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An existence result for nonlinear elliptic problems involving critical Sobolev exponent

by

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ABSTRACT. — In this paper we consider the following problem:

$$(1) \quad \begin{cases} -\Delta u - \lambda u = |u|^{2^*-2} \cdot u \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \quad 2^* = 2n/(n-2)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\lambda \in \mathbb{R}$.

We prove the existence of a nontrivial solution of (1) for any $\lambda > 0$, if $n \geq 4$.

RÉSUMÉ. — Soient Ω un sous-ensemble ouvert borné de \mathbb{R}^n et λ un nombre positif, le but de cette note c'est de montrer que le problème suivant :

$$\begin{cases} -\Delta u - \lambda u = |u|^{2^*-2} \cdot u \\ u|_{\partial\Omega} = 0 \end{cases} \quad 2^* = 2n/(n-2)$$

admet, au moins, une solution non triviale, si $n \geq 4$.

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0. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an open bounded set with smooth boundary. Consider the problem

$$(0.1) \quad \begin{cases} -\Delta u - \lambda u - u \cdot |u|^{2^*-2} = 0 \\ u \in H_0^1(\Omega) \end{cases}$$

where λ is a real parameter and $2^* = 2n/(n - 2)$ is the critical Sobolev exponent for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$.

The solutions of (0.1) are the critical points of the energy functional

$$(0.2) \quad f_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx.$$

Since the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact the functional f_λ does not satisfy the Palais-Smale condition in the energy range $]-\infty, +\infty[$ (cfr. remark 2.3 of [4]).

Moreover if $\lambda \leq 0$ and Ω is starshaped (0.1) has only the trivial solution (cf. [6]).

Recently Brezis and Nirenberg in [2] have proved that if $n \geq 4$ and $0 < \lambda < \lambda_1$ (λ_1 is the first eigenvalue of $-\Delta$) then (0.1) has a positive solution. In [4] Cerami, Fortunato and Struwe have obtained multiplicity results for (0.1) in the case in which λ belongs to a suitable left neighbourhood of an arbitrary eigenvalue of $-\Delta$ (cf. also [3]).

In this paper we prove the following theorem:

THEOREM 0.1. — *If $n \geq 4$ the problem (0.1) possesses at least one non trivial solution for any $\lambda > 0$.*

A weaker result related to theorem 0.1 has been announced in [5].

We observe that if $n = 3$ and Ω is a ball, Brezis and Nirenberg [2] have proved that the problem (0.1) does not have nontrivial radial solutions if $0 < \lambda < \frac{\lambda_1}{4}$.

1. SOME PRELIMINARIES

Let $\|\cdot\|, |\cdot|_p$ denote respectively the norms in $H_0^1(\Omega)$ and $L^p(\Omega)$ ($1 \leq p \leq \infty$), and let

$$S = \inf \{ \|u\|^2 / |u|_{2^*}^2 : u \in H_0^1(\Omega) \setminus \{0\} \}$$

denote the best constant for the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

The following lemma shows that f_λ satisfies a local P. S. condition.

LEMMA 1.1. — For any $\lambda \in \mathbb{R}$ the functional f_λ (see (0.2)) satisfies the Palais-Smale condition in $\left] -\infty, \frac{1}{n} S^{n/2} \right[$ in the following sense:

If $c < \frac{1}{n} S^{n/2}$ and $\{u_m\}$ is a sequence in $H_0^1(\Omega)$ such that
 (P. S.) $\left\{ \begin{array}{l} \text{as } m \rightarrow \infty f_\lambda(u_m) \rightarrow c, f'_\lambda(u_m) \rightarrow 0 \text{ strongly in } H^{-1}(\Omega), \text{ then } \{u_m\} \\ \text{contains a subsequence converging strongly in } H_0^1(\Omega). \end{array} \right.$

The proof of this lemma is in [2] and in [4]. We recall that a deeper compactness result has been proved in [7].

We recall a critical point Theorem (cf. [1, Theorem 2.4]) which is a variant of some results contained in [0].

