others constant stationary points and are saddle points with 2 or more negative eigenvalues.

connections) is given by:

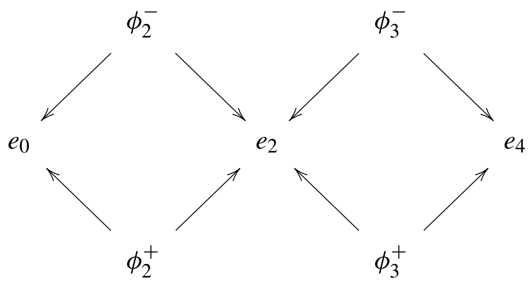


(2.33)

In order to apply the Eyring–Kramers formula given by Theorem 2.6, one has to determine which are the saddle points of minimum potential S, as well as the ordering of the minima. For the minima, from Fig. 1, we see that

$$S(e_2) = 0 > S(e_0) = -2 > S(e_4) = -3. \tag{2.34}$$

The value of the potential at the saddle could be more difficult to compute. Let us suppose that we want to compute the transition time from e_0 to (a small ball, defined by (2.26), around) e_4. We make the assumption that $S(\phi_1^+) > S(\phi_2^+) = S(\phi_3^+)$. Remark that due to the symmetry ($x \mapsto 1 - x$), we have $S(\phi_i^+) = S(\phi_i^-)$ for $i = 1, 2, 3$. In this case the graph constructed in order to apply Theorem 2.6 takes the form:



(2.35)

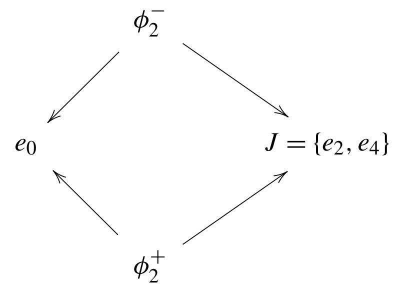
The capacity can be simply evaluated as the equivalent conductance between e_0 and e_4 where the individual conductances (or weights w) at the saddles are given by Eq. (2.23). Due to the symmetry, the conductances are the same for ϕ_i^+ and ϕ_i^- for $i = 2, 3$. Therefore we obtain:

$$\mathcal{C}^*(e_0, e_4) = \left(\frac{1}{2w(\phi_2^+)} + \frac{1}{2w(\phi_3^+)} \right)^{-1} = 2\left(\frac{\sqrt{|\mathrm{Det}\,\mathcal{H}_{\phi_2^+} S|}}{|\lambda^-(\phi_2^+)|} + \frac{\sqrt{|\mathrm{Det}\,\mathcal{H}_{\phi_3^+} S|}}{|\lambda^-(\phi_3^+)|} \right)^{-1}. \tag{2.36}$$

The expected transition time is

$$\mathbb{E}_{e_0}\big[\tau_\varepsilon\big(\mathcal{B}_\rho(e_4)\big)\big] = \frac{2\pi \mathrm{e}^{(S(\phi_2^+) - S(e_0))/\varepsilon}}{\mathcal{C}^*(e_0, e_4)\sqrt{\mathrm{Det}\,\mathcal{H}_{e_0} S}}\big(1 + \mathrm{O}\big(\sqrt{\varepsilon}\big|\ln(\varepsilon)\big|^{3/2}\big)\big). \tag{2.37}$$

If we suppose now that $S(\phi_1^+) > S(\phi_2^+) > S(\phi_3^+)$, then $\widehat{S}(e_0, e_4) = S(\phi_2^+)$ and we have a graph with two vertices, since e_2 and e_4 are in the same equivalence class. The corresponding graph is simply



(2.38)

Therefore we obtain:

$$\mathcal{C}^*(e_0, e_4) = 2w(\phi_2^+) = \frac{2|\lambda^-(\phi_2^+)|}{\sqrt{|\mathrm{Det}\,\mathcal{H}_{\phi_2^+} S|}}. \tag{2.39}$$

The transition time is given by Eq. (2.37) with $\mathcal{C}^*(e_0, e_4)$ defined by (2.39).

2.5. *Sketch of proof of Theorem* 2.6

We first introduce the discretization we consider. The finite dimensional approximation of the SPDE is constructed as in the work of Funaki [26] and the work of Gyöngy [29]. The approximation is defined via a spatial finite difference approximation of Eq. (1.1).

We denote by S_N the discretized potential, for $y \in \mathbb{R}^{N+2}$

$$S_N(y) = h_N \sum_{i=0}^{N} \frac{\gamma}{2h_N^2}(y_{i+1} - y_i)^2 + V(y_i), \tag{2.40}$$

where $h_N > 0$ is the step of discretization. We set $X_0^i = u_0(x_i)$ where $u_0 \in C_{bc}([0, 1])$ is the initial condition and the x_i are the discretization points on $[0, 1]$. Let us denote by $x_{i-1/2}$ the middle point of $[x_{i-1}, x_i]$. We construct an N-dimensional Brownian motion B from the white noise W. Doing so we will be able to prove the convergence of u^N to u in L^p and almost surely. Thus we define, for $1 \le i \le N$

$$B_t^i = \frac{1}{\sqrt{h_N}} W\big([x_{i-1/2}, x_{i+1/2}] \times [0, t]\big). \tag{2.41}$$

The properties of the white noise imply that (B^i) are independent Brownian motions.

The N-dimensional process $(X_t)_t$ is the solution of

$$\mathrm{d}X_t^i = -\frac{1}{h_N}\nabla S_N(X_t)^i\,\mathrm{d}t + \sqrt{\frac{2\varepsilon}{h_N}}\,\mathrm{d}B_t^i \quad \text{for } i = 1, \dots, N. \tag{2.42}$$

X^0 and X^{N+1} are defined by the boundary conditions

- for Dirichlet boundary conditions:

$$X_t^0 = X_t^{N+1} = 0, \quad \forall t \ge 0, \tag{2.43}$$

- for Neumann boundary conditions:

$$X_t^0 = X_t^1 \quad \text{and} \quad X_t^{N+1} = X_t^N, \quad \forall t \ge 0. \tag{2.44}$$

The discretized system u^N is the linear interpolation between the points (x_i, X^i). To simplify, it is easier to adapt the parameters to the boundary conditions.

- For Dirichlet boundary conditions, we choose

$$h_N = \frac{1}{N+1}, \qquad x_i = \frac{i}{N+1}, \quad \forall 0 \le i \le N+1. \tag{2.45}$$

- For Neumann boundary conditions, we choose

$$h_N = \frac{1}{N}, \qquad x_i = \frac{i}{N} - \frac{1}{2N}, \quad \forall 0 \le i \le N+1. \tag{2.46}$$

We set $\tau_\varepsilon^N(B)$ the hitting time of a set B for the discretized system

$$\tau_\varepsilon^N(B) = \inf\{t > 0, u^N(N^{-1}t) \in B\}. \tag{2.47}$$

We decompose the proof of Theorem 2.6 in several steps:

(1) for a given ε and a sequence of initial conditions $\phi_{l_0}^N$, each being a minimum of S^N, converging to ϕ_{l_0} (see Proposition 5.6), we prove that the expectation of $\tau_\varepsilon^N(\mathcal{B}_\rho(\mathcal{M}_l))$ converges to the expectation of the hitting time for the SPDE:

$$\lim_{N\to\infty} \mathbb{E}_{\phi_{l_0}^N}\big[\tau_\varepsilon^N\big(\mathcal{B}_\rho(\mathcal{M}_l)\big)\big] = \mathbb{E}_{\phi_{l_0}}\big[\tau_\varepsilon\big(\mathcal{B}_\rho(\mathcal{M}_l)\big)\big]. \tag{2.48}$$

To this aim, we use the convergence of u^N to the solution u. This is done in Section 3.

(2) For a fixed N, we compute the asymptotics of the transition time uniformly on the dimension. We get a prefactor $a_N(\varepsilon)$ such that

$$\left|\frac{1}{a_N(\varepsilon)}\mathbb{E}_{\phi_{l_0}^N}\big[\tau_\varepsilon^N\big(\mathcal{B}_\rho(\mathcal{M}_l)\big)\big] - 1\right| = \psi(\varepsilon, N) < \Psi(\varepsilon) = \mathrm{O}\big(\sqrt{\varepsilon}\left|\ln(\varepsilon)\right|^{3/2}\big), \tag{2.49}$$

where the error term $\Psi(\varepsilon)$ does not depend on N. This step is the main estimate and is detailed below.

(3) The limit $N \to \infty$ of $a_N(\varepsilon)$ gives us the correct asymptotics for the transition time in the infinite dimensional case:

$$a(\varepsilon) = \lim_{N\to\infty} a_N(\varepsilon). \tag{2.50}$$

This is done in Section 5.

The estimate (2.49) is proved in two steps.

(i) First we start from a probability measure (the equilibrium probability: ν^N) on the boundary of a chosen neighborhood of the minimum $\phi_{l_0}^N$, which allows us to do the computation of $a_N(\varepsilon)$:

$$\left|\frac{1}{a_N(\varepsilon)}\mathbb{E}_{\nu^N}\big[\tau_\varepsilon\big(\mathcal{B}_\rho(\mathcal{M}_0)\big)\big] - 1\right| = \psi_1(\varepsilon, N) < \Psi_1(\varepsilon) = \mathrm{O}\big(\sqrt{\varepsilon}\left|\ln(\varepsilon)\right|^{3/2}\big). \tag{2.51}$$

This is done in Section 6.

(ii) Then we have to control the error made by starting on the boundary of the minimum and not precisely at the minimum:

$$\frac{1}{a_N(\varepsilon)}\left|\mathbb{E}_{\nu^N}\big[\tau_\varepsilon\big(\mathcal{B}_\rho(\mathcal{M}_0)\big)\big] - \mathbb{E}_{\phi_{l_0}^N}\big[\tau_\varepsilon^N\big(\mathcal{B}_\rho(\mathcal{M}_0)\big)\big]\right| = \psi_2(\varepsilon, N) < \Psi_2(\varepsilon) \tag{2.52}$$

with $\Psi_2(\varepsilon) = \mathrm{O}(\sqrt{\varepsilon}|\ln(\varepsilon)|^{3/2})$. This result comes from the loss of memory of the initial condition adapted from Martinelli in [35]. This is exposed in Section 4.

3. Discretization

In this section, we present the convergence of the discretization u^N to the solution of the SPDE and prove the convergence of the hitting times.

3.1. *Finite dimensional model*

We write the discretized system u^N in a mild form. We define a function κ_N, with $\lfloor x\rfloor$ the integer part of x,

$$\kappa_N(x)=\frac{\lfloor (N+1)x+1/2\rfloor}{N+1},\quad \text{for Dirichlet boundary conditions,} \tag{3.1}$$

$$\kappa_N(x)=\frac{\lfloor Nx\rfloor+1}{N}-\frac{1}{2N},\quad \text{for Neumann boundary conditions.} \tag{3.2}$$

We define g^N the semi-group associated with the discretized Laplacian. The discretized Laplacian is an N-dimensional matrix, denoted by Δ_d^N for Dirichlet boundary conditions and by Δ_n^N for Neumann boundary conditions:

$$\Delta_d^N=\frac{1}{h_N^2}\begin{pmatrix}-2 & 1 & 0 & \dots & 0\\ 1 & -2 & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 1\\ 0 & \dots & 0 & 1 & -2\end{pmatrix},\qquad \Delta_n^N=\frac{1}{h_N^2}\begin{pmatrix}-1 & 1 & 0 & \dots & 0\\ 1 & -2 & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 1\\ 0 & \dots & 0 & 1 & -1\end{pmatrix}. \tag{3.3}$$

We consider the matrix $p^N(t)=h_N^{-1}\mathrm{e}^{t\gamma\Delta^N}$. Therefore $p^N(t)_{i,j}$ is the solution of

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}p^N(t)_{i,j}=\left(\gamma\Delta^N p^N(t)\right)_{i,j},\\ p^N(0)_{i,j}=\frac{1}{h_N}\delta_{ij}.\end{cases} \tag{3.4}$$

The semi-group g^N is the linear interpolation of $p^N(t)$ on $[0,1]\times[0,1]$ along the discretization points.

Let us now prove the convergence of u^N to the solution of Eq. (1.1).

Theorem 3.1. *For all initial condition $u_0\in C^3_{bc}([0,1])$, $T>0$, and $p\geq 1$, we get the convergence*

$$u^N\xrightarrow[N\to\infty]{} u\quad \text{on } [0,1]\times[0,T] \tag{3.5}$$

in the following senses:

- *in $L^p(\Omega,C([0,1]\times[0,T]))$, i.e., $\mathbb{E}[\|u^N-u\|^p_{\infty,T}]^{1/p}\xrightarrow[N\to\infty]{}0$,*
- *almost surely in $C([0,1]\times[0,T])$, i.e., for every $\eta\in]0,\frac{1}{2}[$, there exists Ξ a random variable almost surely finite such that*

$$\left\|u^N-u\right\|_{\infty,T}\leq\frac{\Xi}{N^\eta}. \tag{3.6}$$

Remark 5. *Let us denote*

$$\|u\|_{q,T}=\sup_{t\in[0,T]}\left[\int_0^1\left|u(x,t)\right|^q\,\mathrm{d}x\right]^{1/q}=\sup_{t\in[0,T]}\left\|u(t)\right\|_{L^q}. \tag{3.7}$$

We have $\|u\|_{q,T}\leq\|u\|_{\infty,T}$. As a consequence we get convergence in Theorem 3.1 in the L^q norm instead of the uniform norm.

The convergence of the finite discretization was proved in [29] if V' is globally Lipschitz. We proved that the result holds in the case that V' satisfies (2.3) via a localization argument. The idea, notably used by Funaki in [26], is to rewrite the finite dimensional system u^N in a "mild for" and prove the convergence of this finite dimensional mild form to the infinite dimensional mild form (2.4).

