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ABSTRACT. — It is proved that if $u$ is a $C^2(\mathbb{R}^n \sim \Omega)$ solution of the minimal surface equation, if $\Omega$ is bounded, and if $n \leq 7$, then $Du(x)$ has a limit (in $\mathbb{R}^n$) as $|x| \to \infty$. This extends a result of L. Bers for the case $n = 2$. The result here is actually obtained as a special application of a more general result valid for all $n$.

Key-words: Minimal surface, Tangent cone at $\infty$.

RESUME. — On démontre que si $u$ est une solution $C^2(\mathbb{R}^n \sim \Omega)$ de l’équation de la surface minimale, si $\Omega$ est borné $n \leq 7$, alors $Du(x)$ a une limite (dans $\mathbb{R}^n$) telle que $|x| \to \infty$. Ceci étend un résultat de L. Bers dans le cas $n = 2$. Notre résultat est en fait un corollaire d’un résultat plus général, valable quel que soit $n$.

A well known result of L. Bers [BL] says that if $u$ is a $C^2$ solution of the minimal surface equation over $\mathbb{R}^2 \sim \Omega$, where $\Omega$ is a bounded open subset of $\mathbb{R}^2$, then $Du(x)$ has a limit $a \in \mathbb{R}^2$ as $|x| \to \infty$. A more geometric proof, valid for a solution $u$ of any equation of minimal surface type, was given in [SL5].

Here we want to show that Bers’ original result is also valid in dimension $n$, $3 \leq n \leq 7$; specifically, we shall prove

THEOREM 1. — If $u$ is a $C^2$ solution of the minimal surface equation over $\mathbb{R}^n \sim \Omega$, $\Omega$ bounded open in $\mathbb{R}^n$, $3 \leq n \leq 7$, then $Du(x)$ is bounded and has a limit as $|x| \to \infty$. 
(Since $n \geq 3$ it in fact follows from this that there is a constant $b$ such that
\[
\lim_{\rho \to \infty} |u - l|_{C^2(\mathbb{R}^n \sim B, \rho)} = 0,
\]
where $l(x) = a \cdot x + b$, $a = \lim_{|x| \to \infty} Du(x)$, and $B_\rho = \{ x \in \mathbb{R}^n : |x| < \rho \}$.)

Of course in case $\Omega = \mathbb{R}$, the fact that $Du$ is bounded implies that it is
constant (so that $u$ is linear + constant), because each partial derivative $D_i u$
satisfies a uniformly elliptic divergence-form equation. (Cf. [MJ] [BDM].)
Thus Theorem 1 may be viewed as an extension of this "Bernstein" result
(for $\Omega = \phi$, $3 \leq n \leq 7$), which was due originally to Bernstein, Fleming,
De Giorgi, Almgren, and J. Simons (see [SJ]).

We actually here derive Theorem 1 as a special consequence of a more
general result, valid in all dimensions $n \geq 3$. Specifically we shall prove
(in § 2, 3 below):

**Theorem 2.** — If $u$ is a $C^2$ solution of the minimal surface equation on
$\mathbb{R}^n \sim \Omega$, $\Omega$ bounded, then either $Du(x)$ is bounded and has a limit as $|x| \to \infty$
or else all tangent cones of graph $u$ at $\infty$ are cylinders of the form $C \times \mathbb{R}$,
where $C$ is an $(n - 1)$-dimensional minimizing cone in $\mathbb{R}^n$ with $\partial C = 0$
and with $0 \in \text{sing } C$. (In particular $\text{spt } C$ is not a hyperplane in this latter case.)

For the meaning of "tangent cone at $\infty \)\), we refer to § 1 below. Of
course here $\text{spt } C$ is the support of $C$ and $\text{sing } C$ (the singular set of $C$) is
the set of points $\xi \in \text{spt } C$ such that $\text{spt } C \cap B_\sigma(\xi)$ fails to be an embedded $C^2$
submanifold for each $\sigma > 0$. It will also be shown in § 2, 3 that $C$ has the
form $C = \partial[\text{V}]$, with $\text{V}$ an open conical domain in $\mathbb{R}^n$. (That is, $\text{V}$ is open
in $\mathbb{R}^n$ and $\text{V} = \{ \lambda y : y \in \text{V} \}$ for each $\lambda > 0$.)

