Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2015, Vol. 51, No. 1, 349–375
DOI: 10.1214/13-AIHP578

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

An algebraic construction of quantum flows with unbounded generators

Alexander C. R. Belton[a] and Stephen J. Wills[b]

[a]*Department of Mathematics and Statistics, Lancaster University, United Kingdom. E-mail: a.belton@lancaster.ac.uk*
[b]*School of Mathematical Sciences, University College Cork, Ireland. E-mail: s.wills@ucc.ie*

Received 12 September 2012; revised 1 August 2013; accepted 6 August 2013

Abstract. It is shown how to construct $*$-homomorphic quantum stochastic Feller cocycles for certain unbounded generators, and so obtain dilations of strongly continuous quantum dynamical semigroups on C^* algebras; this generalises the construction of a classical Feller process and semigroup from a given generator. Our construction is possible provided the generator satisfies an invariance property for some dense subalgebra $\mathcal{A}_0$ of the C^* algebra $\mathcal{A}$ and obeys the necessary structure relations; the iterates of the generator, when applied to a generating set for $\mathcal{A}_0$, must satisfy a growth condition. Furthermore, it is assumed that either the subalgebra $\mathcal{A}_0$ is generated by isometries and $\mathcal{A}$ is universal, or $\mathcal{A}_0$ contains its square roots. These conditions are verified in four cases: classical random walks on discrete groups, Rebolledo's symmetric quantum exclusion process and flows on the non-commutative torus and the universal rotation algebra.

Résumé. Des cocycles de Feller stochastiques quantiques $*$-homomorphes sont construits pour certains générateurs non bornés, et ainsi nous obtenons des dilatations pour des semigroupes dynamiques quantiques fortement continus sur des C^* algèbres. Ceci généralise la construction d'un processus de Feller classique et de son semigroupe à partir d'un générateur donné. Notre construction est possible à condition que le générateur satisfasse une propriété d'invariance pour une sous-algèbre dense $\mathcal{A}_0$ de la C^* algèbre $\mathcal{A}$ et obéisse aux relations de structure nécessaires; les itérations du générateur, lorsqu'elles sont appliquées à une famille génératrice de $\mathcal{A}_0$, doivent satisfaire à une condition de croissance. De plus, il est supposé que soit la sous-algèbre $\mathcal{A}_0$ est engendrée par les isométries et $\mathcal{A}$ est universelle, ou bien $\mathcal{A}_0$ contient ses racines carrées. Ces conditions sont vérifiées dans quatre cas: marches aléatoires classiques sur les groupes discrets, le processus d'exclusion quantique symétrique introduit par Rebolledo et des flux sur le tore non commutatif et l'algèbre de rotation universelle.

MSC: Primary 81S25; secondary 46L53; 46N50; 47D06; 60J27

Keywords: Quantum dynamical semigroup; Quantum Markov semigroup; Cpc semigroup; Strongly continuous semigroup; Semigroup dilation; Feller cocycle; Higher-order Itô product formula; Random walks on discrete groups; Quantum exclusion process; Non-commutative torus

1. Introduction

The connexion between time-homogeneous Markov processes and one-parameter contraction semigroups is an excellent example of the interplay between probability theory and functional analysis. Given a measurable space $(E, \mathcal{E})$, a *Markov semigroup* T with state space E is a family $(T_t)_{t\geq 0}$ of positive contraction operators on $L^\infty(E)$ such that

$$T_{s+t} = T_s \circ T_t \quad \text{for all } s, t \geq 0 \quad \text{and} \quad T_0 f = f \quad \text{for all } f \in L^\infty(E);$$

the semigroup is *conservative* if $T_t 1 = 1$ for all $t \geq 0$. Typically, such a semigroup is defined by setting

$$(T_t f)(x) = \int_E f(y) p_t(x, \mathrm{d}y)$$

for a family of transition kernels $p_t : E \times \mathcal{E} \to [0, 1]$. Given a time-homogeneous Markov process $(X_t)_{t \geq 0}$ with values in E, the associated Markov semigroup is obtained from the prescription

$$(T_t f)(x) = \mathbb{E}\big[f(X_t) | X_0 = x\big], \tag{1.1}$$

so that $p_t(x, A) = \mathbb{P}(X_t \in A | X_0 = x)$ is the probability of moving from x into A in time t. When the state space E is a locally compact Hausdorff space we may specialise further: a *Feller semigroup* is a Markov semigroup T such that

$$T_t\big(C_0(E)\big) \subseteq C_0(E) \quad \text{for all } t \geq 0 \quad \text{and} \quad \|T_t f - f\|_\infty \to 0 \quad \text{as } t \to 0 \text{ for all } f \in C_0(E).$$

Any sufficiently nice Markov process, such as a Lévy process, gives rise to a Feller semigroup; conversely, if E is separable then any Feller semigroup gives rise to a Markov process with *càdlàg* paths.

A celebrated theorem of Gelfand and Naimark states that every commutative C^* algebra is of the form $C_0(E)$ for some locally compact Hausdorff space E. Thus the first step in generalising Feller semigroups, and so Markov processes, to a non-commutative setting is to replace the commutative algebra $C_0(E)$ with a general C^* algebra $\mathcal{A}$. Moreover, a strengthening of positivity, called *complete positivity*, is required for a satisfactory theory: a map $\phi : \mathcal{A} \to \mathcal{B}$ between C^* algebras is completely positive if the ampliation

$$\phi^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B}); \qquad (x_{ij}) \mapsto \big(\phi(x_{ij})\big)$$

is positive for all $n \geq 1$. This property is justified on physical grounds and is equivalent to the usual form of positivity when either algebra $\mathcal{A}$ or $\mathcal{B}$ is commutative. The resulting object, a semigroup of completely positive contractions on a C^* algebra $\mathcal{A}$, is known as a *quantum dynamical semigroup* or, when conservative, a *quantum Markov semigroup*. Such semigroups appear in the mathematical physics literature, being used to describe the evolution of open quantum-mechanical systems which interact irreversibly with their environment. They also arise in non-commutative geometry.

Any strongly continuous quantum dynamical semigroup T is characterised by its infinitesimal generator τ, the closed linear operator such that

$$\operatorname{dom} \tau = \left\{ f \in \mathcal{A}\colon \lim_{t \to 0} \frac{T_t f - f}{t} \text{ exists} \right\} \quad \text{and} \quad \tau f = \lim_{t \to 0} \frac{T_t f - f}{t}.$$

For a Feller semigroup, the form of the generator τ may reveal properties of the corresponding process; for instance, a classical Lévy process may be specified, *via* the Lévy–Khintchine formula, by the characteristics of its generator, *viz.* a drift vector, a diffusion matrix describing the Brownian-motion component and a Lévy measure characterising its jumps. If we start with a putative generator τ then operator-theoretic methods may be used to construct the semigroup, although there are often considerable analytical challenges to be met. Verifying that τ satisfies the hypotheses of the Hille–Yosida theorem, the key analytical tool for this construction, is often difficult. In this paper we provide, for a suitable class of generators, another method of constructing quantum dynamical semigroups and the corresponding non-commutative Markov processes, continuing a line of research initiated by Accardi, Hudson and Parthasarathy, Meyer, and others.

To understand how the relationship between semigroups and Markov processes generalises to the non-commutative framework, recall first that any locally compact Hausdorff space E may be made compact by adjoining a point at infinity, which corresponds to adding an identity to the algebra $C_0(E)$ or adding a coffin state for an E-valued Markov process; it is sufficient, therefore, to restrict our attention to compact Hausdorff spaces or, equivalently, unital C^* algebras. The correct analogue of an E-valued random variable X is then a unital $*$-homomorphism j from $\mathcal{A}$ to some unital C^* algebra $\mathcal{B}$; classically, j is the map $f \mapsto f \circ X$, where $f \in \mathcal{A} = C_0(E)$ and $\mathcal{B}$ is $L^\infty(\mathbb{P})$ for some probability measure $\mathbb{P}$. A family of unital $*$-homomorphisms $(j_t : \mathcal{A} \to \mathcal{B})_{t \geq 0}$, i.e., a non-commutative stochastic process, is said to *dilate* the quantum dynamical semigroup T if $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and $\mathbb{E} \circ j_t = T_t$ for all $t \geq 0$, where $\mathbb{E}$ is a conditional expectation from $\mathcal{B}$ to $\mathcal{A}$; the relationship to (1.1) is clear. Thus finding a dilation for a given semigroup is analogous to constructing a Markov process from a family of transition kernels.

The tool used here for constructing semigroups and their dilations, quantum Markov processes, is a stochastic calculus: the quantum stochastic calculus introduced by Hudson and Parthasarathy in their 1984 paper [16]. In its simplest form, this is a non-commutative theory of stochastic integration with respect to three operator martingales which correspond to the creation, annihilation and gauge processes of quantum field theory. It generalises simultaneously the

Itô–Doob L^2 integral with respect to either Brownian motion or the compensated Poisson process; as emphasised by Meyer [26] and Attal [3], the L^2 theory of any normal martingale having the chaotic-representation property, such as Brownian motion, the compensated Poisson process or Azéma's martingale, gives a classical probabilistic interpretation of Boson Fock space, the ambient space of quantum stochastic calculus. An excellent introduction to this subject, with the same philosophy as here, is [4]; the use of these quantum noises to produce quantum Markov processes is discussed in its final chapter.

We develop below new techniques for obtaining $*$-homomorphic solutions to the Evans–Hudson quantum stochastic differential equation (QSDE)

$$\mathrm{d} j_t = (j_t \otimes \iota_{\mathcal{B}(\widehat{\mathsf{k}})}) \circ \phi \, \mathrm{d}\Lambda_t, \tag{1.2}$$

where the solution j_t acts on a unital C^* algebra $\mathcal{A}$. In this way, we obtain the process j and the quantum dynamical semigroup T simultaneously. The components of the *flow generator* ϕ include τ, the restriction of a semigroup generator, and δ, a bimodule derivation, which are related to one another through the Bakry–Émery *carré du champ* operator: see Remark 2.4. In Theorem 3.19 we show that restricting j to a suitable commutative subalgebra $\mathcal{A}_c$ of $\mathcal{A}$ yields a classical process, in the sense that the algebra generated by $\{j_t(a)\colon t \geq 0, a \in \mathcal{A}_c\}$ is also commutative.

The recent expository paper [5] on quantum stochastic methods, written for an audience of probabilists, includes Parthasarathy and Sinha's method [27] for constructing continuous-time Markov chains with finite state spaces by solving quantum stochastic differential equations. To quote Biane,

> "It may seem strange to the classical probabilist to use noncommutative objects in order to describe a perfectly commutative situation, however, this seems to be necessary if one wants to deal with processes with jumps ... The right mathematical notion ..., which generalizes to the noncommutative situation, is that of a derivation into a bimodule ... Using this formalism, we can use the Fock space as a uniform source of noise, and construct general Markov processes (both continuous and discontinuous) using stochastic differential equations."

The results herein give a further illustration of this philosophy.

The use of quantum stochastic calculus to produce dilations has now been studied for nearly thirty years. Most results, by Hudson and Parthasarathy, Fagnola, Mohari, Sinha et cetera, are obtained in the case that $\mathcal{A} = \mathcal{B}(\mathsf{h})$ by first solving an operator-valued QSDE, the Hudson–Parthasarathy equation, to obtain a unitary process U, and defining j through conjugation by U; see [11] and references therein. The corresponding theory for the Heisenberg rather than the Schrödinger viewpoint, solving the Evans–Hudson equation (1.2), has mainly been developed under the standing assumption that the generator ϕ is completely bounded, which is necessary if the corresponding semigroup T is norm continuous [22]. When one deviates from this assumption, which is analytically convenient but very restrictive, there are few results. The earliest general method is due to Fagnola and Sinha [12], with later results by Goswami, Sahu and Sinha for a particular model [15] and a more general method developed by Goswami and Sinha in [30]. Another approach based on semigroup methods has yet to yield existence results for the Evans–Hudson equation: see [1] and [25].

Our method here has an attractive simplicity, imposing minimal conditions on the generator ϕ. It must be a $*$-linear map

$$\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}(\mathbb{C} \oplus \mathsf{k}),$$

where $\mathcal{A}_0$ is a dense $*$-subalgebra of the unital C^* algebra $\mathcal{A} \subseteq \mathcal{B}(\mathsf{h})$ which contains $1 = 1_{\mathsf{h}}$ and k is a Hilbert space, called the *multiplicity space*, the dimension of which measures the amount of noise available in the system. This incorporates an assumption that, if ϕ is viewed as a matrix of maps, its components leave $\mathcal{A}_0$ invariant, a hypothesis also used in [12]. Furthermore, ϕ must be such that $\phi(1) = 0$ and the *first-order Itô product formula* holds:

$$\phi(xy) = \phi(x)(y \otimes 1_{\widehat{\mathsf{k}}}) + (x \otimes 1_{\widehat{\mathsf{k}}})\phi(y) + \phi(x)\Delta\phi(y) \quad \text{for all } x, y \in \mathcal{A}_0, \tag{1.3}$$

where $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$ and $\Delta \in \mathcal{A}_0 \otimes \mathcal{B}(\widehat{\mathsf{k}})$ is the orthogonal projection onto $\mathsf{h} \bar{\otimes} \mathsf{k}$. Both these conditions are known to be necessary if ϕ is to generate a family of unital $*$-homomorphisms. Finally, a growth bound must be established for the iterates of ϕ applied to elements taken from a suitable subset of $\mathcal{A}_0$.

Our approach is an elementary one for those adept in quantum stochastic calculus, relying on familiar techniques such as representing the solution to the Evans–Hudson QSDE as a sum of quantum Wiener integrals. An essential tool is the higher-order Itô product formula, presented in Section 2. This formula was first stated, for finite-dimensional noise, in [9], was proved for that case in [17] and reached its definitive form in [24]. In that last paper it was shown that (1.3) is but the first of a sequence of identities that must be satisfied in order to show that the solution j of the QSDE is weakly multiplicative. However, there are many situations in which the validity of (1.3) implies that the other identities hold [24], Corollary 4.2, and this is the case for ϕ as above. Moreover, one of our main observations, Corollary 2.12, is that, by exploiting the algebraic structure imposed by this sequence of identities, it is sufficient to establish pointwise growth bounds on a $*$-generating set of $\mathcal{A}_0$; this is a major simplification when compared with [12]. Also, by using the coordinate-free approach to quantum stochastic analysis given in [20], we can take k to be any Hilbert space, removing the restriction in [12] that k be finite dimensional.