THEOREM 1.2. — Let H be a real Hilbert space and $f \in C^1(H, \mathbb{R})$ be a functional satisfying the following assumptions:

- (f₁) $f(u) = f(-u), f(0) = 0$ for any $u \in H$
- (f₂) there exists $\beta > 0$ such that f satisfies (P. S.) in $]0, \beta[$
- (f₃) there exist two closed subspaces $V, W \subset H$ and positive constants ρ, δ such that
 - (i) $f(u) < \beta$ for any $u \in W$
 - (ii) $f(u) \geq \delta$ for any $u \in V, \|u\| = \rho$
 - (iii) $\text{codim } V < +\infty$.

Then there exist at least m pairs of critical points, with

$$m = \dim(V \cap W) - \text{codim}(V + W).$$

2. PROOF OF THEOREM 0.1

Our aim is to define two suitable closed subspaces V and W , with $V \cap W \neq \{0\}$ and $V + W = H$, such that f_λ satisfies the assumptions f_2 and f_3 of Theorem 1.2 with $\beta = \frac{1}{n} S^{n/2}$.

In the sequel we denote by λ_j the eigenvalues of $-\Delta$ and by $M(\lambda_j)$ the corresponding eigenspaces.

Given $\lambda > 0$, we set

$$(2.1) \quad \lambda^+ = \min \{ \lambda_j \mid \lambda < \lambda_j \}$$

$$H_1 = \overline{\bigoplus_{\lambda_j \geq \lambda^+} M(\lambda_j)} \quad H_2 = \bigoplus_{\lambda_j < \lambda^+} M(\lambda_j)$$

where the closure is taken in $H_0^1(\Omega)$.

If $r > 0$ we set

$$N_r(0) = \{ x \in \mathbb{R}^n \mid \|x\| < r \}.$$

Without loss of generality we can suppose that $0 \in \Omega$ and that $N_1(0) \subset \Omega$. Given $\mu > 0$ we set (cf. [2] [7])

$$\psi_\mu(x) = \phi(x) \cdot u_\mu(x)$$

where $\phi \in C_0^\infty(N_1(0))$, $\phi(x) = 1$ on $N_{\frac{1}{2}}(0)$, and

$$u_\mu(x) = \frac{|n(n-2)\mu|^{(n-2)/4}}{|\mu + |x||^{(n-2)/2}}.$$

The following lemma holds:

LEMMA 2.1. — *If $\psi_\mu(x)$ is defined as in (2.1), then for any μ*

$$(2.2) \quad \|\psi_\mu\|^2 = S^{n/2} + O(\mu^{(n-2)/2}) \quad (1)$$

$$(2.3) \quad |\psi_\mu|_{2^*}^{2^*} = S^{n/2} + O(\mu^{n/2})$$

$$(2.4) \quad |\psi_\mu|_2^2 = \begin{cases} K_1\mu + O(\mu^{(n-2)/2}) & \text{if } n \geq 5 \\ K_1\mu |\log \mu| + O(\mu) & \text{if } n = 4 \end{cases}$$

$$(2.5) \quad |\psi_\mu|_1 \leq K_2\mu^{(n-2)/4}$$

$$(2.6) \quad |\psi_\mu|_{2^*-1}^{2^*-1} \leq K_3\mu^{(n-2)/4}$$

where K_1, K_2, K_3 are suitable positive constants.

Proof. — The proof of (2.2), (2.3), (2.4) is contained in [2], moreover (2.5) and (2.6) can be straightforwardly verified.