Notice that Theorem 1 follows immediately from Theorem 2 because
there are no $(n - 1)$-dimensional minimizing cones $C$ in $\mathbb{R}^n$ with $\partial C = 0$
and $0 \in \text{sing } C$ for $3 \leq n \leq 7$. (Indeed the regularity theory for minimizing
currents guarantees that $\text{sing } T = \phi$ whenever $T$ is an $(n - 1)$-dimensional
mass minimizing current with $\partial T = 0$ and $n \leq 7$; see e.g. [FH, 5.3.18]
or [SL1, § 37].)

§ 1. **Preliminaries, Tangent Cones at $\infty$**

In this section $n \geq 3$ is arbitrary and throughout we assume that $u$
is a $C^2(\mathbb{R}^n \sim \Omega)$ solution of the minimal surface equation
\[
(*) \quad \sum_{i,j=1}^n \left( \delta_{ij} - (1 + |\text{grad } u|^2)^{-1}(D_i u)(D_j u) \right) D_i D_j u = 0,
\]
with $\Omega$ a bounded open subset of $\mathbb{R}^n$.
As a preliminary result, we establish the following lemma.

1.1. LEMMA. — Either $|Du|_\infty$ is bounded on $\mathbb{R}^n \sim \Omega$ or else

$$\lim_{j \to \infty} (\rho_j^{-1} \sup_{B_{\rho_j} \sim \Omega} |u|) = \infty$$

for each sequence $\{\rho_j\} \uparrow \infty$.

(Here, and subsequently, $B_\rho$ is the open ball of radius $\rho$ and centre $0$ in $\mathbb{R}^n$.)

Proof. — Suppose there is a sequence $\{\rho_j\} \uparrow \infty$ with

$$\sup_{j \geq 1} (\rho_j^{-1} \sup_{B_{\rho_j} \sim \Omega} |u|) < \infty.$$ 

By the standard gradient estimates for solutions of the minimal surface equation (the version of [SL6; Theorem 1] is particularly convenient here, because $|Du|_\infty$ by the assumption that $u$ is $C^2$ on $\mathbb{R}^n \sim \Omega$), we have

$$\sup_{j \geq 1} \sup_{B_{\rho_j} \sim \Omega} |Du| < \infty;$$

that is, $\sup |Du| < \infty$ as required. ■

Next we note that (since (*) asserts exactly that $G = \text{graph } u$ has zero mean curvature) we have the formula (see [SL2] or [MS] or [AW] for discussion)

$$\mathcal{H}^n G = 0,$$

where $\nabla_i = e_i \cdot \nabla$, $\nabla$ = gradient operator on $G$, $\phi^1, \ldots, \phi^{n+1} \in C^1_c (\mathbb{R}^{n+1} \sim \overline{\Omega} \times \mathbb{R})$. Notice that if $\nu$ is the upward unit normal for $G$ and if $f$ is $C^1$ in some neighbourhood of $G$, then

$$\nabla_i (x) \mathcal{H}^n G = \sum_{j=1}^{n+1} (\delta_{ij} - \nu_i(x) \nu_j(x)) D_j f(x), \quad x \in G,$$

where $D_j f = \partial f / \partial x^j$ are the usual partial derivatives of $f$ taken in $\mathbb{R}^{n+1}$.

We also have the standard fact (see e. g. [SL2, § 3]) that

1.4 $G$ is mass minimizing in $\mathbb{R}^{n+1} \sim (\overline{\Omega} \times \mathbb{R}),$

in the sense that if we equip $G$ with a smooth orientation, so that it becomes a multiplicity 1 current, then

1.5 $M(G \downarrow W) \leq M(T \downarrow W)$

for any open $W \subset \subset \mathbb{R}^{n+1} \sim (\overline{\Omega} \times \mathbb{R})$ and for any integer multiplicity locally rectifiable current $T$ in $\mathbb{R}^{n+1}$ with $(\partial T) \downarrow W = 0$ and $\text{spt } (T - G) \subset \subset W$.

