Claude Viterbo

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A proof of Weinstein’s conjecture
in $\mathbb{R}^{2n}$

by

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ABSTRACT. — We prove that a hypersurface of contact type in
$(\mathbb{R}^{2n}, \sum dx_i \wedge dy^i)$ has a closed characteristic. A geometric trick is used to
reduce this problem to finding $T$-periodic solutions of a Hamiltonian
system. This system is studied using the Clarke-Ekeland-Lasry dual action
functional.

RÉSUMÉ. — On démontre que toute hypersurface de genre contact de
$(\mathbb{R}^{2n}, \sum dx_i \wedge dy^i)$admet au moins une caractéristique fermée. Une astuce
géométrique ramène notre problème à la recherche d’orbites $T$ périodiques
d’un système hamiltonien. Ce système est analysé en utilisant la fonctionnelle
d’action duale de Clarke-Ekeland-Lasry.
In his paper "On the hypothesis of Rabinowitz' periodic orbit theorem" (denoted by [W. 2] in the sequel) A. Weinstein made the following conjecture.

**Conjecture.** If \( E \subset (M, \omega) \) is a compact hypersurface of contact type in a symplectic manifold, satisfying \( H^1(\Sigma; \mathbb{R}) = 0 \), then \( \Sigma \) has a closed characteristic.

The definition of being of contact type is given in section one (Definition 1.1). We recall that a characteristic is a curve everywhere tangent to the line field \( \ker \omega|_\Sigma \).

As the reader might have hinted from the title, our aim is to prove this conjecture for \((\mathbb{R}^{2n}, \omega_0)\). Thus, we state.

**Theorem.** If \( \Sigma \subset (\mathbb{R}^{2n}, \omega_0) \) is a compact hypersurface of contact type, then \( \Sigma \) has at least one closed characteristic. \( \square \)

Let us mention that we dropped the hypothesis \( H^1(\Sigma; \mathbb{R}) = 0 \); for our proof we only need that \( \Sigma \) has an interior in \( \mathbb{R}^{2n} \), which is automatic.

Recall that if \( J \) is the standard symplectic matrix, and \( N(x) \) denotes the outward normal to \( \Sigma \) at \( x \), then the closed characteristics of \( \Sigma \) correspond to periodic solutions of

\[
\begin{align*}
\dot{x} &= JN(x) \\
x(0) &= x(T)
\end{align*}
\]

(\( \mathcal{N} \))

The standard approach to (\( \mathcal{N} \)) is to transform it into a fixed period Hamiltonian system, that is, to find a function \( H \) on \( \mathbb{R}^{2n} \) such that the non-trivial solutions of

\[
\begin{align*}
\dot{x} &= J \nabla H(x) \\
x(0) &= x(T)
\end{align*}
\]

(\( \mathcal{H} \))
correspond to periodic solutions of (\( \mathcal{N} \)) (here \( T \) is fixed).

Let us recall shortly the historical background. In 1948, Seifert proved existence of closed characteristics for some special class of convex hypersurfaces (cf. [S]). Thirty years later, Weinstein (cf. [W. 1]) extended this result to general \( C^2 \) convex hypersurfaces, and Rabinowitz (in [R]) to strictly starshaped hypersurfaces. Let us mention also the work of Bahri (cf. [B. 2] or a sketchy description in [B. 1]).
Rabinowitz' idea to construct $H$ is, assuming $\Sigma$ is star-shaped with respect to the origin, to set

$$H = 1 \text{ on } \Sigma$$

$$H(\lambda, x) = \varphi(\lambda) \text{ for all } x \in \Sigma, \lambda \geq 0$$

where $\varphi$ is some well chosen function.

In chapter one, by a modification of this idea we get a function $H$ such that non-trivial solutions of $(\mathcal{M})$ yield periodic solutions of $(\mathcal{N})$.

In chapter two, we define the dual action functional, according to an idea of Clarke and Ekeland (cf. [C-E]) later modified by Berestycki, Lasry, Mancini and Ruf (cf. [B-L-M-R]).

The finite dimensional reduction seems to be needed in order to prove the Palais-Smale condition: all the known proofs of the (P.S.) condition use the fact that $\nabla H(z). z \geq \alpha H(z) > 0$ for some positive $\alpha$, that is the level hypersurfaces of $H$ are strictly starshaped. Let us remark that this condition is needed whether the direct action functional or the dual one are used.

Chapter three is concerned with the proof of the (P.S.) condition for the finite dimensional reduction.

Finally by a cohomological argument, we prove that our functional has non-trivial critical points, that is done in chapter four.

I am glad to thank Leila Lassoued for interesting discussions during a stay at the University of Tunis. François Laudenbach for introducing me to symplectic geometry, and for attempting (sometimes unsuccessfully as it is the case in this paper) to get from me "geometric proofs". Abbas Bahri for useful comments, Helmut Hofer for reading the manuscript, finding mistakes, and simplifying the proof (cf. [H-Z]).

And of course special thanks to Ivar Ekeland. He introduced me to Hamiltonian systems. The reader will easily trace his influence in this work. Let me mention how enjoyable it is to work under his direction.

