

An ergodic theorem for the extremal process of branching Brownian motion

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Abstract. In a previous paper, the authors proved a conjecture of Lalley and Sellke that the empirical (time-averaged) distribution function of the maximum of branching Brownian motion converges almost surely to a Gumbel distribution. The result is extended here to the entire system of particles that are extremal, i.e. close to the maximum. Namely, it is proved that the distribution of extremal particles under time-average converges to a Poisson cluster process.

Résumé. Dans un article précédent, les auteurs ont démontré une conjecture de Lalley et Sellke stipulant que la loi empirique (en faisant la moyenne sur les temps) du maximum du mouvement brownien branchant converge presque sûrement vers une loi de Gumbel. Ce résultat est généralisé ici au système de particules extrémales, c'est-à-dire celles se situant près du maximum. Précisément, il est démontré que la loi conjointe empirique des positions des particules extrémales converge vers la loi d'un processus poissonnien de nuages.

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1. Introduction and main result

Let $x(t) = (x_v(t), v \in \Sigma(t))$ be a standard branching Brownian motion (BBM) on \mathbb{R} defined on a filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_+})$. The set $\Sigma(t)$ indexes the particles at time t and $x_v(t)$ is the position of the *particle* v at time t . We recall the construction of the process: at time 0 we start with a single standard Brownian motion $x_1(t)$ that splits after an exponential random time T of mean 1 into k particles with probability p_k , where $\sum_{k=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} k p_k = 2$, and $\sum_k k(k-1)p_k < \infty$. The positions of the k particles are independent Brownian motions starting at $x_1(T)$. The k particles branch independently and with the same law as the first Brownian particle. At time $t > 0$, there will be a random number $n(t) \equiv |\Sigma(t)|$ of particles located at $x(t) = (x_v(t), v \in \Sigma(t))$. Note that $\mathbb{E}[n(t)] = e^t$.

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A fundamental link between the maximum of BBM and partial differential equations was observed by McKean [17]. If $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is such that $0 \leq \phi \leq 1$, then the function

$$u(t, x) \equiv 1 - \mathbb{E} \left[\prod_{v \in \Sigma(t)} \phi(x + x_v(t)) \right] \tag{1.1}$$

solves the Kolmogorov–Petrovsky–Piscounov equation [KPP], also referred to as the Fisher–KPP equation [11],

$$u_t = \frac{1}{2} u_{xx} + (1 - u) - \sum_{k=1}^{\infty} p_k (1 - u)^k, \tag{1.2}$$

with initial condition $u(0, x) = 1 - \phi(x)$. For the case $\phi(x) = \mathbb{1}_{[0, \infty)}(x)$, $1 - u(t, x) = \mathbb{P}(\max_{v \in \Sigma(t)} x_v(t) \leq x)$ is the distribution function of the maximum of BBM.

Results of Kolmogorov, Petrovsky, and Piscounov [14] and of Bramson [6] established the convergence of the distribution under appropriate recentering. Namely, for the initial condition $\phi(x) = \mathbb{1}_{[0, \infty)}(x)$,

$$\lim_{t \rightarrow \infty} u(t, m(t) + x) = 1 - w(x) \quad \text{uniformly in } x, \tag{1.3}$$

with the recentering term

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t, \tag{1.4}$$

and $w(x)$ is the unique solution (up to translation) of a certain ODE. Convergence for other initial conditions was also proved by Bramson in [7]; see Theorem 16 in the Appendix for a precise statement. A probabilistic interpretation of $w(x)$ of BBM was given by Lalley and Sellke [15]. They proved that the martingale $t \mapsto \sum_{v \in \Sigma(t)} e^{-\sqrt{2}(\sqrt{2}t - x_v(t))}$ converges to 0 almost surely, whereas the *derivative martingale*

$$Z(t) \equiv \sum_{v \in \Sigma(t)} (\sqrt{2}t - x_v(t)) e^{-\sqrt{2}(\sqrt{2}t - x_v(t))} \tag{1.5}$$

converges almost surely to a random variable $Z > 0$. Moreover,

$$w(x) = \mathbb{E} \left[\exp(-C_{\max} Z e^{-\sqrt{2}x}) \right], \tag{1.6}$$

for an explicitly known constant $C_{\max} > 0$ that depends on the specific choice of initial condition ϕ for the maximum.

In this paper we study properties of the so-called *extremal process* of BBM, i.e. the point process

$$\mathcal{E}_{t, \omega} = \sum_{v \in \Sigma(t)} \delta_{x_v(t) - m(t)}. \tag{1.7}$$

Our objective is to prove weak convergence of the random point measure $\mathcal{E}_{t, \omega}$ with respect to *time-averaging* for \mathbb{P} -almost all realizations ω .

Remark 1. *The natural topology for point measures is that of vague convergence. Weak convergence of random elements of this space is implied by the convergence of Laplace functionals,*

$$\mathbb{E} \left[\exp \left(- \int f(y) \mu_n(dy) \right) \right] \tag{1.8}$$

for every $f \in C_c^+(\mathbb{R})$ i.e. the set of non-negative continuous functions with compact support, see e.g. [13].

To this aim, for all $f \in C_c(\mathbb{R})^+$, we analyze the convergence of the Laplace functional

$$\left\langle \exp\left(-\int f(y)\mathcal{E}_{t,\omega}(dy)\right) \right\rangle_T \equiv \frac{1}{T} \int_0^T \exp\left(-\int f(y)\mathcal{E}_{t,\omega}(dy)\right) dt. \tag{1.9}$$

The limit point process, denoted $\mathcal{E}_{Z(\omega)}$, is described precisely in (1.13) after the statement of the result. It is a Poisson cluster process whose law depends on the realization ω through the value of the derivative martingale. We write E for the expectation of this point process for a fixed $Z(\omega)$. The main result proves in effect an ergodic theorem for the system of extremal particles of BBM. However, the system has more than one ergodic components since the limit distribution of the particles do depend on the realization ω through $Z(\omega)$.