Next we recall that (from 1.2—see e. g. [GT, Ch. 16] and note
that the arguments easily modify to take account of the fact that we need
\(\text{spt } \phi^j \cap (\Omega \times \mathbb{R}) = \emptyset\) in 1.2) there are the volume bounds.

1.6
\[ \mathcal{H}^n(G \cap B_\rho(y)) \leq c \rho^n, \quad 1 \leq \rho < \infty, \]
for suitable constant \(c\), where \(B_\rho(y)\) is the ball of radius \(\rho\) and centre \(y\)
in \(\mathbb{R}^{n+1}(\rho, y\text{ arbitrary}).

Recall also that one of the versions of the monotonicity formula can be written

1.7
\[ \mathcal{H}^n(G \cap B_\rho \sim B_\sigma) = \rho b_\rho - \sigma b_\sigma, \quad R_0 \leq \sigma < \rho < \infty, \]
where \(R_0\) is large enough to ensure \(\partial G (\equiv \text{graph}(u | \partial \Omega)) \subset B_{R_0}\) (all balls have centre 0 unless explicitly indicated otherwise), and where

\[ b_\rho = \frac{d}{d\rho} \int_{G \cap B_\rho} |\nabla r|^2 d\mathcal{H}^n = \int_{G \cap \partial B_\rho} |\nabla r| d\mathcal{H}^{n-1}, \]
with \(r(x) \equiv |x|\). (The last equality follows from the co-area formula.)

The identity 1.7 follows from 1.2 simply by substituting \(\phi'(x) = \psi(r)x'\) in 1.2, and then letting \(\psi\) approach the characteristic function of the interval \((\sigma, \rho)\). Notice that 1.7 (with \(\sigma = R_0\)) can be written (since \(|\nabla r|^2 = 1 - (x.r)^2/r^2\))

\[ \frac{d}{d\rho} (\rho^{-n}\mathcal{H}^n(G \cap B_\rho \sim B_{R_0})) = \frac{d}{d\rho} \int_{G \cap B_\rho} \frac{(x.r)^2}{r^{n+2}} d\mathcal{H}^n - \rho^{-n-1}R_0 b_{R_0} \]
(in the sense of distributions) for \(\rho > R_0\), so that by integration we have, for \(\rho > R \geq R_0\),

1.8
\[ \rho^{-n}\mathcal{H}^n(G \cap B_\rho \sim B_{R_0}) - R^{-n}\mathcal{H}^n(G \cap B_R \sim B_{R_0}) = \int_{G \cap B_\rho \sim B_R} \frac{(x.r)^2}{r^{n+2}} d\mathcal{H}^n + E(\rho, R), \]
where
\[ |E(\rho, R)| \leq c R^{-n}. \]
c independent of \(\rho, R\). (Cf. the standard monotonicity identities of \([\text{AW}], [\text{MS}], [\text{SL1}]\).)

Now for \(\lambda > 0\) we let \(u_\lambda\) be the scaled function \(u_\lambda(x) = \lambda u(\lambda^{-1}x), x \in \mathbb{R}^n \sim \Omega_\lambda, \Omega_\lambda = \{ \lambda y: y \in \Omega \}\), and let \(G_\lambda\) be the graph of \(u_\lambda\), so that viewing \(G_\lambda\) (equipped with an appropriate orientation) as a current, we may write

\[ G_\lambda = \mathcal{C}^\mathcal{C}_{U_\lambda} \mathcal{L} (\mathbb{R}^{n+1} \sim (\Omega_\lambda \times \mathbb{R})). \]

where \(U_\lambda = \{ (x, y): y > u_\lambda(x), x \in \mathbb{R}^n \sim \Omega_\lambda \}\), and where \([U_\lambda]\) denotes the current obtained by integration of \((n+1)\)-forms \(\omega \in \mathcal{D}^{n+1}(\mathbb{R}^{n+1})\) over \(U_\lambda\).