NOTATIONS
AND STANDARD DEFINITIONS

$\langle . , . \rangle; | . |$ scalar product and norm in euclidean space;

$\langle . , . \rangle; \| . \|$ scalar product and norm in $L^2$ space;

"strictly convex function": $f$ such that
\[
\forall x, y \in \mathbb{R}^n \ (f'(x) - f'(y), x - y) \geq \epsilon |x - y|^2
\]
for some positive $\epsilon$;

"conformal diffeomorphism": $\varphi$ such that $\varphi^* \omega = c \omega$ ($c$ a non-zero constant) $H_*(A), H^*(A)$: homology group, cohomology ring of $A$ (rational coefficients) $H^1_*, \star(A), H^\sharp_\star(A)$: equivariant homology group and cohomology ring of $A$ in the sense of Borel (cf. [Bo]) (with rational coefficients);

$[x]$ : integer part of $x$;
$\Box$ : end of a statement;
$\circ$ : end of a proof.

1. REDUCTION
TO A HAMILTONIAN SYSTEM IN $\mathbb{R}^{2n}$

Let $(M^{2n}, \omega)$ be a symplectic manifold, $\Sigma$ a hypersurface of contact type of $(M^{2n}, \omega)$, that is:

DEFINITION 1.1. — $\Sigma$ is said to be of contact type if and only if there is a 1-form $\theta$ on $\Sigma$ such that:

(i) $d\theta = j^* \omega$ (where $j: \Sigma \to M$ is the inclusion map);
(ii) $\theta \wedge (d\theta)^{n-1}$ is a volume form on $\Sigma$ (i.e. does not vanish on $\Sigma$). \hfill \Box

We now have (cf. [W. 2], p. 354, Lemma 2).

LEMMA 1.2. — $\Sigma$ is of contact type if and only if there is a vector field $\eta$, defined in a neighborhood of $\Sigma$, which is:

\begin{equation}
\text{transverse to } \Sigma
\end{equation}

satisfies $L_\eta \omega = \omega$. \hfill (1.4)

Proof. — Consider the form $\theta$ of Definition 1. By Poincaré's lemma, we can extend $\theta$ to $\tilde{\theta}$, defined in a neighborhood of $\Sigma$, such that $d\tilde{\theta} = \omega$.

Set $\tilde{\theta} = i_\eta \omega$ ($\eta$ exists since $\omega$ is non-degenerate), then $\eta(x)$ is not in $T_x \Sigma$ otherwise we would have $\eta(x) = dj(x) \xi$, with $\xi \in T_x \Sigma$, and

\[
\begin{align*}
    j^* (\tilde{\theta} \wedge (d\tilde{\theta})^{n-1})(x) &= j^* (i_\eta \omega^n)(x) = i_{j*} (\omega^n)(x) \\
    \end{align*}
\]

which is zero because $j^* (\omega^n)$ is zero. So $\eta$ is transverse to $\Sigma$. 

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Checking (1.4) is straightforward:

$$L_n \omega = d\iota_\eta \omega + \iota_n d\omega = d\theta = \omega.$$  

From the lemma, we infer:

**Proposition 1.5.** — *For some positive $\epsilon$, there is a symplectic diffeomorphism*

$$\varphi: (\Sigma \times ]1-\epsilon, 1+\epsilon[, d(tr^* \theta)) \to (U, \omega)$$

where $r$ (resp. $t$) is the projection of $\Sigma \times ]1-\epsilon, 1+\epsilon[$ on the first (resp. second) factor, and $U$ is a tubular neighborhood of $\Sigma$ in $(M, \omega)$.

Moreover $\varphi |_{\Sigma \times \{1\}} = id_\Sigma$.

**Proof.** — Write $\psi_s$ for the flow of $\eta$ defined in Lemma 1.2, and set

$$\varphi(x, e^s) = \psi_s(x) \quad \text{for} \quad (x, s) \in \Sigma \times ]1-\epsilon, \epsilon[. \quad (1.6)$$

By (1.3), $\varphi$ is a diffeomorphism provided $\epsilon$ is small. To show that it is symplectic, we shall compare $\varphi^*(\theta)$ and $tr^*(\theta)$ (the reader is invited to check that $d(tr^*(\theta))$ is a symplectic form. Now,

$$\left. \frac{\partial}{\partial s} \psi_s^*(\theta) = \psi_s^*(L_n \theta) = \psi_s^*(d\iota_n \theta + \iota_n d\theta). \right\}$$

From $\theta = \iota_n \omega$ and $d\theta = \omega$ we infer $\iota_n \theta = 0$ and $\iota_n d\theta = \theta$, so $\left. \frac{\partial}{\partial s} \psi_s^*(\theta) = \psi_s^*(\theta) \right\} = \psi_s^* \theta$

which implies

$$\psi_s^* \theta = e^s \theta. \quad (1.7)$$

Let us write $\psi(x, s) = \psi_s(x)$, then

$$\psi^* \theta = \theta(\psi(x, s)) d\psi(x, s)$$

$$= \theta(\psi_s(x)) \left( d\psi_s(x) + \frac{\partial}{\partial s} \psi_s(x) ds \right)$$

$$= \psi_s^* \theta + \psi_s^* (i_n \theta) ds$$

$$= \psi_s^* \theta \quad \text{as} \quad i_n \theta = 0$$

$$= e^s \theta \quad \text{by (1.5)}$$

The change of variable $t = e^s$ yields, using (1.6),

$$\varphi^* (\theta) = tr^* (\theta).$$
Finally, the last assertion of (1.5) is obvious.

