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Gradient theory of phase transitions with boundary contact energy


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by

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ABSTRACT. — We study the asymptotic behavior as $\varepsilon \to 0^+$ of solutions of the variational problems for the Van der Waals-Cahn-Hilliard theory of phase transitions in a fluid. We assume that the internal free energy, per unit volume, is given by $\varepsilon^2 |\nabla \rho|^2 + W(\rho)$ and the contact energy with the container walls, per unit surface area, is given by $\varepsilon \sigma(\rho)$, where $\rho$ is the density. The result is that such solutions approximate a two-phases configuration satisfying a variational principle related to the equilibrium configuration of liquid drops.

Key words : Phase transitions, variational thermodynamic principles, variational convergence.

RÉSUMÉ. — Nous étudions ici le comportement asymptotique pour $\varepsilon \to 0^+$ des solutions des problèmes variationnels qui viennent de la théorie de Van der Waals-Cahn-Hilliard sur les transitions de phase des fluides. Nous faisons l'hypothèse que l'énergie libre de Gibbs, pour unité de volume, est donnée par $\varepsilon^2 |\nabla \rho|^2 + W(\rho)$ et que l'énergie de contact avec la surface intérieure du conteneur, pour unité de surface, est donnée par $\varepsilon \sigma(\rho)$, où $\rho$ est la densité. Le résultat est que ces solutions approchent...
INTRODUCTION

We continue in this paper the asymptotic analysis of the Van der Waals-Cahn-Hilliard theory of phase transitions in a fluid, by taking also into account, with respect to our earlier results [10], the contact energy between the fluid and the container walls. Our results give a positive answer to some conjectures by M. E. Gurtin [8].

Let us describe briefly the problem we are concerned with; we refer to [10] for further information and bibliography. Consider a fluid, under isothermal conditions and confined to a bounded container $\Omega \subset \mathbb{R}^n$, and assume that the Gibbs free energy, per unit volume, $W = W(u)$ and the contact energy, per unit surface area, $\sigma = \sigma(u)$ between the fluid and the container walls $\partial \Omega$ are prescribed functions of the density distribution (or composition) $u \geq 0$ of the fluid. According to the Van der Waals-Cahn-Hilliard theory, and in particular to the Cahn's approach [2], the stable configurations of the fluid are determined by solving the variational problem

$$
(*) \quad \min \left\{ \int_{\Omega} [\varepsilon^2 |D u|^2 + W(u)] \, dx + \int_{\partial \Omega} \varepsilon \theta(u) \, d\mathcal{H}^{n-1}, \right. \\
\quad \left. \int_{\Omega} u \, dx = m, \right. 
$$

where $\varepsilon > 0$ is a small parameter, and the minimum is taken among all functions $u \geq 0$ satisfying the constraint

$$
\int_{\Omega} u \, dx = m,
$$

$m$ being the prescribed total mass of the fluid. The function $W(t)$ is supposed to vanish only at two points $t = \alpha$ and $t = \beta$ ($\alpha < \beta$), and to be strictly positive everywhere else. Of course, $\mathcal{H}^{n-1}$ denotes the Hausdorff $(n-1)$-dimensional measure.

Our goal is to study the asymptotic behavior as $\varepsilon \to 0^+$ of solutions $u_\varepsilon$ of (*) by looking for a variational problem solved by the limit point (or points) of $u_\varepsilon$ in $L^1(\Omega)$. As conjectured by Gurtin [8], this limit problem does exist and agrees with the so-called liquid-drop problem.
Namely (cf. Theorem 2.1 for a precise statement), if the function \( u_0 \) is the limit of \( u_\varepsilon \) in \( L^1(\Omega) \) as \( \varepsilon \to 0^+ \), then \( u_0 \) takes only the values \( \alpha \) and \( \beta \) (i.e., \( u_0 \) corresponds to a two-phases configuration of the fluid), and the portion \( E_0 \) of the container occupied by the phase \( u_0 = \alpha \) minimizes the geometric area-like quantity

\[
\mathcal{H}^{n-1}(\partial E \cap \Omega) + \gamma \mathcal{H}^{n-1}(\partial E \cap \partial \Omega)
\]

among all subsets \( E \) of \( \Omega \) having the same volume as \( E_0 \). The number \( \gamma \) depends only on \( W \) and \( \sigma \), and it can be explicitly calculated:

\[
\gamma = \frac{\hat{\sigma}(\alpha) - \hat{\sigma}(\beta)}{2c_0},
\]

where

\[
c_0 = \int_{\alpha}^{\beta} W^{1/2}(s) \, ds,
\]

and \( \hat{\sigma} \) represents a modified contact energy between the fluid and the container walls, whose definition involves the values of \( \sigma(t) \) and \( W(t) \) for every \( t \geq 0 \). One has \( |\gamma| \leq 1 \) in correspondence with the geometrical meaning of \( \gamma \), which is the cosine of the contact angle between the fluid phase \( \alpha \) and the walls of the container.