By virtue of 1.5, 1.6 we can conclude from standard compactness.

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results (see e. g. [FH] or [SL1, Ch. 7]) that for any sequence \( \{ \lambda_j \} \downarrow 0 \) there is a subsequence \( \{ \lambda_{j'} \} \) and a current \( T = \partial \mu[U] \) such that

\[
1.9 \quad T = \lim G_{\lambda_{j'}}.
\]

in the weak sense of currents, and in the sense that \( U_{\lambda_{j'}} \) converges to \( U \) in the \( \text{L}^1_{\text{loc}}(\mathbb{R}^{n+1}) \) sense,

\[
1.10 \quad \mathcal{H}^n \llcorner G_{\lambda_{j'}} \to \mathcal{H}^n \llcorner S
\]

in the sense of Radon measures on \( \mathbb{R}^{n+1} \), where

\[
S = \{ x \in \mathbb{R}^{n+1} : \limsup_{\rho \downarrow 0} \rho^{-n} \mu(T \llcorner B_\rho(x)) > 0 \},
\]

\[
1.11 \quad G_{\lambda_{j'}} \to \text{spt} \ T
\]

locally in the Hausdorff distance sense in \( \mathbb{R}^{n+1} \sim (\{ 0 \} \times \mathbb{R}) \),

\[
1.12 \quad T \text{ is minimizing in } \mathbb{R}^{n+1} \sim (\{ 0 \} \times \mathbb{R}),
\]

\[
1.13 \quad \mu(T \llcorner B_\rho(y)) \leq c \rho^n, \quad \rho > 0, \quad y \in \mathbb{R}^{n+1}
\]

(notice that this includes \( y \in (\{ 0 \} \times \mathbb{R}) \)).

Since \( n \geq 3 \) it is easy to check that 1.12 and 1.13 imply

\[
1.14 \quad T \text{ is minimizing in } \mathbb{R}^{n+1}.
\]

It is also standard that then (since \( T = \partial \mu[U] \) implies that \( T \) has multiplicity 1 \( \mathcal{H}^n \) a.e. in \( S \) (\( S \) as in 1.10), and since the density function of a minimizing current is upper semi-continuous—see e. g. [FH] or [SL1, Ch. 7]) \( \liminf_{\rho \downarrow 0} \omega_\rho^{-1} \rho^{-n} \mu(T \llcorner B_\rho(y)) \geq 1 \) at each point of \( \text{spt} \ T \), and hence

\[
1.15 \quad S = \text{spt} \ T \quad (S \text{ as in 1.10})
\]

and we can (and shall) take \( U \) to be open with

\[
1.16 \quad \text{spt} \ T = \partial U.
\]

From the De Giorgi regularity theorem (see e. g. [SL1, § 24] or [G]) we have furthermore that for each \( y \in \text{spt} \ T \) with \( \lim_{\rho \downarrow 0} \omega_\rho^{-1} \rho^{-n} \mu(T \llcorner B_\rho(y)) = 1 \) there is \( \sigma > 0 \) such that

\[
1.17 \quad \text{spt} \ T \cap B_\sigma(y) = \partial U \cap B_\sigma(y) \text{ is an embedded } C^\infty \text{ submanifold of } \mathbb{R}^{n+1}.
\]

This guarantees in particular that the points of sing \( T \) (i. e. the points \( y \in \text{spt} \ T \) such that 1.17 fails for each \( \sigma > 0 \)) form a closed set of \( \mathcal{H}^n \)-measure zero.