We now assume the origin to be an interior point of \( \Sigma \). Our aim is to prove:

**Proposition 1.8.** — For any \( a \), positive constant, there is a \( C^\infty \) function \( H \) on \( \mathbb{R}^{2n} \) satisfying:

(i) \( H(0)>0 \) is the absolute minimum of \( H \), and \( H \) is constant in a neighborhood of the origin.

(ii) \( H'' \) is bounded.

(iii) \( H(z) \geq \frac{a}{2} |z|^2 \) for \( |z| \) large enough.

(iv) If \( (\mathcal{M}) \) has a nonconstant solution, then \( (\mathcal{N}) \) has a periodic orbit.

\( \square \)

**Proof.** — We construct \( H \) explicitly.

Let \( k>1 \) be some sufficiently large number so that \( U \) and \( k.U \) are disjoint [\( U \) is as in (1.5), \( k.U \) is the image of \( U \) by a dilation of ratio \( k \)].

This implies that the \( k^p.U \), for \( p \) positive integers, will be pairwise disjoint.

We first define \( H \) on \( \bigcup_{p \geq 1} k^p.U \) by

\[
H(k^p.\varphi(x, t)) = k^{2p}\lambda(t)
\]  

(1.9)

where \( \lambda \) is some increasing function on \( [1-\varepsilon, 1+\varepsilon] \) that shall be defined more precisely later on.

From (1.9) we can check that in \( k^p.U \) the level hypersurfaces of \( H \) are the \( k^p\psi_s(\Sigma) \). Since by (1.7) \( \psi_s \) is a conformal map, and the dilations are also conformal, the \( k^p\psi_s(\Sigma) \) are conformally diffeomorphic to \( \Sigma \). Hence on \( \bigcup_{p \geq 1} k^p.U \), (iv) holds.

Also on \( \bigcup_{p \geq 1} k^p.U \) (i) to (iii) hold provided

\[
a R^2 \geq \inf_{t} \lambda(t) = \lambda(1-\varepsilon)
\]  

(1.10)

where \( R \) is a real number such that \( U \) is contained in the ball of radius \( R \), centered at the origin.

Now let us extend \( H \) to \( kD-\Delta \) where \( D \) is the union of \( U \) and the interior of \( \Sigma \), and \( \Delta = D - U \).
We can assume that $H$ is constant on $k\Delta - \Delta$ provided $k^2 \lambda (1 - \varepsilon) = \lambda (1 + \varepsilon)$ which we will assume henceforth.

We then extend $H$ to $\mathbb{R}^{2n} - \Delta$ by setting

$$H(k^p z) = k^{2p} H(z) \quad \text{for} \quad z \in k \Delta - \Delta, \quad p \in \mathbb{N}^*.$$  \hspace{1cm} (1.11)

We finally set $H(z) = \lambda (1 - \varepsilon)$ for $z \in \Delta$. It is now easy to check properties (i) to (iv):

(i) is obvious,

(ii) follows from (1.11), for this implies

$$H''(k^p z) = H''(z) \quad \text{for} \quad z \in k \Delta - \Delta, \quad p \in \mathbb{N}^*$$

since $H''$ is continuous, it is bounded on $k \Delta - \Delta$, hence on $\mathbb{R}^{2n}$.

(iii) follows also from (1.11) for it implies

$$H(z) \geq \lambda (1 - \varepsilon) \frac{|z|^2}{k^2 \mathbb{R}^2}$$

and we can assume $\lambda (1 - \varepsilon) \geq \frac{\alpha}{2} \cdot k^2 \mathbb{R}^2$.

(iv) Consider a non-constant solution, its trajectory has to be contained in $\bigcup_{p \geq 1} k^p U$, thus yielding a periodic solution of $(\mathcal{N})$. $\bigcirc$

2. THE DUAL ACTION FUNCTIONAL AND ITS FINITE DIMENSIONAL REDUCTION

Let $H$ be a function on $\mathbb{R}^{2n}$ such that $H''$ is bounded. Then we can find some positive real number, $K$, such that $H'' + K I$ is everywhere greater than $\varepsilon I$ for some positive $\varepsilon$, thus $H_K(z) = H(z) + \frac{K}{2} |z|^2$ is strictly convex, and we can consider its dual function in the sense of Fenchel (see [E-T])

$$H_K^*(y) = \sup_{z \in \mathbb{R}^{2n}} [(z, y) - H_K(z)]$$

which main virtue is to satisfy

$$\nabla H_K^*(\nabla H_K(z)) = \nabla H_K(\nabla H_K^*(z)) = z.$$  \hspace{1cm} (2.1)
For $x \in X = W^{1,2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$, we define

$$F_K(x) = \int_0^T \left[ \frac{1}{2} (J \dot{x} - K x, x) + H_K^*(\dot{x} + K x) \right] ds \quad (2.2)$$

Assume $\frac{KT}{2\pi} \notin \mathbb{Z}$, then $x \to -J \dot{x} + K x$ is an Hilbert space isomorphism from $W^{1,2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$ to $L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$, whose inverse we denote by $M_K$. Then critical points of $F_K$ are precisely the solutions of $(\mathcal{H})$.

Our goal is to find critical points of $F_K$, but as we cannot prove that it satisfies condition (C) of Palais and Smale (to prove that this condition is satisfied, one usually needs some condition like $\nabla H(x). x \geq \gamma H(x) > 0$, hence the level hypersurfaces of $H$ are starshaped; (cf [B-L-M-R] or [R]) we shall use a finite dimensional reduction of $F_K$ that we shall now describe.