Lemma 3.2. *For every $u_0 \in C_{bc}([0,1])$ and $N > 0$, the function u^N defined on $[0,1] \times \mathbb{R}^+$ satisfies the equation*

$$u^N(x,t) = \int_0^1 g_t^N\big(x, \kappa_N(y)\big)u_0\big(\kappa_N(y)\big)\,\mathrm{d}y - \int_0^t \int_0^1 g_{t-s}^N\big(x, \kappa_N(y)\big)V'\big(u^N\big(\kappa_N(y), s\big)\big)\,\mathrm{d}y\,\mathrm{d}s$$
$$+ \sqrt{2\varepsilon}\int_0^t \int_0^1 g_{t-s}^N\big(x, \kappa_N(y)\big)W(\mathrm{d}y, \mathrm{d}s). \tag{3.8}$$

For all $p \geq 1$ and $T > 0$, we have

$$\sup_N \mathbb{E}\Big[\sup_{[0,T]\times[0,1]} \big|u^N(x,t)\big|^p\Big] \leq C(T,p). \tag{3.9}$$

Proof. This lemma is just a reformulation of the system of stochastic differential equations. We use the variation of the constant to integrate the linear part and then interpolate linearly the system to obtain a mild formulation of the function u^N (see [26,29]). To obtain the uniform moment bound, we proceed classically using a truncation procedure. We define u_R^N solution of Eq. (3.8) in which we have replaced the function V' by b_R defined, for $R > 0$ by

$$b_R(u) = V'(u)\mathbb{1}_{[-R,R]} + V'(R)\mathbb{1}_{]R,+\infty[} + V'(-R)\mathbb{1}_{]-\infty,-R[}. \tag{3.10}$$

b_R is continuous, bounded and globally Lipschitz. Firstly, using the uniform estimates of the semi-group and the boundedness of b_R, we prove that for all T, all $p > 1$, there exists $C(p,T,R)$ independent of N such that

$$\sup_{[0,1]\times[0,T]} \mathbb{E}\big[\big|u_R^N(x,t)\big|^p\big] \leq C(p,T,R) < +\infty. \tag{3.11}$$

Secondly, there exists $C(p,T,R)$ independent of N, such that

$$\sup_N \mathbb{E}\Big[\sup_{[0,1]\times[0,T]} \big|u_R^N(x,t)\big|^p\Big] \leq C(p,T,R) < +\infty. \tag{3.12}$$

We use regularity of the solution (Kolmogorov's theorem) to prove (3.12). Thirdly, we use a comparison theorem from Geiss and Manthey ([28], Theorem 1.2) to obtain uniform bounds on u^N from bounds on $u_{R_0}^N$ where R_0 is fixed and sufficiently large. We proceed by using two additional processes $u_R^{N,+}$ (resp. $u_R^{N,-}$) solutions of Eq. (3.8) in which we have replaced the function V' by b_R^+ (resp. b_R^-) defined, for $R > 0$ by

$$b_R^+(u) = V'(u)\mathbb{1}_{[-R,+\infty[} + V'(-R)\mathbb{1}_{]-\infty,-R[}, \tag{3.13}$$
$$b_R^-(u) = V'(u)\mathbb{1}_{]-\infty,R]} + V'(R)\mathbb{1}_{]R,+\infty[}. \tag{3.14}$$

We have the following comparison: $b_R^+(u) \geq b_R(u) \geq b_R^-(u)$. Using the comparison Theorem 2.1 from [28] and taking care of the fact that the actual drift coefficients have a minus sign, we obtain that for all $(x,t) \in [0,1] \times [0,T]$

$$u_R^{N,+}(x,t) \leq u_R^N(x,t) \leq u_R^{N,-}(x,t). \tag{3.15}$$

Using Assumption 2.1 we have that, for some $R_0 > 0$ sufficiently large $|b_{R_0}^+(u)| < A(1 + |u|^q\mathbb{1}_{u>0})$ where $A, q > 0$ are some constants. Therefore, from Eq. (3.12) and (3.15) we get

$$\sup_N \mathbb{E}\Big[\sup_{[0,1]\times[0,T]} \big|b_{R_0}^+\big(u_{R_0}^{N,+}(x,t)\big)\big|^p\Big] \leq 2^p A^p\big(1 + C(pq,T,R_0)\big) < +\infty. \tag{3.16}$$

This equation allows us to prove the equivalent of Eq. (3.12) for $u_{R_0}^{N,+}$.

The same argument can be applied to obtain Eq. (3.12) for $u_{R_0}^{N,-}$ (here we have to use the fact that $|b_{R_0}^-(u)| < A(1+|u|^q \mathbb{1}_{u<0})$).

At last, using the comparison theorem for $b_{R_0}^+(u) \geq V'(u) \geq b_{R_0}^-(u)$, we obtain that for all $(x,t) \in [0,1] \times [0,T]$

$$\left|u^N(x,t)\right| \leq \left|u_{R_0}^{N,-}(x,t)\right| + \left|u_{R_0}^{N,+}(x,t)\right|. \tag{3.17}$$

We conclude by using the bounds obtained on $u_{R_0}^{N,-}$ and $u_{R_0}^{N,+}$ to get Eq. (3.9). □

Let us recall that u_R^N is the solution of Eq. (3.8) in which we have replaced the function V' by b_R defined by Eq. (3.10). Similarly u_R is the solution of Eq. (2.4) with b_R instead of V'.

Proposition 3.3 (Propositions 2.3.4 and 2.3.5 from [2]). *For all $R>0$, $T>0$, and $0<\eta<\frac{1}{2}$ and u_0 in $C_{bc}^3[0,1]$, there exists a random variable ξ_R almost surely finite such that*

$$\left\|u_R^N - u_R\right\|_{\infty,T} \leq \frac{\xi_R}{N^\eta}. \tag{3.18}$$

For $p>24$, there exists $C>0$, such that

$$\mathbb{E}\left[\left\|u_R^N - u_R\right\|_{\infty,T}^p\right] \leq \frac{C}{N^{1/2-4/p}}. \tag{3.19}$$

The proof is quite technical but follows the same method as in the proof of Theorem 3.1 in [29] since b_R is bounded, continuous and globally Lipschitz. We refer to the author's thesis ([2]) in which the details are carefully stated. Let us remark that in particular, by a simple convexity argument, we obtain that $\mathbb{E}[\|u_R^N - u_R\|_{\infty,T}^p]$ converges to 0 for all $p \geq 1$.

Proof of Theorem 3.1. Let $R>0$, we define the stopping times

$$\tau_R = \inf\left\{t, \left\|u_R(t)\right\|_\infty > R\right\} = \inf\left\{t, \exists x \in [0,1], \left|u_R(x,t)\right| > R\right\}, \tag{3.20}$$

$$\tau_R^N = \inf\left\{t, \left\|u_R^N(t)\right\|_\infty > R\right\} = \inf\left\{t, \exists x \in [0,1], \left|u_R^N(x,t)\right| > R\right\}. \tag{3.21}$$

Let us choose $0<\delta<1$. For $R>1$, we define

$$\Omega_R = \left\{\tau_{R-\delta} > T \text{ and } \liminf_{N\to\infty} \tau_R^N > T\right\}. \tag{3.22}$$

First we show that $\mathbb{P}[\Omega_R] \xrightarrow[R\to\infty]{} 1$. Let $M>0$. For $\omega \in \{\xi_R < M\} \cap \{\tau_{R-\delta} \geq T\}$, by Proposition 3.3, for N sufficiently large,

$$\left\|u_R^N\right\|_{\infty,T}(\omega) < \|u_R\|_{\infty,T}(\omega) + \delta < R \tag{3.23}$$

which means that $\liminf_{N\to\infty} \tau_R^N(\omega) \geq T$. Then by taking the complement relatively to $\{\xi_R < M\}$ we get

$$\mathbb{P}\left[\liminf_{N\to\infty} \tau_R^N < T; \xi_R < M\right] \leq \mathbb{P}[\tau_{R-\delta} < T; \xi_R < M] \leq \mathbb{P}[\tau_{R-\delta} < T]. \tag{3.24}$$

By definition of the time $\tau_{R-\delta}$, we have by the Markov inequality for $p>1$ and from Eq. (2.5)

$$\mathbb{P}\left[\liminf_{N\to\infty} \tau_R^N < T; \xi_R < M\right] \leq \mathbb{P}[\tau_{R-\delta} \leq T] \leq \mathbb{P}\left[\|u\|_{\infty,T} \geq R-\delta\right] \leq \frac{\mathbb{E}[\|u\|_{\infty,T}^p]}{(R-\delta)^p}. \tag{3.25}$$

Finally we get

$$\mathbb{P}\big[\Omega_R^c\big] = \mathbb{P}\Big[\tau_{R-\delta} \le T \text{ or } \liminf_{N\to\infty} \tau_R^N \le T\Big]$$
$$\le \mathbb{P}[\tau_{R-\delta} \le T] + \mathbb{P}\Big[\liminf_{N\to\infty} \tau_R^N < T; \xi_R < M\Big] + \mathbb{P}[\xi_R \ge M]$$
$$\le \frac{2\mathbb{E}[\|u\|_{\infty,T}^p]}{(R-\delta)^p} + \mathbb{P}[\xi_R \ge M]. \tag{3.26}$$

Since ξ_R is finite almost surely, we take first the limit $M \to +\infty$ then $R \to +\infty$.

Let us define $\widetilde{\Omega}_R = \Omega_R \cap \{\xi_R < \infty\}$. Since τ_R and τ_R^N are increasing in $R \in \mathbb{N}$, the sets Ω_R are also increasing in R. Then we have

$$\mathbb{P}\Bigg[\bigcup_{R>1}^{\infty} \widetilde{\Omega}_R\Bigg] = \mathbb{P}\Bigg[\bigcup_{R\in\mathbb{N}} \Omega_R\Bigg] = \lim_{R\to\infty} \mathbb{P}[\Omega_R] = 1. \tag{3.27}$$

Let $\omega \in \widetilde{\Omega}_R$. By definition of τ_R^N, there exists $N_0(\omega)$ such that for all $N \ge N_0(\omega)$, $\tau_R^N(\omega) > T$ and $\tau_{R-\delta}(\omega) > T$. By using Proposition 3.3, for all $N \ge N_0(\omega)$,

$$\|u^N - u\|_{\infty,T}(\omega) = \|u_R^N - u_R\|_{\infty,T}(\omega) \le \xi_R(\omega) N^{-\eta}. \tag{3.28}$$

We define $\xi'_R(\omega)$ by

$$\xi'_R(\omega) = \sup_{N \le N_0(\omega)} N^{\eta} \|u_R^N - u_R\|_{\infty,T}(\omega) + \xi_R(\omega). \tag{3.29}$$

$\xi'_R(\omega)$ is finite on $\widetilde{\Omega}_R$ and is such that $\|u^N - u\|_{\infty,T} \le \xi'_R N^{-\eta}$. Let us define the random variable Ξ by

$$\Xi(\omega) = \xi'_R(\omega) \quad \text{on } \widetilde{\Omega}_R \setminus \widetilde{\Omega}_{R-1} \text{ for } R \ge 2,$$
$$\Xi(\omega) = \xi'_1(\omega) \quad \text{on } \widetilde{\Omega}_1. \tag{3.30}$$

Then on $\bigcup_{R\ge1} \widetilde{\Omega}_R$, set of probability 1, Ξ is almost surely finite and $\|u^N - u\|_{\infty,T} \le \Xi N^{-\eta}$ which finishes the proof of the almost sure convergence.

To conclude, we show that $\mathbb{E}[\|u^N - u\|_{\infty,T}^p]$ converges to 0. Since $\|u^N\|_{\infty,T}$ has moments uniformly bounded in N (Lemma 3.2), we define

$$\Omega_{R,N_0} = \bigcap_{N \ge N_0} \big\{\tau_{R-\delta} > T \text{ and } \tau_R^N > T\big\}. \tag{3.31}$$

We have $\Omega_R = \bigcup_{N_0} \Omega_{R,N_0}$. For all $N \ge N_0$, we get by definition

$$\|u^N - u\|_{\infty,T}^p = \mathbb{1}_{\Omega_{R,N_0}} \|u_R^N - u_R\|_{\infty,T}^p + \mathbb{1}_{\Omega_{R,N_0}^c} \|u^N - u\|_{\infty,T}^p. \tag{3.32}$$

Thus using Cauchy–Schwarz inequality and the bound (3.9), we get

$$\mathbb{E}\big[\|u^N - u\|_{\infty,T}^p\big] \le \mathbb{E}\big[\|u_R^N - u_R\|_{\infty,T}^p\big] + \mathbb{P}\big[\Omega_{R,N_0}^c\big]^{1/2} C(2p,T)^{1/2}. \tag{3.33}$$

Using the convergence of u_R^N to u^R (Proposition 3.3), we obtain

$$\limsup_{N\to\infty} \mathbb{E}\big[\|u^N - u\|_{\infty,T}^p\big] \le C(2p,T)^{1/2} \mathbb{P}\big[\Omega_{R,N_0}^c\big]^{1/2}. \tag{3.34}$$

Let us fix $\eta > 0$. Since $\mathbb{P}[\Omega_R]$ tends to 1 and Ω_R is increasing, we choose R such that $\mathbb{P}[\Omega_R^c] \le \eta$. Similarly, Ω_{R,N_0} is increasing in N_0, thus $\mathbb{P}[\Omega_R^c] = \lim_{N_0\to\infty} \mathbb{P}[\Omega_{R,N_0}^c] \le \eta$. Let us choose N_0 such that $\mathbb{P}[\Omega_{R,N_0}^c] \le 2\eta$. Inserting this bound in (3.34), we obtain the result. □

3.2. *Convergence of the transition times*

We conclude this section by proving the convergence of the transition times.

Let us denote by u_0 the initial condition of the solution of Eq. (1.1) and ϕ a continuous function. We define the hitting times: for $\rho > 0$

$$\tau_\varepsilon(\rho) = \inf\{t > 0, \|u(t) - \phi\|_\infty < \rho\}, \tag{3.35}$$

$$\tau_\varepsilon^N(\rho) = \inf\{t > 0, \|u^N(t) - \phi^N\|_\infty < \rho\}, \tag{3.36}$$

where ϕ^N is the linear approximation of ϕ.

Proposition 3.4. *Suppose that $\|\phi^N - \phi\|_\infty$ converges to 0 and that there exists ρ_0 such that for every $\rho_0 > \rho > 0$,*

$$\mathbb{E}_{u_0}\big[\tau_\varepsilon(\rho)\big] < \infty. \tag{3.37}$$

Then for almost every $\rho > 0$,

$$\tau_\varepsilon^N(\rho) \xrightarrow[N\to\infty]{} \tau_\varepsilon(\rho) \quad a.s. \quad and \quad \mathbb{E}_{u_0^N}\big[\tau_\varepsilon^N(\rho)\big] \xrightarrow[N\to\infty]{} \mathbb{E}_{u_0}\big[\tau_\varepsilon(\rho)\big]. \tag{3.38}$$

Proof. For the sake of simplicity we omit ε in the proof. First we prove that for all $\delta > 0$, $T > 0$, we have

$$\tau(\rho+\delta)\wedge T \le \liminf_{N\to\infty} \tau^N(\rho)\wedge T \le \limsup_{N\to\infty} \tau^N(\rho)\wedge T \le \tau(\rho-\delta)\wedge T \quad \text{a.s.} \tag{3.39}$$

From Theorem 3.1, $\|u^N - u\|_{\infty,T}$ converges to 0 almost surely. Therefore with probability 1, there exists $N_0(\omega)$ such that for all $N \ge N_0(\omega)$

$$\sup_{t\in[0,T]} \|u^N(t) - u(t)\|_\infty(\omega) < \frac{\delta}{2} \quad \text{and} \quad \|\phi^N - \phi\|_\infty < \frac{\delta}{2}. \tag{3.40}$$

Then for $t \le \tau(\rho+\delta)\wedge T$ and $N \ge N_0(\omega)$, using the triangle inequality we get

$$\begin{aligned}\rho + \delta \le \|u(t) - \phi\|_\infty &\le \|u(t) - u^N(t)\|_\infty + \|u^N(t) - \phi^N\|_\infty + \|\phi^N - \phi\|_\infty \\ &\le \delta + \|u^N(t) - \phi^N\|_\infty\end{aligned} \tag{3.41}$$

which means that $t \le \tau^N(\rho)\wedge T$. Thus, we obtain $\tau(\rho+\delta)\wedge T \le \liminf_{N\to\infty}[\tau^N(\rho)\wedge T]$ almost surely. By the same arguments for $t \le \tau^N(\rho)\wedge T$ and $N \ge N_0(\omega)$, we get

$$\rho \le \|u^N(t) - \phi^N\|_\infty \le \delta + \|u(t) - \phi\|_\infty. \tag{3.42}$$

Therefore $\limsup_{N\to\infty}[\tau^N(\rho)\wedge T] \le \tau(\rho-\delta)\wedge T$ which proves the inequality (3.39).