Finally we note that \( T \) is a cone; that is, if \( \eta \) is any homothety \( x \mapsto \lambda x \) (\( \lambda > 0 \) fixed), then \( \eta \# T = T \). Indeed using 1.8, 1.10, 1.15 it is easy to see that

\[
\rho^{-n} \mu(B_\rho) = \sigma^{-n} \mu(B_\sigma), \quad 0 < \sigma < \rho < \infty,
\]
where $\mu = \mathcal{H}^n \lfloor \text{spt } T$, and then (since $T$ is minimizing) that an identity like 1.8 holds for $T$ with $R_o = 0$ and $E \equiv 0$, thus giving

$$\int_{\text{spt } T} (v \cdot x)^2 d\mu(x) = 0.$$ 

where $v$ is the unit normal of $\partial U$ (which is well defined on $\text{reg } T = \text{spt } T \sim \text{sing } T$). The fact that $\eta \cdot T = T$ for any homothety $\eta$ now readily follows from this and the homotopy formula for currents. (See for example [SL1, Ch. 7] or [G] for similar arguments.)

Subsequently, any $T$ obtained as described above will be called a tangent cone for graph $u$ at infinity. In case $|Du|$ is bounded we can prove that there is a unique such $T$ and it is a hyperplane. In fact we have the following result:

1.18. Lemma. — If $Du$ is bounded (see Lemma 1.1) then it has a limit at infinity.

Proof. — Since $|Du|$ is bounded, every tangent cone $T$ of graph $u$ at $\infty$ (obtained as above) is the graph of a Lipschitz weak solution of the minimal surface equation. From standard elliptic regularity theory, such solutions are smooth (see e.g. [GT, Ch. 13]). Hence since the graphs of these solutions are cones, they must all be linear functions.

It follows that for any given $\varepsilon > 0$ there is $R(\varepsilon) \geq 1$ such that if $R \geq R(\varepsilon)$ then there is a linear function $l$ (possibly depending on $R$) such that

$$\sup_{B_R^n \setminus B_{R/2}^n} |u - l| \leq \varepsilon R.$$ 

Combining this with the Schauder theory ([GT, Ch. 6]), applied to $u - l$ (which we may do since $u - l$ satisfies a linear elliptic equation with coefficients having finite $C^1$ norm—this follows from the fact that the $C^2$ norm of $u$ is finite). we deduce

$$\sup_{B_R^n \setminus B_{R/2}^n} D^2 u = \sup_{B_R^n \setminus B_{R/2}^n} |D^2(u - l)| \leq c\varepsilon R.$$ 

Hence, by integration along paths in $B_R^n \sim B_{R/2}^n$, we have

$$|Du(x) - Du(y)| \leq c\varepsilon \quad \forall x, y \in B_R^n \sim B_{R/2}^n.$$ 

On the other hand each component $\phi = D_{jk}u$ of $Du$ satisfies an equation of the form

$$D_i(a_{ik}D_k\phi) = 0,$$

where

$$a_{ik} = (1 + |Du|^2)^{-\frac{1}{2}}(\delta_{ik} - (1 + |Du|^2)^{-1}(D_iu)(D_ku))$$

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(as we see by differentiating the divergence form version
\[ \sum_{i=1}^{n} D_i(D_j u/\sqrt{1 + |Du|^2}) = 0 \]
of the minimal surface equation). In particular $D_j u$ satisfies a maximum/minimum principle on bounded domains in $\mathbb{R}^n \sim \overline{\Omega}$; then in view of the arbitrariness of $\varepsilon$ in (1), it follows that $\lim_{j \to \infty} D_j u$ exists.

\section*{§ 2. TANGENT CYLINDERS AT $\infty$}

In this section we show that, unless $Du$ is bounded as $|x| \to \infty$, every tangent cone $T$ of graph $u$ at $\infty$ (obtained as described in § 1) is a vertical cylinder:

\begin{equation}
T = C \times \mathbb{R},
\end{equation}

where $C = \partial [V]$, $V$ open in $\mathbb{R}^n$ with $\partial V = \text{spt} C$, and where $C$ is minimizing in $\mathbb{R}^n$.

In case $\Omega = \emptyset$ this was already known (a proof appears in [MM] for example). The extension here to case $\Omega \neq \emptyset$ is given mainly for the reader's convenience, since no really new ideas are involved. Note however that the fact (observed by Fleming in case $\Omega = \emptyset$) that $\text{sing } C \sim \mathbb{R}$ is not so easy to prove in case $\Omega \neq \emptyset$; this will be done in § 3.