Let us first set for $u \in E = L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$, $\psi_K(u) = F_K(M_K u)$, that is

$$\psi_K(u) = \int_0^T \left[ \frac{1}{2} (-M_K u, u) + H_K^*(u) \right] ds$$

and since $M_K$ is an Hilbert space isomorphism, we can as well look for the critical points of $\psi_K$, and build a finite dimensional reduction of $\psi_K$.

The main point is to remark that $\psi_K$ is convex in the direction orthogonal to some finite dimensional vector space: consider

$$\langle \psi_K'(u) - \psi_K'(v), u - v \rangle = \langle -M_K(u - v) + \nabla H_K^*(u) - \nabla H_K^*(v), u - v \rangle$$

since $H_K^*$ is strictly convex

$$\langle \nabla H_K^*(u) - \nabla H_K^*(v), u - v \rangle \geq \varepsilon \| u - v \|^2.$$

Let us mention that what we here denoted by $M_K$ is in fact the composition of $M_K$ and the Sobolev compact inclusion from $W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^{2n})$ into $L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$ so that, as an endomorphism of $L^2$, $M_K$ is self adjoint and compact. Thus if $G$ is the finite dimensional subspace of $E$ generated by the eigenvectors of $M_K$, the eigenvalues of which are greater than $\frac{\varepsilon}{2}$, we get for $u - v \in G$

$$\langle \psi_K(u) - \psi_K(v), u - v \rangle \geq \frac{\varepsilon}{2} \| u - v \|^2 \quad (2.3)$$
and $\psi_k$ is strictly convex in the direction of $G^\perp$ (i.e. for any $g \in G$, $h \rightarrow \psi_k(g + h)$, defined on $G^\perp$, is strictly convex).

We now prove.

**Proposition 2.4.** For any $g \in G$, the function $h \rightarrow \psi_k(g + h)$ defined on $G^\perp$ has a unique minimum: $h(g)$. The map from $G$ to $G^\perp$ given by $g \rightarrow h(g)$ has its image in $G^\perp \subset W^{1,2}$, and is continuous as a map from $G$ to $G^\perp \subset W^{1,2}$.

Set $\psi_k(g) = \psi_k(g + h(g))$. Then $\psi_k$ is a $C^\infty$ function on $G$, whose critical points are in a one to one correspondence with those of $\psi_k$.

**Remark.** The main feature of $\psi_k$ is that it satisfies condition (C), that we shall prove in Chapter 3.

**Proof.** As $h \rightarrow \psi_k(g + h)$ is strictly convex, it has a unique minimum $h(g)$ satisfying $\psi_k'(g + h(g)).h = 0$ for any $h \in G^\perp$. Let us first prove that, as a map from $G$ to $G^\perp$, $h$ is Lipschitz.

Take $g_1, g_2 \in G$ and set $h_1 = h(g_1), h_2 = h(g_2)$, then

$$\langle \psi_k'(g_1 + h_1) - \psi_k'(g_2 + h_2), h_1 - h_2 \rangle = 0 \tag{2.5}$$

since $h_1 - h_2 \in G^\perp$.

But

$$\langle \psi_k'(g_1 + h_1) - \psi_k'(g_1 + h_1), h_2 - h_1 \rangle \geq \frac{\epsilon}{2} \| h_1 - h_2 \|^2 \tag{2.6}$$

by (2.4), and

$$\| \psi_k(g_2 + h_2) - \psi_k(g_1 + h_2) \| \leq C \| g_1 - g_2 \| \tag{2.7}$$

because $M$ is linear (hence Lipschitz) and $\nabla H_k^*$ is Lipschitz (because

$$H_k^{**}(\nabla H_k(z)) = [H_k'(z)]^{-1}$$

and $H^{**}$ is bounded).

From (2.7) we get,

$$\langle \psi_k'(g_1 + h_2) - \psi_k'(g_2 + h_2), h_2 - h_1 \rangle \geq C \| g_1 - g_2 \| \| h_1 - h_2 \|$$

which compared to (2.5) and (2.6) yields

$$\frac{\epsilon}{2} \| h_1 - h_2 \|^2 \leq C \| g_1 - g_2 \| \| h_1 - h_2 \|$$

hence $h$ is Lipschitz of ratio $\frac{2C}{\epsilon}$. 

Recall now that \( h(\cdot) \) is defined by

\[
\frac{\partial}{\partial h} \Psi_K(g + h(\cdot)) = 0
\]  

(2.8)

that we can write

\[
-M_K h = Q \nabla H^*_{\mathcal{K}}(g + h) \quad \text{with} \quad h = h(\cdot)
\]  

(2.9)

and \( Q \) the orthogonal projection on \( G^\perp \).

We wish to prove that \( h \in W^{1,2} \).

First \( M_K h \) is in \( W^{1,2} \), hence \( Q \nabla H^*_{\mathcal{K}}(g + h) \) also. Now for \( z \in L^2 \), \( z - Qz \in G \subseteq W^{1,2} \), so if \( Qz \in W^{1,2} \) then \( z \in W^{1,2} \), whence we see that \( \nabla H^*_{\mathcal{K}}(g + h) \in W^{1,2} \).