Theorem 2 (Ergodic theorem for the extremal process). *There exists a set $\Omega_0 \subset \Omega$ of \mathbb{P} -probability one on which $\mathcal{E}_{t,\omega}$ converges weakly under time-average to a Poisson cluster process $\mathcal{E}_{Z(\omega)}$, where $Z(\omega)$ is the limit of the derivative martingale. That is, for $\omega \in \Omega_0$,*

$$\lim_{T \rightarrow \infty} \left\langle \exp\left(-\int f(y)\mathcal{E}_{t,\omega}(dy)\right) \right\rangle_T = E\left[\exp\left(-\int f(y)\mathcal{E}_{Z(\omega)}(dy)\right)\right] \quad \forall f \in C_c^+(\mathbb{R}). \tag{1.10}$$

The main result has to be compared with the convergence in *space-average* of the extremal process $\mathcal{E}_{t,\omega}$. From this perspective, one considers the law of $\mathcal{E}_{t,\omega}$ under \mathbb{P} when averaging over the realizations ω instead of under $\langle \cdot \rangle_T$ for ω fixed. The weak convergence of the extremal process under space-average has been studied by several authors: Brunet and Derrida [8,9], Aidekon, Berestycki, Brunet, and Shi [1], and the present authors in [3,4]. The reader is also referred to the excellent review by Gou  r   [12]. A description of the extremal process in the limit has been proved independently in [4] and [1]. In [4], the existence of the following process needed for the description of the limit is proved.

Theorem 3 ([4]). *Under the law \mathbb{P} conditioned on the event $\{\max_{v \in \Sigma(t)} x_v(t) - \sqrt{2}t > 0\}$, the point process*

$$\bar{\mathcal{E}}_t \equiv \sum_{v \in \Sigma(t)} \delta_{x_v(t) - \sqrt{2}t} \tag{1.11}$$

converges weakly as $t \rightarrow \infty$ to a well-defined point process $\bar{\mathcal{E}}$.

If we write $\bar{\mathcal{E}} = \sum_{k \in \mathbb{N}} \delta_{y_k}$, then the process of the gaps given by

$$\mathcal{D} \equiv \sum_{k \in \mathbb{N}} \delta_{y_k - \max_{j < k} y_j}, \tag{1.12}$$

also exists. This process is referred to as *the cluster process*.

Now, let \mathcal{E}_Z be the Poisson cluster process constructed as follows (see [18] for general results on Poisson cluster process): for $Z > 0$ fixed, let $(p_i, i \in \mathbb{N})$ be a Poisson random measure with intensity $C_{\max} Z \sqrt{2} e^{-\sqrt{2}x} dx$ where C_{\max} is the constant appearing in (1.6). Let $\mathcal{D}^{(i)} = (\Delta_j^{(i)}, j \in \mathbb{N})$ be i.i.d. copies of the cluster point process \mathcal{D} defined in (1.12). Then, for a given $Z > 0$, take,

$$\mathcal{E}_Z = \sum_{i,j \in \mathbb{N}} \delta_{p_i + \Delta_j^{(i)}}. \tag{1.13}$$

The convergence when $t \rightarrow \infty$ under \mathbb{P} of the random measure $\mathcal{E}_{t,\omega}$ was proved in [4] and in [1]. The difference with Theorem 2 is that the dependence on $Z(\omega)$ is averaged.

Theorem 4 ([4]). *The random measure $\mathcal{E}_{t,\cdot}$ converges weakly under \mathbb{P} and as $t \rightarrow \infty$ to a mixture of Poisson cluster processes. More precisely, for every $f \in C_c^+(\mathbb{R})$,*

$$\lim_{t \uparrow \infty} \mathbb{E} \left[\exp\left(-\int f(y)\mathcal{E}_{t,\omega}(dy)\right) \right] = \mathbb{E} \left[E \left[\exp\left(-\int f(y)\mathcal{E}_Z(dy)\right) \right] \right]. \tag{1.14}$$

(In the right side of (1.14), the expectation E is over the point process \mathcal{E}_Z defined in (1.13) for Z fixed, whereas the expectation \mathbb{E} is over the random variable Z .)

Remark 5. In the statements of Theorems 2 and 4, we have explicitly written the dependence on Z to highlight the connection with Birkhoff’s ergodic theorem where the limit is measurable with respect to the tail σ -field. Here, the analogue of this σ -field is given by the early times of the system. This analogy is made precise in the analysis of Lalley and Sellke on which Proposition 10 below is based. The dependence on $Z(\omega)$ in the limit can be eliminated by considering the recentered extremal process $\bar{\mathcal{E}}_{t,\omega} \equiv \sum_{v \in \Sigma(t)} \delta_{x_v(t) - m(t) + (1/\sqrt{2}) \log Z(t)}$. Theorem 4 then becomes

$$\lim_{t \uparrow \infty} \mathbb{E} \left[\exp \left(- \int f(y) \bar{\mathcal{E}}_{t,\omega}(\mathrm{d}y) \right) \right] = E \left[\exp \left(- \int f(y) \mathcal{E}(\mathrm{d}y) \right) \right]$$

and similarly for Theorem 2 under time-average. Here \mathcal{E} is the process (1.13) with $Z = 1$.

The proof of Theorem 2 goes along the line of the proof of the convergence of the law of the maximum under time-average to a Gumbel distribution proved in [5] and first conjectured in [15].

Theorem 6 ([5]). The random variable $\max_{v \in \Sigma(t)} x_v(t) - m(t)$ converges weakly under $\langle \cdot \rangle_T$ to a Gumbel distribution for \mathbb{P} -almost all ω . Precisely, for \mathbb{P} -almost all ω ,

$$\lim_{T \uparrow \infty} \langle \mathbb{1}_{\{\max_{v \in \Sigma(t)} x_v(t) - m(t) \leq x\}} \rangle_T = \exp(-C_{\max} Z(\omega) e^{-\sqrt{2}x}), \quad \text{for all } x \in \mathbb{R}. \tag{1.15}$$

Theorem 2 extends this result. In particular, a precise result on the branching times of extremal particles at different time scales is needed, cf. Theorem 7. This result is of independent interest.