We first note that (in the notation of § 1) by 1.9 and 1.15

\begin{equation}
\lim_{j \to \infty} \int_{G_{\lambda_j}} v_{f_j} \cdot \phi d \mathcal{H}^n = \int_{\text{spt } T} v \cdot \phi d \mathcal{H}^n
\end{equation}

for each fixed $\phi = (\phi^1, \ldots, \phi^{n+1}) \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, where $v_j$ is the upward pointing unit normal for $G_{\lambda_j}$ and where $v$ is the outward pointing normal of $\partial U$ at regular points of $\partial U = \text{spt } T$. (U as in 1.16.) Thus in particular (since $v_{f_j} e_{n+1} > 0$ on $G_{\lambda_j}$) we have

\begin{equation}
v \cdot e_{n+1} \geq 0 \quad \text{on } \text{reg } T.
\end{equation}

We already remarked in § 1 that $\mathcal{H}^n(\text{sing } T) = 0$. We also need to recall the further regularity theory for minimizing currents $T = \partial [U]$:

\begin{equation}
\begin{cases}
\text{sing } T = \phi, & 3 \leq n \leq 6 \\
\text{sing } T \text{ is discrete, } & n = 7 \\
\mathcal{H}^{n-7+a}(\text{sing } T) = 0 & \forall \alpha > 0 \quad \text{in case } n \geq 8
\end{cases}
\end{equation}
(See e.g. [G] or [SL1, Ch. 7]), so that, since $T$ is a cone, in particular

\[
\begin{align*}
T \text{ is a hyperplane,} & \quad n \leq 6 \\
\text{Sing } T \subset \{0\}, & \quad n = 7.
\end{align*}
\]

Notice that in case $n = 7$ we then also trivially have that \( \text{reg } T \) is connected; otherwise \( \text{reg } T \cap S^n \) would contain smooth compact disjoint embedded minimal surfaces \( \Sigma_1, \Sigma_2, \) and we could rotate \( \Sigma_1 \) until it touched \( \Sigma_2, \) thus contradicting the Hopf maximum principle. (Actually \( \text{reg } T \) is connected for all \( n \), by a result Bombieri and Giusti [BG].)

Next we claim, under the present assumption that \( |Du| \) is not bounded, that

\[ L \subseteq \text{spt } T, \]

where \( L \) is a vertical ray from 0; either \( L = \{ \hat{x} e_{n+1} : \hat{z} > 0 \} \) or \( \{ \hat{x} e_{n+1} : \hat{z} < 0 \} \). To see this we note that if we let \( u_j = u_{x_j} \), then for each fixed \( \sigma > 0 \),

\[ \lim_{j \to \infty} \sup_{|x| = \sigma} |u_j| = \infty, \]

because otherwise Lemma 1.1 tells us that \( |Du| \) is bounded on \( \mathbb{R}^n \sim \Omega \), contrary to hypothesis. Thus 2.7 is established, and 2.6 clearly follows from this due to 1.11 and the fact that \( \lim \inf_{|x| > J} |u| < \infty \) by a standard barrier argument involving the catenoid.

Now we use the standard fact that \( \Delta v \cdot e_{n+1} + |A|^2 v \cdot e_{n+1} = 0 \) on \( \text{reg } T \), where \( A \) is the second fundamental form of \( \text{reg } T \), so that by 2.3

\[ \Delta v \cdot e_{n+1} \leq 0 \quad \text{on } \text{reg } T. \]

In case \( n \leq 7 \) we can use connectedness of \( \text{reg } T \), 2.3, 2.5, 2.8 and 2.6 (which guarantees that \( v \cdot e_{n+1} = 0 \) at some points of \( \text{reg } T \)) to deduce by the Hopf maximum principle that \( v \cdot e_{n+1} \equiv 0 \) on \( \text{reg } T \). Hence, again using 2.5, we have 2.1 as required.