The growth bounds obtained Section 2 are employed in Section 3 to produce a family of weakly multiplicative $*$-linear maps from the algebra $\mathcal{A}_0$ into the space of linear operators in $\mathsf{h} \bar{\otimes} \mathcal{F}$, where $\mathcal{F}$ is the Boson Fock space over $L^2(\mathbb{R}_+; \mathsf{k})$. It is shown that these maps extend to unital $*$-homomorphisms in two distinct situations. Theorem 3.9, which includes the case of AF algebras, exploits a square-root trick that is well known in the literature; Theorem 3.12, which applies to universal C^* algebras such as the non-commutative torus or the Cuntz algebras, is believed to be novel. Uniqueness of the solution is proved, and it is also shown that j is a cocycle, i.e., it satisfies the evolution equation

$$j_{s+t} = (j_s \bar{\otimes} \iota_{\mathcal{B}(\mathcal{F}_{[s,\infty)})}) \circ \sigma_s \circ j_t \quad \text{for all } s, t \geq 0, \tag{1.4}$$

where $(\sigma_t)_{t \geq 0}$ is the shift semigroup on the algebra of all bounded operators on $\mathcal{F}$. At this point we see another novel feature of our work in contrast to previous results, all of which start with a particular quantum dynamical semigroup T. In these other papers the generator τ of T is then augmented to produce ϕ, and the QSDE solved to give a dilation of T. For example, in [12] it is assumed that T is an analytic semigroup and that the composition of τ with the other components of ϕ is well behaved in a certain sense; in [30] it is assumed that T is covariant with respect to some group action on $\mathcal{A}$. For us, the starting point is the map ϕ, which yields the cocycle j, and hence, by compression, a quantum dynamical semigroup T generated by the closure of τ, which has core $\mathcal{A}_0$; this semigroup, a fortiori, is dilated by j. Thus we do not have to check that τ is a semigroup generator with good properties at the outset, thereby rendering our method easier to apply.

Our first application of Theorem 3.9, in Section 4, is to construct the Markov semigroups which correspond to certain random walks on discrete groups. Theorem 3.9 is also employed in Section 5 to produce a dilation of the symmetric quantum exclusion semigroup. This object, a model for systems of interacting quantum particles, was introduced by Rebolledo [28] as a non-commutative generalisation of the classical exclusion process [19] and has generated much interest: see [14] and [13]. The multiplicity space k is required to be infinite dimensional for this process, as in previous work on processes arising from quantum interacting particle systems, e.g., [15].

In Section 6 we use Theorem 3.12 to obtain flows on some universal C^* algebras, namely the non-commutative torus and the universal rotation algebra [2]; the former is a particularly important example in non-commutative geometry. Quantum flows on these algebras have previously been considered by Chakraborty, Goswami and Sinha [8] and by Hudson and Robinson [18], respectively.

1.1. *Conventions and notation*

The quantity $:=$ is to be read as 'is defined to be' or similarly. The quantity $\mathbb{1}_P$ equals 1 if the proposition P is true and 0 if P is false, where 1 and 0 are the appropriate multiplicative and additive identities. The set of natural numbers is denoted by $\mathbb{N} := \{1, 2, 3, \ldots\}$; the set of non-negative integers is denoted by $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. The linear span of the set S in the vector space V is denoted by $\operatorname{lin} S$; all vector spaces have complex scalar field and inner products are linear on the right. The *algebraic* tensor product is denoted by $\otimes$; the Hilbert-space tensor product is denoted by $\bar{\otimes}$, as is the ultraweak tensor product. The domain of the linear operator T is denoted by $\operatorname{dom} T$. The identity transformation on the vector space V is denoted by 1_V. If P is an orthogonal projection on the inner-product space V then the complement $P^\perp := 1_V - P$, the projection onto the orthogonal complement of the range of P. The Banach space of bounded operators from the Banach space X_1 to the Banach space X_2 is denoted by $\mathcal{B}(X_1; X_2)$, or by $\mathcal{B}(X_1)$ if X_1 and X_2 are equal. The identity automorphism on the algebra $\mathcal{A}$ is denoted by $\iota_{\mathcal{A}}$. If a and b are elements in an algebra $\mathcal{A}$ then $[a, b] := ab - ba$ and $\{a, b\} := ab + ba$ denote their commutator and anti-commutator, respectively. If $\mathcal{A}_0$ is a $*$-algebra, H_1 and H_2 are Hilbert spaces and $\alpha : \mathcal{A}_0 \to \mathcal{B}(\mathsf{H}_1; \mathsf{H}_2)$ is a linear map then the *adjoint* map $\alpha^\dagger : \mathcal{A}_0 \to \mathcal{B}(\mathsf{H}_2; \mathsf{H}_1)$ is such that $\alpha^\dagger(a) := \alpha(a^*)^*$ for all $a \in \mathcal{A}_0$.

2. A higher-order product formula

Notation 2.1. *The Dirac bra-ket notation will be useful*: *for any Hilbert space* H *and vectors* $\xi,\ \chi \in \mathsf{H}$, *let*

$$|\mathsf{H}\rangle := \mathcal{B}(\mathbb{C}; \mathsf{H}), \qquad |\xi\rangle : \mathbb{C} \to \mathsf{H}; \qquad \lambda \mapsto \lambda\xi, \tag{ket}$$

and

$$\langle\mathsf{H}| := \mathcal{B}(\mathsf{H}; \mathbb{C}), \qquad \langle\chi| : \mathsf{H} \to \mathbb{C}; \qquad \eta \mapsto \langle\chi, \eta\rangle. \tag{bra}$$

In particular, we have the linear map $|\xi\rangle\langle\chi| \in \mathcal{B}(\mathsf{H})$ such that $|\xi\rangle\langle\chi|\eta = \langle\chi, \eta\rangle\xi$ for all $\eta \in \mathsf{H}$.

Let $\mathcal{A} \subseteq \mathcal{B}(\mathsf{h})$ be a unital C^* algebra with identity $1 = 1_{\mathsf{h}}$, whose elements act as bounded operators on the *initial space* h, a Hilbert space. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a norm-dense $*$-subalgebra of $\mathcal{A}$ which contains 1.

Let the *extended multiplicity space* $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$, where the *multiplicity space* k is a Hilbert space, and distinguish the unit vector $\omega := (1, 0)$. For brevity, let $\mathcal{B} := \mathcal{B}(\widehat{\mathsf{k}})$.

Let $\Delta := 1 \otimes P_{\mathsf{k}} \in \mathcal{A}_0 \otimes \mathcal{B}$, where $P_{\mathsf{k}} := |\omega\rangle\langle\omega|^{\perp} \in \mathcal{B}$ is the orthogonal projection onto $\mathsf{k} \subset \widehat{\mathsf{k}}$.

Lemma 2.2. *The map* $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ *is* $*$*-linear, such that* $\phi(1) = 0$ *and such that*

$$\phi(xy) = \phi(x)(y \otimes 1_{\mathsf{k}}) + (x \otimes 1_{\mathsf{k}})\phi(y) + \phi(x)\Delta\phi(y) \quad \textit{for all } x, y \in \mathcal{A}_0 \tag{2.1}$$

if and only if

$$\phi(x) = \begin{bmatrix} \tau(x) & \delta^{\dagger}(x) \\ \delta(x) & \pi(x) - x \otimes 1_{\mathsf{k}} \end{bmatrix} \quad \textit{for all } x \in \mathcal{A}_0, \tag{2.2}$$

where $\pi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}(\mathsf{k})$ *is a unital* $*$*-homomorphism*, $\delta : \mathcal{A}_0 \to \mathcal{A}_0 \otimes |\mathsf{k}\rangle$ *is a* π*-derivation*, *i.e., a linear map such that*

$$\delta(xy) = \delta(x)y + \pi(x)\delta(y) \quad \textit{for all } x, y \in \mathcal{A}_0,$$

and $\tau : \mathcal{A}_0 \to \mathcal{A}_0$ *is a* $*$*-linear map such that*

$$\tau(xy) - \tau(x)y - x\tau(y) = \delta^{\dagger}(x)\delta(y) \quad \textit{for all } x, y \in \mathcal{A}_0. \tag{2.3}$$

Proof. This is a straightforward exercise in elementary algebra. □

Definition 2.3. *A* $*$*-linear map* $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ *such that* $\phi(1) = 0$ *and such that* (2.1) *holds is a* flow generator.

Remark 2.4. *Condition* (2.3) *may be expressed in terms of the Bakry–Émery carré du champ operator*

$$\Gamma : \mathcal{A}_0 \times \mathcal{A}_0 \to \mathcal{A}_0; \qquad (x, y) \mapsto \frac{1}{2}\big(\tau(xy) - \tau(x)y - x\tau(y)\big);$$

for (2.3) *to be satisfied, it is necessary and sufficient that* $2\Gamma(x, y) = \delta^{\dagger}(x)\delta(y)$ *for all* $x, y \in \mathcal{A}_0$.

The π*-derivation* δ *becomes a bimodule derivation if* $\mathcal{A}_0 \otimes |\mathsf{k}\rangle$ *is made into an* $\mathcal{A}_0$*-*$\mathcal{A}_0$ *bimodule by setting* $x \cdot z \cdot y := \pi(x)zy$ *for all* $x, y \in \mathcal{A}_0$ *and* $z \in \mathcal{A}_0 \otimes |\mathsf{k}\rangle$.

Lemma 2.5. *Let* $\mathcal{A}_0 = \mathcal{A}$, *let* $\pi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathsf{k})$ *be a unital* $*$*-homomorphism, let* $z \in \mathcal{A} \otimes |\mathsf{k}\rangle$ *and let* $h \in \mathcal{A}$ *be self adjoint. Define*

$$\delta : \mathcal{A} \to \mathcal{A} \otimes |\mathsf{k}\rangle; \qquad x \mapsto zx - \pi(x)z$$

and

$$\tau : \mathcal{A} \to \mathcal{A}; \qquad x \mapsto \mathrm{i}[h, x] - \frac{1}{2}\big\{z^*z, x\big\} + z^*\pi(x)z.$$

Then the map $\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ defined in terms of π, δ and τ through (2.2) is a flow generator.

Proof. This is another straightforward exercise. $\square$

Remark 2.6. *Modulo important considerations regarding tensor products and the ranges of δ and τ, the above form for ϕ is, essentially, the only one possible* [21], *Lemma* 6.4. *The quantum exclusion process in Section 5 has a generator of the same form but with unbounded z and h.*

Definition 2.7. *Given a flow generator $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$, the* quantum random walk $(\phi_n)_{n\in\mathbb{Z}_+}$ *is a family of $*$-linear maps*

$$\phi_n : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}^{\otimes n}$$

defined by setting

$$\phi_0 := \iota_{\mathcal{A}_0} \quad \text{and} \quad \phi_{n+1} := (\phi_n \otimes \iota_{\mathcal{B}}) \circ \phi \quad \text{for all } n \in \mathbb{Z}_+.$$

The following identity is useful: if $\xi_1, \chi_1, \ldots, \xi_n, \chi_n \in \widehat{\mathsf{k}}$ and $x \in \mathcal{A}_0$ then

$$\big(1_{\mathsf{h}} \otimes \langle\xi_1| \otimes \cdots \otimes \langle\xi_n|\big)\phi_n(x)\big(1_{\mathsf{h}} \otimes |\chi_1\rangle \otimes \cdots \otimes |\chi_n\rangle\big) = \phi^{\xi_1}_{\chi_1} \circ \cdots \circ \phi^{\xi_n}_{\chi_n}(x), \tag{2.4}$$

where

$$\phi^{\xi}_{\chi} : \mathcal{A}_0 \to \mathcal{A}_0; \qquad x \mapsto \big(1_{\mathsf{h}} \otimes \langle\xi|\big)\phi(x)\big(1_{\mathsf{h}} \otimes |\chi\rangle\big)$$

is a linear map for each choice of $\xi, \chi \in \widehat{\mathsf{k}}$.

Remark 2.8. *The paper* [24], *results from which will be employed below, uses a different convention to that adopted in Definition 2.7: the components of the product $\mathcal{B}^{\otimes n}$ appear in the reverse order to how they do above.*

Notation 2.9. *Let $\alpha \subseteq \{1, \ldots, n\}$, with elements arranged in increasing order, and denote its cardinality by $|\alpha|$. The unital $*$-homomorphism*

$$\mathcal{A}_0 \otimes \mathcal{B}^{\otimes|\alpha|} \to \mathcal{A}_0 \otimes \mathcal{B}^{\otimes n}; \qquad T \mapsto T(n, \alpha)$$

is defined by linear extension of the map

$$A \otimes B_1 \otimes \cdots \otimes B_{|\alpha|} \mapsto A \otimes C_1 \otimes \cdots \otimes C_n,$$

where

$$C_i := \begin{cases} B_j & \text{if } i \text{ is the } j\text{th element of } \alpha, \\ 1_{\widehat{\mathsf{k}}} & \text{if } i \text{ is not an element of } \alpha. \end{cases}$$

For example, if $\alpha = \{1, 3, 4\}$ and $n = 5$ then

$$(A \otimes B_1 \otimes B_2 \otimes B_3)(5, \alpha) = A \otimes B_1 \otimes 1_{\widehat{\mathsf{k}}} \otimes B_2 \otimes B_3 \otimes 1_{\widehat{\mathsf{k}}}.$$

Given a flow generator $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$, for all $n \in \mathbb{Z}_+$ and $\alpha \subseteq \{1, \ldots, n\}$, let

$$\phi_{|\alpha|}(x; n, \alpha) := \big(\phi_{|\alpha|}(x)\big)(n, \alpha) \quad \text{for all } x \in \mathcal{A}_0$$

and let

$$\Delta(n, \alpha) := \big(1_{\mathsf{h}} \otimes P_{\mathsf{k}}^{\otimes|\alpha|}\big)(n, \alpha),$$

so that, in the latter, P_{k} acts on the components of $\widehat{\mathsf{k}}^{\otimes n}$ which have indices in α and $1_{\widehat{\mathsf{k}}}$ acts on the others.

Theorem 2.10. *Let $(\phi_n)_{n\in\mathbb{Z}_+}$ be the quantum random walk given by the flow generator ϕ. For all $n \in \mathbb{Z}_+$ and x, $y \in \mathcal{A}_0$,*

$$\phi_n(xy) = \sum_{\alpha\cup\beta=\{1,\dots,n\}} \phi_{|\alpha|}(x;n,\alpha)\Delta(n,\alpha\cap\beta)\phi_{|\beta|}(y;n,\beta), \tag{2.5}$$

where the summation is taken over all sets α and β whose union is $\{1,\dots,n\}$.

Proof. This may be established inductively: see [24], Proof of Theorem 4.1. □

Definition 2.11. *The set $S \subseteq \mathcal{A}_0$ is $*$-*generating for $\mathcal{A}_0$ *if $\mathcal{A}_0$ is the smallest unital $*$-algebra which contains S.*

Corollary 2.12. *For a flow generator $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$, let*

$$\mathcal{A}_\phi := \big\{ x \in \mathcal{A}_0 : \text{there exist } C_x, M_x > 0 \text{ such that } \|\phi_n(x)\| \le C_x M_x^n \text{ for all } n \in \mathbb{Z}_+ \big\}. \tag{2.6}$$

Then $\mathcal{A}_\phi$ is a unital $*$-subalgebra of $\mathcal{A}_0$, which is equal to $\mathcal{A}_0$ if $\mathcal{A}_\phi$ contains a $*$-generating set for $\mathcal{A}_0$.