2. Outline of the proof

To prove Theorem 2, one has to find Ω_0 of probability one on which

$$\left\langle \exp \left(- \int f(y) \mathcal{E}_{t,\omega}(\mathrm{d}y) \right) \right\rangle_T \tag{2.1}$$

converges simultaneously for all $f \in \mathcal{C}_c^+(\mathbb{R})$. As explained in Section 3.1, the convergence on countable set of functions in $\mathcal{C}_c^+(\mathbb{R})$ is in fact sufficient. Thus one only needs to prove almost sure convergence for a given function f . Moreover, due to the fact that the recentered maximum of BBM is stochastically bounded, one can introduce a cutoff on large values of y .

Take $\varepsilon > 0$ and R_T such that $R_T \ll T$. For a given $f \in \mathcal{C}_c^+(\mathbb{R})$, consider the decomposition

$$\left\langle \exp \left(- \int f(y) \mathcal{E}_{t,\omega}(\mathrm{d}y) \right) \right\rangle_T = \frac{1}{T} \int_0^{\varepsilon T} \exp \left(- \int f(y) \mathcal{E}_{t,\omega}(\mathrm{d}y) \right) \mathrm{d}t \tag{2.2}$$

$$+ \frac{1}{T} \int_{\varepsilon T}^T \mathbb{E} \left[\exp \left(- \int f(y) \mathcal{E}_{t,\omega}(\mathrm{d}y) \right) \middle| \mathcal{F}_{R_T} \right] \mathrm{d}t \tag{2.3}$$

$$+ \frac{1}{T} \int_{\varepsilon T}^T Y_t(\omega) \mathrm{d}t, \tag{2.4}$$

where

$$Y_t(\omega) \equiv \exp \left(- \int f(y) \mathcal{E}_{t,\omega}(\mathrm{d}y) \right) - \mathbb{E} \left[\exp \left(- \int f(y) \mathcal{E}_{t,\omega}(\mathrm{d}y) \right) \middle| \mathcal{F}_{R_T} \right]. \tag{2.5}$$

The term (2.2) can be made arbitrarily small uniformly in T by taking ε small. The term (2.3) is shown to converge almost surely to the right side of (1.10) in Section 3.2. The treatment is based on the convergence result of Lalley and

Sellke [15] and is a generalization of Theorem 2 in [5]. The condition $t \in [\varepsilon T, T]$ is needed there, because one needs $R_T \ll t$. The term (2.4) is shown to converge to 0 almost surely in Section 3.4. This is similar in spirit to the law of large numbers proved in [5]. However, the proof is simplified here by the explicit use of a new theorem of independent interest about the common ancestor of extremal particles at two different times. Precisely, for a fixed compact interval I , denote by $\Sigma_I(t)$ the particles at time t that are in the set $I + m(t)$,

$$\Sigma_I(t) \equiv \{v \in \Sigma(t): x_v(t) - m(t) \in I\}. \quad (2.6)$$

Take $t > 0$ and $t' > 0$ such that $t' > t$. Let $v \in \Sigma_I(t)$ and $v' \in \Sigma_I(t')$. Define the *branching time* between v and v' as

$$\text{for } v \in \Sigma_I(t) \text{ and } v' \in \Sigma_I(t'), \quad Q(v, v') = \sup\{s \leq t: x_v(s) = x_{v'}(s)\}. \quad (2.7)$$

Note that if v' is a descendant of v then $Q(v, v') = t$.

Theorem 7. *Let I be a compact interval of \mathbb{R} . There exist $C > 0$ and $\kappa > 0$ such that for all $r > 0$*

$$\sup_{\substack{t > 3r \\ t' > t+r}} \mathbb{P}(\exists v \in \Sigma_I(t), v' \in \Sigma_I(t'): Q(v, v') \in [r, t]) \leq C e^{-r^\kappa}. \quad (2.8)$$

This is a generalization of Theorem 2.1 in [2]. The technique of proof is similar and is based on the localization of the paths of extremal particles. The theorem means that with large probability and for two (sufficiently distant) times, the extremal particles come from different ancestors at time r . Hence, they are conditionally independent given \mathcal{F}_r . This gives enough independence between the variable Y_s to derive the desired convergence of (2.4) using standard arguments.

3. Proofs

3.1. Approximation in $\mathcal{C}_c^+(\mathbb{R})$

We first state a lemma that shows that our task is reduced to prove almost sure convergence for a given function f with an additional cutoff.

Lemma 8. *Theorem 2 holds if, for any given $f \in \mathcal{C}_c^+(\mathbb{R})$, and for any $\delta \in \mathbb{R}$,*

$$\lim_{T \rightarrow \infty} \left\langle \exp\left(-\int f_\delta(y) \mathcal{E}_{t,\omega}(dy)\right) \right\rangle_T = E \left[\exp\left(-\int f_\delta(y) \mathcal{E}_{Z(\omega)}(dy)\right) \right] \quad \mathbb{P}\text{-a.s.}, \quad (3.1)$$

where f_δ is defined by $\exp(-f_\delta(y)) = \exp(-f(y)) \mathbb{1}_{(-\infty, \delta]}(y)$.

Proof. The Stone–Weierstrass theorem implies the existence of a countable set of dense functions in $\mathcal{C}_c^+(\mathbb{R})$ under the uniform topology. This reduces the proof to almost sure convergence for a given $f \in \mathcal{C}_c^+(\mathbb{R})$. There is also no loss of generalities in introducing the cutoff $\mathbb{1}_{(-\infty, \delta]}(y)$ since the support of $\mathcal{E}_{\omega,t}$ is stochastically bounded from above (by Theorem 6). \square

3.2. A convergence result of Lalley and Sellke

For a given $f \in \mathcal{C}_c^+(\mathbb{R})$ and $\delta \in \mathbb{R}$, define

$$u_{f,\delta}(t, x) = 1 - \mathbb{E} \left[\prod_{v \in \Sigma(t)} \exp(-f_\delta(-x + x_v(t))) \right]. \quad (3.2)$$