Since \( x \to \nabla H_{\mathcal{K}}(x) \) has a bounded differential, it maps \( W^{1,2} \) in \( W^{1,2} \), hence \( \nabla H_{\mathcal{K}}(\nabla H^*_{\mathcal{K}}(g + h) = g + h \) is in \( W^{1,2} \), and eventually \( h \) is in \( W^{1,2} \).

We finally prove that \( \psi \) is \( C^1 \) and that \( d\psi_K(g) = d\Psi_K(g + h(\cdot)) \). We shall not prove here that \( \psi \) is \( C^\infty \), since using a pseudo gradient vector field, \( C^1 \) is sufficient in order to perform min-max theory.

Let us compute

\[
\Psi_K(g + \delta g) - \Psi_K(g) = \Psi_K(g + \delta g + h(g + \delta g)) - \Psi_K(g + h(g))
\]

\[
= \frac{\partial}{\partial g} \Psi_K(g + h(g)) \delta g + \frac{\partial}{\partial h} \Psi_K(g + h(g)) (h(g + \delta g) - h(g))
\]

\[+ O(\| \delta g \| + \| h(g + \delta g) - h(g) \|).
\]

Since \( g \to h(g) \) is Lipschitz,

\[
o(\| h(g + \delta g) - h(g) \|) = o(\| \delta g \|)
\]

and because \( \frac{\partial}{\partial h} \Psi_K(g + h(g)) = 0 \), we see that

\[
d\psi_K(g) = \frac{\partial}{\partial g} \Psi_K(g + h(g)).
\]

Since \( \Psi_K \) is \( C^1 \) and \( g \to h(g) \) is continuous, this implies that \( \psi_K \) is \( C^1 \). 

Remark. — We write \( \psi_K \) for the finite dimensional reduction of \( \Psi_K \), and \( f_K \) for the corresponding reduction of \( F_K \): since \( M_K \) preserves \( G \) and \( G^\perp \) we set \( \psi_K(M_K y) = f_K(y) \).
3. \( \psi_K \) satisfies condition (C)

The aim of this chapter is to prove

**Proposition 3.1.** \(- \psi_K \) satisfies condition (C) of Palais and Smale.

**Proof.** Let \( g_n \in G \) be a sequence such that \( \psi_K (g_n) \to 0 \) and \( \psi_K (g_n) \) is bounded. Then if \( g_n \) is bounded it has a converging subsequence and there is nothing to prove, so we assume that \( |g_n| \) goes to infinity.

Set \( u_n = g_n + h(g_n) \), then \( |u_n|_{C^0} \to +\infty \) since

\[
\int_0^T \exp \left( \frac{2\pi k J s}{T} \right) u_n(s) \, ds \to +\infty
\]

for some \( k \), and by the same argument, if we set \( u_n = M_K z_n |z_n|_{C^0} \to +\infty \).

Now by assumption

\[-M_K u_n + \nabla H^*_K(u_n) = \varepsilon_n \quad \text{where} \quad \varepsilon_n \in G \text{ goes to zero.} \quad (3.2)

In terms of \( z_n \), (3.2) is equivalent to

\[z_n - \varepsilon_n = \nabla H^*_K(-J \dot{z}_n + K z_n)\]

hence

\[\nabla H(z_n - \varepsilon_n) + K (z_n - \varepsilon_n) = -J \dot{z}_n + K z_n\]

yielding

\[J \dot{z}_n + \nabla H(z_n - \varepsilon_n) = K \varepsilon_n. \quad (3.3)\]

Assume that for large values of \( n \), there exists \( t_0 \in \mathbb{R}/T\mathbb{Z} \) such that

\[z_n(t_0) \notin \bigcup_{k \geq 1} k^p \cdot U \quad (cf. \text{ section 1}). \quad (3.4)

Let \( W \subset \subset U \) defined by

\[W = \{ x \in U / |\nabla H(x)| > 0 \}\]

\[V = \{ x \in U / d(x, W) < \varepsilon \}.

By modifying our choice of the function \( \lambda \), we can take \( W \) to be contained in an arbitrarily small neighborhood of \( \Sigma \times \{1\} \).
We now prove that (3.4) implies, for a good choice of $\varepsilon$, that
\[
\begin{array}{c}
\forall t \in \mathbb{R}/\mathbb{T}.
\end{array}
\]
(3.5)

Let us argue by contradiction, and assume that $t_1$ is the smallest value of $t$ larger than $t_0$ such that $z_n(t_1) \not\in \bigcup_{p \geq 1} k^p \cdot V$.

For $n$ large enough, $|\varepsilon_n(t)| < \varepsilon/2\mathbb{T}$, so if
\[
\begin{array}{c}
z_n(t) \not\in \bigcup_{p \geq 1} k^p \cdot V,
\end{array}
\]
\[
\begin{array}{c}
z_n(t) - \varepsilon_n(t) \not\in \bigcup_{p \geq 1} k^p \cdot W.
\end{array}
\]
Hence $\nabla H(z_n(t) - \varepsilon_n(t)) = 0$, so by (3.3)
\[
\left| \dot{z}_n(t) \right| < \varepsilon/2\mathbb{T}
\]
(3.6)
and by the mean value theorem
\[
\left| z_n(t_1) - z_n(t_0) \right| < \varepsilon/2
\]
since we should have $|t_1 - t_0| < \mathbb{T}$.