Proof. It suffices to demonstrate that $\mathcal{A}_\phi$ is closed under products. To see this, let x, $y \in \mathcal{A}_\phi$ and suppose C_x, M_x and C_y, M_y are as in (2.6). Then (2.5) implies that

$$\begin{aligned}
\|\phi_n(xy)\| &\le \sum_{\alpha\cup\beta=\{1,\dots,n\}} \|\phi_{|\alpha|}(x)\| \, \|\phi_{|\beta|}(y)\| \\
&\le C_x C_y \sum_{k=0}^{n} \binom{n}{k} M_x^k \sum_{l=0}^{k} \binom{k}{l} M_y^{n-k+l} \quad \big(k = |\alpha|, l = |\alpha\cap\beta|\big) \\
&= C_x C_y \sum_{k=0}^{n} \binom{n}{k} M_x^k M_y^{n-k} (1+M_y)^k \\
&= C_x C_y (M_x + M_x M_y + M_y)^n
\end{aligned}$$

for all $n \in \mathbb{Z}_+$, as required. □

Lemma 2.13. *If the flow generator ϕ is as defined in Lemma 2.5 then $\mathcal{A}_\phi = \mathcal{A}_0$.*

Proof. This follows immediately, since ϕ is completely bounded and $\|\phi_n\| \le \|\phi_n\|_{\mathrm{cb}} \le \|\phi\|_{\mathrm{cb}}^n$ for all $n \in \mathbb{Z}_+$. □

The following result shows that, given a flow generator ϕ and vectors χ, $\xi \in \widehat{\mathsf{k}}$, the elements of $\mathcal{A}_\phi$ are entire vectors for ϕ_χ^ξ.

Lemma 2.14. *Let $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ be a flow generator. For all ξ, $\chi \in \widehat{\mathsf{k}}$ we have $\phi_\chi^\xi(\mathcal{A}_\phi) \subseteq \mathcal{A}_\phi$, and the series*

$$\exp\big(z\phi_\chi^\xi\big) := \sum_{n=0}^{\infty} \frac{z^n (\phi_\chi^\xi)^n}{n!} \tag{2.7}$$

is strongly absolutely convergent on $\mathcal{A}_\phi$ for all $z \in \mathbb{C}$.

Proof. Suppose $\|\phi_n(x)\| \le C_x M_x^n$ for all $n \in \mathbb{Z}_+$. It follows from (2.4) that

$$\big(1_{\mathsf{h}\bar{\otimes}\widehat{\mathsf{k}}^{\bar{\otimes} n}} \otimes \langle\xi|\big)\phi_{n+1}(x)\big(1_{\mathsf{h}\bar{\otimes}\widehat{\mathsf{k}}^{\bar{\otimes} n}} \otimes |\chi\rangle\big) = \phi_n\big(\phi_\chi^\xi(x)\big), \tag{2.8}$$

so

$$\left\|\phi_n\big(\phi_\chi^\xi(x)\big)\right\| \le \|\xi\| C_x M_x^{n+1} \|\chi\| = \big(\|\xi\|\|\chi\| C_x M_x\big) M_x^n$$

and $\phi_\chi^\xi(x) \in \mathcal{A}_\phi$. Moreover (2.4) also gives that

$$\left\|\big(\phi_{\chi_1}^{\xi_1} \circ \cdots \circ \phi_{\chi_n}^{\xi_n}\big)(x)\right\| \le \|\xi_1\| \cdots \|\xi_n\| \|\chi_1\| \cdots \|\chi_n\| C_x M_x^n, \tag{2.9}$$

hence the series (2.7) converges as claimed. □

3. Quantum flows

Notation 3.1. *Let $\mathcal{F}$ denote Boson Fock space over $L^2(\mathbb{R}_+; \mathsf{k})$, the Hilbert space of k-valued, square-integrable functions on the half line, and let*

$$\mathcal{E} := \operatorname{lin}\big\{\varepsilon(f) : f \in L^2(\mathbb{R}_+; \mathsf{k})\big\}$$

denote the linear span of the total set of exponential vectors in $\mathcal{F}$. As is customary, elementary tensors in $\mathsf{h} \otimes \mathcal{F}$ are written without a tensor-product sign: in other words, $u\varepsilon(f) := u \otimes \varepsilon(f)$ for all $u \in \mathsf{h}$ and $f \in L^2(\mathbb{R}_+; \mathsf{k})$, et cetera.

If $f \in L^2(\mathbb{R}_+; \mathsf{k})$ and $t \ge 0$ then $\widehat{f}(t) := \widehat{f(t)}$, where $\widehat{\xi} := \omega + \xi \in \widehat{\mathsf{k}}$ for all $\xi \in \mathsf{k}$.

Given $f \in L^2(\mathbb{R}_+; \mathsf{k})$ and an interval $I \subseteq \mathbb{R}_+$, let $f_I \in L^2(\mathbb{R}_+; \mathsf{k})$ be defined to equal f on I and 0 elsewhere, with $f_{t)} := f_{[0,t)}$ and $f_{[t} := f_{[t,\infty)}$ for all $t \ge 0$.

Definition 3.2. *A family of linear operators $(T_t)_{t\ge0}$ in $\mathsf{h} \,\bar{\otimes}\, \mathcal{F}$ with domains including $\mathsf{h} \otimes \mathcal{E}$ is* adapted *if*

$$\big\langle u\varepsilon(f), T_t v\varepsilon(g)\big\rangle = \big\langle u\varepsilon(f_{t)}), T_t v\varepsilon(g_{t)})\big\rangle \big\langle \varepsilon(f_{[t}), \varepsilon(g_{[t})\big\rangle$$

for all $u, v \in \mathsf{h}$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ and $t \ge 0$.

Theorem 3.3. *For all $n \in \mathbb{N}$ and $T \in \mathcal{B}(\mathsf{h} \,\bar{\otimes}\, \widehat{\mathsf{k}}^{\bar{\otimes} n})$ there exists a family $\Lambda^n(T) = (\Lambda_t^n(T))_{t\ge0}$ of linear operators in $\mathsf{h} \,\bar{\otimes}\, \mathcal{F}$, with domains including $\mathsf{h} \otimes \mathcal{E}$, that is adapted and such that*

$$\big\langle u\varepsilon(f), \Lambda_t^n(T) v\varepsilon(g)\big\rangle = \int_{D_n(t)} \big\langle u \otimes \widehat{f}^{\otimes n}(\mathbf{t}), T v \otimes \widehat{g}^{\otimes n}(\mathbf{t})\big\rangle \,\mathrm{d}\mathbf{t} \big\langle \varepsilon(f), \varepsilon(g)\big\rangle \tag{3.1}$$

for all $u, v \in \mathsf{h}$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ and $t \ge 0$. Here the simplex

$$D_n(t) := \big\{\mathbf{t} := (t_1, \ldots, t_n) \in [0, t]^n : t_1 < \cdots < t_n\big\}$$

and

$$\widehat{f}^{\otimes n}(\mathbf{t}) := \widehat{f}(t_1) \otimes \cdots \otimes \widehat{f}(t_n), \quad \textit{et cetera.}$$

We extend this definition to include $n = 0$ by setting $\Lambda_t^0(T) := T \otimes 1_{\mathcal{F}}$ for all $t \ge 0$.

If $f \in L^2(\mathbb{R}_+; \mathsf{k})$ then

$$\big\|\Lambda_t^n(T) u\varepsilon(f)\big\| \le \frac{K_{f,t}^n}{\sqrt{n!}} \|T\| \big\|u\varepsilon(f)\big\| \quad \textit{for all } t \ge 0 \textit{ and } u \in \mathsf{h}, \tag{3.2}$$

where $K_{f,t} := \sqrt{(2 + 4\|f\|^2)(t + \|f\|^2)}$, and the map

$$\mathbb{R}_+ \to \mathcal{B}(\mathsf{h}; \mathsf{h} \,\bar{\otimes}\, \mathcal{F}); \qquad t \mapsto \Lambda_t^n(T)\big(1_{\mathsf{h}} \otimes \big|\varepsilon(f)\big\rangle\big)$$

is norm continuous.

Proof. This is an extension of Proposition 3.18 of [20], from which we borrow the notation; as for Remark 2.8, the ordering of the components in tensor products is different but this is no more than a convention. For each $f \in L^2(\mathbb{R}_+; \mathsf{k})$ define $C_f \geq 0$ so that

$$C_f^2 = \big(\|f\| + \sqrt{1 + \|f\|^2}\big)^2 \leq 2 + 4\|f\|^2,$$

and note that, by inequality (3.21) of [20],

$$\begin{aligned}\big\|\Lambda_t^n(T)u\varepsilon(f)\big\|^2 &\leq (C_{f_{t)}})^{2n}\int_{D_n(t)}\big\|Tu\otimes\widehat{f}^{\otimes n}(\mathbf{t})\big\|^2\,\mathrm{d}\mathbf{t}\big\|\varepsilon(f)\big\|^2\\ &\leq \frac{K_{f,t}^{2n}}{n!}\|T\|^2\big\|u\varepsilon(f)\big\|^2.\end{aligned}$$

To show continuity, let $\widetilde{T}$ denote T considered as an operator on $(\mathsf{h}\mathbin{\bar{\otimes}}\widehat{\mathsf{k}})\mathbin{\bar{\otimes}}\widehat{\mathsf{k}}^{\bar{\otimes}(n-1)}$, where the right-most copy of $\widehat{\mathsf{k}}$ in the n-fold tensor product has moved next to the initial space h. Then

$$\Lambda_t^n(T) - \Lambda_s^n(T) = \Lambda_t\big(1_{(s,t]}(\cdot)\Lambda_\cdot^{n-1}(\widetilde{T})\big),$$

and so, using Theorem 3.13 of [20],

$$\begin{aligned}\big\|\big(\Lambda_t^n(T) - \Lambda_s^n(T)\big)u\varepsilon(f)\big\|^2 &\leq 2\big(t + C_f^2\big)\int_s^t\big\|\Lambda_r^{n-1}(\widetilde{T})\big(u\otimes\widehat{f}(r)\big)\varepsilon(f)\big\|^2\,\mathrm{d}r\\ &\leq 2\big(t + C_f^2\big)\bigg(\int_s^t\big\|\widehat{f}(r)\big\|^2\,\mathrm{d}r\bigg)\frac{K_{f,t}^{2n-2}}{(n-1)!}\|T\|^2\big\|u\varepsilon(f)\big\|^2.\end{aligned}$$

□

The family $\Lambda^n(T)$ is the *n-fold quantum Wiener integral* of T.

Remark 3.4. *It may be shown* [24], *Proof of Theorem* 2.2, *that*

$$\operatorname{dom}\Lambda_t^l(S)^* \supseteq \Lambda_t^m(T)(\mathsf{h}\otimes\mathcal{E})$$

for all $l, m \in \mathbb{Z}_+$, $S \in \mathcal{B}(\mathsf{h}\mathbin{\bar{\otimes}}\widehat{\mathsf{k}}^{\bar{\otimes}l})$, $T \in \mathcal{B}(\mathsf{h}\mathbin{\bar{\otimes}}\widehat{\mathsf{k}}^{\bar{\otimes}m})$ *and* $t \geq 0$.

Theorem 3.5. *Let* $\phi: \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ *be a flow generator. If* $x \in \mathcal{A}_\phi$ *then the series*

$$j_t(x) := \sum_{n=0}^{\infty}\Lambda_t^n\big(\phi_n(x)\big) \tag{3.3}$$

is strongly absolutely convergent on $\mathsf{h}\otimes\mathcal{E}$ *for all* $t \geq 0$, *uniformly so on compact subsets of* $\mathbb{R}_+$. *The map*

$$\mathbb{R}_+ \to \mathcal{B}(\mathsf{h}; \mathsf{h}\mathbin{\bar{\otimes}}\mathcal{F}); \qquad t \mapsto j_t(x)\big(1_\mathsf{h}\otimes\big|\varepsilon(f)\big\rangle\big)$$

is norm continuous for all $f \in L^2(\mathbb{R}_+; \mathsf{k})$, *the family* $(j_t(x))_{t\geq 0}$ *is adapted and*

$$\big\langle u\varepsilon(f), j_t(x)v\varepsilon(g)\big\rangle = \big\langle u\varepsilon(f), (xv)\varepsilon(g)\big\rangle + \int_0^t\big\langle u\varepsilon(f), j_s\big(\phi_{\widehat{g}(s)}^{\widehat{f}(s)}(x)\big)v\varepsilon(g)\big\rangle\,\mathrm{d}s \tag{3.4}$$

for all $u, v \in \mathsf{h}$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$, $x \in \mathcal{A}_\phi$ *and* $t \geq 0$. *Furthermore,*

$$\big(1_\mathsf{h}\otimes\big\langle\varepsilon(f)\big|\big)j_t(x)\big(1_\mathsf{h}\otimes\big|\varepsilon(g)\big\rangle\big) \in \mathcal{A} \tag{3.5}$$

for all $x \in \mathcal{A}_\phi$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ *and* $t \geq 0$.

Proof. The first two claims are a consequence of the estimate (3.2), the definition of $\mathcal{A}_\phi$ and the continuity result from Theorem 3.3; adaptedness is inherited from the adaptedness of the quantum Wiener integrals. Lemma 2.14 implies that the integrand on the right-hand side of (3.4) is well defined and, by (2.8),

$$\begin{aligned}
&\big\langle u\varepsilon(f), \Lambda_s^n\big(\phi_n\big(\phi_{\widehat{g}(s)}^{\widehat{f}(s)}(x)\big)\big)v\varepsilon(g)\big\rangle \\
&\quad = \int_{D_n(s)} \big\langle u\otimes \widehat{f}^{\otimes n}(\mathbf{t}), \phi_n\big(\phi_{\widehat{g}(s)}^{\widehat{f}(s)}(x)\big)v\otimes \widehat{g}^{\otimes n}(\mathbf{t})\big\rangle \,\mathrm{d}\mathbf{t}\big\langle \varepsilon(f), \varepsilon(g)\big\rangle \\
&\quad = \int_{D_n(s)} \big\langle u\otimes \widehat{f}^{\otimes n}(\mathbf{t})\otimes \widehat{f}(s), \phi_{n+1}(x)v\otimes \widehat{g}^{\otimes n}(\mathbf{t})\otimes \widehat{g}(s)\big\rangle \,\mathrm{d}\mathbf{t}\big\langle \varepsilon(f), \varepsilon(g)\big\rangle;
\end{aligned}$$

integrating with respect to s then taking the sum of these terms gives (3.4). For the final claim, note that for any f, $g\in L^2(\mathbb{R}_+;\mathsf{k})$, the $\mathcal{A}_0$-valued map

$$D_n(t)\ni \mathbf{t}\mapsto \phi_{\widehat{g}(t_1)}^{\widehat{f}(t_1)}\circ\cdots\circ\phi_{\widehat{g}(t_n)}^{\widehat{f}(t_n)}(x) = \big(1_{\mathsf{h}}\otimes\big\langle \widehat{f}^{\otimes n}(\mathbf{t})\big|\big)\phi_n(x)\big(1_{\mathsf{h}}\otimes\big|\widehat{g}^{\otimes n}(\mathbf{t})\big\rangle\big)$$

is Bochner integrable, hence

$$\big(1_{\mathsf{h}}\otimes\big\langle\varepsilon(f)\big|\big)\Lambda_t^n\big(\phi_n(x)\big)\big(1_{\mathsf{h}}\otimes\big|\varepsilon(g)\big\rangle\big) = \mathrm{e}^{\langle f,g\rangle}\int_{D_n(t)}\big(\phi_{\widehat{g}(t_1)}^{\widehat{f}(t_1)}\circ\cdots\circ\phi_{\widehat{g}(t_n)}^{\widehat{f}(t_n)}\big)(x)\,\mathrm{d}\mathbf{t}\in\mathcal{A}. \tag{3.6}$$

By (2.9), we may sum (3.6) over all $n\in\mathbb{Z}_+$, with the resulting series being norm convergent, and so the final claim follows. □

Remark 3.6. *For all $t\geq 0$, let j_t be as in Theorem 3.5. Since $\mathcal{A}_\phi$ is a subspace of $\mathcal{A}_0$ containing 1, and each ϕ_n is linear with $\phi_n(1)=0$, it follows from (3.3) and Theorem 3.3 that each j_t is linear and unital, as a map into the space of operators with domain $\mathsf{h}\otimes\mathcal{E}$. Moreover, the maps j_t are weakly $*$-homomorphic in the following sense.*

Lemma 3.7. *Let $\phi:\mathcal{A}_0\to\mathcal{A}_0\otimes\mathcal{B}$ be a flow generator and let j_t be as in Theorem 3.5 for all $t\geq 0$. If x, $y\in\mathcal{A}_\phi$ then $x^*y\in\mathcal{A}_\phi$, with*

$$\big\langle j_t(x)u\varepsilon(f), j_t(y)v\varepsilon(g)\big\rangle = \big\langle u\varepsilon(f), j_t\big(x^*y\big)v\varepsilon(g)\big\rangle \tag{3.7}$$

for all u, $v\in\mathsf{h}$ and f, $g\in L^2(\mathbb{R}_+;\mathsf{k})$. In particular, if $x\in\mathcal{A}_\phi$ then $j_t(x)^*\supseteq j_t(x^*)$.

