ALBERTO VALLI

On the existence of stationary solutions to compressible Navier-Stokes equations

Annales de l'I. H. P., section C, tome 4, nº 1 (1987), p. 99-113
<http://www.numdam.org/item?id=AIHPC_1987__4_1_99_0>

Ann. Inst. Henri Poincaré,
Vol. 4, n° 1, 1987, p. 99-113. *Analyse non linéaire*

On the existence of stationary solutions to compressible Navier-Stokes equations

by

Alberto VALLI (*)
Dipartimento di Matematica, Universita' di Trento,
38050, POVO (Trento), Italy

ABSTRACT. — We prove the existence of a stationary solution to the Navier-Stokes equations for compressible fluids, under the assumption that the external force field is small in a suitable sense. The proof is based over an existence result for a linearized problem, followed by a fixed point argument.

Key-words: Navier-Stokes equations, Compressible fluids, Stationary solutions, Modified Stokes system.

RÉSUMÉ. — Nous démontrons l'existence d'une solution stationnaire pour les équations de Navier-Stokes dans le cas de fluides compressibles, en supposant que le champ de force extérieur est petit en un sens convenable. La démonstration repose sur un résultat d'existence pour un problème linéarisé combiné avec un argument de point fixé.

1. INTRODUCTION AND MAIN THEOREM

Aim of this paper is to present a new method for showing the existence of a solution to the system of equations which describe the motion of a viscous compressible barotropic fluid in the stationary case.

(*) Work partially supported by G. N. A. F. A. of C. N. R.

The equations of motion can be written in the form

$$
(1.1)\quad
\begin{aligned}
\rho[(v\cdot\nabla)v - f] - \mu\Delta v - (\zeta + \mu/3)\nabla \operatorname{div} v + \nabla[p(\rho)] &= 0 \qquad \text{in } \Omega\,,\\
\operatorname{div}(\rho v) &= 0 \qquad \text{in } \Omega,\\
v_{|\partial\Omega} &= 0 \qquad \text{on } \partial\Omega,\\
\int_\Omega \rho = \bar\rho\,|\,\Omega\,| \qquad (\,|\,\Omega\,| \equiv \operatorname{meas}(\Omega))\,,&
\end{aligned}
$$

where $\Omega \subset \mathbf{R}^3$ is a bounded domain, with smooth boundary (it is enough to suppose $\partial\Omega \in C^3$, by choosing local coordinates on $\partial\Omega$ as in [*10*]; however, for the sake of simplicity, we assume $\partial\Omega \in C^4$ and we consider as local coordinates the isothermal coordinates, see § 2); v and ρ are the velocity and the density of the fluid, respectively; p is the pressure, which is assumed to be a known (increasing) function of ρ; f is the (assigned) external force field; the constants $\mu > 0$ and $\zeta \geq 0$ are the viscosity coefficients; $\bar\rho > 0$ is the mean density of the fluid, i. e. the total mass of fluid divided by $|\,\Omega\,|$.

The external force field f is assumed to be *small* in a suitable norm (see the statement of the theorem at the end of this Introduction).

In the last years this problem has been studied by several authors. At first Matsumura-Nishida [*4*] showed that there exists a solution when $f = \nabla\varphi$. This case is much simpler than the general one, since the fluid is at rest, i. e. $v = 0$, and one has only to determine the density ρ.

Later Padula [*5*] found a solution when the bulk viscosity coefficient ζ is much larger than the shear viscosity coefficient μ. Her proof is strictly based on the structure of equation $(1.1)_1$ (more precisely, on the presence of the term $\nabla \operatorname{div} v$; some ideas of her proof are also used in this paper, see § 2 *b ii*). However, in the general physical situation the coefficients ζ is not larger than μ (for monoatomic gases the Stokes relation $\zeta = 0$ is reasonably correct; see Serrin [*6*], p. 239). Moreover, from the mathematical point of view one expects that the principal term in $(1.1)_1$ is given by $-\mu\Delta v$, and that the term $-(\zeta + \mu/3)\nabla \operatorname{div} v$ could be omitted without changing the result.

Actually, in [*9*] it was proved that a stationary solution does exist for each pair of viscosity coefficients satisfying the thermodynamic restrictions $\mu > 0$, $\zeta \geq 0$. However, the proof is not a direct one, since the solution is obtained (by a stability argument) by taking the limit as $t \to +\infty$ of each non-stationary solution starting in a sufficiently small neighbourhood of $v = 0$, $\rho = \bar\rho$.

Let us remark that it is not surprising that the non-stationary case is in some sense « simpler » than the stationary one. In fact, the term which gives more troubles in the proof of the existence of a stationary solution is the term $v\cdot\nabla\rho$ in $(1.1)_2$ (without it, (1.1) would be a non-linear elliptic system; moreover, it would be possible to get a solution by using the implicit function theorem). On the contrary, in the non-stationary case one easily observes that $(\partial \rho/\partial t + v \cdot \nabla \rho)$ is the material derivative of ρ, hence the method of characteristics is an useful tool for constructing a solution.

In this paper we present a new method for showing the existence of a stationary solution in the general case. Our proof is based on an existence theorem for a system which is a natural linearization of (1.1) (see (2.1)), followed by a fixed point argument. The main part of the paper is devoted to the study of the linear system (2.1). At first we find some *a priori* estimates in Sobolev spaces of sufficiently high order, then we show the existence of a solution in a particular case and finally we solve the general case by using the continuity method. The existence of a fixed point is then an easy consequence of Schauder's theorem.