Note that $u_{f,\delta}(0, x)$ is 0 for x large enough because of the indicator function. Convergence of $u_{f,\delta}(t, x + m(t))$ as $t \rightarrow \infty$ has been established by Bramson [7], see Theorem 16 in the Appendix for a complete statement. Using the representation of Lalley and Sellke and arguments of Chauvin and Rouault, one can show that (see Lemma 4.9 in [4]),

$$\lim_{t \rightarrow \infty} u_{f,\delta}(t, x + m(t)) = \mathbb{E}[\exp(-C(f, \delta)Z(\omega)e^{-\sqrt{2}x})], \quad (3.3)$$

where

$$C_r(f, \delta) = \sqrt{\frac{2}{\pi}} \int_0^\infty u_{f,\delta}(r, y + \sqrt{2}r) y e^{y\sqrt{2}} dy \quad (3.4)$$

and $C(f, \delta) \equiv \lim_{r \rightarrow \infty} C_r(f, \delta)$. Moreover, it was established in [4] (see proof of Theorem 3.6 in [4]) that if \mathcal{D} is the cluster process with expectation E introduced in (1.12), then

$$C(f, \delta) = \int \left(1 - E \left[\exp \left(- \int f_\delta(y+z) \mathcal{D}(dz) \right) \right] \right) \sqrt{2} e^{-\sqrt{2}y} dy. \quad (3.5)$$

The proof of this in [4] is written for f without the cutoff but it extends in a straightforward way. Note from (3.3) and Theorem 4 that $\exp(-C(f, \delta)Z)$ is the Laplace functional of the process \mathcal{E}_Z for the test function f with the cutoff.

The proof of the next lemma is similar to that of Lemma 4 in [5]. We present it for completeness. It is based on an estimate of Bramson, see Proposition 8.3 and its proof in [7], that were adapted to the extremal process setting in Proposition 4.3 of [4].

Lemma 9. *Consider $t \geq 0$ and $X(t) \geq 0$ such that $\lim_{t \uparrow \infty} X(t) = +\infty$ and $X(t) = o(\sqrt{t})$. Then, for any fixed r and t large enough such that $t \geq 8r$ and $X(t) \geq 8r - \frac{3}{2\sqrt{2}} \log(t)$,*

$$\gamma(r)^{-1} C_r(f, \delta) X(t) e^{-\sqrt{2}X(t)} (1 + o(1)) \leq u_{f,\delta}(t, X(t) + m(t)) \leq \gamma(r) C_r(f, \delta) X(t) e^{-\sqrt{2}X(t)}, \quad (3.6)$$

where $o(1)$ is a term that tends to 0 as $t \rightarrow \infty$ for r fixed.

Proof. Define

$$\begin{aligned} \psi(r, t, x + \sqrt{2}t) &\equiv \frac{e^{-\sqrt{2}x}}{\sqrt{t-r}} \int_0^\infty \frac{dy'}{\sqrt{2\pi}} \cdot u_{f,\delta}(r, y' + \sqrt{2}r) \cdot e^{y'\sqrt{2}} \\ &\times \left\{ 1 - \exp \left(-2y' \frac{x + (3/(2\sqrt{2})) \log t}{t-r} \right) \right\} \exp \left(-\frac{(y'-x)^2}{2(t-r)} \right). \end{aligned} \quad (3.7)$$

Note that

$$\int_0^\infty y e^{\sqrt{2}y} u_{f,\delta}(0, y) dy < \infty. \quad (3.8)$$

Proposition 4.3 in [4] implies that for r large enough, $t \geq 8r$, and $x \geq 8r - \frac{3}{2\sqrt{2}} \log t$,

$$\gamma(r)^{-1} \psi(r, t, x + \sqrt{2}t) \leq u_{f,\delta}(t, x + \sqrt{2}t) \leq \gamma(r) \psi(r, t, x + \sqrt{2}t) \quad (3.9)$$

for some $\gamma(r) \downarrow 1$ as $r \rightarrow \infty$. As $\sqrt{2}t = m(t) + \frac{3}{2\sqrt{2}} \log(t)$, by taking $x = X(t) - \frac{3}{2\sqrt{2}} \log t$, and $X = X(t)$

$$\gamma(r)^{-1} \psi(r, t, X + m(t)) \leq u_{f,\delta}(t, X + m(t)) \leq \gamma(r) \psi(r, t, X + m(t)). \quad (3.10)$$

It remains to estimate $\psi(r, t, X + m(t))$. By Taylor expansion, since y' is positive and so is X , t large enough,

$$\frac{2y'X}{t-r} - \frac{2y'^2X^2}{(t-r)^2} \leq 1 - e^{-2y'X/(t-r)} \leq \frac{2y'X}{t-r} \quad (3.11)$$

and

$$1 - \frac{(y' - X + (3/(2\sqrt{2})) \log t)^2}{2(t - r)} \leq e^{-(y' - X + (3/(2\sqrt{2})) \log t)^2 / (2(t - r))} \leq 1. \tag{3.12}$$

The upper bound (3.6) follows from plugging the upper bounds of (3.11) and (3.12) into (3.7). As for the lower bound, note that the lower order terms of the lower bounds (3.11) and (3.12) are all of order $o(1)$, when $t \rightarrow \infty$ for r fixed. To complete the proof, one has to prove that for r fixed,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty (y')^n e^{y' \sqrt{2}} u_{f,\delta}(r, y' + \sqrt{2}r) dy' < \infty, \quad \text{for } n = 0, 1, 2, 3. \tag{3.13}$$

The integrability is shown by bounding $u_{f,\delta}$ above by the solution of the linearized KPP equation. This is done exactly as in Proposition 4.4, equation (4.17), in [4] following the argument of [10]. We refer the reader to these papers for the details. \square

Proposition 10. Fix $\varepsilon > 0$. Let $R_T = o(\sqrt{T})$ with $\lim_{T \rightarrow \infty} R_T = +\infty$. Then for any $t \in [\varepsilon T, T]$,

$$\lim_{T \uparrow \infty} \mathbb{E} \left[\exp \left(- \int f_\delta(y) \mathcal{E}_{t,\omega}(dy) \right) \middle| \mathcal{F}_{R_T} \right] = \exp(-C(f, \delta)Z(\omega)) \quad \mathbb{P}\text{-a.s.} \tag{3.14}$$

Remark 11. The right side of (3.14) equals the right side of (1.10) in Theorem 2. The connection is through (3.5).