From the definitions of $\tau(\rho)$ and $\tau^N(\rho)$, the functions $\rho \mapsto \tau(\rho)$ and $\rho \mapsto \tau^N(\rho)$ are left continuous and have right limits. Then using the fact that $\tau(\rho)$ is finite almost surely, we get

$$\tau(\rho^+) \le \liminf_{N\to\infty} \tau^N(\rho) \le \limsup_{N\to\infty} \tau^N(\rho) \le \tau(\rho) < +\infty \quad \text{a.s.}, \tag{3.43}$$

where $\tau(\rho^+) = \lim_{\delta\to 0^+} \tau(\rho+\delta)$.

At a point of continuity of $\rho \mapsto \tau(\rho)$, we obtain $\tau(\rho) = \lim_{N\to\infty} \tau^N(\rho)$. Let us fix $\rho_1 > 0$. There exists $\mathscr{N} \subset \Omega$ a null set such that for $\omega \notin \mathscr{N}$, $\rho \mapsto \tau(\rho)(\omega)$ is bounded, decreasing, left continuous on $[\rho_1, +\infty[$. We define the set of discontinuities, $\mathscr{P}$:

$$\mathscr{P} = \{(\omega,\rho) \in \mathscr{N}^c \times [\rho_1, +\infty[, \tau(\rho^+)(\omega) \ne \tau(\rho)(\omega)\} \subset \Omega \times \mathbb{R}. \tag{3.44}$$

Then we consider the projection $\Pi_\omega^{\mathbb{R}}$ from $\Omega \times \mathbb{R}$ on $\mathbb{R}$ along $\{\omega\} \times \mathbb{R}$. For $\omega \in \mathscr{N}^c$ we define

$$\mathcal{D}(\omega) = \Pi_\omega^{\mathbb{R}}(\mathscr{P}) = \{\rho \in [\rho_1, +\infty[, \tau(\rho^+)(\omega) \neq \tau(\rho)(\omega)\} \subset \mathbb{R}. \tag{3.45}$$

$\mathcal{D}(\omega)$ is at most countable since $\rho \mapsto \tau(\rho)(\omega)$ is a bounded decreasing function.

We define $\mathcal{N}(\rho) = \Pi_\rho^{\Omega}(\mathscr{P})$ with Π_ρ^{Ω} the projection from $\Omega \times \mathbb{R}$ on Ω along $\Omega \times \{\rho\}$. $\mathcal{N}(\rho)$ is the set of Ω for which $\tau(\rho)$ is not continuous at ρ. Therefore, we have

$$\mathscr{P} = \bigcup_{\omega \in \Omega} \{\omega\} \times \mathcal{D}(\omega) = \bigcup_{\rho > \rho_1} \mathcal{N}(\rho) \times \{\rho\}. \tag{3.46}$$

Then, using Fubini–Tonelli Theorem

$$\int_{\rho_1}^{+\infty} \mathbb{P}[\mathcal{N}(\rho)] \, \mathrm{d}\rho = \int_\Omega \int_{\rho_1}^{+\infty} \mathbb{1}_{\mathscr{P}}(\omega, \rho) \, \mathrm{d}\rho \, \mathrm{d}\mathbb{P}(\omega) = \int_\Omega \int_{\rho_1}^{+\infty} \mathbb{1}_{\mathcal{D}(\omega)}(\rho) \, \mathrm{d}\rho \, \mathrm{d}\mathbb{P}(\omega) = 0. \tag{3.47}$$

We get a null set $\mathcal{E}(\rho_1)$ on $[\rho_1, +\infty[$ such that $\mathbb{P}[\mathcal{N}(\rho)] = 0$ for all $\rho \in \mathcal{E}(\rho_1)$, i.e., the convergence is almost sure. To conclude, we consider a sequence $(\rho_n)_{n \geq 0}$ converging to 0, then $\mathcal{E} = \bigcup_{n \geq 0} \mathcal{E}(\rho_n)$ is a null set of $\mathbb{R}$ on which the convergence is almost sure.

By using dominated convergence, we obtain the convergence of the expectations. □

4. Initial condition

4.1. *Large deviation control*

For $0 < \alpha < 1$, we set $C^\alpha([0,1])$ the set of α-Hölder continuous functions on $[0,1]$ equipped with the norm $\|\cdot\|_{C^\alpha}$

$$\|f\|_{C^\alpha} = \|f\|_\infty + \sup_{x,y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}. \tag{4.1}$$

We also define $D^\alpha([0,1])$ the separable subset of this Hölder space which is the closure of C^∞ in C^α.

Let $0 < \alpha < \frac{1}{2}$ and $\rho > 0$, we consider the neighborhood $B_\rho^\alpha(\phi)$ of $\phi \in D_{bc}^\alpha([0,1])$

$$B_\rho^\alpha(\phi) = \{\psi \in D_{bc}^\alpha([0,1]), \|\phi - \psi\|_{C^\alpha} < \rho\}. \tag{4.2}$$

We also have $B_\rho^\alpha(\mathcal{M}_l) = \bigcup_{\phi \in \mathcal{M}_l} B_\rho^\alpha(\phi)$.

With this large deviation principle, Chenal and Millet [18] derived exponential asymptotic estimates for the exit time of domains with a unique stable stationary point. Using their evaluations and the procedure developed by Freidlin and Wentzell [25] in the finite dimensional case, we have the following result.

Lemma 4.1. *For $0 < \alpha < \frac{1}{2}$, there exists ρ_0 such that for all $\rho < \rho_0$, we have for all $\phi \in B_\rho^\alpha(\phi_{l_0})$ and $\eta > 0$*

$$\lim_{\varepsilon \to 0} \mathbb{P}_\phi\left[\exp\left(\varepsilon^{-1}\left(\widehat{S} - S(\phi_{l_0}) + \eta\right)\right) > \tau_\varepsilon\left(B_\rho^\alpha(\mathcal{M}_l)\right) > \exp\left(\varepsilon^{-1}\left(\widehat{S} - S(\phi_{l_0}) - \eta\right)\right)\right] = 1, \tag{4.3}$$

where $\widehat{S} = \widehat{S}(\phi_{l_0}, \mathcal{M}_l)$. Let $\tau_\varepsilon = \tau_\varepsilon(B_\rho^\alpha(\mathcal{M}_l))$. Then

$$\frac{\tau_\varepsilon}{\mathbb{E}_\phi[\tau_\varepsilon]} \xrightarrow[\varepsilon \to 0]{\mathscr{L}} \mathcal{E}, \tag{4.4}$$

where $\mathcal{E}$ is an exponential variable of parameter 1. Moreover for all $\phi \in B_\rho^\alpha(\phi_{l_0})$

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_\phi[\tau_\varepsilon] = \widehat{S} - S(\phi_{l_0}) \quad \text{and} \quad \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_\phi\left[\tau_\varepsilon^2\right] = 2\left(\widehat{S} - S(\phi_{l_0})\right). \tag{4.5}$$

These estimates are the infinite dimensional version of the Freidlin–Wentzell theory. In our case, the proof is exactly a generalization of their approach but using estimates in the proof of Theorem 4.1 in [18] which is exactly the analogous in infinite dimension of Theorem 2.1, Theorem 4.1 and 4.2, Chapter 4 in [25]. We are able to estimate the transition probabilities (probability to reach the neighborhood of a minima starting from a point) uniformly in the initial conditions (around a small ball centered on a minima) using the rate function given by Theorem 3.2 in [18]. Then by taking an approximation via a Markov chain, one can proceed as in the finite dimensional case. Note that in our case, since the deterministic system is gradient, the variation of rate function along the paths can be expressed simply as a difference of the potential S.

4.2. *Exponential contractivity*

For a given $\psi^N = (\psi_1, \dots, \psi_N) \in \mathbb{R}^N$, we consider equivalently the point in $\mathbb{R}^N$ and the function in $C([0,1])$ obtained by the linear interpolation between the points (x_i, ψ_i). Reciprocally, for $\psi \in C_{bc}([0,1])$, we let $\widehat{\psi}^N$ be the linear interpolation of ψ along the discretization. $\widehat{\psi}^N$ is the linear interpolation between the points $(x_i, \psi(x_i))$.

We set

$$B^{\infty}_{\rho}(\phi) = \bigl\{\psi \in C_{bc}\bigl([0,1]\bigr), \|\psi - \phi\|_{\infty} < \rho\bigr\}. \tag{4.6}$$

We adapt trajectorial results of contractivity for the localized process from Martinelli and Scoppola [35]. As usual, since the proof need a globally Lipschitz drift term, we localize the processes. We denote $u(\phi), u_R(\phi)$ the solutions of Eq. (1.1) with respectively V' and b_R, starting from ϕ. Accordingly, we denote $u^N(\phi^N), u^N_R(\phi^N)$ the solutions of Eq. (3.8) with V' and b_R, starting from $\phi^N \in \mathbb{R}^N$.

Lemma 4.2. *Let ϕ be a minimum of S and $R \geq R_0$. There exists $m, C_R > 0$ and $\varepsilon_0, \rho_0 > 0$, such that for all $\rho < \rho_0$ and every $\psi \in B^{\infty}_{\rho}(\phi)$ we have, for all $\varepsilon_0 > \varepsilon > 0$*

$$\mathbb{P}\Bigl[\sup_{N \geq N_0} \bigl\| u^N_R\bigl(\widehat{\psi}^N\bigr)(t) - u^N_R\bigl(\widehat{\phi}^N\bigr)(t)\bigr\|_{\infty} \leq \mathrm{e}^{-mt}\|\psi - \phi\|_{\infty}, \forall t > 0\Bigr] \geq 1 - \mathrm{e}^{-C_R/\varepsilon}. \tag{4.7}$$

R_0 is just an arbitrary constant large enough such that outside $[-R_0, R_0]$, V' is increasing. Equation (4.7) means that, uniformly in N, two discretized processes, with the same noise, but starting from two different initial conditions sufficiently close will be contracting exponentially. In references [36] and [35], since they start from two arbitrary points, they need to take $t > T_0$ for some T_0 large enough. In this statement this is unnecessary since we start from two close initial conditions.

This result can be proved via an adaptation of the arguments of [36] and [35]. Lemma 4.2 asserts that the solutions of Eq. (1.1) and (3.8) depend slightly on the initial condition. Martinelli and Scoppola called that the loss of memory of the initial condition because the specific initial condition is not relevant for the evolution of the process.

4.3. *Uniformity in the initial condition*

Let us recall that ϕ_{l_0} is a minimum and $\mathcal{M}_l$ is a set of lower minima. We denote

$$\begin{aligned} \tau^N_{\varepsilon}(\phi_{l_0}) &= \tau^N_{\varepsilon}\bigl(B^{\alpha}_{\rho}(\phi_{l_0})\bigr) = \inf\bigl\{t, u^N(t) \in B^{\alpha}_{\rho}(\phi_{l_0})\bigr\}, \\ \tau^N_{\varepsilon}(\mathcal{M}_l) &= \tau^N_{\varepsilon}\bigl(B^{\alpha}_{\rho}(\mathcal{M}_l)\bigr) = \inf\bigl\{t, u^N(t) \in B^{\alpha}_{\rho}(\mathcal{M}_l)\bigr\}. \end{aligned} \tag{4.8}$$

Similarly, we denote by $\tau^{N,R}_{\varepsilon}$ the hitting time associated with the localized process u^N_R.

Proposition 4.3. *For all $\rho_0 > \rho > 0$, there exists $\eta > 0$ such that for a sequence $\phi^N_{l_0}$ of minima of S^N, converging to ϕ_{l_0} in L^2,*

$$\sup_{N \geq N_0} \sup_{\|\phi^N - \phi^N_{l_0}\|_{\infty} < \rho} \bigl|\mathbb{E}_{\phi^N}\bigl[\tau^N_{\varepsilon}(\mathcal{M}_l)\bigr] - \mathbb{E}_{\phi^N_{l_0}}\bigl[\tau^N_{\varepsilon}(\mathcal{M}_l)\bigr]\bigr| \leq \mathrm{e}^{(\widehat{S}-\eta)/\varepsilon}. \tag{4.9}$$

For any sequence $\phi_i^N \in H^1$ of minima of S^N converging to $\phi_i \in H^1$ in L^2, we also have

$$\sup_{N\geq N_0}\sup_{\|\phi_i^N-\phi^N\|_\infty<\rho}\left|\mathbb{P}_{\phi_i^N}\left[\tau_\varepsilon^N(\phi_{l_0})<\tau_\varepsilon^N(\mathcal{M}_l)\right]-\mathbb{P}_{\phi^N}\left[\tau_\varepsilon^N(\phi_{l_0})<\tau_\varepsilon^N(\mathcal{M}_l)\right]\right|\leq \mathrm{e}^{-\eta/\varepsilon}. \tag{4.10}$$

The proof comes from a comparison between the deterministic process (i.e., $\varepsilon=0$) and the stochastic process starting from the moment of the hitting time.

Proof of Proposition 4.3. Since the minima are not degenerate, we can assume ρ small enough to get

$$\left\langle\frac{\delta S}{\delta\phi}\phi,\phi-\phi_i\right\rangle_{L^2}\leq -b\|\phi-\phi_i\|_{L^2}^2, \tag{4.11}$$

for some $b>0$, all $1<i<l$, and all $\phi\in\mathcal{B}_{2\rho}(\phi_i)$.

First, let us prove similar estimates on the expectations of transition times for the localized process u_R^N. We denote by $\sigma^N(\phi^N)$ the hitting time $\tau_\varepsilon^{N,R}(\mathcal{M}_l)$ for the process u_R^N starting from ϕ^N. We set

$$\Omega_R=\left\{\sup_{N\geq N_0}\sup_{\|u_0-\phi\|_\infty<\rho}\left\|u_N^R(u_0)(t)-u_N^R(\phi)(t)\right\|_\infty\leq\rho\mathrm{e}^{-mt},\forall t>0\right\}. \tag{4.12}$$

From Lemma 4.2, we get $\mathbb{P}(\Omega_R)>1-\mathrm{e}^{-C_R/\varepsilon}$.