In case \( n \geq 8 \), the argument is only slightly more complicated: by [BG] and 2.8 we have

\[ (\ast) \quad \inf_{\text{reg } T \cap B_{\rho}(y)} e_{n+1} \cdot v \geq \rho^{-n} \int_{\text{reg } T \cap B_{\rho}(y)} e_{n+1} \cdot v d\mathcal{H}^n \]

for any \( \rho > 0 \) and \( y \in \text{spt } T \). However we showed above that \( \text{spt } T \) contains a vertical \( \frac{1}{2} \)-line, and evidently \( \inf_{\text{reg } T \cap B_{\rho}(y)} e_{n+1} \cdot v = 0 \) for any \( y \) in this \( \frac{1}{2} \)-line and any \( \rho > 0 \), thus by (\ast) \( e_{n+1} \cdot v \equiv 0 \).

Thus we have established \( v \cdot e_{n+1} \equiv 0 \) on \( \text{reg } T \). Since \( \partial T = 0 \) it then easily follows (e.g. by using the homotopy formula for currents), that \( T \) is invariant under translations parallel to \( e_{n+1} \). Thus (with \( U \) as in 1.16)
§ 3. PROOF OF THEOREM 2

In view of Lemmas 1.1, 1.17 and the fact 2.1, Theorem 2 of the introduction will be proved if we can establish that

3.1 \text{sing } C \neq \emptyset \quad \text{(i.e. } C \text{ is not a hyperplane)}

for any \( C \) as in 2.1.

Suppose for contradiction that \( C \times \mathbb{R} \), as in 2.1, is indeed a hyperplane \( H \) in \( \mathbb{R}^{n+1} \). We first claim that in this case \( H \) is the unique tangent cone for graph \( u \) at \( \infty \) and that in fact, if \( \eta \) is a unit normal for \( H \), there exist \( R_2 > R_1 \) such that

3.2 \[ G \sim B_{R_2} = \{ x + h(x)\eta \colon x \in H \sim B_{R_1} \} \sim B_{R_2}, \]

with \( h \in C^2(H \sim B_{R_1}) \) satisfying

3.3 \[ |h(x)| + |x| + |Dh(x)| \leq c \cdot |x|^{1-\alpha}, \quad x \in H \sim B_{R_1}, \]

for some constant \( \alpha > 0 \). This is actually a special case of the general unique tangent cone result of [AA]. For a somewhat simpler proof, see [SL3, II, § 6].

Now suppose without loss of generality that \( e_n = (0, 0, \ldots, 0, 1, 0) \) is normal to \( H \) (so that we can take \( \eta = e_n \) in 3.2), and introduce new coordinates \( (y', \ldots, y^{n+1}) \) for \( \mathbb{R}^{n+1} \) according to the transformation \( Q \) given by

\[
\begin{align*}
y' &= x' \quad (y' = (y', \ldots, y^{n-1}), \quad x' = (x^1, \ldots, x^{n-1})) \\
y^n &= x^{n+1} \\
y^{n+1} &= x^n.
\end{align*}
\]

Then for suitable compact \( K \) and suitable \( R \) we have

\[ G \sim K = Q \left( \text{graph } h \mid \mathbb{R}^n \sim B_R \right), \]

so that we have a diffeomorphism \( \psi \colon \mathbb{R}^n \sim \tilde{K} \rightarrow \mathbb{R}^n \sim B^n_R \),

\[ x = (x^1, \ldots, x^n) \mapsto y = (y^1, \ldots, y^n), \]

defined by \( y' = x' \), \( y^n = u(x', x^n) \), where \( \tilde{K} = \mathbb{R}^n \sim \pi(G \sim K) \), \( \pi \) the projection taking \( z \in \mathbb{R}^{n+1} \) onto its first \( n \)-coordinates. The inverse is given by \( x' = y', x^n = h(y', y^n) \), so in particular we have \( \frac{\partial u(x)}{\partial x^n} \frac{\partial h(y)}{\partial y^n} \equiv 1 \) for \( x \in \mathbb{R}^n \sim \tilde{K} \).
\( y = \psi(x) \). Assuming without loss of generality that \( u(te_n) \to \infty \) (rather than \(-\infty\)) as \( t \to \infty \), we thus deduce

\[
\frac{\partial h(y)}{\partial y^n} > 0, \quad y \in \mathbb{R}^n \sim B^R_y.
\]