But if $\varepsilon$ is small enough, $d(V, (U)) > \varepsilon$, thus
\[
\begin{array}{c}
z_n(t_0) \not\in \bigcup_{p \geq 1} k^p \cdot U
\end{array}
\]
implies
\[
\begin{array}{c}
z_n(t_1) \not\in \bigcup_{p \geq 1} k^p \cdot \bar{V}
\end{array}
\]
which contradicts our assumption.

Now we prove that if $z_n$ is such that $z_n(t) \not\in \bigcup_{p \geq 1} k^p \cdot \bar{V}$ for all $t$'s, and $|z_n|_{C^0} \to +\infty$, then $F_k(z_n)$ is unbounded.

Let us first compute
\[
\begin{array}{c}
\int_0^T \frac{1}{2} (J \dot{z}_n - K_z z_n) \, dt = -\frac{K}{2} \|z_n\|^2 + \frac{1}{2} \int_0^T (J \ddot{z}_n, z_n) \, dt.
\end{array}
\]
Using (3.6)
\[
\left| \int_0^T (J \ddot{z}_n, z_n) \, dt \right| \leq \varepsilon \|z_n\|.
\]

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thus

$$\frac{1}{2} \int_0^T (\dot{z}_n - K z_n, z_n) dt + \frac{K}{2} \| z_n \|^2 \leq \varepsilon \| z_n \|. \quad (3.7)$$

Also, since

$$\frac{(K + a)}{2} |z|^2 - C \leq H_k(z) \leq \frac{(K + a')}{2} |z|^2 + C$$

for some $a' > a$. (The right hand side inequality follows from the boundedness of $H^*$.)

$$\frac{1}{2(K + a')} |z|^2 - C \leq H_k^*(z) \leq \frac{1}{2(K + a)} |z|^2 + C$$

so

$$H_k^*(-J \dot{z}_n + K z_n) \leq \frac{1}{2(K + a)} | -J \dot{z}_n + K z_n |^2 + C,$$

and using (3.6) again

$$\leq \frac{1}{2(K + a)} (\varepsilon + K)^2 \| z_n \|^2 + C.$$

Eventually

$$F_k(z_n) \leq -\frac{K}{2} \| z_n \|^2 + \frac{(\varepsilon + K)^2}{2(K + a)} \| z_n \|^2 + \varepsilon \| z_n \| + C. \quad (3.7)$$

Consider now

$$-\frac{K}{2} + \frac{(\varepsilon + K)^2}{2(K + a)}$$

for $\varepsilon = 0$ this equals $-\frac{K}{2} \left( 1 - \frac{K}{K + a} \right) = -\frac{K a}{2(K + a)}$, so for $\varepsilon$ small enough, this quantity is smaller than $-\frac{K a}{4(K + a)}$. So the right hand side of (3.6) goes to minus infinity with $\| z_n \|$, and $F_k(z_n) = \psi_k(u_n)$ is not bounded.

We now see that the only possibility is that \( z^n(t) \in \bigcup_{p \geq 1} k^p \cdot U \) for all \( t \), hence by a trivial connectedness argument

\[
z^n(t) \in k^p \cdot U \quad \text{for all } t's.
\]

Set \( w_n = \frac{1}{k^p} z_n \in U \). From (3.3) and \( |\nabla H(x)| \leq C |x| \) we infer

\[
|\dot{z}_n| \leq C |z_n|
\]

so \( w_n \) is bounded in \( W^{1,2} \), hence there is a converging subsequence, still denoted \( w_n \), such that \( w_n \rightarrow w \) in the \( C^0 \) topology.

Let us remark that since \( w_n(t) \in U \), \( w(t) \in U \) for all \( t\)'s.

Rewriting (3.3) and using the equation \( \nabla H(k^p z) = k^p \nabla H(z) \) for \( z \in U \), yields

\[
J \dot{w}_n + \nabla H \left( w_n - \frac{\varepsilon_n}{k^p} \right) = K \frac{\varepsilon_n}{k^p}.
\] (3.8)

Let \( n \) go to infinity, we thus obtain

\[
J \dot{w} + \nabla H(w) = 0
\] (3.9)

so \( w \) is a solution of (\( \mathcal{M} \)) such that \( w(t) \in U \) for all \( t\)'s.

Now, let us show that \( F_K(z_n) \sim k^{2p} F_K(w) \). Since if \( w \) is a constant in \( U \), \( F_K(w) \) is non zero, this will imply that \( w \) is a nontrivial solution of (\( \mathcal{M} \)).

Obviously,

\[
\int_0^T \frac{1}{2} (J \dot{z}_n - K z_n, z_n) dt = k^{2p} \int_0^T \frac{1}{2} (J \dot{w}_n - K w_n, w_n) dt
\]

On the other hand, by (3.3)

\[
-J \dot{z}_n + K z_n = \nabla H_K(z_n - \varepsilon_n),
\]

therefore

\[
H_K^* \left( -J \dot{z}_n + K z_n \right) = (z_n - \varepsilon_n), \nabla H_K(z_n - \varepsilon_n) - H_K(z_n - \varepsilon_n).
\] (3.10)

As before \( z_n - \varepsilon_n = k^p \left( w_n - \frac{\varepsilon_n}{k^p} \right) \), and for \( n \) large enough, \( w_n - \frac{\varepsilon_n}{k^p} \) is in \( U \) since \( |\varepsilon_n|_{C^0} \) goes to zero.
Thus, using (1.11),

$$H_k^*(\mathbf{z}_n^* + Kz_n^*) = k^{2p_n} \left[ \left( w_n - \frac{\varepsilon_n}{k^{p_n}} \right) \cdot \nabla H_k \left( w_n - \frac{\varepsilon_n}{k^{p_n}} \right) + H_k \left( w_n - \frac{\varepsilon_n}{k^{p_n}} \right) \right]$$

the last equality follows from (3.8).