We want to remark that it is not possible to prove the existence of a solution by using the usual implicit function theorem. In fact, due to the term $v . \nabla \rho$, there is a loss of regularity and the nonlinear operator and its Frechet differential at the point $(0, \bar{\rho})$ don't operate between the same spaces. On the other hand, one of the basic assumptions of the Nash-Moser implicit function theorem is that there exists a right inverse for the Frechet differential at a point *near* $(0, \bar{\rho})$. One easily sees that this linear operator is more complicated than the linear operator expressed by (2.1), hence our proof is simpler and more precise than an eventual one based on the Nash-Moser theorem.

At last, let us underline that we can obtain the same existence result also in the non-barotropic case (i. e. the pressure p is a function of the density ρ and of the temperature θ), and in the case of non-constant viscosity (and heat conductivity) coefficients (see Remark 3.3).

As a final comment, we must also mention that, during the preparation of this paper, Beirão da Veiga [*1*] has obtained an analogous result, by means of a completely different method.

Let us state now the main theorem. Since we are searching for a solution in a neighbourhood of the equilibrium solution $v = 0$, $\rho = \bar{\rho}$, it is useful to introduce the new unknown

$$\sigma \equiv \rho - \bar{\rho} . \tag{1.2}$$

The equations of motion thus become

$$\begin{aligned} (\sigma+\bar{\rho})[(v.\nabla)v-f]-\mu\Delta v-(\zeta+\mu/3)\nabla \operatorname{div} v+\nabla[p(\sigma+\bar{\rho})] &=0 \quad \text{in } \Omega, \\ \bar{\rho} \operatorname{div} v+\operatorname{div}(\sigma v) &=0 \quad \text{in } \Omega, \\ v_{|\partial\Omega} &=0 \quad \text{on } \partial\Omega, \\ \int_\Omega \sigma &=0 . \end{aligned} \tag{1.3}$$

It is clear that (1.3) and (1.1) are equivalent problems.

Denoting by $H^k(\Omega)$ the usual Sobolev space (with norm $\|\cdot\|_k$) for $k+1 \in N$, we can at last state the main theorem of this paper.

THEOREM. — Suppose that the pressure p is a C^2-function in a neighbourhood of $\bar{\rho}$, and that $p'(\bar{\rho}) > 0$. Suppose moreover that $f \in H^1(\Omega)$ with $\| f \|_1$ sufficiently small. Then there exists a unique solution

$$(v, \sigma) \in H^3(\Omega) \times H^2(\Omega) \qquad \text{to} \quad (1.3)$$

in a neighbourhood of the origin, and moreover

$$\| v \|_3 + \| \sigma \|_2 \leq c \| f \|_1 .$$

2. THE LINEAR PROBLEM

We want to find a solution to the linear problem

$$(2.1) \qquad \begin{aligned} - \mu\Delta w - (\zeta + \mu/3)\nabla \operatorname{div} w + p_1 \nabla\eta &= F \quad \text{in } \Omega, \\ \bar{\rho} \operatorname{div} w + \operatorname{div}(v\eta) &= G \quad \text{in } \Omega, \\ w_{|\partial\Omega} &= 0 \quad \text{on } \partial\Omega \\ \int_\Omega \eta &= 0 \end{aligned}$$

where $p_1 \equiv p'(\bar{\rho}) > 0$, v is a given vector field satisfying $v_{|\partial\Omega} = 0$, and $\int_\Omega G = 0$.

It is clear that if we take $F = (\sigma + \bar{\rho})[f - (v.\nabla)v] + [p_1 - p'(\sigma + \bar{\rho})]\nabla\sigma$, $G = 0$, then a fixed point of the map

$$(2.2) \qquad \Phi : (v, \sigma) \to (w, \eta)$$

is a solution of problem (1.3).

2a) *A priori* estimates.

We shall begin by proving some *a priori* estimates for a solution of (2.1). More precisely, we want to show that a solution $(w, \eta) \in H^3(\Omega) \times H^2(\Omega)$ satisfies

$$(2.3) \qquad \| w \|_3 + \| \eta \|_2 \leq c_1(\| F \|_1 + \| G \|_2)$$

for $\| v \|_3 \leq A$, A *small enough.* The constant c_1 depends in a continuous way on μ and ζ (but it is independent of v; of course, here and in the sequel the constant depends also on Ω, p_1 and $\bar{\rho}$).

Observe that we are interested in obtaining (2.3) since we need to control the behaviour of the non-linear terms contained in F. Analogous estimates in Sobolev spaces of lower order would not be sufficient to get a fixed point of Φ (see § 3).

By using well-known results about Stokes problem (see for instance Temam [*8*], p. 33) we easily get

LEMMA 2.1. — A solution $(w, \eta) \in H^3(\Omega) \times H^2(\Omega)$ of (2.1) satisfies

$$\| w \|_2^2 + \| \eta \|_1^2 \leq c(\| F \|_0^2 + \| \operatorname{div} w \|_1^2), \tag{2.4}$$

$$\| w \|_3^2 + \| \eta \|_2^2 \leq c(\| F \|_1^2 + \| \operatorname{div} w \|_2^2). \tag{2.5}$$

Hence our aim is to estimate $\| \operatorname{div} w \|_2^2$. We can proceed now by following a method which is essentially due (in a slightly different context) to Matsumura-Nishida [*4*] (see also Valli [*9*], § 4).

LEMMA 2.2. — A solution $(w, \eta) \in H^3(\Omega) \times H^2(\Omega)$ of (2.1) satisfies

$$\| w \|_1^2 \leq c(\| F \|_{-1}^2 + \varepsilon^{-1} \| G \|_0^2 + \| v \|_3 \| \eta \|_0^2) + \varepsilon \| \eta \|_0^2, \tag{2.6}$$

for each $\varepsilon > 0$.