Similarly if \( G^- = \text{graph} (-u) \), then we have, for suitable compact \( K_1 \) and \( R > 0 \), that

\[
G^- \sim K_1 = Q(\text{graph} h^-) \quad \text{where} \quad h^- \in C^2(H \sim B^R_R).
\]

Notice that, for suitably large \( \rho > R \) and any \( c \in \mathbb{R} \),

\[
\{ (y', h(y', c)) : |(y', c)| > \rho \} \quad \text{and} \quad \{ (y', h^-(y', c)) : |(y', c)| > \rho \}
\]

coincide with \( \{ x : |(x', c)| > \rho, u(x) = c \} \) and \( \{ x : |(x', c)| > \rho, u(x) = -c \} \)
espectively, so that

\[
h(y', y^n) = h^-(y', -y^n), \quad |y| > \rho.
\]

Writing

\[
w(y) = h(y) - h^-(y), \quad |y| > \rho,
\]

we see that (using 3.3, 3.6 and the fact that \( h, h^- \) satisfy the minimal surface equation)

\[
\Delta w = \text{div} (A \cdot Dw), \quad |y| > \rho,
\]

where the matrix \( A \) is smooth and

\[
|A| + |DA| |y| \leq c |y|^{-\alpha}, \quad |y| > \rho.
\]

Also by 3.4, 3.6 we have

\[
\frac{\partial w(y)}{\partial y^n} > 0, \quad |y| > \rho
\]

and

\[
w(y', y^n) = -w(y', -y^n).
\]

Now let \( \{ t_j \} \uparrow \infty \) be arbitrary, and define

\[
w_j(y) = \frac{h(t_j y)}{h(t_j e_n)}.
\]

Since \( w(y) > 0 \) for \( y^n > 0 \) (by 3.9, 3.10), in view of 3.7, 3.8, 3.9, 3.10 we can use Harnack's inequality and Schauder estimates in order to deduce that there is a subsequence \( \{ t_{j_k} \} \) such that

\[
w_{j_k} \to w_* \text{ locally in } C^1 \text{ on } \mathbb{R}^n \sim \{ 0 \},
\]

where \( w_* \) is harmonic on \( \mathbb{R}^n \sim \{ 0 \}, \hat{\partial}w_*/\hat{\partial}y^n \geq 0 \), and \( w_*(y', y^n) = -w_*(y', -y^n) \), \( y \neq 0 \). Thus \( w_* \) is bounded on \( B_1(0) \sim \{ 0 \} \), and hence the singularity at 0 is removable; that is, \( w_* \) extends to a harmonic function on \( \mathbb{R}^n \). But then
\[ \frac{\partial w_*/\partial y^n}{\partial y^n} \text{ extends to be a non-negative harmonic function on all of } \mathbb{R}^n \]

and hence must be constant by Liouville's theorem. Thus \( w_*(y', y^n) = cy^n \)

for some constant \( c \). Since \( w_*(e_n) = 1 \) (by construction) we then have

\[ w_*(y', y^n) = y^n. \]

In view of the arbitrariness of the sequence \( \{ t_j \} \) in the above argument, it follows that for each given \( \varepsilon > 0 \) there is a \( T = T(\varepsilon) \geq \rho \) such that

\[ w(2te_n) \geq 2(1 - \varepsilon)w(te_n) \]

for each \( t \geq T \). Taking \( t = 2T(\varepsilon) \) for \( \varepsilon \) small, and iterating, we then deduce that for any given \( \beta \in (0, 1) \) there is \( c = c(\beta) \) such that

\[ w(te_n) \geq ct^{1-\beta}, \quad t \geq T. \]

However, taking \( \beta < \alpha \) (\( \alpha \) as in 3.3), this contradicts 3.3. This completes the proof of Theorem 2.

REFERENCES


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