The presence of such \( \hat{\sigma} \) instead of \( \sigma \) disproves a part of the Gurtin's conjecture but, what is more interesting, it is perfectly in accord with theory and experiments by J. W. Cahn and R. B. Heady ([2], [3]) about critical point wetting. They discovered that, in a range of temperatures below the critical one for a binary system, the phase \( \alpha \) does not wet the container (i.e. \( \gamma = 1 \)) and a layer of phase \( \beta \), which is, on the contrary, perfectly wetting, appears between the phase \( \alpha \) and the container walls. A theoretical explanation of such phenomenon was given by Cahn in the case \( \varepsilon > 0 \).

We confirm in this paper, under very general assumptions and by a fully mathematical proof, the existence of the critical point wetting phenomenon in the asymptotic case \( \varepsilon \to 0 \). Indeed, we show that \( \gamma = 1 \) and \( \hat{\sigma}(\alpha) = \hat{\sigma}(\beta) + \sigma_{\alpha\beta} \) (\( \sigma_{\alpha\beta} \) denotes the energy, per unit surface area, associated to the interface between the phases \( \alpha \) and \( \beta \)), for \( \sigma \) and \( W \) having the same global behavior exhibited in the semi-empirical figures of [2]. It now suffices to remark that the balance of energy \( \hat{\sigma}(\alpha) = \hat{\sigma}(\beta) + \sigma_{\alpha\beta} \) can be interpreted as the contact energy on \( \partial E_0 \cap \partial \Omega \) coming from an infinitely.
thin layer of the phase \( \beta \) interposed between the phase \( \alpha \) and the container walls (cf. Section 3 for details).

We think that other very interesting experimental evidences, discussed by Cahn in [2], would deserve a similar careful mathematical treatment. Finally, we would like to thank Morton Gurtin for stimulating and friendly discussions.

1. SOME PRELIMINARY RESULTS

Throughout this paper \( \Omega \) will be an open, bounded subset of \( \mathbb{R}^n \) \((n \geq 2)\) with smooth boundary \( \partial \Omega \); \( W \) and \( \sigma \) will be two non-negative continuous functions defined on \([0, +\infty[\). The function \( W(t) \) is supposed to have exactly two zeros at the points \( t = \alpha \) and \( t = \beta \), with \( 0 < \alpha < \beta \).

For every \( \varepsilon > 0 \) and for every non-negative function \( u \) in the Sobolev space \( H^1(\Omega) \), we define

\[
\mathcal{E}_\varepsilon(u) = \int_{\Omega} \left[ \varepsilon^2 |D u(x)|^2 + W(u(x)) \right] dx + \varepsilon \int_{\partial \Omega} \sigma(\tilde{u}(x)) d\mathcal{H}^{n-1}(x) \quad (1)
\]

where \( D u \) denotes the gradient of \( u \), \( \tilde{u} \) denotes the trace of \( u \) on \( \partial \Omega \), and \( \mathcal{H}^{n-1} \) denotes the \((n-1)\)-dimensional Hausdorff measure.

1.1. PROPOSITION. — For every \( \varepsilon > 0 \) and for every \( m \geq 0 \) the minimization problem

\[
(P_\varepsilon) \quad \min \left\{ \mathcal{E}_\varepsilon(u) : u \in H^1(\Omega), \int_{\Omega} u(x) dx = m \right\}
\]

admits (at least) one solution.

Proof. — The proof is standard. Let

\[
U = \left\{ u \in H^1(\Omega) : u \geq 0, \mathcal{E}_\varepsilon(u) \leq c, \int_{\Omega} u(x) dx = m \right\},
\]

with \( c \in \mathbb{R} \) large enough so that \( U \neq \emptyset \). It suffices to prove that \( \mathcal{E}_\varepsilon \) is lower semicontinuous on \( U \) and \( U \) is compact with respect to the topology of \( L^2(\Omega) \).
Let $u_\infty \in U$ and $(u_h)$ be a sequence in $U$ converging to $u_\infty$ in $L^2(\Omega)$: we have to prove that

$$\mathcal{E}_\varepsilon(u_\infty) \leq \liminf_{h \to +\infty} \mathcal{E}_\varepsilon(u_h).$$