Let us fix $\delta_1>0$. We define $T(\varepsilon)=\mathrm{e}^{(\widehat{S}-\delta_1)/\varepsilon}$ and we take $\varepsilon<\varepsilon_0$ such that $\mathrm{e}^{-mT(\varepsilon)}<\rho$. On the set $\{\sigma^N(\phi_{l_0})>T(\varepsilon)\}$, setting $\psi=u_N^R(\phi)(\sigma^N(\phi_{l_0}))$, we get

$$\left\|\psi-u_N^R(\phi_{l_0})\big(\sigma^N(\phi_{l_0})\big)\right\|_\infty<\mathrm{e}^{-mT(\varepsilon)}<\rho \tag{4.13}$$

with probability at least $1-\mathrm{e}^{-C_R/\varepsilon}$. Let us suppose that $\sigma^N(\phi)-\sigma^N(\phi_{l_0})\geq 0$ and that $u_N^R(\phi_{l_0})(\sigma^N(\phi_{l_0}))\in\mathcal{B}_\rho(\phi_i)$.

The deterministic process $u_R^{N,0}$ is the solution of (3.8) for the drift b_R and $\varepsilon=0$. ϕ_i^N is a minimum of S^N, so ϕ_i^N is an equilibrium point of $u_R^{N,0}$. Then using Eq. (4.11), we get for $t\geq 0$

$$\left\|u_R^{N,0}(\psi)(t)-\phi_i\right\|_{L^2}^2\leq\mathrm{e}^{-bt}\|\psi-\phi_i\|_{L^2}^2\leq\mathrm{e}^{-bt}\left(\mathrm{e}^{-mt}\rho+\rho\right)^2\leq 4\rho^2\mathrm{e}^{-bt} \tag{4.14}$$

by the triangle inequality. For $t>t_0=\frac{1}{b}\ln(16)$, we obtain $\|u_R^{N,0}(\psi)(t)-\phi_i\|_{L^2}\leq\frac{\rho}{2}$.

From the large deviation principle, we can compare the deterministic solution with the perturbed one. We obtain $C>0$ such that

$$\mathbb{P}\left[\left\{\left\|u_R^{N,0}\big(\psi^N\big)-u_R^N\big(\psi^N\big)\right\|_{\infty,2t_0}<\frac{\rho}{3}\right\}\right]\geq 1-\mathrm{e}^{-C/\varepsilon}. \tag{4.15}$$

Therefore, with probability at least $1-\mathrm{e}^{-C/\varepsilon}-\mathrm{e}^{-C_R/\varepsilon}$, we get $\|u_R^N(\psi)(2t_0)-\phi_i\|_{L^2}<\frac{5\rho}{6}$ which implies

$$\big(\sigma^N(\phi)-\sigma^N(\phi_{l_0})\big)_+\leq 2t_0. \tag{4.16}$$

We proceed similarly if $\sigma^N(\phi)-\sigma^N(\phi_{l_0})\leq 0$. In this case, we stop the process at $\sigma^N(\phi)$. Finally we get $|\sigma^N(\phi)-\sigma^N(\phi_{l_0})|\leq 2t_0$ with probability at least $1-\mathrm{e}^{-C'/\varepsilon}$, for some $C'>0$.

We obtain

$$\begin{aligned}\mathbb{E}\big[\big|\sigma^N(\phi)-\sigma^N(\phi_{l_0})\big|\big]&\leq\mathbb{E}\big[\big|\sigma^N(\phi)-\sigma^N(\phi_{l_0})\big|\mathbb{1}_{\Omega_R}\mathbb{1}_{\{\sigma^N(\phi_{l_0})>T(\varepsilon)\}}\big]\\&\quad+\mathbb{E}\big[\big|\sigma^N(\phi)-\sigma^N(\phi_{l_0})\big|\big(\mathbb{1}_{\Omega_R^c}+\mathbb{1}_{\{\sigma^N(\phi_{l_0})>T(\varepsilon)\}^c}\big)\big]\\&\leq 2t_0\big(1-\mathrm{e}^{-C'/\varepsilon}\big)\mathbb{P}\big[\Omega_R\cap\big\{\sigma^N(\phi_{l_0})>T(\varepsilon)\big\}\big]\\&\quad+\mathbb{E}\big[\big|\sigma^N(\phi)-\sigma^N(\phi_{l_0})\big|^2\big]^{1/2}\big(\mathbb{P}\big[\Omega_R^c\big]^{1/2}+\mathbb{P}\big[\big\{\sigma^N(\phi_{l_0})\leq T(\varepsilon)\big\}\big]^{1/2}\big).\end{aligned} \tag{4.17}$$

By using Proposition 4.2, we have $\mathbb{P}[\Omega_R^c] < \mathrm{e}^{-C_R/\varepsilon}$. From Proposition 4.1, we deduce that for $\varepsilon \le \varepsilon_0$

$$\mathbb{P}\big[\sigma^N(\phi_{l_0}) \le T(\varepsilon)\big] < 1 - \mathrm{e}^{-\mathrm{e}^{-\delta_1/\varepsilon}} < \mathrm{e}^{-\delta_1/\varepsilon}. \tag{4.18}$$

Moreover, we have for all $\delta_2 > 0$

$$\mathbb{E}\big[\big|\sigma^N(\phi) - \sigma^N(\phi_{l_0})\big|^2\big] < \mathrm{e}^{2(\widehat{S}+\delta_2)/\varepsilon}. \tag{4.19}$$

So we finally get

$$\mathbb{E}\big[\big|\sigma^N(\phi) - \sigma^N(\phi_{l_0})\big|\big] \le 2t_0\big(1 - \mathrm{e}^{-C_R/\varepsilon} - \mathrm{e}^{-\delta_1/\varepsilon}\big) + \mathrm{e}^{(\widehat{S}+\delta_2)/\varepsilon}\big(\mathrm{e}^{-C/2\varepsilon} + \mathrm{e}^{-\delta_1/2\varepsilon}\big) \le \mathrm{e}^{(\widehat{S}-\eta)/\varepsilon}. \tag{4.20}$$

By choosing δ_1, δ_2 and η small enough, we prove the proposition for the localized process.

Let us now choose R such that $\widehat{S}(B_R^\infty(0), B_\rho^\infty(\phi_{l_0})) > \widehat{S} + 1$, then from Proposition 4.1, we have

$$\sup_{\phi \in B_\rho^\infty(\phi_{l_0})} \mathbb{P}_\phi\big[\tau_\varepsilon\big(B_R^\infty(0)\big) \le \exp\big((\widehat{S} + 1 - \delta_3)/\varepsilon\big) = T_2(\varepsilon)\big] \le \mathrm{e}^{-C/\varepsilon} \tag{4.21}$$

$$\sup_{\phi \in B_\rho^\infty(\phi_{l_0})} \mathbb{P}_\phi\big[\tau_\varepsilon^N(\mathcal{M}_l) \ge T_2(\varepsilon)\big] \le \mathrm{e}^{-C/\varepsilon}. \tag{4.22}$$

We consider the process u starting from ϕ and ϕ_{l_0}. Before $T_2(\varepsilon)$, with high probability, the processes are in $B_R^\infty(0)$ and coincide with u_R up to this time. Moreover $T_2(\varepsilon)$ is much larger than the transition time, so the transition already occurs when the processes reach $B_R^\infty(0)^c$. Therefore, with very high probability, the transition time for the localized process is exactly the correct transition time.

For Eq. (4.10), we follow a similar method, by using Proposition 4.2 for the localized process and then comparing the deterministic and stochastic processes in the neighborhood of a minimum. □

5. Approximation of the potential

In this section, we prove (or refer to) results about the convergence of the potential and its related quantities. Let us consider the norms for $y \in \mathbb{R}^N$ and $p \ge 1$

$$\|y\|_{p,N}^p = \sum_{i=1}^N |y_i|^p, \qquad \|y\|_{\infty,N} = \max_{i=1,\dots,N} |y_i|. \tag{5.1}$$

5.1. *Convergence of the potential*

Let us recall from Section 4.2 that for a point $u^N \in \mathbb{R}^N$, we denote also by u^N the linear interpolation between the points (x_i, u_i^N). For a function $u \in C_{bc}([0,1])$, we denote by $\widehat{u}^N$ the linear interpolation between the points $(x_i, u(x_i))$. We say that the sequence $u^N \in \mathbb{R}^N$ converges to $u \in H^1$ if the sequence of linear interpolations associated to u^N (also denoted u^N) converges to u in the H^1 norm.

Let us recall that $HS^N(u^N)$ is the Hessian matrix of S^N at u^N and can be interpreted as a bilinear form. We prove the following proposition.

Proposition 5.1. *For any sequence $u^N \in \mathbb{R}^N$ converging to $u \in H^1$, we have*

- $S^N(u^N) \xrightarrow[N\to\infty]{} S(u) < \infty$,
- *for any sequence h^N converging to h*: $\nabla S^N(u^N) \cdot h^N \xrightarrow[N\to\infty]{} D_u S(h)$,
- *for any sequences h^N, k^N converging to h, k*:

$$HS^N\big(u^N\big)\big(h^N, k^N\big) \xrightarrow[N\to\infty]{} D_u^2 S(h,k).$$

If u is twice differentiable $D_u S(h) = \int_0^1 \frac{\delta S}{\delta \phi}(u)h$ and if k is twice differentiable $D_u^2 S(h,k) = \int_0^1 h\mathcal{H}_u Sk$.

Proof. Let $u^N \in \mathbb{R}^N$ be a sequence converging to $u \in H^1$, then u^N converges uniformly on $[0,1]$ to u, so by dominated convergence,

$$\frac{1}{N}\sum_{i=1}^{N} V\big(u_i^N\big) \xrightarrow[N\to\infty]{} \int_0^1 V\big(u(x)\big)\,\mathrm{d}x. \tag{5.2}$$

The convergence in H^1 directly ensures us that

$$\frac{1}{N}\sum_{i=1}^{N} N^2\big(u_{i+1}^N - u_i^N\big)^2 = \int_0^1 \big|\big(u^N\big)'(x)\big|^2\,\mathrm{d}x \xrightarrow[N\to\infty]{} \int_0^1 \big|u'(x)\big|^2\,\mathrm{d}x. \tag{5.3}$$

Let $h^N \in \mathbb{R}^N$ be some sequence converging to $h \in H^1$ then we have

$$\begin{aligned}\nabla S^N\big(u^N\big)\cdot h^N &= \sum_{i=1}^{N} \frac{\partial S^N}{\partial x_i}\big(u_i^N\big)h_i^N = \frac{1}{N}\sum_{i=1}^{N} \gamma N^2\big(u_{i+1}^N - u_i^N\big)\big(h_{i+1}^N - h_i^N\big) + V'\big(u_i^N\big)h_i^N \\ &\xrightarrow[N\to\infty]{} \int_0^1 \gamma u'h' + V'(u)h \end{aligned}\tag{5.4}$$

by L^2 convergence of the derivatives and dominated convergence. Lastly, the convergence of the Hessian is completely similar. □

5.2. *Convergence of the eigenvalues*

Let us consider a sequence of points $u^N \in \mathbb{R}^N$ converging to u in H^1. We need to estimate the convergence of the eigenvalues $(N\lambda_{k,N})_{1\le k\le N}$ of $N \cdot HS^N(u^N)$ to the eigenvalues $(\lambda_k)_{1\le k}$ of $\mathcal{H}_u S$.

The convergence of a single eigenvalue $N\lambda_{k,N}$ for k fixed, is obvious from Proposition 5.1. The control of the convergence for all the eigenvalues is complex because of the higher eigenvalues (e.g., $\lambda_{N,N}$). This problem is closely related to the discrepancy between the eigenvalues of $\frac{\gamma}{N}\Delta^N$ and $\gamma\Delta$, the discrete Laplacian (defined by (3.3)) and the Laplacian. We denote $\lambda_{N,k}^0$, λ_k^0 their respective eigenvalues in the increasing order. For Dirichlet boundary conditions, we have

$$e_{k,N} = N\lambda_{N,k}^0 - \lambda_k^0 = \gamma\left[4N^2\sin^2\left(\frac{k\pi}{2N}\right) - \pi^2k^2\right]. \tag{5.5}$$

Then $e_{N,N} = \gamma N^2(4-\pi^2)$ does not converge to 0. The following proposition adapted from [31] gives us a control of the approximation of the eigenvalues and eigenvectors.

Proposition 5.2. *Let us consider a sequence $u_N \in \mathbb{R}^N$ converging to $u \in C^2$ and such that $\|u_N - u\|_\infty = \mathrm{O}(\frac{1}{N^2})$. We have*:

(i) *there exist $\alpha \in [0,1[$ and a constant C_1 such that for all N and $k < \alpha N$*

$$|N\lambda_{N,k} - \lambda_k - e_{k,N}| \le \frac{C_1}{N^2}, \tag{5.6}$$

(ii) *there exists a constant C_2 such that $|e_{N,k}| \le C_2k^4N^{-2}$,*

(iii) *for a fixed $k \le N$, the normalized (in H^1) eigenvector $\phi_{k,N}$ of $HS^N(u^N)$ associated to $\lambda_{k,N}$ converges in H^1 to the eigenvector ϕ_k of $\mathcal{H}_u S$ associated to λ_k and we have, for all k*

$$\frac{\|\phi_{k,N}\|_\infty}{\|\phi_{k,N}\|_{2,N}} \le \frac{C}{\sqrt{N}}, \tag{5.7}$$

where the norm $\|\cdot\|_{2,N}$ *is defined by* (5.1).

Proof. The proposition is an adaptation of the results of [31] in our case since $NHS^N(u^N)$ is the finite difference approximation of the Sturm–Liouville operator $\mathcal{H}_u S$. The original statement in [31] concerns an approximating sequence u^N which is precisely the sequence $\widehat{u}^N$ of linear interpolations of u. If we take a sequence u^N, then for all $y \in \mathbb{R}^N$

$$N\left|HS^N\left(u^N\right)(y) - HS^N\left(\widehat{u}^N\right)(y)\right| = \sum_{i=1}^{N}\left|V''\left(u_i^N\right) - V''\left(u(x_i)\right)\right|y_i^2 \le C\left\|u^N - u\right\|_\infty \|y\|_2^2. \tag{5.8}$$

Since $\|u_N - u\|_\infty = \mathrm{O}(\frac{1}{N^2})$, we deduce that the difference between the eigenvalues of $NHS^N(u^N)$ and $NHS^N(\widehat{u}^N)$ is bounded by $\mathrm{O}(\frac{1}{N^2})$ which gives us the result. A similar control holds for the convergence of the eigenvectors. The last result (5.7) comes from the fact that for the eigenvectors of $\mathcal{H}_u S$ ([19], pp. 334–335), we have a constant C such that $\|\phi_k\|_\infty \le C\|\phi_k\|_{L^2}$. Then, since $\phi_{k,N}$ converges in H^1, it converges in L^∞ and L^2, then the result comes from the fact that $\|\phi_{k,N}\|_{2,N} \ge C\sqrt{N}\|\phi_{k,N}\|_{L^2}$. □

Remark 6. *The normalized eigenvector* $e_N = \frac{\phi_N}{\|\phi_N\|_{2,N}}$ *satisfies*

$$\|e_N\|_{\infty,N}^2 = \frac{\|\phi_N\|_{\infty,N}^2}{\|\phi_N\|_{2,N}^2} \le \frac{\|\phi_N\|_{L^\infty}^2}{N\|\phi_N\|_{L^2}^2} \le \frac{C}{N}\frac{\|\phi_N\|_{H^1}^2}{\|\phi_N\|_{L^2}^2} \le \frac{C}{N}. \tag{5.9}$$

Thus, this proves that the coordinates of the normalized eigenvectors in $\mathbb{R}^N$ *for the Euclidean norm are uniformly bounded by* $\mathrm{O}(\frac{1}{\sqrt{N}})$.