Proof. As $\mathcal{A}_\phi$ is a $*$-algebra, so $x^*y\in\mathcal{A}_\phi$. Let $N\in\mathbb{Z}_+$ and note that, by [24], Theorem 2.2,

$$\sum_{l,m=0}^{N}\Lambda_t^l\big(\phi_l(x)\big)^*\Lambda_t^m\big(\phi_m(y)\big) = \sum_{n=0}^{2N}\Lambda_t^n\big(\phi_{n,N]}\big(x^*y\big)\big)\quad\text{on } \mathsf{h}\otimes\mathcal{E}, \tag{3.8}$$

where

$$\phi_{n,N]}\big(x^*y\big) := \sum_{\substack{\alpha\cup\beta=\{1,\ldots,n\}\\ |\alpha|,|\beta|\leq N}}\phi_{|\alpha|}\big(x^*;n,\alpha\big)\Delta(n,\alpha\cap\beta)\phi_{|\beta|}(y;n,\beta).$$

Working as in the proof of Corollary 2.12 yields the inequality

$$\big\|\phi_{n,N]}\big(x^*y\big)\big\|\leq C_{x^*}C_y(M_{x^*}+M_{x^*}M_y+M_y)^n,$$

and so, by (3.2),

$$\big|\big\langle u\varepsilon(f), \Lambda_t^n\big(\phi_{n,N]}\big(x^*y\big)\big)v\varepsilon(g)\big\rangle\big|\leq\frac{K_{g,t}^n(M_{x^*}+M_{x^*}M_y+M_y)^n}{\sqrt{n!}}C_{x^*}C_y\big\|u\varepsilon(f)\big\|\,\big\|v\varepsilon(g)\big\|. \tag{3.9}$$

As $\phi_{n,N]} = \phi_n$ if $n \in \{0, 1, \ldots, N\}$, it follows that

$$
\begin{aligned}
\langle j_t(x)u\varepsilon(f), j_t(y)v\varepsilon(g)\rangle &= \lim_{N\to\infty} \sum_{l,m=0}^{N} \langle u\varepsilon(f), \Lambda_t^l(\phi_l(x))^* \Lambda_t^m(\phi_m(y))v\varepsilon(g)\rangle \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \langle u\varepsilon(f), \Lambda_t^n(\phi_n(x^*y))v\varepsilon(g)\rangle \\
&\quad + \lim_{N\to\infty} [t] \sum_{n=N+1}^{2N} \langle u\varepsilon(f), \Lambda_t^n(\phi_{n,N]}(x^*y))v\varepsilon(g)\rangle \\
&= \langle u\varepsilon(f), j_t(x^*y)v\varepsilon(g)\rangle,
\end{aligned}
$$

since the final limit is zero by (3.9). □

Lemma 3.8. *If $\mathcal{A}_\phi$ is dense in $\mathcal{A}$ then there is at most one family of $*$-homomorphisms $(\bar{j}_t)_{t\geq 0}$ from $\mathcal{A}$ to $\mathcal{B}(\mathsf{h}\bar{\otimes}\mathcal{F})$ that satisfies* (3.4).

Proof. Suppose that $j^{(1)}$ and $j^{(2)}$ are two families of $*$-homomorphisms from $\mathcal{A}$ to $\mathcal{B}(\mathsf{h}\otimes\mathcal{F})$ that satisfy (3.4). Set $k_t := j_t^{(1)} - j_t^{(2)}$ and note we have that

$$
\langle u\varepsilon(f), k_t(x)v\varepsilon(g)\rangle = \int_0^t \langle u\varepsilon(f), k_s(\phi_{\widehat{g}(s)}^{\widehat{f}(s)}(x))v\varepsilon(g)\rangle \,\mathrm{d}s
$$

for all $u, v \in \mathsf{h}$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ and $x \in \mathcal{A}_\phi$. Iterating the above, and using the fact that $\|k_t\| \leq 2$ for all $t \geq 0$, we obtain the inequality

$$
|\langle u\varepsilon(f), k_t(x)v\varepsilon(g)\rangle| \leq 2 \int_{D_n(t)} \|\phi_{\widehat{g}(t_1)}^{\widehat{f}(t_1)} \circ \cdots \circ \phi_{\widehat{g}(t_n)}^{\widehat{f}(t_n)}(x)\| \,\mathrm{d}\mathbf{t} \|u\varepsilon(f)\| \|v\varepsilon(g)\|.
$$

However (2.9) now gives that

$$
|\langle u\varepsilon(f), k_t(x)v\varepsilon(g)\rangle| \leq 2C_x \frac{(M_x\|\widehat{f}_{t)}\|\|\widehat{g}_{t)}\|)^n}{n!} \|u\varepsilon(f)\| \|v\varepsilon(g)\|
$$

and the result follows by letting $n \to \infty$. □

Theorem 3.9. *Let $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ be a flow generator and suppose $\mathcal{A}_0$ contains its square roots: for all non-negative $x \in \mathcal{A}_0$, the square root $x^{1/2}$ lies in $\mathcal{A}_0$. If $\mathcal{A}_\phi = \mathcal{A}_0$ then, for all $t \geq 0$, there exists a unital $*$-homomorphism*

$$
\bar{j}_t : \mathcal{A} \to \mathcal{B}(\mathsf{h}\bar{\otimes}\mathcal{F})
$$

such that $\bar{j}_t(x) = j_t(x)$ on $\mathsf{h}\otimes\mathcal{E}$ for all $x \in \mathcal{A}_0$, where $j_t(x)$ is as defined in Theorem 3.5.

Proof. Let $x \in \mathcal{A}_0$ and suppose first that $x \geq 0$. If $y := (\|x\|1 - x)^{1/2}$, which lies in $\mathcal{A}_0$ by assumption, then Lemma 3.7 and Remark 3.6 imply that

$$
0 \leq \|j_t(y)\theta\|^2 = \langle\theta, j_t(y^2)\theta\rangle = \|x\|\|\theta\|^2 - \langle\theta, j_t(x)\theta\rangle \quad \text{for all } \theta \in \mathsf{h}\otimes\mathcal{E}.
$$

If x is now an arbitrary element of $\mathcal{A}_0$, it follows that

$$
\|j_t(x)\theta\|^2 = \langle\theta, j_t(x^*x)\theta\rangle \leq \|x^*x\|\|\theta\|^2 = \|x\|^2\|\theta\|^2.
$$

Thus $j_t(x)$ extends to $\bar{j}_t(x) \in \mathcal{B}(\mathsf{h} \bar{\otimes} \mathcal{F})$, which has norm at most $\|x\|$, and the map

$$\mathcal{A}_0 \to \mathcal{B}(\mathsf{h} \bar{\otimes} \mathcal{F}); \qquad x \mapsto \bar{j}_t(x)$$

is a $*$-linear contraction, which itself extends to a $*$-linear contraction

$$\bar{j}_t : \mathcal{A} \to \mathcal{B}(\mathsf{h} \bar{\otimes} \mathcal{F}).$$

Furthermore, if $x, y \in \mathcal{A}_0$ and $\theta, \zeta \in \mathsf{h} \otimes \mathcal{E}$ then, by Lemma 3.7,

$$\langle \theta, \bar{j}_t(x)\bar{j}_t(y)\zeta \rangle = \langle \bar{j}_t(x^*)\theta, \bar{j}(y)\zeta \rangle = \langle j_t(x^*)\theta, j_t(y)\zeta \rangle = \langle \theta, j_t(xy)\zeta \rangle = \langle \theta, \bar{j}_t(xy)\zeta \rangle,$$

so $\bar{j}_t$ is multiplicative on $\mathcal{A}_0$. An approximation argument now gives that $\bar{j}_t$ is multiplicative on the whole of $\mathcal{A}$. □

Remark 3.10. *If $\mathcal{A}$ is an AF algebra, i.e., the norm closure of an increasing sequence of finite-dimensional $*$-subalgebras, then its local algebra $\mathcal{A}_0$, the union of these subalgebras, contains its square roots, since every finite-dimensional C^* algebra is closed in $\mathcal{A}$.*

Definition 3.11. *The unital C^* algebra $\mathcal{A}$ has* generators $\{a_i : i \in I\}$ *if $\mathcal{A}$ is the smallest unital C^* algebra which contains $\{a_i : i \in I\}$. These generators* satisfy the relations $\{p_k : k \in K\}$ *if each p_k is a complex polynomial in the non-commuting indeterminate $\langle X_i, X_i^* : i \in I\rangle$ and, for all $k \in K$, the algebra element $p_k(a_i, a_i^* : i \in I)$, obtained from p_k by replacing X_i by a_i and X_i^* by a_i^* for all $i \in I$, is equal to* 0.

Suppose $\mathcal{A}$ has generators $\{a_i : i \in I\}$ which satisfy the relations $\{p_k : k \in K\}$. Then $\mathcal{A}$ is *generated by isometries* if $\{X_i^* X_i - 1 : i \in I\} \subseteq \{p_k : k \in K\}$ and is *generated by unitaries* if $\{X_i^* X_i - 1, X_i X_i^* - 1 : i \in I\} \subseteq \{p_k : k \in K\}$. The algebra $\mathcal{A}$ is *universal* if, given any unital C^* algebra $\mathcal{B}$ containing a set of elements $\{b_i : i \in I\}$ which satisfies the relations $\{p_k : k \in K\}$, i.e., $p_k(b_i, b_i^* : i \in I) = 0$ for all $k \in K$, there exists a unique $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ such that $\pi(a_i) = b_i$ for all $i \in I$.

Theorem 3.12. *Let $\mathcal{A}$ be the universal C^* algebra generated by isometries $\{s_i : i \in I\}$ which satisfy the relations $\{p_k : k \in K\}$, and let $\mathcal{A}_0$ be the $*$-algebra generated by $\{s_i : i \in I\}$. If $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ is a flow generator such that $\mathcal{A}_\phi = \mathcal{A}_0$ then, for all $t \geq 0$, there exists a unital $*$-homomorphism*

$$\bar{j}_t : \mathcal{A} \to \mathcal{B}(\mathsf{h} \bar{\otimes} \mathcal{F})$$

such that $\bar{j}_t(x) = j_t(x)$ on $\mathsf{h} \otimes \mathcal{E}$ for all $x \in \mathcal{A}_0$, where $j_t(x)$ is as defined in Theorem 3.5.

Proof. Remark 3.6 and Lemma 3.7 imply that $j_t(s_i)$ is isometric and that $j_t(s_i^*)$ is contractive for all $i \in I$. Repeated application of (3.7) then shows that $j_t(x)$ is bounded for each $x \in \mathcal{A}_0$, and that j_t extends to a unital $*$-homomorphism from $\mathcal{A}_0$ to $\mathcal{B}(\mathsf{h} \bar{\otimes} \mathcal{F})$. Furthermore, the set $\{j_t(s_i) : i \in I\}$ satisfies the relations $\{p_k : k \in K\}$ so, by the universal nature of $\mathcal{A}$, there exists a $*$-homomorphism π from $\mathcal{A}$ into $\mathcal{B}(\mathsf{h} \bar{\otimes} \mathcal{F})$ such that $\pi(s_i) = j_t(s_i)$ for all $i \in I$ and $\bar{j}_t := \pi$ is as required. □

Corollary 3.13. *The family $(\bar{j}_t : \mathcal{A} \to \mathcal{B}(\mathsf{h} \bar{\otimes} \mathcal{F}))_{t \geq 0}$ constructed in Theorems 3.9 and 3.12 is a strong solution of the QSDE (1.2).*

Proof. Fix $x \in \mathcal{A}_\phi$ and let

$$L_t := \Sigma\big((\bar{j}_t \otimes \iota_{\mathcal{B}})(\phi(x))\big) \tag{3.10}$$

for all $t \geq 0$, where $\Sigma : \mathcal{B}(\mathsf{h} \bar{\otimes} \mathcal{F} \bar{\otimes} \widehat{\mathsf{k}}) \to \mathcal{B}(\mathsf{h} \bar{\otimes} \widehat{\mathsf{k}} \bar{\otimes} \mathcal{F})$ is the isomorphism that swaps the last two components of simple tensors. If $f \in L^2(\mathbb{R}_+; \mathsf{k})$ then

$$\|L_t u \otimes \widehat{f}(t) \otimes \varepsilon(f)\| \leq \|\phi(x)\| \, \|\widehat{f}(t)\| \, \|u\varepsilon(f)\|,$$

so if $t \mapsto L_t u \otimes \widehat{f}(t) \otimes \varepsilon(f)$ is strongly measurable then $t \mapsto L_t$ is quantum stochastically integrable [20], p. 232, and $\bar{j}$ satisfies the QSDE in the strong sense, since we already have from (3.4) that it is a weak solution.

Now, Theorem 3.5 implies that for each $x \in \mathcal{A}_\phi = \mathcal{A}_0$ and $\theta \in \mathsf{h} \otimes \mathcal{E}$ the map $t \mapsto \bar{j}_t(x)\theta$ is continuous, hence so is

$$t \mapsto (\bar{j}_t \otimes \iota_{\mathcal{B}})(y \otimes T)(\theta \otimes \xi) = \bar{j}_t(y)\theta \otimes T\xi$$

for all $y \in \mathcal{A}_0$, $T \in \mathcal{B}(\widehat{\mathsf{k}})$ and $\xi \in \widehat{\mathsf{k}}$. As $\|L_t\| = \|\phi(x)\|$ for all $t \geq 0$, it follows that $t \mapsto L_t$ and $t \mapsto L_t^*$ are strongly continuous on $\mathsf{h} \bar{\otimes} \widehat{\mathsf{k}} \bar{\otimes} \mathcal{F}$. Hence $t \mapsto L_t(u \otimes \widehat{f}(t) \otimes \varepsilon(f))$ is separably valued and weakly measurable, so Pettis's theorem gives the result. □

Remark 3.14. *Property* (3.5) *implies that the homomorphism $\bar{j}_t$ given by Theorems* 3.9 *and* 3.12 *takes values in the matrix space $\mathcal{A} \otimes_{\mathrm{M}} \mathcal{B}(\mathcal{F})$* [20].