Now we shall prove some technical lemmas. We set

$$\overline{W}(\mu) = \{u \in H_0^1 \mid u = u^- + t\psi_\mu, u^- \in H_2, t \in \mathbb{R}\}.$$

The following lemma holds:

LEMMA 2.2. — *If $u \in \overline{W}(\mu)$, then for any $\mu > 0$*

$$(2.7) \quad |u|_{2^*}^{2^*} \geq |t\psi_\mu|_{2^*}^{2^*} + \frac{1}{2}|u^-|_{2^*}^{2^*} - K_4 t^{2^*} \mu^{n(n-2)/(2n+4)} \quad \text{for any } t \in \mathbb{R}.$$

Proof. — By the identity

$$(2.8) \quad |u|_{2^*}^{2^*} = 2^* \int_\Omega dx \int_0^u |s|^{2^*-2} s ds$$

(1) In the sequel we denote by $O(\mu^\alpha)$, $\alpha > 0$ a function $|f(\mu)| \leq \text{const } \mu^\alpha$ near $\mu = 0$, and by $o(\mu)$, a function such that $f(\mu)/\mu \rightarrow 0$ as $\mu \rightarrow 0$.

it follows that

$$\begin{aligned}
 (2.9) \quad & |u^- + t\psi_\mu|_{2^*}^{2^*} - |t\psi_\mu|_{2^*}^{2^*} - |u^-|_{2^*}^{2^*} = \\
 & = 2^* \int_0^1 d\tau \int_\Omega [|t\psi_\mu + \tau u^-|^{2^*-2} \cdot (t\psi_\mu + \tau u^-) - |\tau u^-|^{2^*-2} \cdot \tau u^-] u^- dx = \\
 & = 2^*(2^* - 1) \int_0^1 d\tau \int_\Omega |\tau u^- + t\psi_\mu \theta|^{2^*-2} \cdot t\psi_\mu \cdot u^- dx
 \end{aligned}$$

where $\theta = \theta(x)$ is a measurable function such that $0 < \theta(x) < 1$.

By (2.9) and by (2.5), (2.6) we have that

$$\begin{aligned}
 (2.10) \quad & | |u|_{2^*}^{2^*} - |t\psi_\mu|_{2^*}^{2^*} - |u^-|_{2^*}^{2^*} | \\
 & \leq c_1 \int_0^1 d\tau \int_\Omega \{ |u^-| \cdot |t\psi_\mu|^{2^*-1} + \tau^{2^*-2} \cdot |t\psi_\mu| \cdot |u^-|^{2^*-1} \} dx \leq \\
 & \leq c_2 \{ |t\psi_\mu|_{2^*-1}^{2^*-1} \cdot |u^-|_\infty + |t\psi_\mu|_1 \cdot |u^-|_\infty^{2^*-1} \} \leq \\
 & \leq c_3 \{ |t\psi_\mu|_{2^*-1}^{2^*-1} \cdot |u^-|_2 + |t\psi_\mu|_1 \cdot |u^-|_{2^*}^{2^*-1} \} \leq
 \end{aligned}$$

$$\begin{aligned}
 (2.10)a \quad & \leq e_3 \cdot t^{2^*-1} \cdot \mu^{(n-2)/4} |u^-|_2 + \frac{1}{4} |u^-|_{2^*}^{2^*} + c_4 \cdot t^{2^*} \cdot \mu^{n/2} \leq \\
 & \leq \frac{1}{2} |u^-|_{2^*}^{2^*} + k_4 t^{2^*} \cdot \mu^{n(n-2)/2n+4}
 \end{aligned}$$

and the lemma is proved.

LEMMA 2.3. — *If μ is sufficiently small, then*

$$(2.11) \quad \frac{\|\psi_\mu\|^2 - \lambda \|\psi_\mu\|_2^2}{\|\psi_\mu\|_{2^*}^{2^*}} = \begin{cases} S - K_5 \mu + O(\mu^{\frac{n-2}{2}}) & \text{if } n \geq 5 \\ S + K_5 \mu \log \mu + O(\mu) & \text{if } n = 4 \end{cases} \quad (2.11)a$$

$$(2.11)b$$

Proof. — The evaluation (2.11) follows immediately by (2.2), (2.3) and (2.4).

REMARK 2.4. — Suppose that $\lambda = \lambda_j$, with $\lambda_j \in \sigma(-\Delta)$ and denote by P_j the projector on the eigenspace M_j corresponding to λ_j .