This proves $F_k^*(z_n^*) = k^{2p_n} F_k(w_n)$, since $w_n$ converges strongly to $w$, we indeed get $F_k(z_n^*) \sim k^{2p_n} F_k(w)$.

Finally, if $w$ is a constant in $U$, solution of $(\mathcal{H})$, then $\nabla H(w) = 0$, hence $\nabla H_k(w) = Kw$, so

$$H_k^*(Kw) = kw - H_k(kw) = \frac{K}{2} |w|^2 - H(w)$$

and

$$F_k(w) = T \left[ -\frac{K}{2} |w|^2 + \frac{K}{2} |w|^2 - H(w) \right] = -TH(0).$$

So if $F_k(z_n^*)$ is bounded $w$ is a non constant solution of $(\mathcal{H})$. $\Box$

*Remark.* - It is easy to see that in order that (P.S.) holds, we only need that $F_k(w) \neq 0$ for all solutions $w$ of $(\mathcal{H})$. Now, a computation yields

$$F_k(w) = w \cdot \nabla H(w) - H(w)$$

and it can be shown that if the set of periods of closed characteristics of $\Sigma$ is discrete, we can choose $\lambda$ so that 0 is not a critical value of $F_k$.

4. PROOF OF THE THEOREM

Now that we proved that $f_k$ satisfies condition (C) we must prove that it has a non trivial critical point.

Let us first remark that if we let $S^1 = \mathbb{R}/T\mathbb{Z}$ act on $X = W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^{2n})$ by

$$\theta \cdot z(\cdot) = z(\theta + \cdot)$$

then $F_k$ is equivariant, as well as $f_k$ since $g \to h(g)$ is equivariant.
Now let $F$ be the set of fixed points of $X$ by the $S^1$ action, that is $F$ is the set of constant paths. We now prove

**Proposition 4.1.** — There are two $S^1$-invariant vector subspaces of $G$, $V$ and $W$ such that $V \supseteq W \supseteq F$ and if we denote by $S$ the unit sphere of $G$,

(i) $\alpha S \cap W^\perp \subseteq G - G^{-\gamma}$

(ii) $G - G^C \cap V = \emptyset$

for $\alpha$ small, $0 < \gamma + TH(0) < \varepsilon$, with $\varepsilon$ small, $C$ large. Moreover

$\dim V - \dim W \geq 1$ for a large enough $\alpha$ was defined in (1.8). $\square$

**Proof.** — Set

$$Q_{\alpha, K}(x) = \frac{1}{2} \int_0^T \left[ (\dot{x} - Kx, x) + \frac{1}{(K + \lambda)} |\dot{x} - Kx|^2 \right] ds$$

Then since

$$H_k^\alpha(x) \leq \frac{1}{2(K + a)} |x|^2 + C'$$

we have for $x \in G$

$$f_k(x) \leq F_k(x) \leq Q_{\alpha, K}(x) + \frac{C'}{2} T$$

so for $C$ large, $X - X^C$ does not meet the negative eigenspace of $Q_{\alpha, K}$, that shall be our space $V$ (remark that indeed $V \subseteq G$).

On the other hand, for $x$ in a neighborhood of the origin, we assumed

$$H(x) = H(0)$$

so that

$$F_k(z) = Q_{\alpha, K}(z) - H(0) \quad \text{for} \quad |z|_{C^0} \text{ small enough},$$

hence since for small $g$, $|g + h(g)|_{C^0}$ is small, $f_k(g) = q_{0, K}(g) - H(0)$ near the origin, where $q_{0, K}$ is obtained from $Q_{0, K}$ in the same way as $f_k$ is obtained from $F_k$.

It is easy to see that $q_{0, K}$ and $Q_{0, K}$ have the same index, and we take for $W$ the non positive eigenspace of $Q_{0, K}$.
An easy computation shows that

\[
\text{index } Q_{\lambda, \kappa} = 2n \left( \left\lfloor \frac{(K + \lambda) T}{2\pi} \right\rfloor + 1 \right)
\]

Set \( y = \sum_{k \in \mathbb{Z}} \exp(2\pi k t J/T) y_k \), then

\[
Q_{\lambda, \kappa}(y, y) = \int_0^T \left( (J \dot{y} - K y, y) + \frac{1}{K + \lambda} |J \dot{y} - K y|^2 \right) dt
\]

\[
= \sum_{k \in \mathbb{Z}} \left( \frac{2\pi k}{T} + K \right) y_k^2 + \frac{1}{K + \lambda} \left( \frac{2\pi k}{T} + K \right)^2 y_k^2
\]

\[
= \sum_{k \in \mathbb{Z}} \left( \frac{2\pi k}{T} + K \right) \left( \frac{1}{K + \lambda} \left( \frac{2\pi k}{T} + K \right) - 1 \right) y_k^2
\]