The following proposition from [31] states uniform estimates in the function ϕ of the eigenvalues of the Hessian operators $\mathcal{H}_\phi S$ and $HS^N(\phi^N)$.

Proposition 5.3. *Let* ϕ_1^N, ϕ_2^N *be sequences converging in* H^1 *to* ϕ_1, ϕ_2, *then for all* N, k

$$\left|\lambda_{k,N}^1 - \lambda_{k,N}^2\right| \le C, \qquad \left|\lambda_k^1 - \lambda_k^2\right| \le C \tag{5.10}$$

and $\lambda_k^i = \pi^2 k^2 + \int_0^1 V''(\phi_i(x))\,\mathrm{d}x + \mathrm{O}(\frac{1}{k^2})$ *for* $i = 1, 2$.

Remark 7. *This proposition shows the convergence of the infinite product of the ratio of eigenvalues denoted by* $D(\phi, \psi)$

$$\prod_{k=1}^{N}\frac{\lambda_k(\phi)}{\lambda_k(\psi)} = \prod_{k=1}^{N}\left[1 + \frac{\lambda_k(\phi) - \lambda_k(\psi)}{\lambda_k(\psi)}\right] \xrightarrow[N\to\infty]{} \prod_{k=1}^{\infty}\frac{\lambda_k(\phi)}{\lambda_k(\psi)} = D(\phi, \psi) \tag{5.11}$$

since

$$\left|\frac{\lambda_k(\phi) - \lambda_k(\psi)}{\lambda_k(\psi)}\right| \le \frac{C}{k^2}. \tag{5.12}$$

5.3. *Product of eigenvalues*

We show the convergence of the product ratio of the eigenvalues of $HS^N(\phi^N)$ and $HS^N(\psi^N)$ to $D(\phi, \psi)$.

Proposition 5.4. *For any* ϕ^N, ψ^N *converging in* H^1 *to* ϕ, ψ *such that* $\mathcal{H}S(\psi)$ *and* $\mathcal{H}S(\phi)$ *do not have a zero eigenvalue, and that*

$$\left\|\phi^N - \phi\right\|_\infty \vee \left\|\psi^N - \psi\right\|_\infty \le \frac{C}{N^2}, \tag{5.13}$$

we have the convergence

$$\frac{\det(HS^N(\phi^N))}{\det(HS^N(\psi^N))} \xrightarrow[N\to\infty]{} D(\phi,\psi) = \prod_{k=1}^{+\infty} \frac{\lambda_k(\phi)}{\lambda_k(\psi)}. \tag{5.14}$$

Proof. The proof of the convergence comes from the fact that for small k the approximated eigenvalues are close to the continuous ones ($\lambda_{k,N} \approx \lambda_k$) whereas this is not the case for k close to N (Proposition 5.2). The eigenvalues $\lambda_{k,N}(\phi)$, $\lambda_{k,N}(\psi)$ are close at the first order in k uniformly on ϕ, ψ (Proposition 5.3). Therefore we decompose the product in two parts for small k (i.e., $k < \alpha N$ from Proposition 5.2) and large k.

Let us denote $\mu_{k,N}(\phi) = N\lambda_{N,k}(\phi^N) - \lambda_k(\phi) - e_{k,N}$. From Proposition 5.2, there exists $0 < \alpha < 1$ such that, for $k \le \alpha N$, $|\mu_{k,N}(\phi)| \le \frac{c}{N^2}$. The same holds for the sequence ψ^N. Then, we get,

$$\frac{N\lambda_{k,N}(\phi)}{\lambda_k(\phi)} \frac{\lambda_k(\psi)}{N\lambda_{k,N}(\psi)} = \frac{1+\theta_{k,N}(\phi)}{1+\theta_{k,N}(\psi)} = 1 + \frac{\theta_{k,N}(\phi)-\theta_{k,N}(\psi)}{1+\theta_{k,N}(\psi)}, \tag{5.15}$$

where $\theta_{k,N}(\phi) = \lambda_k(\phi)^{-1}(e_{k,N} + \mu_{k,N}(\phi))$. Let us remark that for $k \le \alpha N$

$$\left|\theta_{k,N}(\psi)\right| \le \frac{C}{k^2}\left(\frac{k^4}{N^2} + \frac{1}{N^2}\right) \le C\left(\alpha^2 + \frac{1}{N^2}\right) \tag{5.16}$$

thus if we take α small enough and N large enough, we have $|\theta_{k,N}(\psi)| < \frac{1}{2}$. Hence we obtain

$$\left|\ln \prod_{k=1}^{\alpha N} \frac{N\lambda_{k,N}(\phi)}{\lambda_k(\phi)} \frac{\lambda_k(\psi)}{N\lambda_{k,N}(\psi)}\right| \le 2\sum_{k=1}^{\alpha N} \left|\theta_{k,N}(\phi) - \theta_{k,N}(\psi)\right| \le \frac{2C\alpha}{N} \tag{5.17}$$

since from Proposition 5.3, $|\theta_{k,N}(\phi) - \theta_{k,N}(\psi)| \le \frac{C}{N^2}$.

For $k > \alpha N$ we proceed similarly. Let us write

$$\frac{N\lambda_{k,N}(\phi)}{\lambda_k(\phi)} \frac{\lambda_k(\psi)}{N\lambda_{k,N}(\psi)} = \frac{1+\theta'_{k,N}}{1+\theta'_k} = 1 + \frac{\theta'_{k,N} - \theta'_k}{1+\theta'_k}, \tag{5.18}$$

where $\theta'_{k,N} = \lambda_{k,N}(\psi)^{-1}(\lambda_{k,N}(\phi) - \lambda_{k,N}(\psi))$ and alike for θ'_k. From Proposition 5.3, we get for all k and $N > N_0$, that $|\theta'_{k,N}| \vee |\theta'_k| \le \frac{C}{k^2}$. Thus we obtain

$$\left|\ln \prod_{k=\alpha N}^{N} \frac{N\lambda_{k,N}(\phi)}{\lambda_k(\phi)} \frac{\lambda_k(\psi)}{N\lambda_{k,N}(\psi)}\right| \le \sum_{k=\alpha N}^{N} \frac{C}{k^2}\left(1 + \frac{C}{k^2}\right) \le \frac{C}{N} \tag{5.19}$$

which finishes the proof. □

In fact, we need a slightly different convergence.

Corollary 5.5. *Let be ϕ^N, ψ^N converging to ϕ, ψ such that*

$$\|\phi^N - \phi\|_{L^2} \vee \|\psi^N - \psi\|_{L^2} \le \frac{C}{N}. \tag{5.20}$$

Then we have

$$\frac{\det(HS^N(\phi^N))}{\det(HS^N(\psi^N))} \xrightarrow[N\to+\infty]{} D(\phi,\psi). \tag{5.21}$$

Proof. From the previous proposition, we get that

$$\frac{\det(HS^N(\widehat{\phi}^N))}{\det(HS^N(\widehat{\psi}^N))} \xrightarrow[N\to+\infty]{} D(\phi,\psi), \tag{5.22}$$

where $\widehat{\phi}^N$ (resp. $\widehat{\psi}^N$) is the linear interpolation of ϕ (resp. ψ). So we prove

$$D_N = \frac{\det(HS^N(\widehat{\phi}^N))}{\det(HS^N(\widehat{\psi}^N))}\left[\frac{\det(HS^N(\phi^N))}{\det(HS^N(\psi^N))}\right]^{-1} = \prod_{k=1}^{N} \frac{1+\theta_k(\phi)}{1+\theta_k(\psi)} \xrightarrow[N\to\infty]{} 1, \tag{5.23}$$

where $\theta_k(\phi) = \lambda_{k,N}(\phi^N)^{-1}(\lambda_{k,N}(\widehat{\phi}^N) - \lambda_{k,N}(\phi^N))$. From the fact that $\|\phi^N - \phi\|_{L^2} \le \frac{C}{N}$ we obtain $\|\phi^N - \widehat{\phi}^N\|_{L^2} \le \frac{C'}{N}$. Then for all $y \in \mathbb{R}^N$, we have

$$\begin{aligned}\left|HS^N(\phi^N)(y) - HS^N(\widehat{\phi}^N)(y)\right| &= \frac{1}{N}\sum_{i=1}^{N}\left|V''(\phi_i^N) - V''(\phi(x_i))\right||y_i|^2 \\ &\le \frac{C}{N}\sum_{i=1}^{N}\left|\phi_i^N - \phi(x_i)\right||y_i|^2 \le \frac{C}{\sqrt{N}}\|\phi^N - \widehat{\phi}^N\|_{L^2}\|y\|_{4,N}^2 \le \frac{C}{N^{3/2}}\|y\|_{2,N}^2.\end{aligned}$$

Therefore we get that $|\lambda_{k,N}(\phi^N) - \lambda_{k,N}(\widehat{\phi}^N)| \le \frac{C}{N^{3/2}}$. The same holds for ψ.

Then, we obtain

$$\left|\theta_k(\psi)\right| \le \frac{CN}{k^2} \times \frac{1}{N^{3/2}} \le \frac{C}{k^2\sqrt{N}} \le \frac{1}{2} \tag{5.24}$$

for N sufficiently large.

Thus we get

$$\left|\ln[D_N]\right| \le \sum_{k=1}^{N} \frac{|\theta_k(\phi) - \theta_k(\psi)|}{1+\theta_k(\psi)} \le 2\sum_{k=1}^{N}\left|\theta_k(\phi)\right| + \left|\theta_k(\psi)\right| \le 4C\sum_{k=1}^{N}\frac{1}{k^2\sqrt{N}}. \tag{5.25}$$

Then let us fix $\eta > 0$, we have

$$\left|\ln[D_N]\right| \le C\sum_{k=1}^{\eta\sqrt[3]{N}} \frac{1}{k^2\sqrt{N}} + C\sum_{k=\eta\sqrt[3]{N}}^{N}\frac{1}{k^2\sqrt{N}} \le C\eta N^{-1/6} + \frac{C}{\eta^2}N^{-1/6}. \tag{5.26}$$

Therefore we get $\limsup_{N\to\infty}|\ln[D_N]| = 0$ which proves the proposition. □

5.4. *Approximated stationary points*

The last property we need to check is that for each stationary point of S, there exists a unique sequence of stationary points of S^N converging to this stationary point. Moreover, to ensure the limit of the ratio of eigenvalues, this convergence has to be fast enough (see Corollary 5.5). To this aim, we have the following proposition.

Proposition 5.6. *There exist C, N_0, such that for all $N > N_0$, there is for each minimum (resp. saddle point) ϕ of S a unique minimum (resp. saddle point) ϕ^N of S^N such that*

$$\left\|\phi - \phi^N\right\|_{L^2} \le \frac{C}{N}. \tag{5.27}$$

Proof. Since by Assumption 2.4, there is a finite number of saddles and stable points then we only need to prove the proposition for a given saddle or minimum. Let ϕ be a minimum, we prove that there is sequence ϕ^N of minima of S^N such that

$$\|\phi^N - \widehat{\phi}^N\|_{L^2} \leq \frac{C}{N}. \tag{5.28}$$

The result (5.27) follows from (5.28) since we already have that

$$\|\phi - \widehat{\phi}^N\|_{L^2} \leq \|\phi - \widehat{\phi}^N\|_{\infty} \leq \frac{C}{N^2}. \tag{5.29}$$

In order to prove (5.28), we use a fixed point theorem. Let us consider the ball $B_{C/\sqrt{N}}$ of radius $\frac{C}{\sqrt{N}}$ in the $\|\cdot\|_{2,N}$ norm where C is a constant we will fix later. We want to find $z^0 \in B_{C/\sqrt{N}}$ such that $\nabla S^N(\widehat{\phi}^N + z^0) = 0$. In that case we will have $\phi^N = \widehat{\phi}^N + z$ and

$$\|\phi^N - \widehat{\phi}^N\|_{L^2}^2 \leq \frac{1}{N}\sum_{i=1}^N |z_i|^2 \leq \frac{C}{N^2}. \tag{5.30}$$

By a Taylor expansion of the gradient we have

$$\nabla S^N(\widehat{\phi}^N + z)_i = \nabla S^N(\widehat{\phi}^N)_i + (HS^N(\widehat{\phi}^N)z)_i + g_i(z), \tag{5.31}$$

where g_i is the remainder which can take the form

$$g_i(z) = \int_0^1 (1-t)\frac{\partial^3 S^N}{\partial z_i^3}(\widehat{\phi}^N + tz)z_i^2\,\mathrm{d}t = \frac{1}{N}\int_0^1 (1-t)V'''(\phi_i + tz_i)z_i^2\,\mathrm{d}t. \tag{5.32}$$

Then we have for all $z, y \in B_{C/\sqrt{N}}$

$$|g_i(z)| \leq \frac{C_0}{N}z_i^2 \quad \text{and} \quad |g_i(x) - g_i(y)| \leq \frac{C_0}{N}|z_i^2 - y_i^2| \leq \frac{2C_0}{N^{3/2}}|z_i - y_i|. \tag{5.33}$$

Let us also remark that since ϕ is a stationary point for the potential S, thus we have $-\gamma\phi''(x_i) + V'(\phi(x_i)) = 0$. Therefore we get

$$\begin{aligned}|\nabla S^N(\widehat{\phi}^N)_i| &= \left|\nabla S^N(\widehat{\phi}^N)_i - \frac{1}{N}(-\gamma\phi''(x_i) + V''(\phi(x_i)))\right| \\ &= \frac{1}{N}|\gamma N^2(\phi(x_{i+1}) - 2\phi(x_i) + \phi(x_{i-1})) - \gamma\phi''(x_i)| \leq \frac{C_1}{N^2}.\end{aligned} \tag{5.34}$$

For N sufficiently large $HS^N(\widehat{\phi}^N)$ is not degenerate then z^0 is solution of the fixed point equation

$$z^0 = HS^N(\widehat{\phi}^N)^{-1}(-\nabla S^N(\widehat{\phi}^N) - g_i(z^0)) = F(z^0). \tag{5.35}$$

The $(2, N)$-norm (defined by (5.1)) of $HS^N(\widehat{\phi}^N)^{-1}$ is bounded by the inverse of the smallest eigenvalue (in absolute value). Then $\|HS^N(\widehat{\phi}^N)^{-1}\|_{2,N} \leq C_2 N$. For $z \in B_{C/\sqrt{N}}$, we get

$$\begin{aligned}\|F(z)\|_{2,N}^2 &\leq \|HS^N(\widehat{\phi}^N)^{-1}\|_2^2\left(\|\nabla S^N(\widehat{\phi}^N)\|_2^2 + \sum_{i=1}^N |g_i(z)|^2\right) \leq C_2^2 N^2\left(\frac{C_1^2}{N^3} + \frac{C_0^2}{N^2}\|z\|_{4,N}^4\right) \\ &\leq C_1'\left(\frac{1}{N} + \|z\|_{2,N}^4\right) \leq C_1'\left(\frac{1}{N} + \frac{C^4}{N^2}\right) \leq \frac{C^2}{N}\end{aligned} \tag{5.36}$$

for C sufficiently small. Therefore $F(B_{C/\sqrt{N}}) \subset B_{C/\sqrt{N}}$. We also have for $z, y \in B_{C/\sqrt{N}}$, $F(y) - F(z) = HS^N(\widehat{\phi}^N)^{-1}(-g_i(y) + g_i(z))$.