Proof. — Multiply $(2.1)_1$ by w and $(2.1)_2$ by $(p_1/\bar{\rho})\eta$, integrate in Ω and add these two equations. Since

$$\int_\Omega p_1 \nabla\eta \cdot w = - \int_\Omega p_1 \eta \operatorname{div} w ,$$

one has

$$\mu \| \nabla w \|_0^2 + (\zeta + \mu/3) \| \operatorname{div} w \|_0^2 \leq c(\| F \|_{-1}^2 + \varepsilon^{-1} \| G \|_0^2) + (\mu/2) \| \nabla w \|_0^2 + \varepsilon \| \eta \|_0^2 + (p_1/\bar{\rho}) \int_\Omega | v \cdot \nabla\eta\eta + \operatorname{div} v\eta^2 | ,$$

hence (2.6). (Observe that $\| \operatorname{div} v \|_{L^\infty(\Omega)} \leq c \| v \|_3$). □

One can repeat the same procedure for the higher order derivatives in the interior of Ω. Take the gradient of $(2.1)_1$, and $(2.1)_2$, and then multiply the first equation by $\chi_0^2 Dw$ and the second equation by $(p_1/\bar{\rho})\chi_0^2\nabla\eta$, where $\chi_0 \in C_0^\infty(\Omega)$. Finally, integrate in Ω and add these two equations. If the same procedure is applied also for the second order derivatives, we get.

LEMMA 2.3. — A solution $(w, \eta) \in H^3(\Omega) \times H^2(\Omega)$ of (2.1) satisfies

$$\| \chi_0 D^2 w \|_0^2 \leq c(\| F \|_0^2 + \delta^{-1} \| G \|_1^2 + \| v \|_3 \| \eta \|_1^2 + \delta^{-1} \| w \|_1^2) + \delta \| \eta \|_1^2, \tag{2.7}$$

$$\| \chi_0 D^3 w \|_0^2 \leq c(\| F \|_1^2 + \delta^{-1} \| G \|_2^2 + \| v \|_3 \| \eta \|_2^2 + \delta^{-1} \| w \|_2^2) + \delta \| \eta \|_2^2, \tag{2.8}$$

for each $0 < \delta < 1$.

Proof. — By proceeding as indicated above, one obtains

$$\mu \| \chi_0 D^2 w \|_0^2 + (\zeta + \mu/3) \| \chi_0 \nabla \operatorname{div} w \|_0^2 \leq c \Big[\int_\Omega | \chi_0 | \, | \nabla \chi_0 | \, | Dw | (| D^2 w |$$

$$+ | \nabla \operatorname{div} w | + | \nabla \eta | + | F |) + \int_\Omega \chi_0^2 | F | \, | D^2 w | + \int_\Omega \chi_0^2 | \nabla G | \, | \nabla \eta |$$

$$+ \int_\Omega \chi_0^2 | Dv | \, | \nabla \eta |^2 + \int_\Omega \chi_0^2 | D^2 v | \, | \eta | \, | \nabla \eta | + \int_\Omega | \chi_0 | \, | \nabla \chi_0 | \, | v | \, | \nabla \eta |^2 \Big]$$

$$\leq c (\| F \|_0^2 + \delta^{-1} \| G \|_1^2 + \| v \|_3 \| \eta \|_1^2 + \delta^{-1} \| w \|_1^2) + (\mu/2) \| \chi_0 D^2 w \|_0^2 + \delta \| \eta \|_1^2 ,$$

hence (2.7).

Estimate (2.8) is obtained in the same way, since

$$(p_1/\bar{\rho}) \left| \int_\Omega \chi_0^2 D^2 (v \cdot \nabla \eta + \eta \operatorname{div} v) D^2 \eta \right| \leq c \| v \|_3 \| \eta \|_2^2 . \qquad \square$$

We need to obtain now the estimates on the boundary. Choose for instance as local coordinates the isothermal coordinates $\lambda(\psi, \varphi)$ (see e. g. Spivak [7], p. 460; Valli [9]). A point x in an open set Q near $\partial\Omega$ can be written as

$$x = \Lambda(\psi, \varphi, r) \equiv \lambda(\psi, \varphi) + r n(\lambda(\psi, \varphi)) ,$$

where n is the unit outward normal vector to $\partial\Omega$. By setting $y \equiv (\psi, \varphi, r)$, the equations $(2.1)_1$, $(2.1)_2$ become

$$(2.9) \quad -\mu a_{ki} D_k (a_{si} D_s W^j) - (\zeta + \mu/3) a_{kj} D_k (a_{si} D_s W^i) + p_1 a_{kj} D_k H = F^j(\Lambda) \quad \text{in } U \equiv \Lambda^{-1}(Q \cap \Omega),$$

$$(2.10) \qquad \bar{\rho} a_{kj} D_k W^j + a_{kj} D_k (V^j H) = G(\Lambda) \quad \text{in} \quad U ,$$

where

$$W(y) \equiv w(\Lambda(y)), \qquad H(y) \equiv \eta(\Lambda(y)), \qquad V(y) \equiv v(\Lambda(y)) ,$$

$a_{kj} = a_{kj}(y)$ is the entry (k, j) of $(\operatorname{Jac} \Lambda)^{-1} = (\operatorname{Jac} \Lambda^{-1})(\Lambda)$ (Jac Λ has the term $D_j \Lambda^i$ in the i-th row, j-th column), and $D_k \equiv \partial/\partial y_k$. Here and in the sequel we adopt the Einstein convention about summation over repeated indices.