Proof of Proposition 10. This is an application of Lemma 9 and the convergence of the derivative martingale. Enumerate the particles at time R_T by $i = 1, \dots, n(R_T)$, and write $x_i(t)$ for the position of the particular i . For a particle $v \in \Sigma(t)$ at time s , define i_v to be the index of the ancestor of v at time R_T , and $x_v^{(i_v)}(t, R_T) \equiv x_v(t) - x_{i_v}(R_T)$. By the Markov property of BBM, conditionally on \mathcal{F}_{R_T} , the processes $(x_v^{(i)}(t, R_T), v \text{ such that } i_v = i), i = 1, \dots, n(R_T)$ are independent and distributed as the particles of a BBM at time $t - R_T$. We have

$$\exp \left(- \int f_\delta(y) \mathcal{E}_{t,\omega}(dy) \right) = \prod_{i=1}^{n(R_T)} \exp \left(- \sum_{\substack{v \in \Sigma(t) \\ i_v = i}} f_\delta(-y_i(R_T) + x_v^{(i)}(t, R_T) - m(t - R_T)) \right), \tag{3.15}$$

where

$$y_i(R_T) \equiv \sqrt{2}R_T - x_i(R_T) + \frac{3}{2\sqrt{2}} \log \left(\frac{t - R_T}{t} \right) = \sqrt{2}R_T - x_i(R_T) + o(1), \tag{3.16}$$

and $o(1) \downarrow 0$ as $T \uparrow \infty$. Using the independence of the $x^{(i)}$'s and the definition (3.2), the conditional expectation of (3.15) given \mathcal{F}_{R_T} can be written as

$$\exp \left(\sum_{j \leq n(R_T)} \log(1 - u_{f,\delta}(t - R_T, y_j(R_T) + m(t - R_T))) \right). \tag{3.17}$$

Note that the convergence of the martingales defined in (1.5) implies that

$$\lim_{T \uparrow \infty} \min_{j \leq n(R_T)} y_j(R_T) = +\infty. \tag{3.18}$$

Since for $0 < u < 1/2$, one has $-u - u^2 \leq \log(1 - u) \leq -u$, one can pick T large enough so that

$$\mathbb{E} \left[\exp \left(- \int f_\delta(y) \mathcal{E}_{t,\omega}(dy) \right) \middle| \mathcal{F}_{R_T} \right] \leq \exp \left(- \sum_{j \leq n(R_T)} u_{f,\delta}(t - R_T, y_j(R_T) + m(t - R_T)) \right) \tag{3.19}$$

and

$$\begin{aligned} \mathbb{E} \left[\exp \left(- \int f_\delta(y) \mathcal{E}_{t,\omega}(dy) \right) \middle| \mathcal{F}_{R_T} \right] &\geq \exp \left(- \sum_{j \leq n(R_T)} u_{f,\delta}(t - R_T, y_j(R_T) + m(t - R_T)) \right) \\ &\quad \times \exp \left(- \sum_{j \leq n(R_T)} (u_{f,\delta}(t - R_T, y_j(R_T) + m(t - R_T)))^2 \right). \end{aligned} \tag{3.20}$$

To finish the proof, it suffices to show that

$$\lim_{T \rightarrow \infty} \sum_{j \leq n(R_T)} u_{f,\delta}(t - R_T, y_j(R_T) + m(t - R_T)) = C(f, \delta) Z(\omega), \tag{3.21}$$

$$\lim_{T \rightarrow \infty} \sum_{j \leq n(R_T)} (u_{f,\delta}(t - R_T, y_j(R_T) + m(t - R_T)))^2 = 0. \tag{3.22}$$

We claim that $y_j(R_T) = o(\sqrt{t})$, uniformly in j , so that Lemma 9 can be applied. Indeed, since

$$\lim_{T \rightarrow \infty} \frac{\max_{j \leq n(R_T)} x_j(R_T)}{R_T} = \sqrt{2}, \quad \mathbb{P}\text{-a.s.},$$

we have

$$\frac{y_j(R_T)}{t^{1/2}} \rightarrow 0 \quad \text{as } T \rightarrow \infty \text{ } \mathbb{P}\text{-a.s., uniformly in } j \leq n(R_T). \tag{3.23}$$

Equation (3.21) follows by picking a fixed r in Lemma 9, taking first $T \rightarrow \infty$, then $r \rightarrow \infty$, and using the convergence of the derivative martingale. Lemma 9 is also used to establish (3.22). By fixing r , the proof is reduced to show that $\sum_{j \leq n(R_T)} y_j(R_T)^2 e^{-2\sqrt{2}y_j(R_T)}$ goes to zero. This is clear since this sum is bounded above by

$$\max_{j \leq n(R_T)} (y_j(R_T)^2 e^{-\sqrt{2}y_j(R_T)}) \times \sum_{j \leq n(R_T)} e^{-\sqrt{2}y_j(R_T)} \tag{3.24}$$

and both terms tend to zero almost surely as $T \uparrow \infty$. □

3.3. Proof of Theorem 7

Throughout the proof, C and κ denote generic constants that do not depend on $t, t',$ and r and that are not necessarily the same at different occurrences. We recall the result on the localization of the paths of extremal particles established in [2]. Let $t > 0$ and $\gamma > 0$ and define

$$\begin{aligned} f_{\gamma,t}(s) &\equiv \begin{cases} s^\gamma & 0 \leq s \leq t/2, \\ (t-s)^\gamma & t/2 \leq s \leq t, \end{cases} \\ F_{\alpha,t}(s) &\equiv \frac{s}{t} m(t) - f_{\alpha,t}(s), \quad 0 \leq s \leq t. \end{aligned} \tag{3.25}$$

Fix $0 < \alpha < 1/2 < \beta < 1$. By definition, $F_{\beta,t}(s) < F_{\alpha,t}(s)$, $F_{\beta,t}(0) = F_{\alpha,t}(0) = 0$, and $F_{\beta,t}(t) = F_{\alpha,t}(t) = m(t)$.

The following proposition gives strong bounds for the probability of finding particles that are close to the level of the maximum at given times but whose paths are bounded by $F_{\beta,t}$ and $F_{\alpha,t}$. This was stated as Proposition 6 in [5].