Then

$$\|F(y) - F(z)\|_{2,N}^2 \leq C_2 N^2 \sum_{i=1}^{N} \left|-g_i(y) + g_i(z)\right|^2 \leq \frac{C_2'}{N} \|y - z\|_{2,N}^2.$$

Thus F is a contraction for N sufficiently large. By the fixed point theorem, there exists a unique $z^0 \in B_{C/\sqrt{N}}$ solution of $z^0 = F(z^0)$ which proves Proposition 5.6. □

6. Estimates

6.1. *Description*

In this section, we compute uniformly in the dimension the expectation of the transition times. We proceed as in [3] and use the potential theory developed in [10]. Let us consider the N-dimensional diffusion

$$\mathrm{d}Y_t = -\nabla S_N(Y_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}B_t \tag{6.1}$$

which comes from (2.42) with the time change $Y_{h_N t} = X_t$. We denote by μ^N the invariant measure for the process Y

$$\mu^N(\mathrm{d}x) = \mathrm{e}^{-S^N(x)/\varepsilon}\,\mathrm{d}x. \tag{6.2}$$

Let us recall for clarity the norms defined for $y \in \mathbb{R}^N$ and $p \geq 1$

$$\|y\|_{p,N}^p = \sum_{i=1}^{N} |y_i|^p, \qquad \|y\|_{\infty,N} = \max_{i=1,\ldots,N} |y_i|. \tag{6.3}$$

Remark 8. *As in the previous section, we associate to a point $y \in \mathbb{R}^N$ its linear interpolation on* $[0, 1]$ *between the points (x_i, y_i) (x_i is given by* (2.45),(2.46)) *that we denote by y. Let us consider the L^p norm of y on* $[0, 1]$, *we have for all $p \in [1, +\infty]$*

$$\frac{1}{(4N)^{1/p}} \|y\|_{p,N} \leq \|y\|_{L^p} = \left[\int_0^1 \left|y(x)\right|^p \mathrm{d}x\right]^{1/p} \leq \frac{1}{N^{1/p}} \|y\|_{p,N}. \tag{6.4}$$

This can be done using the Riesz–Thorin Theorem, recalling that

$$\frac{1}{4N} \|y\|_{1,N} \leq \|y\|_{L^1} \leq \frac{1}{N} \|y\|_{1,N} \quad \text{and} \quad \|y\|_{\infty,N} = \|y\|_{L^\infty}. \tag{6.5}$$

In order to introduce the other norms, we need the following a priori estimates on the eigenvalues of the Hessian of S^N. Let us recall the Hessian of S^N at a point $\phi^N \in \mathbb{R}^N$ is

$$HS^N(\phi^N)(h)_j = -\frac{1}{N}\left(\Delta^N h\right)_j + \frac{1}{N} V''\left(\phi^N(x_j)\right) h_j \quad \text{for } h \in \mathbb{R}^N \tag{6.6}$$

with the suitable boundary conditions.

Lemma 6.1 [31]. *For all $\phi^N \in \mathbb{R}^N$ such that $\|\phi^N\|_\infty < A$, the eigenvalues $(\lambda_{k,N}(\phi^N))_{k=1}^N$ of $HS^N(\phi^N)$ arranged in increasing order satisfy the bound*

$$m(A)k^2 - 1 \leq N\lambda_{k,N}\left(\phi^N\right) \leq M(A)k^2 + 1, \tag{6.7}$$

where $m(A)$ and $M(A)$ do not depend on N or ϕ^N (only on A).

Let us fix $\phi^N \in \mathbb{R}^N$. We consider the orthonormal eigenvectors $(v_l)_l$ of $HS^N(\phi^N)$. The decomposition of $h \in \mathbb{R}^N$ in this orthonormal basis is given by $h = \sum_{l=1}^N \widetilde{h}_i v_l$. For $p \in [1, \infty]$, we define the norms $\|h\|_{p,\mathcal{F}}$

$$\|h\|_{p,\mathcal{F}}^p = \sum_{i=1}^N |\widetilde{h}_i|^p, \qquad \|h\|_{\infty,\mathcal{F}} = \max_{i=1,\dots,N} |\widetilde{h}_i|. \tag{6.8}$$

As in [3], these are the norms we use to control the approximations of the potential around our stationary points. Let us note that the norms depend on the point ϕ^N.

Remark 9. *As in Section* 4.1.1 *in* [3], *the Hausdorff–Young Theorem can be adapted to the norms* $\|\cdot\|_{p,\mathcal{F}}$ *and* $\|\cdot\|_{p,N}$. *For all* $2 \le p \le +\infty$ *and* q *such that* $q^{-1} + p^{-1} = 1$, *we obtain*

$$\frac{1}{N}\|x\|_{p,N}^p \le C\left(\frac{1}{\sqrt{N}}\|x\|_{q,\mathcal{F}}\right)^p. \tag{6.9}$$

In fact, let $T : \mathbb{R}^N \to \mathbb{R}^N$ *be the linear mapping* $T(y) = \sum_{k=0}^{N-1} y_k v_{N,l}(z_i^*)$. *By definition,* $\|Ty\|_{p,\mathcal{F}} = \|y\|_{p,N}$. *The proof of* (6.9) *is an application of the Riesz–Thorin Theorem, between* $p = 2$ *and* $p = \infty$. *On one hand, we have* $\|Ty\|_{2,N}^2 = \|y\|_{2,N}^2$ *since the eigenvectors form a orthonormal basis. On the other hand, we have* $\|Ty\|_{\infty,N} \le \frac{C}{\sqrt{N}}\|y\|_{1,N}$ *since the coordinates of the eigenvectors of the basis are bounded by* $\frac{C}{\sqrt{N}}$ (*see Lemma* 5.2, *Eq.* (5.7)).

Let us recall the infinite dimensional situation. The process u starts from a minimum ϕ_{l_0} of S and reaches the set of minima $\mathcal{M}_l$. We denote by $\widehat{S}_0 = \widehat{S}(\phi_{l_0}, \mathcal{M}_l)$ the height of the saddle points defined by (2.14). The idea is to construct the same graph as in the infinite dimensional case with the approximated stationary points (minima and saddles) given by Proposition 5.6. The difficulty here is that, even if we know that the approximated saddle points converge towards the infinite dimensional saddle points, their potential are not strictly equal. So we construct this graph from the infinite dimensional graph defined in Section 2.3.

By Assumption 2.4, for all N sufficiently large, we have a finite set $\mathcal{M}^N = \{x_i^*(N)\}$ of minima of S^N. From Proposition 5.2 and Proposition 5.6, we deduce that a unique sequence of minima $(x_{l_0}^*(N))_N$ converges to ϕ_{l_0}. Similarly, there is a subset $\mathcal{M}_l^N$ of $\mathcal{M}^N$ such that each minimum of $(\mathcal{M}_l^N)$ converges to a minimum in $\mathcal{M}_l$. For each edge $\widehat{\sigma}_k$ of the infinite dimensional graph, by Proposition 5.6, there is a unique saddle point z_k^* of S^N satisfying (5.27).

We construct a graph for the finite dimensional case. We replace formally in the infinite dimensional graph (defined in Section 2.3) the minima and saddle points by their (unique) finite dimensional approximation. The vertices are subsets (K_j) of the minima $\mathcal{M}^N$. The edges are saddle points of the finite dimensional potential S^N.

To each saddle point z_k^*, we associate a weight

$$w_k^* = \frac{|\lambda_N^-(z_k^*)| \mathrm{e}^{-S^N(z_k^*)/\varepsilon}}{\sqrt{|\det HS^N(z_k^*)|}}. \tag{6.10}$$

To each vertex K_j, we associate a variable $a_j = a(K_j) \in \mathbb{R}$. We denote by a_{i+} and a_{i-} the two variables associated to the vertices connected by the saddle point z_i^*.

We associate to this graph a quadratic form $Q^N(a)$, for a a real vector indexed by the vertices (K_j)

$$Q^N(a) = \sum_{z_l^*} w_l^* (a_{l+} - a_{l-})^2. \tag{6.11}$$

We distinguish K_0 the subset of $\mathcal{M}^N$ containing $x_{l_0}^*(N)$. The equivalent conductance, $C^*(N, \varepsilon)$, between the sets $x_{l_0}^*$ and $\mathcal{M}_l^N$ is defined by

$$C^*(N, \varepsilon) = \inf\bigl\{Q^N(a), a(K_0) = 1, a(J) = 0, \text{ for all } J \text{ such that } J \cap \mathcal{M}_l^N \ne \varnothing\bigr\}. \tag{6.12}$$

Remark 10. *The graph for the finite dimensional case is constructed from the infinite dimensional case, since by Proposition* 5.6 *finite stationary points converge to the infinite dimensional stationary points. All the other quantities* *are defined analogously, except for the weight of the saddle points (Eq. (6.10)) for which we must take care of the fact that the values of the potential at these saddle points are not rigorously the same. Therefore this leads to incorporate the exponential factor into the weight, whereas in the infinite dimensional case (Eq. (2.23)) we can omit it by factorizing it directly in the transition time. It is this exponential term which gives us the exponential order of the transition time already known via large deviation estimates.*

We recall the fundamental formula (6.15) proved in [10]. The expression of the expectation of the hitting time $\tau_\varepsilon^N(\mathcal{B}_\rho^N(x_{l_0}^*))$ is based on two quantities: the equilibrium potential and the capacity with respect to the sets $\mathcal{B}_\rho^N(x_{l_0}^*)$ and $\mathcal{B}_\rho^N(\mathcal{M}_l^N)$. The equilibrium potential, h^*, is defined by $h^*(x) = \mathbb{P}_x[\tau_\varepsilon^N(\mathcal{B}_\rho^N(x_{l_0}^*)) < \tau_\varepsilon^N(\mathcal{B}_\rho^N(\mathcal{M}_l^N))]$. The Dirichlet form, $\mathscr{E}^N$, associated with the diffusion process Y on $\mathbb{R}^N$ is

$$\mathscr{E}^N(h) = \varepsilon \int_{\mathbb{R}^N} \left\| \nabla h(x) \right\|_{2,N}^2 \mu^N(\mathrm{d}x). \tag{6.13}$$

The capacity is the evaluation of the Dirichlet form on h^*. The capacity also satisfies a variational principle. We have

$$\begin{aligned} \operatorname{cap}\bigl(\mathcal{B}_\rho^N(x_{l_0}^*), \mathcal{B}_\rho^N(\mathcal{M}_l^N)\bigr) &= \mathscr{E}^N(h^*) \\ &= \inf\bigl\{\mathscr{E}^N(h), h \in H^1(\mathbb{R}^N), h = 1 \text{ on } \mathcal{B}_\rho^N(x_{l_0}^*), h = 0 \text{ on } \mathcal{B}_\rho^N(\mathcal{M}_l^N)\bigr\}. \end{aligned} \tag{6.14}$$

The expectation of the hitting time is expressed by

$$\mathbb{E}_{\nu^N}\bigl[\tau_\varepsilon^N\bigl(\mathcal{B}_\rho^N(\mathcal{M}_l^N)\bigr)\bigr] = \frac{\int_{\mathbb{R}^N} h^*(x)\,\mathrm{d}\mu^N(x)}{\operatorname{cap}(\mathcal{B}_\rho^N(x_{l_0}^*), \mathcal{B}_\rho^N(\mathcal{M}_l^N))}, \tag{6.15}$$

where ν^N is a probability measure on $\partial\mathcal{B}_\rho^N(x_{l_0}^*)$.

6.2. *Capacity*

We prove that the capacity defined in (6.14) can be estimated by the equivalent conductance $C^*(N,\varepsilon)$ defined in (6.12).

Proposition 6.2. *For all* $\varepsilon < \varepsilon_0$ *and* ρ, *we have*

$$\operatorname{cap}\bigl(\mathcal{B}_\rho^N(x_{l_0}^*), \mathcal{B}_\rho^N(\mathcal{M}_l^N)\bigr) = \varepsilon\sqrt{2\pi\varepsilon}^{N-2} C^*(N,\varepsilon)\bigl(1 + \psi_1(\varepsilon, N)\bigr), \tag{6.16}$$

where $\limsup_{N\to+\infty} |\psi_1(\varepsilon, N)| < \sqrt{\varepsilon}|\ln(\varepsilon)|^{3/2}$ *for all* $N > N_0$.

The proof of this result is an adaptation to the case of a finite number of saddle points of Proposition 4.3 in [3]. The estimate of the capacity is made in two steps: an upper bound and a lower bound.

6.2.1. *Upper bound*

We have the following proposition.

Proposition 6.3. *For all* $\varepsilon < \varepsilon_0$ *and* ρ, *we have*

$$\operatorname{cap}\bigl(\mathcal{B}_\rho^N(x_{l_0}^*), \mathcal{B}_\rho^N(\mathcal{M}_l^N)\bigr) \le \varepsilon\sqrt{2\pi\varepsilon}^{N-2} C^*(N,\varepsilon)\bigl(1 + \psi_u(\varepsilon, N)\bigr), \tag{6.17}$$

where $\limsup_{N\to\infty} |\psi_u(\varepsilon, N)| < \sqrt{\varepsilon}|\ln(\varepsilon)|^{3/2}$.

Proof. The proof of this upper bound follows the proof of Lemma 4.4 in [3]. To obtain an upper bound for the capacity, we just estimate the Dirichlet form on a test function h^+. h^+ is defined on some neighborhood $C_\delta^N(z_i^*)$ of each saddle point z_i^* for some $\delta > 0$ small enough.

In the local orthonormal basis (given by coordinates $y^{(i)} \in \mathbb{R}^N$) of the saddle point z_i^*, the neighborhood $C_\delta^N(z_i^*)$ is defined by

$$C_\delta^N(z_i^*) = \left\{ y^{(i)} \in \mathbb{R}^N \colon |y_l^{(i)}| \le \delta \frac{r_l}{\sqrt{|\lambda_{N,l}|}}, 0 \le l \le N-1 \right\} + z_i^*, \tag{6.18}$$

where (r_l) is a sequence satisfying $\sum_l \frac{r_l^{3/2}}{l^{3/2}} < \infty$ and $(\lambda_{N,l})_l$ are the eigenvalues in the increasing order of $HS^N(z_i^*)$. Let us denote $C_\delta^N = \bigcup_i C_\delta^N(z_i^*)$.