Notation 3.15. *For all $t \geq 0$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ and $a \in \mathcal{A}$, let*

$$\bar{j}_t[f, g](a) := \big(1_{\mathsf{h}} \otimes \langle \varepsilon(f_{t)})|\big)\bar{j}_t(a)\big(1_{\mathsf{h}} \otimes |\varepsilon(g_{t)})\rangle\big).$$

Theorem 3.16. *The family of $*$-homomorphisms $(\bar{j}_t)_{t \geq 0}$ given by Theorems* 3.9 *and* 3.12 *forms a Feller cocycle* [23], Section 2.4, *for the shift semigroup on $\mathcal{B}(\mathcal{F})$: for all $s, t \geq 0$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ and $a \in \mathcal{A}$,*

(i) $\bar{j}_0[0, 0](a) = a$,
(ii) $\bar{j}_t[f, g](a) \in \mathcal{A}$,
(iii) $t \mapsto \bar{j}_t[f, g](a)$ *is norm continuous and*
(iv) $\bar{j}_{s+t}[f, g] = \bar{j}_s[f, g] \circ \bar{j}_t[f(\cdot + s), g(\cdot + s)]$.

Consequently, setting

$$T_t(a) := \bar{j}_t[0, 0](a) = \big(1_{\mathsf{h}} \otimes \langle \varepsilon(0)|\big)\bar{j}_t(a)\big(1_{\mathsf{h}} \otimes |\varepsilon(0)\rangle\big) \quad \textit{for all } a \in \mathcal{A}$$

gives a strongly continuous semigroup $T = (T_t)_{t \geq 0}$ of completely positive contractions on $\mathcal{A}$ such that $T_t(x) = \exp(t\phi_\omega^\omega)(x)$ for all $x \in \mathcal{A}_0$ and $t \geq 0$. In particular, $T_t(1) = 1$ for all $t \geq 0$ and $\mathcal{A}_0$ is a core for the generator of T.

Proof. Properties (i) and (ii) are immediate consequences of (3.4) and (3.5) respectively. For (iii), note that if $x \in \mathcal{A}_0$ and $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ then Theorem 3.5 implies that

$$t \mapsto \bar{j}_t[f, g](x) = \big(1_{\mathsf{h}} \otimes \langle \varepsilon(f)|\big) j_t(x)\big(1_{\mathsf{h}} \otimes |\varepsilon(g)\rangle\big) \exp\left(-\int_t^\infty \langle f(s), g(s)\rangle \,\mathrm{d}s\right)$$

is norm continuous; the general case follows by approximation.

In order to establish (iv), fix $s \geq 0$ and continuous functions $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$, and let

$$J_t := \bar{j}_s[f, g] \circ \bar{j}_t\big[f(\cdot + s), g(\cdot + s)\big] \quad \text{for all } t \geq 0.$$

We will show that $J_t = \bar{j}_{s+t}[f, g]$.

First note that for any $x \in \mathcal{A}_0$ and $t > 0$, the map

$$F : [0, t] \to \mathcal{A}; \qquad r \mapsto \bar{j}_r\big[f(\cdot + s), g(\cdot + s)\big]\big(\phi_{\widehat{g}(r+s)}^{\widehat{f}(r+s)}(x)\big)\big\langle \varepsilon(f_{[s+r,s+t)}), \varepsilon(g_{[s+r,s+t)})\big\rangle$$

is continuous, hence Bochner integrable, and so

$$x\big\langle \varepsilon(f_{[s,s+t)}), \varepsilon(g_{[s,s+t)})\big\rangle + \int_0^t F(r)\,\mathrm{d}r \in \mathcal{A}.$$

By the adaptedness of $\bar{\jmath}_t(x)$ and (3.4),

$$\begin{aligned}
&\Big\langle u, \Big(x\big\langle\varepsilon(f_{[s,s+t)}), \varepsilon(g_{[s,s+t)})\big\rangle + \int_0^t F(r)\,\mathrm{d}r\Big)v\Big\rangle \\
&\quad = \langle u, xv\rangle\big\langle\varepsilon\big(f(\cdot+s)_{t)}\big), \varepsilon\big(g(\cdot+s)_{t)}\big)\big\rangle \\
&\qquad + \int_0^t \big\langle u\varepsilon\big(f(\cdot+s)_{r)}\big), j_r\big(\phi^{\widehat{f}(r+s)}_{\widehat{g}(r+s)}(x)\big)v\varepsilon\big(g(\cdot+s)_{r)}\big)\big\rangle\big\langle\varepsilon\big(f(\cdot+s)_{[r,t)}\big), \varepsilon\big(g(\cdot+s)_{[r,t)}\big)\big\rangle\,\mathrm{d}r \\
&\quad = \big\langle u\varepsilon\big(f(\cdot+s)_{t)}\big), j_t(x)v\varepsilon\big(g(\cdot+s)_{t)}\big)\big\rangle \\
&\quad = \big\langle u, \bar{\jmath}_t\big[f(\cdot+s), g(\cdot+s)\big](x)v\big\rangle.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\big\langle u, J_t(x)v\big\rangle &= \big\langle u, \bar{\jmath}_s[f,g](x)v\big\rangle\big\langle\varepsilon(f_{[s,s+t)}), \varepsilon(g_{[s,s+t)})\big\rangle \\
&\quad + \int_0^t \big\langle u, \bar{\jmath}_s[f,g]\circ\bar{\jmath}_r\big[f(\cdot+s), g(\cdot+s)\big]\big(\phi^{\widehat{f}(r+s)}_{\widehat{g}(r+s)}(x)\big)v\big\rangle\big\langle\varepsilon(f_{[s+r,s+t)}), \varepsilon(g_{[s+r,s+t)})\big\rangle\,\mathrm{d}r \\
&= \big\langle u, \bar{\jmath}_s[f,g](x)v\big\rangle\big\langle\varepsilon(f_{[s,s+t)}), \varepsilon(g_{[s,s+t)})\big\rangle \\
&\quad + \int_0^t \big\langle u, J_r\big(\phi^{\widehat{f}(r+s)}_{\widehat{g}(r+s)}(x)\big)v\big\rangle\big\langle\varepsilon(f_{[s+r,s+t)}), \varepsilon(g_{[s+r,s+t)})\big\rangle\,\mathrm{d}r.
\end{aligned}$$

On the other hand, by (3.4),

$$\begin{aligned}
\big\langle u, \bar{\jmath}_{s+t}[f,g](x)v\big\rangle &= \langle u, xv\rangle\big\langle\varepsilon(f_{s+t)}), \varepsilon(g_{s+t)})\big\rangle + \int_0^s \big\langle u\varepsilon(f_{s+t)}), j_r\big(\phi^{\widehat{f}(r)}_{\widehat{g}(r)}(x)\big)v\varepsilon(g_{s+t)})\big\rangle\,\mathrm{d}r \\
&\quad + \int_s^{s+t} \big\langle u\varepsilon(f_{s+t)}), j_r\big(\phi^{\widehat{f}(r)}_{\widehat{g}(r)}(x)\big)v\varepsilon(g_{s+t)})\big\rangle\,\mathrm{d}r \\
&= \big\langle u\varepsilon(f_{s+t)}), j_s(x)v\varepsilon(g_{s+t)})\big\rangle + \int_0^t \big\langle u\varepsilon(f_{s+t)}), j_{q+s}\big(\phi^{\widehat{f}(q+s)}_{\widehat{g}(q+s)}(x)\big)v\varepsilon(g_{s+t)})\big\rangle\,\mathrm{d}q \\
&= \big\langle u, \bar{\jmath}_s[f,g](x)v\big\rangle\big\langle\varepsilon(f_{[s,s+t)}), \varepsilon(g_{[s,s+t)})\big\rangle \\
&\quad + \int_0^t \big\langle u, \bar{\jmath}_{q+s}[f,g]\big(\phi^{\widehat{f}(q+s)}_{\widehat{g}(q+s)}(x)\big)v\big\rangle\big\langle\varepsilon(f_{[s+q,s+t)}), \varepsilon(g_{[s+q,s+t)})\big\rangle\,\mathrm{d}q.
\end{aligned}$$

Now set $K_t := J_t - \bar{\jmath}_{s+t}[f,g]$, so that

$$\big\langle u, K_t(x)v\big\rangle = \int_0^t \big\langle u, K_r\big(\phi^{\widehat{f}(r+s)}_{\widehat{g}(r+s)}(x)\big)v\big\rangle G(r)\,\mathrm{d}r,$$

where $G: r \mapsto \langle\varepsilon(f_{[s+r,s+t)}), \varepsilon(g_{[s+r,s+t)})\rangle$ is continuous. As

$$\|K_t\| \le 2\exp\Big(\frac{1}{2}\big(\|f\|^2 + \|g\|^2\big)\Big) \quad \text{for all } t \ge 0,$$

iterating the above and estimating as in the proof of Lemma 3.8 shows that $K \equiv 0$. The density of $\mathcal{A}_0$ in $\mathcal{A}$ and of continuous functions in $L^2(\mathbb{R}_+; \mathsf{k})$ now gives (iv).

That T is a semigroup follows from this cocycle property (iv): note that

$$T_{s+t} = \bar{\jmath}_{s+t}[0,0] = \bar{\jmath}_s[0,0]\circ\bar{\jmath}_t[0,0] = T_s\circ T_t \quad \text{for all } s, t \ge 0.$$

Contractivity, complete positivity and strong continuity of T are immediate; the exponential identity holds because

$$\langle u, T_t(x)v\rangle = \langle u, xv\rangle + \int_0^t \langle u, T_s\big(\phi_\omega^\omega(x)\big)v\rangle \,\mathrm{d}s \tag{3.11}$$

for all $u, v \in \mathsf{h}$, $t \geq 0$ and $x \in \mathcal{A}_0$, by (3.4). That $\mathcal{A}_0$ is a core for the generator of T follows from Lemma 2.14 and [6], Corollary 3.1.20. □

Remark 3.17. *A $*$-homomorphic Feller cocycle as in Theorem 3.16 is called a* quantum flow *or* quantum stochastic flow; *a strongly continuous semigroup $(T_t)_{t\geq 0}$ of completely positive contractions is known as a* quantum dynamical semigroup, *and the condition $T_t(1) = 1$ for all $t \geq 0$ means that the semigroup is* conservative; *conservative quantum dynamical semigroups are also known as* quantum Markov semigroups. *Hence Theorem 3.16 gives the existence of a quantum flow which dilates a quantum Markov semigroup on the C^* algebra $\mathcal{A}$.*

Remark 3.18. *By Theorem 3.16, the component $\phi_\omega^\omega = \tau$ of the flow generator ϕ is closable, with $\overline{\tau}$ being the generator of the quantum Markov semigroup T. However, closability of the bimodule map δ seems to be a much more delicate issue and remains an open question.*

Theorem 3.19. *Consider the family of $*$-homomorphisms $(\bar{\jmath}_t)_{t\geq 0}$ constructed in Theorems 3.9 and 3.12. If $\mathcal{A}_c$ is a commutative $*$-subalgebra of $\mathcal{A}$ such that*

(i) *$\phi(\mathcal{A}_c \cap \mathcal{A}_0) \subseteq \mathcal{A}_c \otimes \mathcal{B}$ and*
(ii) *$\mathcal{A}_c \cap \mathcal{A}_0$ is dense in $\mathcal{A}_c$*

then the family $\{\bar{\jmath}_t(a) : t \geq 0, a \in \mathcal{A}_c\}$ is commutative, i.e., the commutator $[\bar{\jmath}_s(a), \bar{\jmath}_t(b)] = 0$ for all $s, t \geq 0$ and a, $b \in \mathcal{A}_c$.

Proof. The result is immediate when $s = t$, so assume without loss of generality that $s < t$ and let $b \in \mathcal{A}_c \cap \mathcal{A}_0$; if

$$K_t(b) := \langle u\varepsilon(f), \big[\bar{\jmath}_s(a), \bar{\jmath}_t(b)\big]v\varepsilon(g)\rangle = 0,$$

where $u, v \in \mathsf{h}$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ and $a \in \mathcal{A}_c$ are arbitrary, then the result follows by (ii) and the continuity of $\bar{\jmath}_t$.

Write $\bar{\jmath}_t(b) = \bar{\jmath}_s(b) + \int_s^t L_r \,\mathrm{d}\Lambda_r$, where $L = (L_r)_{r\geq 0}$ is the process defined in (3.10) with x changed to b. It is straightforward, using adaptedness, to show that

$$A \int_s^t L_r \,\mathrm{d}\Lambda_r B = \int_s^t \Sigma(A \otimes 1_{\widehat{\mathsf{k}}}) L_r \Sigma(B \otimes 1_{\widehat{\mathsf{k}}}) \,\mathrm{d}\Lambda_r$$

for any $A, B \in \mathcal{B}(\mathsf{h} \mathbin{\bar{\otimes}} \mathcal{F}_{[0,s)}) \mathbin{\bar{\otimes}} 1_{\mathcal{F}_{[s,\infty)}}$, where Σ is the swap isomorphism defined after (3.10). Since $\bar{\jmath}$ is a strong solution of the QSDE (1.2), by Corollary 3.13, it follows that

$$K_t(b) = \int_s^t K_r\big(\phi_{\widehat{g}(r)}^{\widehat{f}(r)}(b)\big) \,\mathrm{d}r.$$

Assumption (i) allows us to iterate this identity; noting also that

$$\big|K_r(c)\big| \leq 2\|u\|\|v\|\big\|\varepsilon(f)\big\|\big\|\varepsilon(g)\big\|\|a\|\|c\| \quad \text{for all } c \in \mathcal{A}_c \cap \mathcal{A}_0,$$

one readily obtains the estimate

$$\big|K_t(b)\big| \leq 2\|u\|\|v\|\big\|\varepsilon(f)\big\|\big\|\varepsilon(g)\big\|\|a\| C_b M_b^n \frac{1}{n!}\bigg(\int_s^t \big\|\widehat{f}(r)\big\|\big\|\widehat{g}(r)\big\| \,\mathrm{d}r\bigg)^n,$$

where C_b and M_b are constants associated to b through its membership of $\mathcal{A}_\phi$. Letting $n \to \infty$ gives the result. □

Remark 3.20. *If $\mathcal{A}$ is commutative then conditions* (i) *and* (ii) *of Theorem* 3.19 *are satisfied automatically when $\mathcal{A}_c = \mathcal{A}$, so Theorems* 3.9 *and* 3.12 *produce classical Markov semigroups in this case. However, Theorem* 3.19 *also allows for the possibility of dealing with different commutative subalgebras that do not commute with one another, a necessary feature of quantum dynamics.*

4. Random walks on groups

Definition 4.1. *Let $\mathcal{A} = C_0(G) \oplus \mathbb{C}1 \subseteq \mathcal{B}(\ell^2(G))$, where G is a discrete group and $x \in C_0(G)$ acts on $\ell^2(G)$ by multiplication, and let $\mathcal{A}_0 = \operatorname{lin}\{1, e_g\colon\ g \in G\}$, where $e_g(h) := \mathbb{1}_{g=h}$ for all $h \in G$. That is, $\mathcal{A}$ is the unitisation of the C^* algebra of functions on G which vanish at infinity and $\mathcal{A}_0$ is the dense unital subalgebra generated by the functions with finite support; as positivity in the C^*-algebraic sense corresponds here to the pointwise positivity of functions, $\mathcal{A}_0$ contains its square roots.*

Let H be a non-empty finite subset of $G \setminus \{e\}$ and let the Hilbert space k have orthonormal basis $\{f_h\colon\ h \in H\}$; the maps

$$\lambda_h : G \to G; \qquad g \mapsto hg \quad (h \in H)$$

correspond to the permitted moves in the random walk constructed on G.