For the tangential derivatives one can essentially proceed as before. One applies D_τ, $\tau = 1, 2$, to (2.9) and (2.10), and then multiplies by $J \chi^2 D_\tau W^j$ and $(p_1/\bar{\rho}) J \chi^2 D_\tau H$ and integrates in U. Here $J \equiv \det \operatorname{Jac} \Lambda$, and $\chi \in C_0^\infty(\Lambda^{-1}(Q))$. By repeating the same procedure for the second order derivatives, one gets as in Lemma 2.3:

LEMMA 2.4. — W and H satisfy

$$(2.11)\quad \int_U J\chi^2 \,|\, D_\tau D_y W \,|^2 \leq c(\| F \|_0^2 + \delta^{-1} \| G \|_1^2 + \| v \|_3 \| \eta \|_1^2 + \delta^{-1} \| w \|_1^2) + \delta \| \eta \|_1^2, \quad \tau = 1, 2,$$

$$(2.12)\quad \int_U J\chi^2 \,|\, D_\tau D_\xi D_y W \,|^2 \leq c(\| F \|_1^2 + \delta^{-1} \| G \|_2^2 + \| v \|_3 \| \eta \|_2^2 + \delta^{-1} \| w \|_2^2) + \delta \| \eta \|_2^2, \quad \tau, \xi = 1, 2,$$

for each $0 < \delta < 1$.

Let us consider now the normal derivatives. Take the normal derivative of $(2.2)_1$, multiplied by $(\zeta + 4\mu/3)/\bar{\rho}$. Then take the scalar product of $(2.1)_1$ by n and add these two equations. One has

$$(2.13)\quad p_1 \frac{\partial \eta}{\partial n} = \mu(\Delta w \cdot n - \nabla \operatorname{div} w \cdot n) + F \cdot n + \bar{\rho}^{-1}(\zeta + 4\mu/3)\nabla G \cdot n - \bar{\rho}^{-1}(\zeta + 4\mu/3)\nabla \operatorname{div}(v\eta) \cdot n,$$

and the term $(\Delta w \cdot n - \nabla \operatorname{div} w . n)$ does not contain second order normal derivatives $\partial^2 w^j/\partial n^2$, as can be easily seen in local coordinates. Multiplying (2.13) (written in local coordinates) by $J\chi^2 D_3 H$ and integrating in U one obtains

LEMMA 2.5. — W and H satisfy

$$(2.14)\quad \int_U J\chi^2 \,|\, D_3 H \,|^2 \leq c \int_U J\chi^2 \,|\, D_\tau D_y W \,|^2 + c(\| w \|_1^2 + \| F \|_0^2 + \| G \|_1^2 + \| v \|_3 \| \eta \|_1^2), \quad \tau = 1, 2.$$

The same can be done for the second order derivatives:

LEMMA 2.6. — W and H satisfy

$$(2.15)\quad \int_U J\chi^2 \,|\, D_\xi D_3 H \,|^2 \leq c \int_U J\chi^2 \,|\, D_\tau D_\xi D_y W \,|^2 + c(\| w \|_2^2 + \| F \|_1^2 + \| G \|_2^2 + \| v \|_3 \| \eta \|_2^2), \quad \tau, \xi = 1, 2,$$

$$(2.16)\quad \int_U J\chi^2 \,|\, D_3 D_3 H \,|^2 \leq c \int_U J\chi^2 \,|\, D_\tau D_3 D_y W \,|^2 + c(\| w \|_2^2 + \| F \|_1^2 + \| G \|_2^2 + \| v \|_3 \| \eta \|_2^2), \quad \tau = 1, 2.$$

On the other hand from $(2.1)_1$ one has

$$(\zeta + 4\mu/3)\nabla \operatorname{div} w \cdot n = p_1 \nabla \eta \cdot n - F \cdot n - \mu(\Delta w \cdot n - \nabla \operatorname{div} w \cdot n),$$

hence (2.14) (2.15) and (2.16) hold not only for D_3H, but also for $D_3(a_{kj}D_kW^j)$ (i. e. $\nabla \operatorname{div} w \cdot n$ in local coordinates).

It remains to estimate

$$\int_U J\chi^2 \, | D_\tau D_3 D_y W |^2 .$$

By considering the following Stokes problem

$$\begin{aligned}
&-\mu\Delta[(\chi D_\tau W)\circ\Lambda^{-1}] + p_1\nabla[(\chi D_\tau H)\circ\Lambda^{-1}] &&= L \quad \text{in} \quad Q\cap\Omega, \\
&\bar\rho \operatorname{div} [\chi D_\tau W)\circ\Lambda^{-1}] &&= M \quad \text{in} \quad Q\cap\Omega, \\
&(\chi D_\tau W)\circ\Lambda^{-1} &&= 0 \quad \text{on} \quad \partial(Q\cap\Omega) \ ,
\end{aligned}$$

where L and M can be calculated by writing the problem in local coordinates, one gets

LEMMA 2.7. — W satisfies

$$(2.17) \quad \int_U J\chi^2 \, | D_\tau D_y^2 W |^2 \le c \int_U J\chi^2 \, | D_\tau D_y(a_{kj}D_kW^j) |^2 + c(\| w \|_2^2 + \| \eta \|_1^2 + \| F \|_1^2), \qquad \tau = 1, 2 .$$