Let us consider

$$S_{N,\delta} = \left\{ x, S^N(x) \ge S^N(z_i^*) + c\delta^2, \forall i \right\}. \tag{6.19}$$

The set $(S_{N,\delta} \cup C_\delta^N)^c$ contains a finite number of connected components denoted D_j since each of them contains at least a minimum x_j^* (which are in finite number by Assumption 2.4). For each connected component D_j, we define h^+ to be the constant $a_j \in [0,1]$. For a saddle z_i^*, we denote D_{i+} and D_{i-} the connected components attained from z_i^* when $y^{(i)} = (\delta\sigma_0, 0)$ and $y^{(i)} = (-\delta\sigma_0, 0)$ respectively.

On $S_{N,\delta} \setminus C_\delta^N$, we take h^+ of class C^1 and such that $\|\nabla h^+\|_{2,N} \le \frac{c_1}{\delta}$. Then we define h^+ on each $C_\delta^N(z_i^*)$ in the local coordinates, by $h^+(y^{(i)}) = f_i(y_0^{(i)})$ where

$$f_i(y_0) = \left(a^{i-} - a^{i+}\right) \frac{\int_{y_0}^{\delta\sigma_0} \mathrm{e}^{-|\lambda_{N,0}|t^2/2\varepsilon}\, \mathrm{d}t}{\int_{-\delta\sigma_0}^{\delta\sigma_0} \mathrm{e}^{-|\lambda_{N,0}|t^2/2\varepsilon}\, \mathrm{d}t} + a^{i+}. \tag{6.20}$$

Therefore, we have to estimate $\mathcal{E}^N(h^+) = \sum_i I_1(i) + I_2$ with

$$I_1(i) = \varepsilon \int_{C_\delta^N(z_i^*)} \left\| \nabla h^+(x) \right\|_{2,N}^2 \mathrm{e}^{-S^N(x)/\varepsilon}\, \mathrm{d}x, \qquad I_2 = \varepsilon \int_{S_{N,\delta} \setminus B_\delta^N} \left\| \nabla h^+(x) \right\|_{2,N}^2 \mathrm{e}^{-\frac{S^N(x)}{\varepsilon}}\, \mathrm{d}x. \tag{6.21}$$

Taking $\delta = K\sqrt{\varepsilon |\ln \varepsilon|}$, the integrals $I_1(i)$ give us the right asymptotics and are estimated by an adaptation of Lemma 4.4 from [3]. The quadratic approximation of the potential on the sets $C_\delta^N(z_i^*)$ is a consequence of Remark 9 and of the choice of the sets $C_\delta^N(z_i^*)$. The integral I_2 is computed by following the same method as in Lemma 4.6 in [3].

Therefore, we obtain that for all $(a_j)_j$, for $N \ge N_0(\varepsilon)$

$$\operatorname{cap}\left(\mathcal{B}_\rho^N(x^*), \mathcal{B}_\rho^N(\mathcal{M}_l^N)\right) \le \sum_i \varepsilon \sqrt{2\pi\varepsilon}^{N-2} \frac{(a^{i-} - a^{i+})^2 |\lambda_{N,0}| \mathrm{e}^{-S^N(z_i^*)/\varepsilon}}{\sqrt{|\det(HS^N(z_i^*))|}} \left(1 + A_1 \sqrt{\varepsilon} \left|\ln(\varepsilon)\right|^{3/2}\right).$$

Taking the minimum of the right-hand side over a, we get the result (6.17). □

6.2.2. *Lower bound*

We now prove the corresponding lower bound.

Proposition 6.4. *For all $\varepsilon < \varepsilon_0$ and ρ, we have*

$$\operatorname{cap}\left(\mathcal{B}_\rho^N(x^*), \mathcal{B}_\rho^N(\mathcal{M}_l^N)\right) \ge \varepsilon \sqrt{2\pi\varepsilon}^{N-2} C^*(N,\varepsilon)\left(1 + \psi_l(\varepsilon, N)\right), \tag{6.22}$$

where $\limsup_{N\to\infty} |\psi_l(\varepsilon, N)| < \sqrt{\varepsilon} |\ln(\varepsilon)|^{3/2}$.

Proof. The proof is adapted from [3]. For a saddle point z_i^*, we take a narrow corridor from one (local) minimum to another one and minimize the Dirichlet form on the union of these corridors. In [3], this corridor was a rectangle because of the particular case considered. In this article, we have to be more precise about their construction. We use the same notations as in the proof of the upper bound.

Let us fix δ_0. We consider the subset of $\mathbb{R}^{N-1}$

$$C_\delta^{N,\perp}(z_i^*) = \left\{ y^{(i)} \in \mathbb{R}^N \colon \left|y_l^{(i)}\right| \le \delta \frac{r_l}{\sqrt{|\lambda_{N,l}|}}, 1 \le l \le N-1 \right\} \tag{6.23}$$

and we define $C_\delta^N(z_i^*) = [-\delta_0, \delta_0] \times C_\delta^{N,\perp}(z_i^*) + z_i^*$. We denote by x_{i-}^* and x_{i+}^* the two minima of the basins surrounding z_i^*.

Let $(\gamma_0(s))_{s\in[-s_-,s_+]}$ be a regular C^2 path from x_{i-} to x_{i+} with $\gamma_0(s) = z_i^* + (s, 0)$ for $s \in [-\delta_0, \delta_0]$. We also suppose that there is $\eta > 0$ for which $S^N(\gamma_0(s)) < S_0^N - 3\eta$ for $|s| \ge \delta_0$ and that $\|\gamma_0'(s)\|_{2,N} = 1$. Let, for all s, $A(s)$ be an isomorphism from $\mathbb{R}^{N-1}$ to $\gamma_0'(s)^\perp \subset \mathbb{R}^N$ of class C^1 in s and such that for $|s| < \delta_0$, $A(s)y = (0, y_1, \dots, y_{N-1})$. Then we construct a family of paths $\gamma(s, y_\perp)$ by

$$\gamma(s, y_\perp) = \gamma_0(s) + A(s) y_\perp. \tag{6.24}$$

Such a construction of a path γ_0 is always possible in the infinite dimensional setting (because of Assumption 2.4). Then taking the finite dimensional projection, it gives us a path for the finite dimensional case.

We define the corridor from x_{i-} to x_{i+}, for $\delta > 0$ small enough

$$C_\delta(z_i^*) = \left\{ x = \gamma(s, y_\perp), y_\perp \in C_\delta^{N,\perp}(z_i^*), \forall s \right\}. \tag{6.25}$$

Let h be the equilibrium potential which realizes the minimum of the Dirichlet form and define $a^{i\pm}(y_\perp) = h(x_{i\pm} + A(\pm s_\pm) y_\perp)$, the values near the minimum.

To estimate a lower bound, we are going to restrict the Dirichlet form on the union of the corridors $C_\delta(z_i^*)$:

$$\mathcal{E}^N(h) = \varepsilon \int_{\mathbb{R}^N} \|\nabla h\|_{2,N}^2 \mu^N(\mathrm{d}x) \ge \sum_i \varepsilon \int_{C_\delta(z_i^*)} \|\nabla h\|_{2,N}^2 \mu^N(\mathrm{d}x) = \varepsilon \sum I_5(i). \tag{6.26}$$

We define the function f_i on $C_\delta(z_i^*)$, by $f_i(s, y_\perp) = h(\gamma(s, y_\perp))$. The change of variable on $C_\delta(z_i^*)$ gives us the Jacobian $g_i(s, y_\perp) = \det(J\gamma)(s, y_\perp)$ and we obtain

$$I_5(i) \ge \int_{B_\delta^{N,\perp}(z_i^*)} \int_{-s_-}^{s_+} \left| \frac{\partial f_i}{\partial s} \right|^2 \mathrm{e}^{-S^N(\gamma(s,y_\perp))/\varepsilon} g_i(s, y_\perp)\, \mathrm{d}s\, \mathrm{d}y_\perp. \tag{6.27}$$

We take $y_\perp$ as a parameter then the second term is bounded below by the minimum over functions f_i of the integral

$$\int_{-s_-}^{s_+} \left| \frac{\partial f_i}{\partial s} \right|^2 \mathrm{e}^{-S^N(\gamma(s,y_\perp))/\varepsilon} g_i(s, y_\perp)\, \mathrm{d}s \tag{6.28}$$

with the conditions $f_i(-s_-, y_\perp) = h(x_{i-} + A(-s_-)y_\perp) = a^{i-}(y_\perp)$ and $f_i(s_+, y_\perp) = h(x_{i+} + A(s_+)y_\perp) = a^{i+}(y_\perp)$. This gives us a lower bound for the capacity.

A simple computation shows that the function f_i realizing this lower bound is

$$f_i(s, y_\perp) = \left(a^{i+}(y_\perp) - a^{i-}(y_\perp)\right) \frac{\int_{-s_-}^{s} \mathrm{e}^{S^N(s,y_\perp)/\varepsilon} g_i(s, y_\perp)^{-1}\, \mathrm{d}s}{\int_{-s_-}^{s_+} \mathrm{e}^{S^N(s,y_\perp)/\varepsilon} g_i(s, y_\perp)^{-1}\, \mathrm{d}s} + a^{i-}(y_\perp). \tag{6.29}$$

Inserting this function in the integral (6.27), we obtain

$$I_5(i) \ge \int_{C_\delta^{N,\perp}(z_i^*)} \left(a^{i+}(y_\perp) - a^{i-}(y_\perp)\right)^2 \left[\int_{-s_-}^{s_+} \mathrm{e}^{S^N(s,y_\perp)/\varepsilon} g_i(s, y_\perp)^{-1}\, \mathrm{d}s \right]^{-1} \mathrm{d}y_\perp. \tag{6.30}$$

The end of the proof comes from an upper bound of the integral uniformly for $y_\perp \in C_\delta^{N,\perp}(z_i^*)$. We write

$$\int_{-s_-}^{s_+} \mathrm{e}^{S^N(s,y_\perp)/\varepsilon} g_i(s, y_\perp)^{-1}\, \mathrm{d}s = I_6(i) + I_7(i), \tag{6.31}$$

where

$$I_6(i) = \int_{-\delta_0}^{\delta_0} \mathrm{e}^{S^N(s,y_\perp)/\varepsilon} g_i(s,y_\perp)^{-1}\,\mathrm{d}s \quad \text{and} \quad I_7(i) = \int_{|s|>\delta_0} \mathrm{e}^{S^N(s,y_\perp)/\varepsilon} g_i(s,y_\perp)^{-1}\,\mathrm{d}s. \tag{6.32}$$

As in Lemma 4.8 in [3], we control the quadratic approximation near the saddle z_i^* with the following lemma for which we omit the proof.

Lemma 6.5. *For all $y = (s, y_\perp) \in C_\delta^N(z_i^*)$, if the sequence $(r_l)_l$ satisfies $\sum_l \frac{r_l^{3/2}}{l^{3/2}} < \infty$, we have for $\delta_0 \geq \delta$*

$$\left| S^N\big(\gamma(s,y_\perp) + z_i^*\big) - S^N\big(\gamma(0,y_\perp) + z_i^*\big) + \frac{1}{2}|\lambda_{0,N}|s^2 \right| \leq A_6\delta_0^3, \tag{6.33}$$

$$\left| S^N\big(z_i^* + \gamma(0,y_\perp)\big) - S^N\big(z_i^*\big) - \frac{1}{2}\sum_{k=1}^{N-1}\lambda_{N,k}y_k^2 \right| < A_8\delta^3. \tag{6.34}$$

Following the proof of Lemma 4.7 in [3], we can also prove the existence of a constant A_6 such that for all N and $y_\perp$

$$I_6(i) \leq \mathrm{e}^{S^N(z_i^*+(0,y_\perp))/\varepsilon}\sqrt{\frac{2\pi\varepsilon}{|\lambda_{N,0}|}}\left(1 + A_6\frac{\delta_0^3}{\varepsilon}\right). \tag{6.35}$$

In addition, we need to prove an upper bound for the integral $I_7(i)$.

Lemma 6.6. *There exists a constant A_7 such that for all N and $y_\perp$*

$$I_7(i) \leq A_7\sqrt{N}\mathrm{e}^{(\widehat{S}-2\eta)/\varepsilon}, \tag{6.36}$$

where $\eta > 0$ is given by the definition of the path γ_0.

Proof. We have to be careful with the change of variable. Let us write the Jacobian matrix $J\gamma(s,y_\perp)$ in the local base $(\gamma_0'(s), \gamma_0'(s)^\perp)$, if we denote P_0 the projection on $\mathrm{Span}(\gamma_0'(s))$, we get the Jacobian matrix (written by blocks)

$$J\gamma(s,y_\perp) = \begin{pmatrix} 1 + P_0(A'(s)y_\perp) & 0 \\ * & A(s) \end{pmatrix} \tag{6.37}$$

since $\mathrm{Im}A(s) = \gamma_0'(s)^\perp$. Then, as $A(s)$ is an isometry, we obtain that

$$g_i(s,y_\perp) = \big|\det\big(J\gamma(s,y_\perp)\big)\big| = \big|1 + P_0\big(A'(s)y_\perp\big)\big| = 1 + \mathrm{O}(\delta). \tag{6.38}$$

Thus, for δ sufficiently small,

$$I_7(i) = \int_{|s|>\delta_0} \mathrm{e}^{S^N(s,y_\perp)/\varepsilon} g_i(s,y_\perp)^{-1}\,\mathrm{d}s \leq (1 + C\delta)\mathrm{e}^{(\widehat{S}-2\eta)/\varepsilon}(s_+ + s_-) \leq 2(s_+ + s_-)\mathrm{e}^{(\widehat{S}-2\eta)/\varepsilon}$$

since $S^N(s,y_\perp) < \widehat{S} - 2\eta$ for all $|s| > \delta_0$, and $y_\perp \in C_\delta^{N,\perp}$. Then by construction of the path we have that

$$s_+ + s_- \leq C\|x_{i-} - x_{i+}\|_{2,N} \leq C\sqrt{N}\|x_{i-} - x_{i+}\|_{L^2}. \tag{6.39}$$

□

We insert (6.35) and (6.36) in Eq. (6.30). Then we proceed as in the proof of Lemma 4.7 from [3] and we obtain

$$I_5(i) \geq \varepsilon\sqrt{\frac{|\lambda_{N,0}|}{2\pi\varepsilon}}\int_{B_\delta^{N,\perp}(z_i^*)} \big(a^{i+}(y_\perp) - a^{i-}(y_\perp)\big)^2 \mathrm{e}^{-S^N(z_i^*+(0,y_\perp))/\varepsilon}\,\mathrm{d}y_\perp\left[1 + A_6\frac{\delta_0^3}{\varepsilon} + A_7'\mathrm{e}^{-\frac{\eta}{\varepsilon}}\right]^{-1}. \tag{6.40}$$

Using Eq. (4.10) from Proposition 4.3, we obtain for all $y_\perp$, $|a^j(y_\perp) - a^j(0)| < \mathrm{e}^{-C/\varepsilon}$. Then using the approximation (6.34) and following the proof of Lemma 4.7 in [3], we obtain for $\delta = \sqrt{K\varepsilon|\ln\varepsilon|}$ and $\delta_0 = K'\varepsilon|\ln\varepsilon|$ with $K' > K$,

$$I_5(i) \geq \varepsilon\bigl(a^{i-} - a^{i+}\bigr)^2 \mathrm{e}^{-S^N(z_i^*)/\varepsilon} \frac{\sqrt{2\pi\varepsilon}^{N-2}|\lambda_{N,0}|}{\sqrt{|\det(HS^N(z_i^*))|}}\bigl(1 - A_5\sqrt{\varepsilon}\bigl|\ln(\varepsilon)\bigr|^{3/2}\bigr). \tag{6.41}$$

Equation (6.22) follows by minimizing along the $(a^j)_j$. □

6.3. *Uniform estimate of the mass of the equilibrium potential*

We prove estimates of the numerator of (6.15). Let us denote $x_{l_0}^* \in \mathbb{R}^N$ to be the closest minimum to ϕ_{l_0} in $L^2([0,1])$. We will prove an adaptation of Proposition 4.9 of [3].