Proposition 12. *Let I be a compact interval. There exist $C > 0$ and $\kappa > 0$ (depending on α, β and I) such that for all $r > 0$*

$$\sup_{t \geq 3r} \mathbb{P}[\exists v \in \Sigma_I(t): x_v(s) \geq F_{\alpha,t}(s) \text{ or } x_v(s) \leq F_{\beta,t}(s) \text{ for some } s \in (r, t-r)] \leq C e^{-r^\kappa}. \tag{3.26}$$

We now prove Theorem 7. Fix α and β as in Proposition 12. Define the set of extremal particles in I at time t that are localized in the interval $(r_1, t - r_1)$

$$\Sigma_I^{\text{loc}}(t) = \{v \in \Sigma(t): x_v(t) \in I + m(t), F_{\beta,t}(s) \leq x_v(s) \leq F_{\alpha,t}(s) \forall s \in (r_1, t - r_1)\}. \quad (3.27)$$

The parameter $r_1 = r_1(r)$ is chosen to be smaller than r (so that $(r_1, t - r_1) \supset (r, t - r)$). The precise choice will be given below. By Proposition 12, to prove Theorem 7, it suffices to show

$$\sup_{\substack{t > 3r \\ t' > t+r}} \mathbb{P}(\exists v \in \Sigma_I^{\text{loc}}(t), v' \in \Sigma_I^{\text{loc}}(t'): Q(v, v') \in [r, t]) \leq C e^{-r^\kappa}. \quad (3.28)$$

By Markov's inequality the probability is smaller than

$$\mathbb{E}[\#\{(v, v'): v \in \Sigma_I^{\text{loc}}(t), v' \in \Sigma_I^{\text{loc}}(t'): Q(v, v') \in [r, t]\}]. \quad (3.29)$$

This expectation can be evaluated using Sawyer's formula [19], see also [6]. Write x for a standard Brownian motion on \mathbb{R} , μ_s for the standard Gaussian measure of variance s . Let also \mathcal{E}^{t_1, t_2} be the set of continuous path on $[0, \infty]$ that satisfy $F_{\beta,t}(s) \leq x_v(s) \leq F_{\alpha,t}(s) \forall s \in (t_1, t_2)$. The formula gives

$$\begin{aligned} & \mathbb{E}[\#\{(v, v'): v \in \Sigma_I^{\text{loc}}(t), v' \in \Sigma_I^{\text{loc}}(t'): Q(v, v') \in [r, t]\}] \\ &= K e^t \int_r^{t'} e^{t'-s} ds \int_{\mathbb{R}} \mu_s(dy) \mathbb{P}(x \in \mathcal{E}^{r_1, t-r_1} | x(s) = y) \mathbb{P}(x \in \mathcal{E}^{s, t'-r_1} | x(s) = y), \end{aligned} \quad (3.30)$$

where $K = \sum_{k=1}^{\infty} k(k-1)p_k$. The second probability is estimated exactly the same way as in [2] using Brownian bridge estimates (from equation (4.5) to (4.14)). One needs to pick $r_1(r) < r/2$ and $r_1(r) < r^{1-\beta}$. The result is

$$\mathbb{P}(x \in \mathcal{E}^{s, t'-r_1} | x(s) = y) \leq C r^{1/2} \frac{e^{-(t'-s)} e^{(3/2)((t'-s)/t') \log t'} F_{\beta, t'}(s) e^{-\sqrt{2}F_{\alpha, t'}(s)}}{(t'-s)^{3/2}}. \quad (3.31)$$

Since this bound is uniform in y , (3.30) is smaller than

$$C e^t \mathbb{P}(x \in \mathcal{E}^{r_1, t-r_1}) r^{1/2} \int_r^{t'-r} \frac{e^{(3/2)((t'-s)/t') \log t'} F_{\beta, t'}(s) e^{-\sqrt{2}F_{\alpha, t'}(s)}}{(t'-s)^{3/2}} ds. \quad (3.32)$$

Note that the integral is now on $[r, t' - r] \supset [r, t]$. The term $e^t \mathbb{P}(x \in \mathcal{E}^{r_1, t-r_1})$ is of order of $r_1 > r$ (cf. equation (4.17) in [2]). It remains to estimate the integral. The domain of integration is split into $[r, t'/2]$, $[t'/2, t' - t'^\nu]$, and $[t' - t'^\nu, t' - r]$, for a fixed $\nu > 0$. On the first interval, one has that (3.32) is smaller than

$$C r^{3/2} \int_r^{t'/2} \frac{e^{(3/2)((t'-s)/t') \log t'} s^\beta e^{-\sqrt{2}s^\alpha}}{(t'-s)^{3/2}} ds \leq C r^{3/2} \int_r^\infty s^\beta e^{-\sqrt{2}s^\alpha} ds \leq C r^{3/2} e^{-r^\alpha}. \quad (3.33)$$

On the second interval, using the change of variable $s \rightarrow t' - s$, one gets the upper bound

$$C r^{3/2} \int_{t'^\nu}^{t'/2} \frac{e^{(3/2)(s/t') \log t'} s^\beta e^{-\sqrt{2}s^\alpha}}{s^{3/2}} ds \leq C r^{3/2} t'^{3/4} \int_{t'^\nu}^{t'/2} s^\beta e^{-\sqrt{2}s^\alpha} ds \leq C r^{3/2} t'^{7/4} e^{-t'^{\nu\alpha}}. \quad (3.34)$$

Note that for $t' > t > 3r$, κ can be chosen for the bound to be of the desired form. Finally on $[t' - t'^\nu, t' - r]$, the upper bound is

$$C r^{3/2} \int_r^{t'^\nu} \frac{e^{(3/2)(s/t') \log t'} s^\beta e^{-\sqrt{2}s^\alpha}}{s^{3/2}} ds \leq C r^{3/2} \int_r^\infty s^\beta e^{-\sqrt{2}s^\alpha} ds \leq C r^{3/2} e^{-r^\alpha}. \quad (3.35)$$

The constants C and κ can be picked such that the bounds (3.33), (3.34), and (3.35) have the desired form. This completes the proof of Theorem 7.