Proof. — One has

$$\int_{Q\cap\Omega} | D^2 [(\chi D_\tau W)\circ\Lambda^{-1}] |^2 \le c\left(\int_{Q\cap\Omega} | L |^2 + \int_{Q\cap\Omega} | \nabla \operatorname{div} [(\chi D_\tau W)\circ\Lambda^{-1}] |^2\right).$$

and by evaluating the integral concerning L (2.17) follows. □

We are now in a position to obtain (2.3). In fact by using (2.4) (2.7) (2.11) (2.14) (for $D_3(a_{kj}D_kW^j)$) and finally (2.6) for $\varepsilon=\delta^2$, δ small enough, we get

$$(2.18) \qquad \| w \|_2^2 + \| \eta \|_1^2 \le c(\| F \|_0^2 + \| G \|_1^2 + \| v \|_3 \| \eta \|_1^2) .$$

Moreover, by using (2.5) (2.8) (2.12) (2.15) and (2.16) (for $D_3(a_{kj}D_kW^j)$), and (2.17) one obtains for each $0 < \delta < 1$

$$\| w \|_3^2 + \| \eta \|_2^2 \le c(\| F \|_1^2 + \delta^{-1} \| G \|_2^2 + \| v \|_3 \| \eta \|_2^2 + \delta^{-1} \| w \|_2^2 + \| \eta \|_1^2) + \delta \| \eta \|_2^2 .$$

Hence, by using (2.18) and by choosing δ small enough, we have

$$(2.19) \qquad \| w \|_3^2 + \| \eta \|_2^2 \le c(\| F \|_1^2 + \| G \|_2^2 + \| v \|_3 \| \eta \|_2^2) .$$

If $\| v \|_3 \le A$, A small enough, then (2.3) follows at once from this estimate.

2*b*) **Existence.**

We want to prove the existence of a solution $(w, \eta) \in H^3(\Omega) \times H^2(\Omega)$ to (2.1), under the assumptions $\partial\Omega \in C^4$, $F \in H^1(\Omega)$, $G \in H^2(\Omega)$, $\int_\Omega G = 0$, $v \in H^3(\Omega)$, $v_{|\partial\Omega} = 0$ and $\| v \|_3 \leq A$, A small enough.

i) Elliptic approximation.

First of all, we want to spend some words about elliptic approximation. Consider the following Neumann problem

$$(2.20)\qquad \begin{aligned} - \mu\Delta w_\varepsilon - (\zeta + \mu/3)\nabla \operatorname{div} w_\varepsilon + p_1\nabla\eta_\varepsilon &= F \quad \text{in} \quad \Omega\,, \\ - \varepsilon\Delta\eta_\varepsilon + \bar{\rho} \operatorname{div} w_\varepsilon + \operatorname{div}(v\eta_\varepsilon) &= G \quad \text{in} \quad \Omega\,, \\ w_{\varepsilon|\partial\Omega} &= 0 \quad \text{on} \quad \partial\Omega\,, \\ (\partial\eta_\varepsilon/\partial n)_{|\partial\Omega} &= 0 \quad \text{on} \quad \partial\Omega\,, \\ \int_\Omega \eta_\varepsilon &= 0\,. \end{aligned}$$

It is possible to show that for each $\varepsilon > 0$ there exists a unique solution $(w_\varepsilon, \eta_\varepsilon) \in H^3(\Omega) \times H^4(\Omega)$. In fact, the bilinear from associated to (2.20) is continuous and weakly-coercive in

$$X \equiv H_0^1(\Omega) \times H_M^1(\Omega) \qquad \left(H_M^1(\Omega) \equiv \left\{ \eta \in H^1(\Omega) \,|\, \int_\Omega \eta = 0 \right\} \right).$$

Hence by following a standard argument, the existence of a solution in X is a consequence of Fredholm's alternative theorem (uniqueness is given by the *a priori* estimate

$$(2.21)\qquad \| w_\varepsilon \|_1^2 + \| \eta_\varepsilon \|_0^2 + \varepsilon \| \eta_\varepsilon \|_1^2 \leq c(\| F \|_{-1}^2 + \| G \|_0^2),$$

which is not difficult to be proved.)

A bootstrap argument shows that the solution indeed belongs to $H^3(\Omega) \times H^4(\Omega)$, and by proceeding as in § 2 *a* one obtains the estimate

$$(2.22)\qquad \| w_\varepsilon \|_2^2 + \| \eta_\varepsilon \|_1^2 + \varepsilon \| \eta_\varepsilon \|_2^2 \leq c(\| F \|_0^2 + \| G \|_1^2),$$

for each $\varepsilon > 0$ small enough.

Hence by a compactness argument one easily proves the existence of a solution $(w, \eta) \in H^2(\Omega) \times H^1(\Omega)$ to problem (2.1).

However, (2.1) is *not* an elliptic system in the sense of Agmon-Douglis-Nirenberg (unless $v = 0$; in this case it is the Stokes system), hence the usual regularization procedures do not work (the « bad » term is $v \cdot \nabla\eta$ in $(2.1)_2$).

On the other hand, due to the boundary condition on w, one cannot prove further regularity by differentiating the equations (as it is generally done for first order hyperbolic equations).

In conclusion, we have thus proved that (2.1) has a solution belonging to $H^2(\Omega) \times H^1(\Omega)$, but we *don't* known if this solution is more regular. (Remark moreover that estimate (2.22) is in some sense the « best » possible: we cannot obtain $\| \eta_\varepsilon \|_2 \leq$ const., as the normal derivative of the limit function η does not vanish on $\partial\Omega$).