Lemma 4.2. *Given a* transition function

$$t : H \times G \to \mathbb{C}; \qquad (h, g) \mapsto t_h(g),$$

the map

$$\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}; \qquad x \mapsto \begin{bmatrix} \sum_{h\in H} |t_h|^2 (x \circ \lambda_h - x) & \sum_{h\in H} \overline{t_h}(x \circ \lambda_h - x) \otimes \langle f_h| \\ \sum_{h\in H} t_h(x \circ \lambda_h - x) \otimes |f_h\rangle & \sum_{h\in H} (x \circ \lambda_h - x) \otimes |f_h\rangle\langle f_h| \end{bmatrix}$$

is a flow generator such that

$$\phi(e_g) = e_g \otimes m_e(g) + \sum_{h\in H} e_{h^{-1}g} \otimes m_h\bigl(h^{-1}g\bigr) \quad \textit{for all } g \in G,$$

where

$$m_e(g) := \begin{bmatrix} -\sum_{h\in H} |t_h(g)|^2 & -\sum_{h\in H} \overline{t_h(g)}\langle f_h| \\ -\sum_{h\in H} t_h(g)|f_h\rangle & -1_{\mathsf{k}} \end{bmatrix} \quad \textit{and} \quad m_h(g) := \begin{bmatrix} |t_h(g)|^2 & \overline{t_h(g)}\langle f_h| \\ t_h(g)|f_h\rangle & |f_h\rangle\langle f_h| \end{bmatrix}.$$

Hence

$$\phi_n(e_g) = \sum_{h_1 \in H\cup\{e\}} \cdots \sum_{h_n \in H\cup\{e\}} e_{h_n^{-1}\cdots h_1^{-1}g} \otimes m_{h_n}\bigl(h_n^{-1}\cdots h_1^{-1}g\bigr) \otimes \cdots \otimes m_{h_1}\bigl(h_1^{-1}g\bigr)$$

for all $n \in \mathbb{N}$ and $g \in G$.

Proof. The first claim is readily verified with the aid of Lemma 2.2; the second is immediate. □

Theorem 4.3. *Let $\mathcal{A}$ be as in Definition* 4.1 *and ϕ as in Lemma* 4.2. *If the transition function t is chosen such that $\mathcal{A}_\phi = \mathcal{A}_0$ then there exists an adapted family of unital $*$-homomorphisms $(\bar{\jmath}_t : \mathcal{A} \to \mathcal{B}(\mathsf{h} \bar{\otimes} \mathcal{F}))_{t\geq 0}$ which forms a Feller cocycle in the sense of Theorem* 3.16 *and satisfies the quantum stochastic differential equation* (1.2) *in the strong sense on $\mathcal{A}_0$ for all $t \geq 0$. Setting*

$$T_t(a) := \bigl(1_{\mathsf{h}} \otimes \langle\varepsilon(0)|\bigr)\bar{\jmath}_t(a)\bigl(1_{\mathsf{h}} \otimes |\varepsilon(0)\rangle\bigr) \quad \textit{for all } a \in \mathcal{A} \textit{ and } t \geq 0$$

gives a classical Markov semigroup T on $\mathcal{A}$ whose generator is the closure of

$$\tau : \mathcal{A}_0 \to \mathcal{A}_0; \qquad x \mapsto \sum_{h \in H} |t_h|^2 (x \circ \lambda_h - x).$$

Proof. This follows from Theorems 3.9 and 3.19 together with Lemma 4.2. □

Remark 4.4. *Given $g \in G$, let $A := [B\,1_{\mathsf{k}}] \in \mathcal{B}(\mathbb{C} \oplus \mathsf{k}; \mathsf{k})$, where $B := \sum_{h \in H} t_h(g)|f_h\rangle$. Then $m_e(g) = -A^*A$ and*

$$\|m_e(g)\| = \|AA^*\| = \|BB^* + 1_{\mathsf{k}}\| = \|B^*B\| + 1 = 1 + \sum_{h \in H} |t_h(g)|^2.$$

It may be shown similarly that $\|m_h(g)\| = 1 + |t_h(g)|^2$ for all $g \in G$ and $h \in H$, so if

$$M_g := \lim_{n \to \infty} \max\bigl\{\bigl|t_h\bigl(h_n^{-1} \cdots h_1^{-1} g\bigr)\bigr| \colon h_1, \ldots, h_n \in H \cup \{e\}, h \in H\bigr\} < \infty \tag{4.1}$$

then

$$\|\phi_n(e_g)\| \le \bigl(1 + |H| + 2|H|M_g^2\bigr)^n \quad \text{for all } n \in \mathbb{Z}_+,$$

where $|H|$ denotes the cardinality of H. Hence $\mathcal{A}_\phi = \mathcal{A}_0$ if (4.1) holds for all $g \in G$.

Remark 4.5. *If t is bounded then clearly (4.1) holds for all $g \in G$. In this case, there exist bounded operators $L \in \mathcal{B}(\mathsf{h}; \mathsf{h} \mathbin{\bar{\otimes}} \mathsf{k})$, $S \in \mathcal{B}(\mathsf{h} \mathbin{\bar{\otimes}} \mathsf{k})$ and $F \in \mathcal{B}(\mathsf{h} \mathbin{\bar{\otimes}} \widehat{\mathsf{k}})$ such that*

$$L = \sum_{h \in H} t_h \otimes |f_h\rangle, \qquad S = \sum_{h \in H} S_h \otimes |f_h\rangle\langle f_h| \quad \text{and} \quad F = \begin{bmatrix} -\frac{1}{2}L^*L & -L^* \\ SL & S - 1_{\mathsf{h} \otimes \mathsf{k}} \end{bmatrix},$$

where t_h acts by multiplication and S_h is the unitary operator on $\ell^2(G)$ such that $e_g \mapsto e_{hg}$.

It follows from [21], *Theorems* 7.1 *and* 7.5, *that the Hudson–Parthasarathy QSDE*

$$U_0 = I_{\mathsf{h} \otimes \mathcal{F}}, \qquad \mathrm{d}U_t = (F \otimes 1_{\mathcal{F}})\Sigma(U_t \otimes I_{\widehat{\mathsf{k}}})\,\mathrm{d}\Lambda_t,$$

where Σ is the swap isomorphism defined after (3.10), *has a unique solution which is a unitary cocycle. Furthermore, by* [21], *Theorem* 7.4, *setting*

$$k_t(a) := U_t^*(a \otimes 1_{\mathcal{F}})U_t \quad \text{for all } a \in \mathcal{B}(\mathsf{h}) \text{ and } t \ge 0$$

defines a quantum flow k with generator

$$\varphi : \mathcal{B}(\mathsf{h}) \to \mathcal{B}(\mathsf{h} \mathbin{\bar{\otimes}} \widehat{\mathsf{k}}); \qquad a \mapsto (a \otimes 1_{\widehat{\mathsf{k}}})F + F^*(a \otimes 1_{\widehat{\mathsf{k}}}) + F^*\Delta(a \otimes 1_{\widehat{\mathsf{k}}})F.$$

A short calculation shows that φ is of the form covered by Lemma 2.5, *with*

$$\pi(a) = S^*(a \otimes 1_{\mathsf{k}})S, \qquad \delta(a) = -La + \pi(a)L \quad \text{and} \quad \tau(a) = -\frac{1}{2}\bigl\{L^*L, a\bigr\} + L^*\pi(a)L$$

for all $a \in \mathcal{B}(\mathsf{h})$. It follows that $\varphi|_{\mathcal{A}_0} = \phi$, where ϕ is the flow generator of Lemma 4.2, *and so the cocycle $\bar{\jmath}$ given by Theorem* 4.3 *is the restriction of k to $\mathcal{A}$. However, this construction by conjugation does not give the Feller property, that $\mathcal{A}$ is preserved by k.*

Example 4.6. *If $G = (\mathbb{Z}, +)$, $H = \{\pm 1\}$ and the transition function t is bounded, with $t_{+1}(g) = 0$ for all $g < 0$ and $t_{-1}(g) = 0$ for all $g \le 0$, then the Markov semigroup T given by Theorem* 4.3 *corresponds to the classical birth–death process with birth and dates rates $|t_{+1}|^2$ and $|t_{-1}|^2$, respectively. The cocycle constructed here is Feller, as it acts on $\mathcal{A} = C_0(\mathbb{Z}) \oplus \mathbb{C}1$, in contrast to* [27], *Example* 3.3, *where the cocycle acts on the whole of $\ell^\infty(\mathbb{Z})$.*

Remark 4.7. *If $G = (\mathbb{Z}, +)$, $H = \{+1\}$ and $t_{+1} : g \mapsto 2^g$ then $M_g = 2^g$ and the condition* (4.1) *holds for all $g \in G$. Thus Theorem* 4.3 *applies to examples where the transition function t is unbounded.*

5. The symmetric quantum exclusion process

This section was inspired by Rebolledo's treatment of the quantum exclusion process: see [28], Examples 2.4.3 and 4.1.3.

Definition 5.1. *Let I be a non-empty set. The* CAR algebra *is the unital C^* algebra $\mathcal{A}$ with generators $\{b_i\colon i \in I\}$, subject to the anti-commutation relations*

$$\{b_i, b_j\} = 0 \quad \text{and} \quad \{b_i, b_j^*\} = \mathbb{1}_{i=j} \quad \text{for all } i, j \in I. \tag{5.1}$$

It follows from (5.1) that the b_i are nonzero partial isometries for all $i \in I$.

As is well known [7], Proposition 5.2.2, $\mathcal{A}$ is represented faithfully and irreducibly on $\mathcal{F}_-(\ell^2(I))$, the Fermionic Fock space over $\ell^2(I)$; in other words, we may (and do) suppose that $\mathcal{A} \subseteq \mathcal{B}(\mathsf{h})$, where $\mathsf{h} := \mathcal{F}_-(\ell^2(I))$, and the algebra identity $1 = 1_{\mathsf{h}}$.

Remark 5.2. *The elements of I may be taken to correspond to sites at which Fermionic particles may exist, with the operators b_i and b_i^* representing the annihilation and creation, respectively, of a particle at site i.*

Notation 5.3. *Let $\mathcal{A}_0$ be the unital algebra generated by $\{b_i, b_i^*\colon i \in I\}$; by definition, this is a norm-dense unital $*$-subalgebra of $\mathcal{A}$.*

Lemma 5.4. *For each $x \in \mathcal{A}_0$ there exists a finite subset $J \subseteq I$ such that x lies in the finite-dimensional $*$-subalgebra*

$$\mathcal{A}_J := \operatorname{lin}\big\{b_{j_1}^* \cdots b_{j_q}^* b_{i_1} \cdots b_{i_p}\colon 0 \le p, q \le |J|, \{i_1, \ldots, i_p\} \in J^{(p)}, \{j_1, \ldots, j_q\} \in J^{(q)}\big\} \subseteq \mathcal{A}_0,$$

where $J^{(p)}$ denote the set of subsets of J with cardinality p et cetera. Consequently, $\mathcal{A}$ is an AF algebra and $\mathcal{A}_0$ contains its square roots.

Proof. By employing the anti-commutation relations (5.1), any finite product of terms from the generating set $\{b_i, b_i^*\colon i \in I\}$ may be reduced to a linear combination of words of the form

$$b_{j_1}^* \cdots b_{j_q}^* b_{i_1} \cdots b_{i_p}, \tag{5.2}$$

where $i_1, \ldots, i_p$ are distinct elements of I, as are $j_1, \ldots, j_q$, and $p, q \in \mathbb{Z}_+$, with an empty product equal to 1. As every element of $\mathcal{A}_0$ is a finite linear combination of such terms, the first claim follows. The second claim holds by Remark 3.10. □

Definition 5.5. *Let $\{\alpha_{i,j}\colon i, j \in I\} \subseteq \mathbb{C}$ be a fixed collection of* amplitudes. *We may view $(I, \{\alpha_{i,j}\colon i, j \in I\})$ as a weighted directed graph, where I is the set of vertices, an edge exists from i to j if $\alpha_{i,j} \neq 0$ and $\alpha_{i,j}$ is a complex weight on the edge from vertex i to vertex j, which may differ from the weight $\alpha_{j,i}$ from j to i.*

For all $i \in I$, let

$$\operatorname{supp}(i) := \{j \in I\colon \alpha_{i,j} \neq 0 \text{ or } \alpha_{j,i} \neq 0\} \quad \text{and} \quad \operatorname{supp}^+(i) := \operatorname{supp}(i) \cup \{i\}.$$

Thus $\operatorname{supp}(i)$ is the set of sites with which site i interacts and $|\operatorname{supp}(i)|$ is the valency of the vertex i. We require that the valencies are finite:

$$\big|\operatorname{supp}(i)\big| < \infty \quad \text{for all } i \in I. \tag{5.3}$$

The transport of a particle from site i to site j with amplitude $\alpha_{i,j}$ is described by the operator

$$t_{i,j} := \alpha_{i,j} b_j^* b_i.$$

Definition 5.6. *Let $\{\eta_i\colon i \in I\} \subseteq \mathbb{R}$ be fixed. The total energy in the system is given by*

$$h := \sum_{i\in I} \eta_i b_i^* b_i,$$

where η_i gives the energy of a particle at site i. If the set $\{i \in I\colon \eta_i \neq 0\}$ is infinite then the proper interpretation of h involves issues of convergence; below it will only appear in a commutator with elements of $\mathcal{A}_0$, which is sufficient to give a well-defined quantity.

Lemma 5.7. *Let*

$$\tau_{i,j}(x) := t_{i,j}^*[t_{i,j}, x] + \big[x, t_{i,j}^*\big]t_{i,j} = |\alpha_{i,j}|^2\big(b_i^* b_j \big[b_j^* b_i, x\big] + \big[x, b_i^* b_j\big] b_j^* b_i\big)$$

for all $i, j \in I$ and $x \in \mathcal{A}$, and let

$$[h, x] := \sum_{i\in I} \eta_i \big[b_i^* b_i, x\big] \tag{5.4}$$

for all $x \in \mathcal{A}_0$. Setting

$$\tau(x) := \mathrm{i}[h, x] - \frac{1}{2}\sum_{i,j\in I} \tau_{i,j}(x) \tag{5.5}$$

defines a $*$-linear map $\tau : \mathcal{A}_0 \to \mathcal{A}_0$.