We set

$$(2.12) \quad \tilde{\psi}_\mu = \psi_\mu - P_j \psi_\mu.$$

Let $\{v_k\}$ an orthonormal family spanning M_j , then by (2.5) we have

$$(2.13) \quad |P_j \psi_\mu|_2^2 = \sum_k \left(\int_\Omega \psi_\mu v_k dx \right)^2 \leq \text{const} \|\psi_\mu\|_1^2 \leq K_6 \mu^{\frac{n-2}{2}}$$

then

$$(2.14) \quad |P_j \psi_\mu|_\infty \leq K_7 \mu^{\frac{n-2}{4}}.$$

Moreover we have

$$\begin{aligned} \left| \int_{\Omega} \{ |\tilde{\psi}_{\mu}|^{2^*} - |\psi_{\mu}|^{2^*} \} dx \right| &= 2^* \left| \int_0^1 d\tau \int_{\Omega} |\psi_{\mu} - \tau P_j \psi_{\mu}|^{2^*-2} (\psi_{\mu} - \tau P_j \psi_{\mu}) P_j \psi_{\mu} dx \right| \leq \\ &\leq 2^* \cdot 2^{2^*-1} \int_0^1 d\tau \int_{\Omega} \{ |\psi_{\mu}|^{2^*-1} + \tau^{2^*-1} |P_j \psi_{\mu}|^{2^*-1} \} |P_j \psi_{\mu}| dx \\ &\leq \text{const} \{ |\psi_{\mu}|_{2^{*}-1}^{2^*-1} \cdot |P_j \psi_{\mu}|_{\infty} + |P_j \psi_{\mu}|_{2^*}^{2^*} \}. \end{aligned}$$

Then by (2.14) and (2.6) it follows that

$$(2.15) \quad \left| |\tilde{\psi}_{\mu}|_{2^*}^{2^*} - |\psi_{\mu}|_{2^*}^{2^*} \right| \leq c_1 \mu^{\frac{n-2}{2}}.$$

Moreover by (2.14) and (2.6) we have

$$(2.16) \quad \left| |\tilde{\psi}_{\mu}|_{2^{*}-1}^{2^{*}-1} - |\psi_{\mu} - P_j \psi_{\mu}|_{2^{*}-1}^{2^{*}-1} \right| \leq \text{const} \{ |\psi_{\mu}|_{2^{*}-1}^{2^{*}-1} + |P_j \psi_{\mu}|_{2^{*}-1}^{2^{*}-1} \} \leq \text{const} \mu^{\frac{n-2}{4}}.$$

Analogously by (2.14) and (2.5) we have

$$(2.17) \quad |\tilde{\psi}_{\mu}|_1 \leq \text{const} \mu^{\frac{n-2}{4}}.$$

By (2.15), (2.16), (2.17) it easily follows that (2.11) holds if we replace ψ_{μ} with $\tilde{\psi}_{\mu}$.

Moreover, by (2.15), (2.16), (2.17), also (2.7) holds (for μ small) if we replace ψ_{μ} with $\tilde{\psi}_{\mu}$ and $\overline{W}(\mu)$ with

$$\overline{\overline{W}}(\mu) = \{ u \in H_0^1 \mid u = u^- + t\tilde{\psi}_{\mu}, u^- \in H_2, t \in \mathbb{R} \}.$$

Now we can prove a crucial lemma:

LEMMA 2.5. — For μ sufficiently small

$$(2.18) \quad \sup_W f(u) < \frac{1}{n} S^{n/2}$$

where $W = \overline{W}(\mu)$ (resp. $\overline{\overline{W}}(\mu)$) if $\lambda \notin \sigma(-\Delta)$ (resp. $\lambda \in \sigma(-\Delta)$).

Proof. — Observe that if we fix $u \in H_0^1(\Omega)$, $u \neq 0$, then

$$(2.19) \quad \max_t f_{\lambda}(tu) = \frac{1}{n} \left(\frac{\|u\|^2 - \lambda |u|_2^2}{|u|_{2^*}^2} \right)^{n/2}.$$

Then in order to prove (2.18) we need to evaluate

$$(2.20) \quad \sup_{\substack{u \in W(\mu) \\ |u|_{2^*} = 1}} \{ \|u\|^2 - \lambda |u|_2^2 \}.$$

We distinguish two cases:

Case i) $\lambda \notin \sigma(-\Delta)$.