Without loss of generality we can assume that there exists the limit of $\mathcal{E}_\varepsilon(u_h)$ as $h \to +\infty$ and it is finite. Since $W \geq 0$ and $\sigma \geq 0$, we have that

$$\int_{\Omega} |Du|^2 \, dx \leq c/\varepsilon^2, \quad \forall u \in U;$$

hence, modulo replacing $(u_h)$ by a subsequence, $(u_h)$ and $(\tilde{u}_h)$ converge pointwise to $u_\infty$ and $\tilde{u}_\infty$, respectively almost everywhere on $\Omega$ and $\mathcal{H}^{n-1}$-almost everywhere on $\partial \Omega$ [recall that the trace operator is compact between $H^1(\Omega)$ and $L^2(\partial \Omega, \mathcal{H}^{n-1})$]. Then (2) follows from lower semicontinuity of the Dirichlet integral and from continuity of $W$ and $\sigma$, by applying Fatou’s Lemma.

Lower semicontinuity of $\mathcal{E}_\varepsilon$ implies now that $U$ is closed in $L^2(\Omega)$; on the other hand, by (3) and by Poincaré Inequality, $U$ is bounded in $H^1(\Omega)$. Then Rellich’s Theorem gives that $U$ is compact in $L^2(\Omega)$ and the proof is complete.

The aim of the present paper is to study the asymptotic behavior as $\varepsilon \to 0^+$ of (P$_\varepsilon$). We shall prove in Section 2 that such asymptotic behavior is related with the following geometric minimization problem:

$$(P_0) \quad \min \{ P_\Omega(E) + \gamma \mathcal{H}^{n-1}(\partial^* E \cap \partial \Omega); E \subseteq \Omega, |E| = m_1 \}. $$

Here $\gamma \in [-1, 1]$, $m_1 \in [0, |\Omega|]$ are fixed real constants; $|E|, P_\Omega(E)$, $\partial^* E$ denote respectively the Lebesgue measure of $E$, the perimeter of $E$ in $\Omega$, and the reduced boundary of $E$. We refer to the book by E. Giusti [6] for these concepts, which go back to the De Giorgi’s approach to the minimal surfaces theory. Anyhow, for reader’s convenience, we recall that $P_\Omega(E) = \mathcal{H}^{n-1}(\partial E \cap \Omega)$ and $\partial^* E = \partial E$, provided that the boundary of $E$ is locally Lipschitz continuous; hence (P$_0$) consists in finding a subset $E$ of $\Omega$, with prescribed volume $m_1$, which minimizes a quantity related with the $(n-1)$-dimensional measure of its boundary.

The problem (P$_0$) is known as the liquid-drop problem (cf. E. Giusti [5]). Since $\Omega$ is bounded and $|\gamma| \leq 1$, it always admits (at least) one solution. Such existence result could also be obtained by the following proposition, which we need later.

1.2. Proposition. — Let \( \tau: \partial \Omega \times \mathbb{R} \to \mathbb{R} \) be a Borel function and define, for \( u \in BV(\Omega) \),

\[
F(u) = \int_{\Omega} |Du| + \int_{\partial \Omega} \tau(x, \tilde{u}(x)) \, d\mathcal{H}^{n-1}(x) \quad (1),
\]

where \( \tilde{u} \) denotes the trace of \( u \) on \( \partial \Omega \). If

\[
(i) \quad \left\{ \begin{array}{l}
|\tau(x, s_1) - \tau(x, s_2)| \leq |s_1 - s_2|, \\
\forall x \in \partial \Omega, \quad \forall s_1, s_2 \in \mathbb{R}
\end{array} \right.
\]

then the functional \( F \) is lower semicontinuous on \( BV(\Omega) \) with respect to the topology of \( L^1(\Omega) \).

Proof. — Fix \( u_\infty \in BV(\Omega) \) and let \( (u_h) \) be a sequence in \( BV(\Omega) \) converging to \( u_\infty \) in \( L^1(\Omega) \). We want to prove that

\[
\limsup_{h \to +\infty} [F(u_\infty) - F(u_h)] \leq 0. \quad (4)
\]

By (i) we deduce that

\[
F(u_\infty) - F(u_h) \leq \int_{\Omega} |Du_\infty| - \int_{\Omega} |Du_h| + \int_{\partial \Omega} |\tilde{u}_\infty - \tilde{u}_h| \, d\mathcal{H}^{n-1}.
\]