Proof. Let $x \in \mathcal{A}_0$ and note that $x \in \mathcal{A}_J$ for some finite set $J \subseteq I$, by Lemma 5.4. Furthermore,

$$\big[b_j^* b_i, x\big] = b_j^*\{b_i, x\} - \big\{b_j^*, x\big\} b_i = 0 \quad \text{whenever } i \notin J \text{ and } j \notin J,$$

so

$$[h, x] = \sum_{i\in J} \eta_i \big[b_i^* b_i, x\big] \in \mathcal{A}_J \quad \text{and} \quad \tau(x) = \mathrm{i}[h, x] - \frac{1}{2}\sum_{i,j\in J^+} \tau_{i,j}(x) \in \mathcal{A}_{J^+},$$

where

$$J^+ := \bigcup_{k\in J} \operatorname{supp}^+(k). \tag{5.6}$$

Hence $\tau(\mathcal{A}_J) \subseteq \mathcal{A}_{J^+}$ and, as (5.3) implies that J^+ is finite, it follows that $\mathcal{A}_0$ is invariant under τ. The $*$-linearity of τ is immediately verified. □

Lemma 5.8. *Let*

$$\delta_{i,j}(x) := [t_{i,j}, x] = \alpha_{i,j}\big(b_j^* b_i x - x b_j^* b_i\big)$$

for all $i, j \in I$ and $x \in \mathcal{A}$, and let k be a Hilbert space with orthonormal basis $\{f_{i,j}\colon i, j \in I\}$. Setting

$$\delta(x) := \sum_{i,j\in I} \delta_{i,j}(x) \otimes |f_{i,j}\rangle \tag{5.7}$$

for all $x \in \mathcal{A}_0$ defines a linear map $\delta : \mathcal{A}_0 \to \mathcal{A}_0 \otimes |\mathsf{k}\rangle$ such that

$$\delta(xy) = \delta(x)y + (x \otimes 1_{\mathsf{k}})\delta(y) \tag{5.8}$$

and

$$\delta^\dagger(x)\delta(y) = \tau(xy) - \tau(x)y - x\tau(y) \tag{5.9}$$

for all $x, y \in \mathcal{A}_0$, *where* τ *is as defined in Lemma* 5.7.

Proof. The series in (5.7) contains only finitely many terms, since if $x \in \mathcal{A}_J$ then

$$\delta_{i,j}(x) = 0 \quad \text{when } \{i, j\} \nsubseteq J^+.$$

Hence δ is well defined, and (5.8) holds because each $\delta_{i,j}$ is a derivation. A short calculation shows that

$$\tau_{i,j}(xy) - \tau_{i,j}(x)y - x\tau_{i,j}(y) = -2\delta^\dagger_{i,j}(x)\delta_{i,j}(y) \tag{5.10}$$

for all $x, y \in \mathcal{A}$. Since $x \mapsto [b_i^* b_i, x]$ is a derivation for all $i \in I$, it follows from (5.10) that

$$\tau(xy) - \tau(x)y - x\tau(y) = \sum_{i,j \in I} \delta^\dagger_{i,j}(x)\delta_{i,j}(y) = \delta^\dagger(x)\delta(y) \quad \text{for all } x, y \in \mathcal{A}_0.$$

□

Lemma 5.9. *The map*

$$\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}; \qquad x \mapsto \begin{bmatrix} \tau(x) & \delta^\dagger(x) \\ \delta(x) & 0 \end{bmatrix}, \tag{5.11}$$

where τ, δ *and* $\delta^\dagger$ *are as defined in Lemmas* 5.7 *and* 5.8, *is a flow generator.*

If the amplitudes satisfy the symmetry condition

$$|\alpha_{i,j}| = |\alpha_{j,i}| \quad \textit{for all } i, j \in I \tag{5.12}$$

then, for all $n \in \mathbb{N}$ *and* $i_0 \in I$,

$$\phi_n(b_{i_0}) = \sum_{i_1 \in \operatorname{supp}^+(i_0)} \cdots \sum_{i_n \in \operatorname{supp}^+(i_{n-1})} b_{i_n} \otimes B_{i_{n-1},i_n} \otimes \cdots \otimes B_{i_0,i_1}, \tag{5.13}$$

where

$$B_{i,j} := \mathbb{1}_{j=i}\lambda_i|\omega\rangle\langle\omega| + |\omega\rangle\langle\alpha_{i,j} f_{i,j}| - |\alpha_{j,i} f_{j,i}\rangle\langle\omega|$$

and

$$\lambda_i := -\mathrm{i}\eta_i - \frac{1}{2}\sum_{j \in \operatorname{supp}(i)} |\alpha_{j,i}|^2$$

for all $i, j \in I$.

Proof. The first claim is an immediate consequence of Lemmas 5.7, 5.8 and 2.2.

If $i, j, k \in I$ then a short calculation shows that

$$\tau_{j,k}(b_i) = \begin{cases} |\alpha_{i,i}|^2 b_i & (j = i, k = i), \\ |\alpha_{j,i}|^2 b_j^* b_j b_i & (j \neq i, k = i), \\ |\alpha_{i,k}|^2 b_k b_k^* b_i & (j = i, k \neq i), \\ 0 & (j \neq i, k \neq i). \end{cases}$$

Since

$$[h, b_i] = \sum_{j \in I} \eta_j \big[b_j^* b_j, b_i\big] = \eta_i \big[b_i^* b_i, b_i\big] = -\eta_i b_i,$$

the symmetry condition (5.12) implies that

$$\tau(b_i) = \lambda_i b_i \quad \text{for all } i \in I.$$

Furthermore, if $i,\ j,\ k \in I$ then

$$\delta_{j,k}(b_i) = \alpha_{j,k}\big(b_k^* b_j b_i - b_i b_k^* b_j\big) = -\alpha_{j,k}\big\{b_k^*, b_i\big\}b_j = -\mathbb{1}_{k=i}\alpha_{j,i} b_j$$

and

$$\delta_{j,k}^\dagger(b_i) = \overline{\alpha_{j,k}}\big(b_i b_j^* b_k - b_j^* b_k b_i\big) = \overline{\alpha_{j,k}}\big\{b_i, b_j^*\big\}b_k = \mathbb{1}_{j=i}\overline{\alpha_{i,k}} b_k;$$

thus

$$\delta(b_i) = \sum_{j,k\in I} \delta_{j,k}(b_i) \otimes |f_{j,k}\rangle = -\sum_{j\in\operatorname{supp}(i)} \alpha_{j,i} b_j \otimes |f_{j,i}\rangle$$

and

$$\delta^\dagger(b_i) = \sum_{j,k\in I} \delta_{j,k}^\dagger(b_i) \otimes \langle f_{j,k}| = \sum_{k\in\operatorname{supp}(i)} \overline{\alpha_{i,k}} b_k \otimes \langle f_{i,k}|.$$

Hence

$$\begin{aligned}\phi(b_i) &= \lambda_i b_i \otimes |\omega\rangle\langle\omega| - \sum_{j\in\operatorname{supp}(i)} \alpha_{j,i} b_j \otimes |f_{j,i}\rangle\langle\omega| + \sum_{k\in\operatorname{supp}(i)} \overline{\alpha_{i,k}} b_k \otimes |\omega\rangle\langle f_{i,k}| \\ &= \sum_{j\in\operatorname{supp}^+(i)} b_j \otimes \big(\mathbb{1}_{j=i}\lambda_i |\omega\rangle\langle\omega| + |\omega\rangle\langle\alpha_{i,j} f_{i,j}| - |\alpha_{j,i} f_{j,i}\rangle\langle\omega|\big)\end{aligned}$$

and the identity (5.13) follows. □

Theorem 5.10. *Let $\mathcal{A}$ be the CAR algebra and let ϕ be defined as in Lemma 5.9. If the amplitudes $\{\alpha_{i,j}\}$ and energies $\{\eta_i\}$ are chosen so that $\mathcal{A}_\phi = \mathcal{A}_0$ then there exists an adapted family of unital $*$-homomorphisms $(j_t : \mathcal{A} \to \mathcal{B}(\mathsf{h} \mathbin{\bar\otimes} \mathcal{F}))_{t\ge 0}$ which forms a Feller cocycle in the sense of Theorem 3.16 and satisfies the quantum stochastic differential equation (1.2) in the strong sense on $\mathcal{A}_0$ for all $t \ge 0$. Setting*

$$T_t(a) := \big(1_\mathsf{h} \otimes \langle\varepsilon(0)|\big) j_t(a) \big(1_\mathsf{h} \otimes |\varepsilon(0)\rangle\big) \quad \textit{for all } a \in \mathcal{A} \textit{ and } t \ge 0$$

gives a quantum Markov semigroup T on $\mathcal{A}$ whose generator is the closure of

$$\tau : \mathcal{A}_0 \to \mathcal{A}_0; \qquad x \mapsto \mathrm{i}\sum_{i\in I} \eta_i\big[b_i^* b_i, x\big] - \frac{1}{2}\sum_{i,j\in I} |\alpha_{i,j}|^2 \big(b_i^* b_j\big[b_j^* b_i, x\big] + \big[x, b_i^* b_j\big] b_j^* b_i\big).$$

Proof. This is an immediate consequence of Theorem 3.9, Theorem 3.16 and Lemma 5.9. □

Example 5.11. *Suppose that the amplitudes satisfy the symmetry condition (5.12), and further that there are uniform bounds on the amplitudes, valencies and energies:*

$$M := \sup_{i,j\in I} |\alpha_{i,j}| < \infty, \qquad V := \sup_{i\in I} \big|\operatorname{supp}(i)\big| < \infty \quad \textit{and} \quad H := \sup_{i\in I} |\eta_i| < \infty. \tag{5.14}$$

It follows that

$$|\lambda_i| \le |\eta_i| + \frac{1}{2} V M^2 \quad \textit{and} \quad \|B_{i,j}\| \le |\lambda_i| + 2M \le H + \frac{1}{2} V M^2 + 2M$$

for all i, $j \in I$. Hence, for all $n \in \mathbb{Z}_+$,

$$\|\phi_n(b_i)\| \leq (V+1)^n \left(H + \frac{1}{2} V M^2 + 2M \right)^n$$

and so $\mathcal{A}_\phi = \mathcal{A}_0$, by Corollary 2.12. Hence there is a flow on $\mathcal{A}$ for this generator.

Example 5.12. *We can lift the boundedness assumptions in Example 5.11 by taking I to be a disjoint union of subsets,*

$$I = \bigsqcup_{k \in K} I_k,$$

such that there is no transport between any of these subsets, i.e.,

$$\alpha_{i,j} \neq 0 \quad \text{only if there is some } k \in K \text{ such that } i, j \in I_k.$$

Assume the symmetry condition (5.12) once again. Suppose that in each I_k the conditions of (5.14) are satisfied, but with respect to constants M_k, V_k and H_k that depend on k. Then, if $i \in I_k$, we get the estimate

$$\|\phi_n(b_i)\| \leq (V_k+1)^n \left(H_k + \frac{1}{2} V_k M_k^2 + 2M_k \right)^n$$

and so $\mathcal{A}_\phi = \mathcal{A}_0$ once more, but now it is possible that $M = \infty$ et cetera.

Example 5.13. *To create an example where the graph associated to I has only one component, but where we do not assume $M < \infty$ as in Example 5.11, assume once again that I is decomposed into a disjoint union:*

$$I = \bigsqcup_{k \in \mathbb{Z}_+} I_k \quad \text{with } |I_k| < \infty \text{ for all } k \in \mathbb{Z}_+.$$

This time assume, as well as the symmetry condition (5.12), that $\alpha_{i,j} = 0$ unless there is some $k \in \mathbb{Z}_+$ such that $i \in I_k$ and $j \in I_{k+1}$, or $j \in I_k$ and $i \in I_{k+1}$, so that there is transport only between neighbouring levels in I. Set

$$a_k = \sup\left\{ |\alpha_{i,j}| \colon i \in I_k, j \in I_{k+1} \right\} \quad \text{for all } k \in \mathbb{Z}_+,$$

and furthermore assume that the energies are bounded, i.e., $H < \infty$.

Now if $k \in \mathbb{N}$ and $i \in I_k$ then

$$\begin{aligned} \sum_{j \in \operatorname{supp}^+(i)} \|B_{i,j}\| &\leq \|B_{i,i}\| + \sum_{j \in I_{k-1}} \|B_{i,j}\| + \sum_{j \in I_{k+1}} \|B_{i,j}\| \\ &\leq |\lambda_i| + 2|I_{k-1}| a_{k-1} + 2|I_{k+1}| a_k, \end{aligned}$$

with a similar estimate holding if $i \in I_0$. Furthermore,

$$|\lambda_i| \leq H + \frac{1}{2}|I_{k-1}| a_{k-1}^2 + \frac{1}{2}|I_{k+1}| a_k^2.$$

As in Example 5.11, if it can be shown that

$$\sum_{j \in \operatorname{supp}^+(i)} \|B_{i,j}\| \leq C$$

for some constant C that does not depend on i, it follows that $\|\phi_n(b_i)\| \leq C^n$ for each $n \in \mathbb{Z}_+$ and $i \in I$, and so $\mathcal{A}_\phi = \mathcal{A}_0$ once more. Here, the previous working shows this will hold if there are constants $a > 0$, $b > 0$ and $p \geq 1$ such that

$$a_k \leq \frac{a}{(k+2)^p} \quad \text{and} \quad |I_k| \leq b(k+1)^p \quad \text{for all } k \in \mathbb{Z}_+.$$

It is clear that this can yield an example where $M = \infty$, i.e., there is no upper bound on the valencies.

6. Flows on universal C^* algebras

6.1. *The non-commutative torus*

Definition 6.1. *Let $\lambda \in \mathbb{T}$, the set of complex numbers with unit modulus. The* non-commutative torus *is the universal C^* algebra $\mathcal{A}$ generated by unitaries U and V which satisfy the relation*

$$UV = \lambda VU.$$

Let $\mathcal{A}_0$ denote the dense $*$-subalgebra of $\mathcal{A}$ generated by U and V.

There is a faithful trace tr on $\mathcal{A}$ such that $\tau(U^m V^n) = \mathbb{1}_{m=n=0}$ for all $m, n \in \mathbb{Z}$; the proof of this in [10], pp. 166–168, is valid for all λ. Consequently $\{U^m V^n : m, n \in \mathbb{Z}\}$ is a basis for $\mathcal{A}_0$.

Lemma 6.2. *Let $\mathsf{h} := \ell^2(\mathbb{Z}^2)$, let*

$$(U_c u)_{m,n} = u_{m+1,n} \quad \text{and} \quad (V_c u)_{m,n} = \lambda^m u_{m,n+1} \quad \text{for all} u \in \mathsf{h} \text{ and } m, n \in \mathbb{Z},$$

and let $\mathcal{A}_c \subseteq \mathcal{B}(\mathsf{h})$ be the C^* algebra generated by U_c and V_c. There is a C^* isomorphism from $\mathcal{A}$ to $\mathcal{A}_c$ such that $U \mapsto U_c$ and $V \mapsto V_c$. Moreover, under this map the trace tr *corresponds to the vector state given by $e \in \mathsf{h}$ such that $e_{m,n} = \mathbb{1}_{m=n=0}$ for all $m, n \in \mathbb{Z}$.*

Proof. Unitarity of U_c and V_c is immediately verified, as is the identity $U_c V_c = \lambda V_c U_c$, so the universality of $\mathcal{A}$ gives a surjective $*$-homomorphism from $\mathcal{A}$ to $\mathcal{A}_c$. Injectivity is a consequence of the final observation, that tr corresponds to the vector state given by e. □

From now on we will identify $\mathcal{A}$ and $\mathcal{A}_c$.