\[
= \sum_{k \in \mathbb{Z}} \left( \frac{2\pi k}{T} + K \right) \frac{1}{(K + \lambda)} \left( \frac{2\pi k}{T} - \lambda \right) y_k^2
\]

whose index is given by

\[
2n \left\{ k \in \mathbb{Z}/ -\frac{KT}{2\pi} < k < \frac{\lambda T}{2\pi} \right\} = 2n \left( \left\lfloor \frac{(\lambda + K) T}{2\pi} \right\rfloor + 1 \right)
\]

so that

\[
\dim V = 2n \left( \left\lfloor \frac{(K + a) T}{2\pi} \right\rfloor \right)
\]

and

\[
\dim W = \text{index } Q_{0, \kappa} + \text{nullity } Q_{0, \kappa}
\]

\[
= 2n \left\lfloor \frac{KT}{2\pi} \right\rfloor + 2n
\]

(because \( \ker Q_{0, \kappa} = F \) has dimension \( 2n \))

\[
\text{so } \dim V - \dim W = 2n \left( \left\lfloor \frac{(K + a) T}{2\pi} \right\rfloor - \left\lfloor \frac{KT}{2\pi} \right\rfloor - 1 \right)
\]

\[\Box\]

**Corollary 4.2:**

\[H^1_0(G - G^r, G - G^c) \neq 0 \quad \text{for } \forall \epsilon \notin \dim V^1, \dim W^1\]

thus \( f_k \) has a critical level in \([\gamma, C]\) whence a non trivial critical circle. \(\square\)
Proof. — The idea is that if \( i(A) \) is the Fadell-Rabinowitz cohomological index of \( A \) (cf. \([F-R]\))\), then by (ii) \( i(G - G^C) \geq \text{codim}_G V \); on the other hand by (i) \( i(G - G^c) \geq \text{codim}_G W \) so that

\[
H^*_G(G - G', G - G^C) \neq 0 \quad \text{for all } q \in \{\text{codim}_G V, \text{codim}_G W\}.
\]

To be more precise, there are maps

\[
\alpha S \cap W^\perp \to G - G' \to G - F,
\]

since as proved at the end of chapter 3, \( F \subset G' \)

\[
H^*_G(G - \{0\}) \to H^*_G(G - G') \to H^*_G(\alpha S \cap W^\perp)
\]

as the map \( H^*_G(G - \{0\}) \to H^*_G(\alpha S \cap W^\perp) \) is surjective for \( * \leq \dim W^\perp \), \( H^*_G(G - G') \to H^*_G(\alpha S \cap W^\perp) \) will also be surjective.

On the other hand there is a homotopy commutative diagram

\[
\begin{array}{ccc}
G - G^C & \xrightarrow{\pi} & V^\perp - \{0\} \\
\downarrow & & \downarrow \\
G - \{0\} & \to & G - \{0\}
\end{array}
\]

where \( \pi \) is the orthogonal projection on \( V^\perp \) and "homotopy commutative" means that the inclusion of \( G - G^C \) in \( G - \{0\} \) is homotopic to \( \pi \) composed with the inclusion of \( V^\perp - \{0\} \) in \( G - \{0\} \); the homotopy being given by \( \phi_t(x) = (1 - t)x + t \pi(x) \).

Thus there is a commutative diagram

\[
\begin{array}{ccc}
H^*_G(G - G^C) & \xleftarrow{\sim} & H^*_G(V^\perp - \{0\}) \\
\downarrow & & \downarrow \\
H^*_G(G - \{0\}) & \to & H^*_G(G - \{0\})
\end{array}
\]

as

\[
H^*_G(V^\perp - \{0\}) = 0 \quad \text{for } * \geq \dim V^\perp,
\]

so

\[
H^*_G(G - \{0\}) \to H^*_G(G - G^C)
\]

is zero in these dimensions.
Finally let us write the cohomology sequences of the pairs \((G - G', G - G^\circ); (G - F, G - F)\) and the map between these sequences induced by the inclusion

\[
\begin{align*}
H^q_\ast (G - G', G - G^\circ) &\rightarrow H^q_\ast (G - G') \rightarrow H^q_\ast (G - G^\circ) \\
\uparrow & \uparrow \beta \quad \uparrow \gamma \\
0 & \rightarrow H^q_\ast (G - F) \rightarrow H^q_\ast (G - F)
\end{align*}
\]

\(\gamma\) is zero for \(\ast \geq \dim V^\perp\), \(\beta\) is non zero for any \(\ast \leq \dim W^\perp\). So let \(\gamma(y) = 0\), \(y \in H^q_\ast (G - \{0\})\) where \(\dim V^\perp \leq q < \dim W^\perp\) such that \(\beta(y) \neq 0\). As \(\gamma(y) = 0\), \(\beta(y)\) is in the image of \(\alpha\) hence

\[H^q_\ast (G - G^\circ) \neq 0.\]


We can now conclude the proof of our theorem:

By corollary 4.2 \(f_k\) has at least one critical value in \([\gamma, C]\). Thus \(F_k\) ha a critical value, also in \([\gamma, C]\) since the critical values of \(f_k\) and \(F_k\) coincide. Since \(\gamma > -\text{TH}(0)\), the critical orbit thus found is non trivial. According to proposition 1.8, this yields a periodic orbit of \(\langle \mathcal{N} \rangle\). 

**REFERENCES**


*(Manuscrit reçu le 23 octobre 1986.)*