REMARK 2.8. — It is also possible to approximate (2.1) by means of a degenerate elliptic problem, by adding to $(2.1)_2$ the terms

$$- \varepsilon \operatorname{div} (d \nabla \eta_\varepsilon) + \varepsilon \eta_\varepsilon, \qquad d \equiv d(x, \partial\Omega).$$

In this way it is not necessary to impose boundary conditions on η_ε, hence in principle one can expect that η_ε converge in $H^2(\Omega)$ to η. However, we have not been able to get the estimates assuring the convergence. We have only obtained

$$\| w_\varepsilon \|_1^2 + \| \eta_\varepsilon \|_0^2 + \varepsilon \| d^{1/2} \nabla \eta_\varepsilon \|_0^2 \leq (\| F \|_{-1}^2 + \| G \|_0^2);$$

it is not sure that better estimates hold. ■

ii) A particular case: ζ/μ large enough.

We shall follow an alternative approach, which is related to the method of Padula [5].

If we define

$$\pi \equiv p_1 \eta/\mu - (\zeta/\mu + 1/3) \operatorname{div} w, \tag{2.23}$$

then (2.1) is transformed into

$$\begin{aligned} - \Delta w + \nabla \pi &= F/\mu && \text{in } \Omega, \\ \operatorname{div} w &= (\zeta/\mu + 1/3)^{-1} (p_1 \eta/\mu - \pi) && \text{in } \Omega, \\ w_{|\partial\Omega} &= 0 && \text{on } \partial\Omega, \\ \int_\Omega \pi &= 0, \end{aligned} \tag{2.24}$$

$$\begin{aligned} \bar{\rho}(\zeta/\mu + 1/3)^{-1} p_1 \eta/\mu + \operatorname{div}(v\eta) &= \bar{\rho}(\zeta/\mu + 1/3)^{-1} \pi + G \quad \text{in} \quad \Omega \\ \int_\Omega \eta &= 0 . \end{aligned} \tag{2.25}$$

We can solve (2.24) (2.25) by means of a fixed point argument. Being assigned (η^*, π^*) in the right hand side of $(2.24)_2$, we get at first (w, π) from the Stokes problem (2.24) ,and consequently η by (2.25). A fixed point of the map

$$\Psi : (\eta^*, \pi^*) \rightarrow (\eta, \pi) \tag{2.26}$$

is a solution of (2.24) (2.25).

Set

$$(2.27)\quad K_1 \equiv \left\{ (\eta^*, \pi^*) \in H^2(\Omega) \times H^2(\Omega) \mid \int_\Omega \eta^* = 0 = \int_\Omega \pi^*, \right.$$
$$\left. \| \eta^* \|_2^2 \le B, \| \pi^* \|_2^2 \le B \right\},$$

where B will be chosen later (see (2.31)).

From well-known estimates on Stokes problem we get

$$(2.28)\quad \| w \|_3^2 + \| \pi \|_2^2 \le c[\mu^{-2} \| F \|_1^2 + (\zeta/\mu + 1/3)^{-2}(p_1^2\mu^{-2} \| \eta^* \|_2^2 + \| \pi^* \|_2^2)]$$
$$\le c_2[\mu^{-2} \| F \|_1^2 + (\zeta/\mu + 1/3)^{-2}(p_1^2\mu^{-2} + 1)B], c_2 = c_2(\Omega).$$

On the other hand, (2.25) is solved by the usual methods concerning hyperbolic first-order equations (see for instance Friedrichs [*2*], Lax-Phillips [*3*]). If

$$(2.29)\quad \| v \|_3 \le c_3\bar{\rho}p_1(\zeta + \mu/3)^{-1}, \qquad c_3 = c_3(\Omega),$$

one easily obtains

$$(2.30)\quad \| \eta \|_2^2 \le c[(\zeta + \mu/3)^2\bar{\rho}^{-2}p_1^{-2} \| G \|_2^2 + \mu^2 p_1^{-2} \| \pi \|_2^2]$$
$$\le c_4 \{ (\zeta + \mu/3)^2\bar{\rho}^{-2}p_1^{-2} \| G \|_2^2 + p_1^{-2}[\| F \|_1^2$$
$$+ (\zeta/\mu + 1/3)^{-2}(p_1^2 + \mu^2)B] \}, \qquad c_4 = c_4(\Omega),$$

having used (2.28).

If we choose

$$(2.31)\quad B > \max \{ c_2\mu^{-2} \| F \|_1^2, c_4[(\zeta + \mu/3)^2\bar{\rho}^{-2}p_1^{-2} \| G \|_2^2 + p_1^{-2} \| F \|_1^2] \}$$

and if we have

$$(2.32)\qquad \zeta/\mu \quad \text{is } \textit{large} \text{ enough},$$

we get at last

$$\| \pi \|_2^2 \le B, \qquad \| \eta \|_2^2 \le B,$$

i. e. $\Psi(K_1) \subset K_1$.

The set K_1 is a convex and compact set in $Y \equiv H^1(\Omega) \times H^1(\Omega)$, and the map $\Psi : K_1 \to K_1$ is easily shown to be continuous in the Y-topology. Hence by Schauder's theorem Ψ has a fixed point.

In this way we have solved (2.1) when (2.29) and (2.32) are satisfied.

iii) The general case.

We want to apply now the continuity method. We can choose μ_0 and ζ_0 so small that

$$A \le c_3\bar{\rho}p_1(\zeta_0 + \mu_0/3)^{-1};$$

at the same time we can obtain that ζ_0/μ_0 is large enough. In this way (2.29) and (2.32) are satisfied for ζ_0 and μ_0.