Let $u = u^- + t\psi_{\mu} \in \overline{W}(\mu)$ with $|u|_{2^*} = 1$.

Observe that t is bounded if μ is small, in fact by (2.7) and (2.3) we get

$$1 = |u|_{2^*}^2 \geq |t\psi_\mu|_{2^*}^2 - K_4 t^{2^*} \mu^{n/2} + \frac{1}{2} |u^-|_{2^*}^2 = t^{2^*} [S^{n/2} + O(\mu^{n/2})] + \frac{1}{2} |u^-|_{2^*}^2.$$

Then by (2.5) we have that

$$\begin{aligned} (2.21) \quad \|u\|^2 - \lambda |u|_2^2 &= |\nabla u^-|_2^2 - \lambda |u^-|_2^2 + |\nabla t\psi_\mu|_2^2 - \lambda |t\psi_\mu|_2^2 - \\ &\quad - 2 \int_{\Omega} \{ |t\psi_\mu| |\Delta u^-| + \lambda |u^-| |t\psi_\mu| \} dx \leq \\ &\leq |\nabla u^-|_2^2 - \lambda |u^-|_2^2 + |\nabla t\psi_\mu|_2^2 - \lambda |t\psi_\mu|_2^2 + c_1 \{ |\Delta u^-|_\infty |t\psi_\mu|_1 + |u^-|_\infty |t\psi_\mu|_1 \} \leq \\ &\leq |\nabla u^-|_2^2 - \lambda |u^-|_2^2 + |\nabla t\psi_\mu|_2^2 - \lambda |t\psi_\mu|_2^2 + c_2 |u^-|_2 \cdot \mu^{\frac{n-2}{4}} \leq \\ &\leq (\bar{\lambda} - \lambda) |u^-|_2^2 + \frac{|\nabla t\psi_\mu|_2^2 - \lambda |t\psi_\mu|_2^2}{|t\psi_\mu|_{2^*}^2} \cdot |t\psi_\mu|_{2^*}^2 + c_2 |u^-|_2 \cdot \mu^{\frac{n-2}{4}} \end{aligned}$$

where $\bar{\lambda} = \max \{ \lambda_j \mid \lambda_j < \lambda \}$.

We set $A(u^-, \mu, c) = (\bar{\lambda} - \lambda) |u^-|_2^2 + C |u^-|_2 \mu^{\frac{n-2}{4}}$ and observe that

$$(2.22) \quad A(u^-, \mu, c) \leq 0 \quad \text{or} \quad A(u^-, \mu, c) \leq c^2 / (\lambda - \bar{\lambda}) \mu^{(n-2)/2}$$

If $|u^-|_{2^*}^2 \leq 2K_4 t^{2^*} \mu^{\frac{n(n-2)}{2n+4}}$, by (2.10)a and the boundness of t ,

$$\begin{aligned} |t\psi_\mu|_{2^*}^2 &\leq \left(1 - \frac{3}{4} |u^-|_{2^*}^2 + c_3 \mu^{\frac{n-2}{4}} |u^-|_2 + c_4 \mu^{\frac{n}{2}} \right)^{\frac{2}{2^*}} \\ &\leq 1 + \frac{2}{2^*} (c_3 \mu^{\frac{n-2}{4}} |u^-|_2 + c_4 \mu^{\frac{n}{2}})^{\frac{2}{2^*}}, \end{aligned}$$

then, if $n \geq 5$, by (2.11)a, (2.21)

$$(2.23) \quad \|u\|^2 - \lambda |u|_2^2 \leq (S - K_5 \mu + O(\mu^{\frac{n-2}{2}}))(1 + c_5 \mu^{n/2}) + A(u^-, \mu, c_6).$$

If $|u^-|_{2^*}^2 > 2K_4 t^{2^*} \mu^{\frac{n(n-2)}{2n+4}}$, by (2.7), $|t\psi_\mu|_{2^*} < 1$, then, by (2.21)

$$(2.24) \quad \|u\|^2 - \lambda |u|_2^2 \leq (S - K_5 \mu + O(\mu^{\frac{n-2}{2}})) + A(u^-, \mu, c_2),$$

then, by (2.22), the conclusion follows in the case $n \geq 5$.