Proposition 6.7. *For all $\varepsilon < \varepsilon_0$ and ρ, we have*

$$\int_{\mathbb{R}^N} h^*(x)\,\mathrm{d}\mu^N(x) = \frac{(2\pi\varepsilon)^N}{\sqrt{\det HS^N(x_{l_0}^*)}} \mathrm{e}^{-S^N(x_{l_0}^*)/\varepsilon}\bigl(1 + \psi_2(\varepsilon, N)\bigr), \tag{6.42}$$

where $|\psi_2(\varepsilon, N)| < \sqrt{\varepsilon}|\ln(\varepsilon)|^{3/2}$ for all $N > N_0$.

Proof. As the previous section, we define around the minimum $x_{l_0}^* \in \mathbb{R}^N$ a neighborhood $C_\delta^N(x_{l_0}^*)$. In the local orthonormal basis of the minimum $x_{l_0}^*$, the neighborhood $C_\delta^N(x_{l_0}^*)$ is defined by

$$C_\delta^N\bigl(x_{l_0}^*\bigr) = \left\{y \in \mathbb{R}^N\colon\ |y_l| \leq \delta \frac{r_l}{\sqrt{|\lambda_{N,l}|}}, 0 \leq l \leq N-1\right\} + x_{l_0}^*, \tag{6.43}$$

where (r_l) is a sequence satisfying $\sum_l \frac{r_l^{3/2}}{l^{3/2}} < \infty$ and $(\lambda_{N,l})_l$ are the eigenvalues in the increasing order of $HS^N(x_{l_0}^*)$.
We need to estimate

$$\int_{\mathbb{R}^N} h^*(x)\,\mathrm{d}\mu^N(x). \tag{6.44}$$

Let us remark that for $x \in \partial C_\delta^N(x^*)$, one of the coordinates is precisely $\delta r_k/\sqrt{\lambda_{k,N}}$ thus

$$S^N(x) > S^N\bigl(x^*\bigr) + \delta^2 r_k^2 - C\delta^3 > S^N\bigl(x^*\bigr) + c\delta^2. \tag{6.45}$$

We consider S' such that the set $\{\phi, S(\phi) \in]S(\phi_{l_0}), S']\}$ contains no stationary point. Then using Proposition 5.1, for all η small enough, there exists N_0 such that for $N > N_0$, $\{x, S^N(x) \in [S^N(x^*) + \frac{1}{2}c\delta^2, S' - \eta]\}$ contains no stationary point. We define the set $A = \{S^N(x) \leq S^N(x^*) + c\delta^2\} \setminus \mathcal{B}_\rho^N(x^*)$. Note also that for δ small enough, $C_\delta^N(x^*) \subset \mathcal{B}_\rho^N(x^*)$. Hence we decompose (6.44) in three parts:

$$\int_{\mathbb{R}^N} h^*(x)\,\mathrm{d}\mu^N(x) = I_8 + \int_{S^N(x) \geq S^N(x_{l_0}^*) + c\delta^2} h^*(x)\,\mathrm{d}\mu^N(x) + \int_A h^*(x)\,\mathrm{d}\mu^N(x). \tag{6.46}$$

To estimate the third integral we need a control on the equilibrium potential on the set A.

Lemma 6.8. *For all $\rho < \rho_0$ and $\eta > 0$ there exists $\varepsilon_0(\rho)$ such that for $\varepsilon < \varepsilon_0$ and $\delta > 0$, let $x \in A$, we have*

$$h_N^*(x) = \mathbb{P}_x\bigl[\tau_\varepsilon^N\bigl(B_\rho^N\bigl(x^*\bigr)\bigr) < \tau_\varepsilon^N\bigl(B_\rho^N\bigl(\mathcal{M}_l^N\bigr)\bigr)\bigr] \leq \mathrm{e}^{-(S' - S^N(x) - 2\eta)/\varepsilon}. \tag{6.47}$$

Proof. By definition of the set A all the paths from $x \in A$ to x^* attain a height of $S' - \eta$ at least. To prove this fact, let us take a path from x to x^*, it must attain its maximum $\widehat{S}$ at some time t_0. This maximum must satisfies $\widehat{S} > S^N(x^*) + c\delta^2$, since if it is not the case then from Eq. (6.45), the path must stay in $C^N_\delta(x^*)$ which contradicts the fact that x is in A. Then the minimal path from x to x^* must attain its maximum at a stationary point of height greater than $S^N(x^*) + c\delta^2$ thus of height greater than $S' - \eta$. This gives us an easy lower bound for the rate function on the set of transition from $x \in A$ to x^*. Then using the method from [25] and the uniform large deviation principle, we prove that

$$h^*(x) = \mathbb{P}_x\big[\tau^N_\varepsilon\big(\mathcal{B}^N_\rho\big(x^*\big)\big) < \tau^N_\varepsilon\big(\mathcal{B}^N_\rho\big(\mathcal{M}^N_l\big)\big)\big] \le \mathrm{e}^{-(S'-2\eta-S^N(x))/\varepsilon} \tag{6.48}$$

uniformly in N. □

We get from (6.46)

$$\int_{\mathbb{R}^N} h^*(x)\,\mathrm{d}\mu^N(x) \le I_8 + \int_{S^N(x)\ge S^N(x^*_{l_0})+c\delta^2} \mathrm{e}^{-S^N(x)/\varepsilon}\,\mathrm{d}x + \int_{S^N(x)\le S^N(x^*_{l_0})+c\delta^2} \mathrm{e}^{-(S'-2\eta)/\varepsilon}\,\mathrm{d}x, \tag{6.49}$$

where we have used the fact that h^* is bounded by one for the second integral and the previous lemma for the third integral. The integral I_8 gives the main contribution and is estimated as in the proof of Proposition 4.9 of [3] using the quadratic approximation of the potential on $C^N_\rho(x^*_{l_0})$. The second integral on the right-hand side is estimated as in the proof of Lemma 4.6 in [3].

We bound the third integral by the volume of the set $\{S^N(x) \le S^N(x^*_{l_0}) + c\delta^2\}$ which is bounded uniformly in N. In fact, from the bound on S^N and the convergence of $S^N(x^*_{l_0})$ to $S(\phi_{l_0})$, we get for δ sufficiently small

$$\big\{S^N(x) \le S^N\big(x^*_{l_0}\big) + c\delta^2\big\} \subset \big\{\big\|\nabla^N x\big\|^2_{2,N} + \|x\|^2_{2,N} < N\big(S(\phi_{l_0}) + c\big)\big\} \tag{6.50}$$

which is a deformed ball. The computation shows that this quantity is uniformly bounded in N.

We obtain the result since the order of magnitude of the two last integrals ($\mathrm{O}(\mathrm{e}^{-(S'-\eta)/\varepsilon})$) of (6.49) is much smaller than $I_8 = \mathrm{O}(\mathrm{e}^{-S^N(x^*_{l_0})/\varepsilon})$. □

6.4. *Finite dimensional formula*

The finite dimensional formula is now obtained with a uniform control in the dimension. From Proposition 5.6, we take $x^* = \phi^N_{l_0}$ where $\phi^N_{l_0}$ is the unique minimum of S^N such that

$$\big\|\phi_{l_0} - \phi^N_{l_0}\big\|_{L^2} \le \frac{C}{N}, \qquad \big\|\widehat{\phi}^N_{l_0} - \phi^N_{l_0}\big\|_\infty \le \frac{C}{\sqrt{N}}, \tag{6.51}$$

where $\widehat{\phi}^N_{l_0}$ is the linear interpolation of ϕ_{l_0}.

Proposition 6.9. *Let τ^N_ε be the transition time from $\mathcal{B}^N_\rho(\phi^N_{l_0})$ to $\mathcal{B}^N_\rho(\mathcal{M}^N_l)$, we have uniformly in N*

$$\mathbb{E}_{\phi^N_{l_0}}\big[\tau^N_\varepsilon\big] = \frac{2\pi \mathrm{e}^{S^N(\phi^N_{l_0})/\varepsilon}}{C^*(N,\varepsilon)\sqrt{\det HS^N(\phi^N_{l_0})}}\big(1 + \Psi(\varepsilon, N)\big), \tag{6.52}$$

where $C^*(N,\varepsilon)$ is the equivalent conductance and

$$\limsup_{N\to+\infty}\big|\Psi(\varepsilon, N)\big| \le C\sqrt{\varepsilon}|\ln\varepsilon|^{3/2}. \tag{6.53}$$

Proof. Inserting the estimates for the capacity (Proposition 6.2) and the numerator (Proposition 6.7) in Eq. (6.15) we conclude that

$$\mathbb{E}_{\nu^N}\big[\tau_\varepsilon^N\big] = \frac{2\pi e^{S^N(\phi_{l_0}^N)/\varepsilon}}{C^*(N,\varepsilon)\sqrt{\det HS^N(\phi_{l_0}^N)}}\big(1+\Psi_1(\varepsilon,N)\big), \tag{6.54}$$

where $\limsup_N |\Psi_1(\varepsilon,N)| < C\sqrt{\varepsilon}|\ln(\varepsilon)|^{3/2}$ and ν^N is a probability measure on $\partial\mathcal{B}_\rho^N(\phi_{l_0}^N)$. Now we use Proposition 4.3 to replace the measure ν^N by the point $\phi_{l_0}^N$. For $y \in \mathcal{B}_\rho^N(\phi_{l_0}^N)$, we have by definition

$$\big\|\phi_{l_0}^N - y\big\|_{L^2}^2 < \rho^2, \qquad \big|S^N\big(\phi_{l_0}^N\big) - S^N(y)\big| < \rho. \tag{6.55}$$

Then from Proposition 5.6, we have N_0 such that for $N \ge N_0$

$$\|\phi_{l_0} - y\|_{L^2}^2 < 2\rho^2, \qquad \big|S(\phi_{l_0}) - S^N(y)\big| < 2\rho. \tag{6.56}$$

Thus since V is regular, we obtain $|\|\phi_{l_0}'\|_{L^2}^2 - \|y'\|_{L^2}^2| < C\rho$.

Let $z = y - \phi_{l_0}$, we have by integration by parts

$$\big|\big\|\phi_{l_0}' + z'\big\|_{L^2}^2 - \big\|\phi_{l_0}'\big\|_{L^2}^2\big| = \big|2\langle\phi_{l_0}', z'\rangle + \big\|z'\big\|_{L^2}^2\big| = \big|-2\langle\phi_{l_0}'', z\rangle + \big\|z'\big\|_{L^2}^2\big| < C\rho \tag{6.57}$$

since ϕ_{l_0} is regular as a classical solution of a differential equation. Then we obtain by the Cauchy–Schwarz inequality

$$\big\|z'\big\|_{L^2}^2 \le C\rho + 2\big\|\phi_{l_0}''\big\|_{L^2}\|z\|_{L^2} \le \big(C + 2\big\|\phi_{l_0}''\big\|_{L^2}\big)\rho. \tag{6.58}$$

Thus we get

$$\big\|y - \phi_{l_0}^N\big\|_\infty \le \|y - \phi_{l_0}\|_\infty \le C'\|y - \phi_{l_0}\|_{H^1} = C'\|z\|_{H^1} \le C''\sqrt{\rho}. \tag{6.59}$$

Using Proposition 4.3, we get that for all $N \ge N_0$

$$\big|\mathbb{E}_{\nu^N}\big[\tau_\varepsilon^N\big] - \mathbb{E}_{\phi_{l_0}{}^N}\big[\tau_\varepsilon^N\big]\big| \le e^{(\widehat{S}-2\eta)/\varepsilon} \tag{6.60}$$

which gives us (6.52) since the exponential asymptotics of (6.54) is greater than $e^{(\widehat{S}-\eta)/\varepsilon}$. □

6.5. *Proof of Theorem* 2.6

From Proposition 6.9 applied to the finite diffusion approximation where the minima and saddle points are given by Proposition 5.6, we have

$$\mathbb{E}_{\phi_{l_0}^N}\big[\tau_\varepsilon^N\big] = \frac{2\pi h_N e^{S^N(\phi_{l_0}^N)/\varepsilon}}{C^*(N,\varepsilon)\sqrt{\det HS^N(\phi_{l_0}^N)}}\big(1+\Psi(\varepsilon,N)\big), \tag{6.61}$$

where the factor h_N comes from the time change (Eq. (2.42)). Using Proposition 5.2 (convergence of the eigenvalues) and Corollary 5.5 (convergence of the ratio of eigenvalues), the quadratic forms Q^N converges to Q:

$$\begin{aligned}
\frac{1}{h_N}Q^N(a)\sqrt{\det HS^N\big(\phi_{l_0}^N\big)} &= \sum_{\phi^*{}_l^N}\frac{|\lambda_N^-(\phi^*{}_l^N)|}{h_N}\sqrt{\frac{\det HS^N(\phi_{l_0}^N)}{|\det HS^N(\phi^*{}_l^N)|}}e^{-S^N(\phi^*{}_l^N)/\varepsilon}(a_{l+}-a_{l-})^2,\\
\frac{1}{h_N}Q^N(a)\sqrt{\det HS^N\big(\phi_{l_0}^N\big)} &\xrightarrow[N\to+\infty]{} \sum_{\phi^*{}_l}\big|\lambda^-\big(\phi^*{}_l\big)\big|\sqrt{\frac{\operatorname{Det}\mathcal{H}_{\phi_{l_0}}S}{|\operatorname{Det}\mathcal{H}_{\phi_l^*}S|}}e^{-S(\phi^*{}_l)/\varepsilon}(a_{l+}-a_{l-})^2\\
&= Q(a)e^{-S(\phi^*{}_l)/\varepsilon}\sqrt{\operatorname{Det}\mathcal{H}_{\phi_{l_0}}S},
\end{aligned} \tag{6.62}$$

where ${\phi^*}_l^N$ are the relevant saddle points given by Proposition 5.6. Then the minimizer converges. For all ε, we get

$$\frac{1}{h_N} C^*(N,\varepsilon)\sqrt{\det HS^N\left(\phi_{l_0}^N\right)} \xrightarrow[N\to\infty]{} \mathcal{C}^*(\phi_{l_0},\mathcal{M}_l)\mathrm{e}^{-S({\phi^*}_l)/\varepsilon}\sqrt{\operatorname{Det}\mathcal{H}_{\phi_{l_0}}S}. \tag{6.63}$$

Therefore, we obtain the result of Theorem 2.6 from Proposition 3.4.

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