3.4. The law of large numbers

In this section, we prove that the term (2.4) goes to zero as T goes to infinity:

Proposition 13. *Let $f \in C_c(\mathbb{R})$, $R_T = o(\sqrt{T})$ with $\lim_{T \rightarrow \infty} R_T = +\infty$, and $\varepsilon > 0$. Consider $Y_t(\omega)$ defined in (2.5). Then*

$$\lim_{T \uparrow \infty} \frac{1}{T} \int_{\varepsilon T}^T Y_t(\omega) dt = 0, \quad \mathbb{P}\text{-a.s.} \tag{3.36}$$

Proof. The proof is based on a Theorem of Lyons [16] that we cite from [5]:

Theorem 14 ([16], Theorem 8 in [5]). *Consider a process $\{Y_s\}_{s \in \mathbb{R}_+}$ such that $\mathbb{E}[Y_s] = 0$ for all s . Assume furthermore that the random variables are uniformly bounded, say $\sup_s |Y_s| \leq 2$ almost surely. If*

$$\sum_{T=1}^{\infty} \frac{1}{T} \mathbb{E} \left[\left| \frac{1}{T} \int_0^T Y_s ds \right|^2 \right] < \infty, \tag{3.37}$$

then

$$\frac{1}{T} \int_0^T Y_s ds \rightarrow 0, \quad \text{a.s.} \tag{3.38}$$

A straightforward decomposition gives

$$\begin{aligned} \sum_{T=1}^{\infty} \frac{1}{T} \mathbb{E} \left[\left| \frac{1}{T} \int_{\varepsilon T}^T Y_s ds \right|^2 \right] &= \sum_{T=1}^{\infty} \frac{1}{T^3} \int_{\varepsilon T}^T \int_{\varepsilon T}^T \mathbb{E}[Y_s Y_{s'}] ds ds' \\ &= \sum_{T=1}^{\infty} \frac{1}{T^3} \int_{\substack{s, s' \in [\varepsilon T, T] \\ |s-s'| \leq R_T}} \mathbb{E}[Y_s Y_{s'}] ds ds' \\ &\quad + \sum_{T=1}^{\infty} \frac{1}{T^3} \int_{\substack{s, s' \in [\varepsilon T, T] \\ |s-s'| \geq R_T}} \mathbb{E}[Y_s Y_{s'}] ds ds'. \end{aligned} \tag{3.39}$$

Since $0 \leq Y_s \leq 2$ for any s , the first term is smaller than a constant times $\sum_{T=1}^{\infty} \frac{R_T}{T^2}$, which is summable for $R_T = o(\sqrt{T})$. Therefore, to prove Proposition 13 using Theorem 14, it remains to show that the second term is summable. This is done in the following lemma. □

Lemma 15. *Let Y_s be as in (2.5). Then for $R_T = o(\sqrt{T})$ with $\lim_{T \rightarrow \infty} R_T = +\infty$,*

$$\mathbb{E}[Y_s Y_{s'}] \leq C e^{-R_T^k} \quad \text{for any } s, s' \in [\varepsilon T, T]. \tag{3.40}$$

Proof. For the given $f \in C_c(\mathbb{R})$, take the compact interval $I = \text{supp } f$. Recall the definitions of $\Sigma_I(s)$ and $Q(v, v')$ in (2.6) and (2.7). Consider the events

$$\begin{aligned} \mathcal{A}_{s,T} &\equiv \{ \exists v \in \Sigma_I(s): x_v(R_T) \geq F_{\alpha,s}(R_T) \text{ or } x_v(R_T) \leq F_{\beta,s}(R_T) \}, \\ \mathcal{B}_{s,s',T} &\equiv \{ \exists v \in \Sigma_I(s), v' \in \Sigma_I(s'): Q(v, v') \in [R_T, s] \} \quad \text{for } s' > s, \\ \mathcal{C}_{s,s',T} &\equiv \mathcal{A}_{s,T} \cup \mathcal{A}_{s',T} \cup \mathcal{B}_{s,s',T}. \end{aligned} \tag{3.41}$$

By Proposition 12 and Theorem 7, one has that for constants $C > 0$ and $\kappa > 0$

$$\mathbb{P}(\mathcal{C}_{s,s',T}) \leq Ce^{-R_T^\kappa} \quad \text{for any } |s - s'| \geq R_T. \quad (3.42)$$

Enumerate (in no particular order) the particles at time R_T by $i = 1, \dots, n(R_T)$ and write $x_i(t)$ for the position of the particular i . Recall the notation introduced at the beginning of the proof of Proposition 10. Define

$$\Phi_i(s) \equiv \exp\left(-\sum_{v \in \Sigma_I(s): i_v = i} f_\delta(-y_i(R_T) + x_v^{(i)}(s, R_T) - m(s - R_T))\right). \quad (3.43)$$

Note that $\Phi_i(s)$ is non-trivial (that is, not equal to one) if and only if the particle i has descendants in the interval $I + m(s)$ at time s (and similarly for the time s'). The crucial step is to notice that on $\mathcal{C}_{s,s',T}^c$ (on $\mathcal{B}_{s,s',T}^c$ in fact), for any $i = 1, \dots, n(R_T)$, no two extremal particles in I at time s and at time s' have branching time in $[R_T, s]$. In particular, two such extremal particles must have two distinct ancestors at time R_T . This implies that

$$\forall i = 1, \dots, n(R_T), \Phi_i(s) \text{ and } \Phi_i(s') \text{ cannot be simultaneously non-trivial on } \mathcal{C}_{s,s',T}^c. \quad (3.44)$$

Therefore, the following identity holds on $\mathcal{C}_{s,s',T}^c$

$$\Phi_i(s)\Phi_i(s') = 1 - (1 - \Phi_i(s)) - (1 - \Phi_i(s')). \quad (3.45)$$

Putting all this together, one gets

$$\left| \mathbb{E} \left[\prod_{i=1}^{n(R_T)} \Phi_i(s)\Phi_i(s') \right] - \mathbb{E} \left[\prod_{i=1}^{n(R_T)} (1 - (1 - \Phi_i(s)) - (1 - \Phi_i(s'))) \right] \right| \leq 2\mathbb{P}(\mathcal{C}_{s,s',T}). \quad (3.46)$$