If $n = 4$ the proof is the same. In this case (2.11)b replaces (2.11)a in (2.22).

Case ii) $\lambda = \lambda_{\bar{j}} \in \sigma(-\Delta)$.

Let $u = u^- + t\tilde{\psi}_\mu \in \overline{W}(\mu)$ with $|u|_{2^*} = 1$. We set $u = u^- + t\tilde{\psi}_\mu = \tilde{u} + P_j u^- + t\tilde{\psi}_\mu$, then

$$\begin{aligned} \|u\|^2 - \lambda_{\bar{j}} |u|_2^2 &= |t\tilde{\psi}_\mu|_2^2 - \lambda_{\bar{j}} |t\tilde{\psi}_\mu|_2^2 + |\nabla \tilde{u}|_2^2 - \lambda_{\bar{j}} |\tilde{u}|_2^2 - 2 \int_{\Omega} (t\tilde{\psi}_\mu \Delta u^- + \lambda_{\bar{j}} \tilde{\psi}_\mu u^-) dx. \end{aligned}$$

Observe that

$$\begin{aligned} \int_{\Omega} (t\tilde{\psi}_{\mu}\Delta u^{-} + \lambda_{\bar{j}}t\tilde{\psi}_{\mu}u^{-})dx &= \int_{\Omega} (t\tilde{\psi}_{\mu}\Delta\tilde{u}_{-} + \lambda_{\bar{j}}t\tilde{\psi}_{\mu}\tilde{u}_{-})dx \leq \\ &\leq |\Delta\tilde{u}_{-}|_{\infty} |t\tilde{\psi}_{\mu}|_1 + \lambda_{\bar{j}}|\tilde{u}_{-}|_{\infty} |t\tilde{\psi}_{\mu}|_1 \leq c_3|\tilde{u}_{-}|_{2\mu}^{\frac{n-2}{4}}. \end{aligned}$$

Now the proof follows by using the previous arguments.

Proof of theorem 0.1. — If $\lambda \notin \sigma(-\Delta)$ ($\lambda > 0$) we set $V = H_1$ and $W = \overline{W}(\mu)$ with μ suitably small in order that (2.18) is verified. We see that the assumptions of Theorem 1.2 are satisfied. Obviously (f_1) and $(f_3.iii)$ are verified. Moreover (f_2) is verified with $\beta = \frac{1}{n}S^{n/2}$ by lemma 1.1 and $(f_3.i)$ (with $\beta = \frac{1}{n}S^{n/2}$) is verified by lemma 2.5.

Finally observe that if $u \in H_1$, then

$$\begin{aligned} (2.25) \quad f_{\lambda}(u) &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} dx \geq \\ &\geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda^+}\right) \|u\|^2 - \frac{1}{2^*} \|u\|^{2^*} \geq \frac{1}{2} \left(\lambda - \frac{\lambda}{\lambda^+}\right) \|u\|^2 - \text{const} \|u\|^{2^*} \geq \delta > 0 \end{aligned}$$

if $\|u\| = \rho$ with ρ suitably small.

Hence by (2.27) also $(f_3.ii)$ is verified. Since $\dim V \cap W = 1$ and $V + W = H_0^1(\Omega)$, then by Theorem 1.2, we deduce that problem (0.1) has at least one non trivial solution.

If $\lambda \in \sigma(-\Delta)$ we set $W = \overline{W}(\mu)$ with μ suitably small in order that (2.18) is verified and, by repeating the above arguments, the conclusion follows.

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