Definition 6.3. *For each $(\mu, \nu) \in \mathbb{T}^2$, let $\pi_{\mu,\nu}$ be the automorphism of $\mathcal{A}$ such that*

$$\pi_{\mu,\nu}\big(U^m V^n\big) = \mu^m \nu^n U^m V^n \qquad \text{for all } m, n \in \mathbb{Z};$$

the existence of $\pi_{\mu,\nu}$ is an immediate consequence of universality.

The proofs of the next two lemmas are a matter of routine algebraic computation.

Lemma 6.4. *For all $a, b \in \mathbb{Z}$, define maps ${}_a\delta : \mathcal{A}_0 \to \mathcal{A}_0$ and $\delta_b : \mathcal{A}_0 \to \mathcal{A}_0$ by linear extension of the identities*

$${}_a\delta\big(U^m V^n\big) = mU^{a+m} V^n \quad \text{and} \quad \delta_b\big(U^m V^n\big) = n\lambda^{-bm} U^m V^{b+n} \quad \text{for all } m, n \in \mathbb{Z}.$$

Then ${}_a\delta$ is a π_{1,λ^a}-derivation and δ_b is a $\pi_{\lambda^{-b},1}$-derivation; moreover, their adjoints are such that

$${}_a\delta^\dagger\big(U^m V^n\big) = -m\lambda^{an} U^{-a+m} V^n \quad \text{and} \quad \delta_b^\dagger\big(U^m V^n\big) = -nU^m V^{-b+n}$$

for all $m, n \in \mathbb{Z}$.

Remark 6.5. *The sufficient condition in Lemma 6.4 is also necessary. It is easy to show that if ${}_a\delta$ is a $\pi_{\mu,\nu}$-derivation then $\mu = 1$ and $\nu = \lambda^a$; similarly, if δ_b is a $\pi_{\mu,\nu}$-derivation then $\mu = \lambda^{-b}$ and $\nu = 1$.*

Lemma 6.6. *With $\mathcal{A}_0$ as in Definition 6.1, and ${}_a\delta$ and δ_b as in Lemma 6.4, fix $c_1, c_2 \in \mathbb{C}$ and let*

$$\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}(\mathbb{C}^3); \qquad x \mapsto \begin{bmatrix} \tau(x) & \overline{c_1}\, {}_a\delta^\dagger(x) & \overline{c_2}\, \delta_b^\dagger(x) \\ c_1\, {}_a\delta(x) & \pi_{1,\lambda^a}(x) - x & 0 \\ c_2\, \delta_b(x) & 0 & \pi_{\lambda^{-b},1}(x) - x \end{bmatrix},$$

where the map

$$\tau : \mathcal{A}_0 \to \mathcal{A}_0; \qquad U^m V^n \mapsto -\frac{1}{2}\big(|c_1|^2 m^2 + |c_2|^2 n^2\big) U^m V^n.$$

Then τ is $*$-linear and ϕ is a flow generator.

Lemma 6.7. *Let ϕ be as in Lemma 6.6. If $a = b = 0$ then $\mathcal{A}_\phi = \mathcal{A}_0$; conversely, if $a \neq 0$ and $c_1 \neq 0$ then $U \notin \mathcal{A}_\phi$, and if $b \neq 0$ and $c_2 \neq 0$ then $V \notin \mathcal{A}_\phi$.*

Proof. When $a = b = 0$, note that $\phi(U) = U \otimes m_U$ and $\phi(V) = V \otimes m_V$, where

$$m_U := \begin{bmatrix} -\frac{1}{2}|c_1|^2 & -\overline{c_1} & 0 \\ c_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_V := \begin{bmatrix} -\frac{1}{2}|c_2|^2 & 0 & -\overline{c_2} \\ 0 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix}.$$

Hence $\phi_n(U) = U \otimes m_U^{\otimes n}$ and $\phi_n(V) = V \otimes m_V^{\otimes n}$, so $U, V \in \mathcal{A}_\phi$, as claimed, and $\mathcal{A}_\phi = \mathcal{A}_0$, by Corollary 2.12.

If $a > 0$ then, by induction, one gets that

$${}_a\delta^n(U) = \prod_{i=0}^{n-1}(ia+1)U^{an+1} \quad \text{for all } n \in \mathbb{N}.$$

Let $e = [1\ 0\ 0]^T$ and $f = [0\ 1\ 0]^T$ be unit vectors in $\mathbb{C}^3$, and note that

$$\big(1_{\mathsf{h}} \otimes \langle f| \otimes \cdots \otimes \langle f|\big)\phi_n(x)\big(1_{\mathsf{h}} \otimes |e\rangle \otimes \cdots \otimes |e\rangle\big) = c_1^n\, {}_a\delta^n(x) \quad \text{for all } x \in \mathcal{A}_0,$$

so

$$\big\|\phi_n(U)\big\| \geq |c_1|^n \prod_{i=0}^{n-1}(ia+1) \geq |c_1|^n n!.$$

If $a < 0$ then, by considering ${}_a\delta^\dagger$ instead, we see that

$$\big\|\phi_n(U)\big\| \geq \big\|\big(1_{\mathsf{h}} \otimes \langle e| \otimes \cdots \otimes \langle e|\big)\phi_n(U)\big(1_{\mathsf{h}} \otimes |f\rangle \otimes \cdots \otimes |f\rangle\big)\big\| \geq |c_1|^n n!.$$

A similar proof shows that $V \notin \mathcal{A}_\phi$ when $b \neq 0$. □

Remark 6.8. *The lower bounds obtained in Lemma 6.7 when $a \neq 0$ or $b \neq 0$ show that our techniques do not apply in these cases. The same problem arises if one attempts to use the results of* [12] *instead.*

The following theorem gives the existence of a quantum flow used by Chakraborty, Goswami and Sinha [8], Theorem 2.1(i).

Theorem 6.9. *Let $\mathcal{A}$ be as in Definition 6.1 and ϕ as in Lemma 6.6 for $a = b = 0$. There exists an adapted family j of unital $*$-homomorphisms from $\mathcal{A}$ to $\mathcal{B}(\mathsf{h} \mathbin{\bar{\otimes}} \mathcal{F})$ such that*

$$\big\langle u\varepsilon(f), j_t(x)v\varepsilon(g)\big\rangle = \big\langle u\varepsilon(f), (xv)\varepsilon(g)\big\rangle + \int_0^t \big\langle u\varepsilon(f), j_s\big(\phi^{\widehat{f}(s)}_{\widehat{g}(s)}(x)\big)v\varepsilon(g)\big\rangle\, \mathrm{d}s$$

for all $u, v \in \mathsf{h}$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$, $x \in \mathcal{A}_0$ and $t \geq 0$.

Proof. This follows from Theorem 3.12, Lemma 6.6 and Lemma 6.7. □

Remark 6.10. *The cocycle constructed in Theorem* 6.9 *is essentially a classical object*: *as noted in* [8], *Theorem* 2.1, *when* $c_1 = c_2 = \mathrm{i}$ *one may take*

$$j_t(x) := \beta\bigl(\exp\bigl(2\mathrm{i}B_t^1\bigr), \exp\bigl(2\mathrm{i}B_t^2\bigr)\bigr)(x) \quad \text{for all } x \in \mathcal{A} \text{ and } t \geq 0,$$

where $\beta : \mathbb{T}^2 \to \mathrm{Aut}(\mathcal{A})$ *is the natural action of the* 2-*torus* $\mathbb{T}^2$ *on* $\mathcal{A}$, *so that*

$$\beta(z, w)\bigl(U^m V^n\bigr) = z^m w^n U^m V^n \quad \text{for all } (z, w) \in \mathbb{T}^2,$$

and the Fock space $\mathcal{F}$ *is identified in the usual manner with the* L^2 *space of the two-dimensional classical Brownian motion* (B^1, B^2).

As noted by Hudson and Robinson [18], the following result makes clear why in Theorem 6.9 it is necessary to use two dimensions of noise to obtain a process whose flow generator includes both of the derivations $c_1\,{}_0\delta$ and $c_2\delta_0$: the linear combination $\delta = c_1\,{}_0\delta + c_2\delta_0$ can appear on the right-hand side of (2.3) only when the coefficients c_1 and c_2 satisfy a particular algebraic relation.

Proposition 6.11. *Let* ${}_0\delta$ *and* δ_0 *be as in Lemma* 6.4, *and let* $\delta = c_1\,{}_0\delta + c_2\delta_0$ *for complex numbers* c_1 *and* c_2. *A necessary and sufficient condition for the existence of a linear map* $\tau : \mathcal{A}_0 \to \mathcal{A}$ *such that*

$$\tau(xy) - \tau(x)y - x\tau(y) = \delta^\dagger(x)\delta(y) \quad \text{for all } x, y \in \mathcal{A}_0$$

is the equality $c_1\overline{c_2} = \overline{c_1}c_2$.

Proof. This may be established by adapting slightly the proof of [29], Theorem 2.2. □

6.2. *The universal rotation algebra*

To avoid the issue of Proposition 6.11, Hudson and Robinson work with the universal rotation algebra.

Definition 6.12. *Let* $\mathcal{A}$ *be the* universal rotation algebra [2]: *this is the universal* C^* *algebra with unitary generators* U, V *and* Z *satisfying the relations*

$$UV = ZVU, \qquad UZ = ZU \quad \text{and} \quad VZ = ZV.$$

It may be viewed as the group C^* *algebra corresponding to the discrete Heisenberg group*

$$\Gamma := \langle u, v, z \mid uv = zvu, uz = zu, vz = zv\rangle;$$

from this perspective, *its universal nature is immediately apparent*.

Letting $\mathcal{A}_0$ *denote the* $*$-*subalgebra generated by* U, V *and* Z, *there are skew-adjoint derivations*

$$\delta_1 : \mathcal{A}_0 \to \mathcal{A}_0; \qquad U^m V^n Z^p \mapsto mU^m V^n Z^p \quad \text{and} \quad \delta_2 : \mathcal{A}_0 \to \mathcal{A}_0; \qquad U^m V^n Z^p \mapsto nU^m V^n Z^p$$

for all $m, n, p \in \mathbb{Z}$.

Remark 6.13. *For a concrete version of the universal rotation algebra*, *let* $\mathsf{h} := \ell^2(\mathbb{Z}^3)$ *and define operators* U_c, V_c *and* Z_c *by setting*

$$(U_c u)_{m,n,p} = u_{m+1,n,p}, \qquad (V_c u)_{m,n,p} = u_{m,n+1,m+p} \quad \text{and} \quad (Z_c u)_{m,n,p} = u_{m,n,p+1}$$

for all $u \in \mathsf{h}$ *and* $m, n, p \in \mathbb{Z}$. *It is readily verified that* U_c, V_c *and* Z_c *are unitary and satisfy the commutation relations as claimed*; *let* $\mathcal{A}_c$ *be the* C^* *algebra generated by these operators*.

Universality gives a surjective $*$-homomorphism from $\mathcal{A}$ to $\mathcal{A}_c$ such that $U \mapsto U_c$, $V \mapsto V_c$ and $Z \mapsto Z_c$, and injectivity may be established in the same manner as for the non-commutative torus: there is a faithful state τ on $\mathcal{A}$ such that $\tau(U^m V^n Z^p) = \mathbb{1}_{m=n=p=0}$ and this corresponds to the vector state given by $e \in \mathsf{h}$ such that $e_{m,n,p} = \mathbb{1}_{m=n=p=0}$.

Lemma 6.14. *With $\mathcal{A}_0$, δ_1 and δ_2 as in Definition 6.12, fix $c_1, c_2 \in \mathbb{C}$, let $\delta = c_1\delta_1 + c_2\delta_2$ and define the* Bellissard map

$$\tau : \mathcal{A}_0 \to \mathcal{A}_0; \qquad U^m V^n Z^p \mapsto -\Big(\frac{1}{2}|c_1|^2 m^2 + \frac{1}{2}|c_2|^2 n^2 + \overline{c_1} c_2 mn + (\overline{c_1} c_2 - c_1 \overline{c_2}) p\Big) U^m V^n Z^p,$$

then τ is $*$-linear and such that

$$\tau(xy) - \tau(x)y - x\tau(y) = \delta^\dagger(x)\delta(y) \quad \textit{for all } x, y \in \mathcal{A}_0,$$

so the map

$$\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}(\mathbb{C}^2); \qquad x \mapsto \begin{bmatrix} \tau(x) & \delta^\dagger(x) \\ \delta(x) & 0 \end{bmatrix}$$

is a flow generator.

Furthermore, $U, V, Z \in \mathcal{A}_\phi$ and $\mathcal{A}_\phi = \mathcal{A}_0$.

Proof. The algebraic statements are readily verified, and a short calculation shows that

$$\phi(U) = U \otimes m_U, \qquad \phi(V) = V \otimes m_V \quad \text{and} \quad \phi(Z) = Z \otimes m_Z,$$

where

$$m_U = \begin{bmatrix} -\frac{1}{2}|c_1|^2 & -\overline{c_1} \\ c_1 & 0 \end{bmatrix}, \qquad m_V = \begin{bmatrix} -\frac{1}{2}|c_2|^2 & -\overline{c_2} \\ c_2 & 0 \end{bmatrix} \quad \text{and} \quad m_Z = \begin{bmatrix} c_1\overline{c_2} - \overline{c_1}c_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$\phi_n(U) = U \otimes m_U^{\otimes n}, \qquad \phi_n(V) = V \otimes m_V^{\otimes n} \quad \text{and} \quad \phi_n(Z) = Z \otimes m_Z^{\otimes n}$$

for all $n \in \mathbb{Z}_+$, so $U, V, Z \in \mathcal{A}_\phi$ and $\mathcal{A}_\phi = \mathcal{A}_0$, by Corollary 2.12. □

The following theorem is an algebraic version of the result presented by Hudson and Robinson in [18], Section 4.

Theorem 6.15. *Let $\mathcal{A}$ be as in Definition 6.12 and ϕ as in Lemma 6.14. There exists an adapted family j of unital $*$-homomorphisms from $\mathcal{A}$ to $\mathcal{B}(\mathsf{h} \bar{\otimes} \mathcal{F})$ such that*

$$\big\langle u\varepsilon(f), j_t(x)v\varepsilon(g)\big\rangle = \big\langle u\varepsilon(f), (xv)\varepsilon(g)\big\rangle + \int_0^t \big\langle u\varepsilon(f), j_s\big(\phi_{\widehat{g}(s)}^{\widehat{f}(s)}(x)\big)v\varepsilon(g)\big\rangle \,\mathrm{d}s$$

for all $u, v \in \mathsf{h}$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$, $x \in \mathcal{A}_0$ and $t \geq 0$.

Acknowledgements

Alexander C. R. Belton thanks Professors Kalyan Sinha and Tirthankar Bhattacharyya for hospitality at the Indian Institute of Science, Bangalore, and in Munnar, Kerala; part of this work was completed during a visit to India supported by the UKIERI research network *Quantum Probability, Noncommutative Geometry and Quantum Information*. Thanks are also due to Professor Martin Lindsay for helpful discussions. Funding from Lancaster University's Research Support Office and Faculty of Science and Technology is gratefully acknowledged.

Stephen J. Wills thanks Professor Rolando Rebolledo for a very pleasant visit to Santiago in 2006 where thoughts about the quantum exclusion process were first encouraged.

Both authors are indebted to the two anonymous referees and the associate editor for their constructive comments on an earlier draft of this paper.

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