This has the right decay by (3.42). By the definition of $u_{f,\delta}(t, x)$ in (3.2)

$$u_i(s) \equiv \mathbb{E}[1 - \Phi_i(s') | \mathcal{F}_{R_T}] = u_f(s - R_T, y_i(R_T) + m(s - R_T)), \quad (3.47)$$

and by the conditional independence property of BBM,

$$\mathbb{E} \left[\prod_{i=1}^{n(R_T)} (1 - (1 - \Phi_i(s)) - (1 - \Phi_i(s'))) \right] = \mathbb{E} \left[\prod_{i=1}^{n(R_T)} (1 - u_i(s) - u_i(s')) \right]. \quad (3.48)$$

It remains to compare the right side of (3.48) with the contribution to the correlation coming out from the second term of (2.5):

$$\mathbb{E} \left[\prod_{i=1}^{n(R_T)} \mathbb{E}[\Phi_i(s) | \mathcal{F}_{R_T}] \prod_{i=1}^{n(R_T)} \mathbb{E}[\Phi_i(s') | \mathcal{F}_{R_T}] \right] = \mathbb{E} \left[\prod_{i=1}^{n(R_T)} (1 - u_i(s))(1 - u_i(s')) \right]. \quad (3.49)$$

Again,

$$\begin{aligned} 0 &\leq \mathbb{E} \left[\prod_{i=1}^{n(R_T)} (1 - u_i(s))(1 - u_i(s')); \mathcal{C}_{s,s',T} \right] - \mathbb{E} \left[\prod_{i=1}^{n(R_T)} (1 - u_i(s) - u_i(s')); \mathcal{C}_{s,s',T} \right] \\ &\leq 2\mathbb{P}(\mathcal{C}_{s,s',T}), \end{aligned} \quad (3.50)$$

so that by (3.42), it suffices to bound the difference of the two terms on $\mathcal{C}_{s,s',T}^c$. By the definition of this event in (3.41), the only i 's whose contribution to the product is not one are those for which $F_{\alpha,s}(R_T) < x_i(R_T) < F_{\beta,s}(R_T)$ and $F_{\alpha,s'}(R_T) < x_i(R_T) < F_{\beta,s'}(R_T)$. Define

$$\Delta \equiv \{i = 1, \dots, n(R_T): F_{\alpha,s}(R_T) < x_i(R_T) < F_{\beta,s}(R_T) \text{ and } F_{\alpha,s'}(R_T) < x_i(R_T) < F_{\beta,s'}(R_T)\}. \quad (3.51)$$

By the definition of the functions F in (3.25), the above condition reduces to $R_T^\alpha + o(1) \leq y_i(R_T) \leq R_T^\beta + o(1)$ where $o(1)$ converges to 0 as $T \rightarrow \infty$ uniformly for $s \in [\varepsilon T, T]$. We are left to bound

$$\mathbb{E} \left[\prod_{i \in \Delta} (1 - u_i(s))(1 - u_i(s')) - \prod_{i \in \Delta} (1 - u_i(s) - u_i(s')); \mathcal{C}_{s,s',T}^c \right]. \tag{3.52}$$

Since both terms are smaller than one, we can use the Lipschitz property $|e^z - e^{z'}| \leq |z - z'|$ for $z, z' \leq 0$ to bound the above by

$$\mathbb{E} \left[\sum_{i \in \Delta} \log(1 - u_i(s) - u_i(s') + u_i(s)u_i(s')) - \log(1 - u_i(s) - u_i(s')) \right]. \tag{3.53}$$

By Lemma 9, for a fixed r ,

$$u_i(s) = u_f(s - R_T, y_i(R_T) + m(s - R_T)) \leq \gamma(r)C_r(f, \delta)y_i(R_T)e^{-\sqrt{2}y_i(R_T)}. \tag{3.54}$$

In particular, by the restrictions on $y_i(R_T)$ for $i \in \Delta$, it is possible to choose T large enough such that $u_i(s) \leq 1/4$ and $u_i(s') \leq 1/4$ for every $i \in \Delta$. Since for $0 < b < 1$ and $a > 1/2$, $\log(a + b) - \log a \leq 2b$, (3.53) is bounded by

$$2\mathbb{E} \left[\sum_{i \in \Delta} u_i(s)u_i(s') \right] \leq C\mathbb{E} \left[\sum_{i \in \Delta} y_i(R_T)^2 e^{-2\sqrt{2}y_i(R_T)} \right], \tag{3.55}$$

for some constant $C > 0$. The variables $(x_i(R_T), i \leq n(R_T))$ are Gaussian. We take advantage of the linearity and that $\mathbb{E}[n(R_T)] = e^{R_T}$ to get the upper bound

$$C e^{R_T} \int_{R_T^\alpha + o(1)}^{R_T^\beta + o(1)} y^2 e^{-2\sqrt{2}y} \frac{e^{-(\sqrt{2}R_T - y)^2 / (2R_T)}}{\sqrt{2\pi R_T}} dy \leq C R_T^{2\beta} e^{-\sqrt{2}R_T^\alpha}. \tag{3.56}$$

This concludes the proof of the lemma. □

Appendix: Convergence of the KPP equation

Bramson settled the question of the convergence of the solutions of the KPP equation for generic initial conditions in [7]. The recentering is a function of the initial conditions. We combine here two of the convergence theorems of [7] as well as the condition on $u(0, x)$ to get the specific form (1.4) for the recentering.

Theorem 16 (Theorem A, Theorem B, and Example 2 in [7]). *Let u be a solution of the F -KPP equation (1.2) with $0 \leq u(0, x) \leq 1$. Then*

$$u(t, x + m(t)) \rightarrow 1 - w(x), \quad \text{uniformly in } x \text{ as } t \rightarrow \infty, \tag{A.1}$$

where w is the unique solution (up to translation) of

$$\frac{1}{2}w'' + \sqrt{2}w' + \sum_{k=1}^{\infty} p_k w^k - w = 0, \tag{A.2}$$

if and only if

1. for some $h > 0$, $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \int_t^{t(1+h)} u(0, y) dy \leq -\sqrt{2}$;
2. and for some $\nu > 0, M > 0, N > 0$, $\int_x^{x+N} u(0, y) dy > \nu$ for all $x \leq -M$.

Moreover, if $\lim_{x \rightarrow \infty} e^{bx} u(0, x) = 0$ for some $b > \sqrt{2}$, then one may choose

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t. \quad (\text{A.3})$$

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