Define now for $t \in [0, 1]$

$$\mu_t \equiv (1-t)\mu_0 + t\mu, \qquad \zeta_t \equiv (1-t)\zeta_0 + t\zeta,$$
$$\mathrm{L}_t(w, \eta) \equiv (-\mu_t \Delta w + (\zeta_t + \mu_t/3)\nabla \operatorname{div} w + p_1 \nabla \eta, \bar{\rho} \operatorname{div} w + \operatorname{div}(v\eta)),$$
$$\mathscr{H} \equiv \left\{ (w, \eta) \in \mathrm{H}^3(\Omega) \times \mathrm{H}^2(\Omega) \mid w_{|\partial\Omega} = 0, \int_\Omega \eta = 0 \right\},$$
$$\mathscr{Y} \equiv \left\{ (\mathrm{F}, \mathrm{G}) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^2(\Omega) \mid \int_\Omega \mathrm{G} = 0 \right\}.$$

We want to prove that the set

$\mathrm{T} \equiv \{ t \in [0, 1] \mid$ for each $(\mathrm{F}, \mathrm{G}) \in \mathscr{Y}$ there exists a unique solution $(w, \eta) \in \mathscr{H}$ of $\mathrm{L}_t(w, \eta) = (\mathrm{F}, \mathrm{G}) \}$

is not empty, open and closed, i. e. $\mathrm{T} \equiv [0, 1]$.

First of all, $0 \in \mathrm{T}$ since (2.29) and (2.32) are satisfied.

Let now $t_0 \in \mathrm{T}$: from (2.3) we have that

$$\| \mathrm{L}_{t_0}^{-1} \|_{\mathscr{L}(\mathscr{Y};\mathscr{H})} \leq c_5,$$

where $c_5 = c_1(\Omega, p_1, \bar{\rho}, \mu_{t_0}, \zeta_{t_0})$. Equation $\mathrm{L}_{t_0+\varepsilon}(w, \eta) = (\mathrm{F}, \mathrm{G})$ can be written in the form

$$[\mathrm{I} - \varepsilon \mathrm{L}_{t_0}^{-1}(\mathrm{L}_0 - \mathrm{L}_1)](w, \eta) = \mathrm{L}_{t_0}^{-1}(\mathrm{F}, \mathrm{G}),$$

hence it can be solved if

$$|\varepsilon| \| \mathrm{L}_{t_0}^{-1}(\mathrm{L}_0 - \mathrm{L}_1) \|_{\mathscr{L}(\mathscr{H};\mathscr{H})} \leq |\varepsilon| c_5 \| \mathrm{L}_0 - \mathrm{L}_1 \|_{\mathscr{L}(\mathscr{H};\mathscr{Y})} < 1,$$

i. e. $|\varepsilon| < c_5^{-1} \| \mathrm{L}_0 - \mathrm{L}_1 \|_{\mathscr{L}(\mathscr{H};\mathscr{Y})}^{-1}$. Observe that in general L_t does not belong to $\mathscr{L}(\mathscr{H}; \mathscr{Y})$ (as $v.\nabla\eta \notin \mathrm{H}^2(\Omega)$), while $(\mathrm{L}_0 - \mathrm{L}_1)$ belongs to it.

It remains to show that T is closed. Let $t_n \to t_0$, $t_n \in \mathrm{T}$. From (2.3) we have that

$$\| \mathrm{L}_{t_n}^{-1} \|_{\mathscr{L}(\mathscr{Y};\mathscr{H})} \leq \max_{t \in [0,1]} c_1(\Omega, p_1, \bar{\rho}, \mu_t, \zeta_t) \equiv c_6, \tag{2.33}$$

(recall that c_1 depends continuously on μ and ζ). Set $(w_n, \eta_n) \equiv \mathrm{L}_{t_n}^{-1}(\mathrm{F}, \mathrm{G})$; from (2.33) we see that we can select a subsequence (w_{n_k}, η_{n_k}) such that $(w_{n_k}, \eta_{n_k}) \to (w, \eta)$ weakly in $\mathscr{H}$, $(w, \eta) \in \mathscr{H}$. Moreover one easily obtains that $\mathrm{L}_{t_{n_k}}(w_{n_k}, \eta_{n_k}) \to \mathrm{L}_{t_0}(w, \eta)$ weakly in $\mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$, i. e. $(\mathrm{F}, \mathrm{G}) = \mathrm{L}_{t_0}(w, \eta)$. This proves that $t_0 \in \mathrm{T}$.

We have thus shown that $\mathrm{T} = [0, 1]$, i. e. for each $(\mathrm{F}, \mathrm{G}) \in \mathscr{Y}$ there exists a unique solution $(w, \eta) \in \mathscr{H}$ to (2.1).

3. THE NON-LINEAR PROBLEM

We are now in a position to prove the existence of a solution to (1.3) when f is *small enough*.

We consider

$$(3.1)\quad K_2 \equiv \left\{ (v, \sigma) \in H^3(\Omega) \times H^2(\Omega) \,|\, v_{|\partial\Omega} = 0, \int_\Omega \sigma = 0, \| v \|_3 + \| \sigma \|_2 \leq D \right\}$$

(D to be chosen later, see (3.3); in any case we require $D \leq A$ in such a way that the results of § 2 can be applied).

Then we set in (2.1)

$$(3.2)\quad F = (\sigma + \bar{\rho})[f - (v \cdot \nabla)v] + [p_1 - p'(\sigma + \bar{\rho})]\nabla\sigma, \quad G = 0.$$

We are searching for a fixed point of the map

$$\Phi : (v, \sigma) \to (w, \eta)$$

defined in (2.2).

First of all, from (2.3) we have

$$\| w \|_3 + \| \eta \|_2 \leq c_1 \| F \|_1 \leq c[(\| \sigma \|_2 + 1)(\| f \|_1 + \| v \|_2^2) + \| \sigma \|_2^2]$$
$$\leq c_7(D + 1)(\| f \|_1 + D^2).$$

Choosing

$$(3.3)\quad D \equiv \| f \|_1^{1/2},$$

and $\| f \|_1$ so small that $D \leq A$ and $2c_7(D+1)D^2 \leq D$, we get that $\Phi(K_2) \subset K_2$. On the other hand K_2 is convex and compact in $Z \equiv H^2(\Omega) \times H^1(\Omega)$. To prove the continuity of Φ in the topology of Z, consider $(v_n, \sigma_n) \to (v, \sigma)$ in Z, $(v_n, \sigma_n) \in K_2$. Define $(w_n, \eta_n) \equiv \Phi(v_n, \sigma_n)$, $(w, \eta) \equiv \Phi(v, \sigma)$. By using (2.18) for $(w_n - w)$ and $(\eta_n - \eta)$ one easily gets that $(w_n, \eta_n) \to (w, \eta)$ in Z. The existence of a fixed point for Φ is now a consequence of Schauder's theorem.

REMARK 3.1. — It is also possible to find a solution of (1.3) by taking the limit of a suitable sequence of successive approximations. More precisely, set $v_0 = 0$, $\sigma_0 = 0$, and choose (v_n, σ_n) to be the solution of (2.1) with F given by (3.2) and $v = v_{n-1}$, $\sigma = \sigma_{n-1}$. In this way one easily obtains that (v_n, σ_n) is bounded in $H^3(\Omega) \times H^2(\Omega)$, and moreover, by looking at $(v_n - v_{n-1}, \sigma_n - \sigma_{n-1})$, (2.18) gives that (v_n, σ_n) converges in Z to a solution $(v, \sigma) \in H^3(\Omega) \times H^2(\Omega)$. ■

REMARK 3.2. — Uniqueness and stability of the solution of (1.3) (in a neighbourhood of the origin) are proved in [*9*] (for some results about uniqueness, see also Padula [*5*]). ■

REMARK 3.3. — An analogous result can be obtained in the case of non-barotropic compressible fluids and non-constant viscosity (and heat conductivity) coefficients.

In this case the linear problem (2.1) must be substituted by

$$
(3.4)\qquad
\begin{aligned}
-\bar{\mu}\Delta w - (\zeta + \mu/3)\nabla \operatorname{div} w + p_1\nabla\eta + p_2\nabla\gamma &= F \quad \text{in } \Omega,\\
\bar{\rho}\operatorname{div} w + \operatorname{div}(v\eta) &= G \quad \text{in } \Omega,\\
-\bar{\chi}\Delta\gamma + \theta_1 p_2 \operatorname{div} w &= H \quad \text{in } \Omega,\\
w_{|\partial\Omega} &= 0 \quad \text{on } \partial\Omega,\\
\gamma_{|\partial\Omega} &= 0 \quad \text{on } \partial\Omega,\\
\int_\Omega \eta &= 0 \quad ,
\end{aligned}
$$

where

$$
p_1 \equiv p_\rho(\bar{\rho}, \theta_1) > 0, \quad p_2 \equiv p_\theta(\bar{\rho}, \theta_1), \qquad \theta_1 \equiv \inf_{\partial\Omega} \bar{\theta} > 0,
$$

$$
\bar{\mu} \equiv \mu(\bar{\rho}, \theta_1) > 0, \qquad \bar{\zeta} \equiv \zeta(\bar{\rho}, \theta_1) \geq 0, \quad \bar{\chi} \equiv \chi(\bar{\rho}, \theta_1) > 0,
$$

and $\bar{\theta}$ is the boundary datum for the absolute temperature θ. (One must have in mind that $\gamma = \theta - \bar{\theta}$). The expression of F, G and H can be easily written explicitly.

By proceeding as in § 2, one shows the existence of a solution to (3.4), satisfying

$$
(3.5)\qquad \| w \|_3 + \| \eta \|_2 + \| \gamma \|_2 \leq c(\| F \|_1 + \| G \|_2 + \| H \|_0)
$$

if $\| v \|_3 \leq A$, A small enough. (For obtaining the analogous of Lemma 2.2, one has to multiply $(3.4)_3$ by $\theta_1^{-1}\gamma$, and to add the result to the other two equations).

By using (3.5) instead of (2.3), one can solve the non-linear problem by means of Schauder's fixed point theorem. More precisely, assume that: $\mu \in C^1$, $\zeta \in C^1$, $\chi \in C^1$ (and the specific heat at constant volume $c_v \in C^1$); the heat sources $r \in L^2(\Omega)$ with $\| r \|_0$ small enough and $\bar{\theta} \in H^{3/2}(\partial\Omega)$ with $\| \bar{\theta} - \theta_1 \|_{3/2}$ small enough. Then there exists a (locally) unique solution $(v, \rho, \theta) \in H^3(\Omega) \times H^2(\Omega) \times H^2(\Omega)$ to the non-linear problem. If $r \in H^1(\Omega)$ and $\bar{\theta} \in H^{5/2}(\partial\Omega)$, then it is easily seen that $\theta \in H^3(\Omega)$. ■

ACKNOWLEDGEMENT

It is a pleasure to thank Prof. Giovanni Prodi for many useful conversations about the subject of this paper.