Let \( \delta > 0 \) and define \( v_\delta = (1 - \chi_\delta) \, (u_\infty - u_h) \), where \( \chi_\delta \) is the usual cut-off function, i.e. \( \chi_\delta \in C_0^1(\Omega) \), \( 0 \leq \chi_\delta \leq 1 \), \( \chi_\delta(x) = 1 \) if \( \text{dist}(x, \partial \Omega) \geq \delta \), \( |D \chi_\delta| \leq 2/\delta \). The trace inequality for BV functions (cf. G. Anzellotti and M. Giaquinta [1]), applied to \( v_\delta \), gives that

\[
\int_{\partial \Omega} |\tilde{u}_\infty - \tilde{u}_h| \, d\mathcal{H}^{n-1} \leq c_1 \int_{\Omega_\delta} |D(u_\infty - u_h)| + (2c_1/\delta) \int_{\Omega_\delta} |u_\infty - u_h| \, dx + c_2 \int_{\Omega_\delta} |u_\infty - u_h| \, dx,
\]

\(^{(1)}\) For \( u \in BV(\Omega) \) and \( E \) measurable subset of \( \Omega \), we denote by \( \int_E |Du| \) the value of the measure \( |Du| \) at the set \( E \). Of course, if \( Du \) is a Lebesgue integrable vector function, then \( \int_E |Du| \) agrees with the ordinary integral \( \int_E |Du(x)| \, dx \).
where $\Omega_\delta = \{ x \in \Omega : \text{dist} (x, \partial \Omega) > \delta \}$ and $\Omega_\delta^c = \Omega \setminus \Omega_\delta$. Let us remark that $c_1 = 1$ because $\partial \Omega$ is smooth (see [1]), and that
\[
\int_{\Omega_\delta} |D(u_\infty - u_h)| \leq \int_{\Omega_\delta} |Du_\infty| + \int_{\Omega_\delta} |Du_h| + \int_{\partial \Omega_\delta} |D(u_\infty - u_h)|.
\]

Since $u_\infty - u_h \in BV(\Omega)$, we have that
\[
\int_{\partial \Omega_\delta} |D(u_\infty - u_h)| = 0, \quad \forall h \in \mathbb{N}
\]
for a set of $\delta > 0$ of full measure; hence
\[
F(u_\infty) - F(u_h) 
\leq \int_{\Omega} |Du_\infty| + \int_{\Omega_\delta} |Du_\infty| - \int_{\Omega_\delta} |Du_h| + \left( \frac{2}{\delta} + c_2 \right) \int_{\Omega_\delta} |u_\infty - u_h| \, dx
\]
and, by lower semicontinuity in $L^1(\Omega_\delta)$ of the functional
\[
u \mapsto \int_{\Omega_\delta} |Du|,
\]
we conclude that
\[
\limsup_{h \to +\infty} [F(u_\infty) - F(u_h)] \leq 2 \int_{\Omega_\delta} |Du_\infty|
\]
for almost all $\delta > 0$. By taking $\delta \to 0^+$, the inequality (4) is proved. \[\square\]

1.3. Remark. — The previous proposition fails to be true if $\partial \Omega$ is not smooth, or if the function $\tau$ has in (i) a Lipschitz constant $L > 1$. For example, in the case $\Omega = [0,1] \times [0,1]$ and $\tau(x, s) = -\lambda s$ with $\lambda > \sqrt{2}/2$, the corresponding functional $F$ is not lower semicontinuous at the point $u_\infty = 0$; it is enough to check lower semicontinuity on the sequence $(u_h)$ given by $u_h(x, y) = 0$ for $x + y \geq 1/h$, $u_h(x, y) = h$ for $x + y < 1/h$. Analogously, in the case $\Omega = \{ x \in \mathbb{R}^2 : |x| < 1 \}$ and $\tau(x, s) = \lambda |s|$ with $\lambda > 1$, the corresponding functional $F$ is not lower semicontinuous at the point $u_\infty(x) = |x|$: one can choose $u_h(x) = \min \{ |x|, (h - 1) (1 - |x|) \}$.

However, it is worth noticing that, in the particular case $\tau(x, s) = |s - \psi(x)|$ with $\psi \in L^1(\partial \Omega, \mathcal{H}_{n-1})$, the functional $F$ defined in Proposition 1.2 is lower semicontinuous on $L^1(\Omega)$ even for Lipschitz
continuous $\partial \Omega$. Indeed, by choosing an open, bounded set $\Omega' \supseteq \overline{\Omega}$ and a function $\hat{\psi} \in \text{BV}(\Omega')$ whose trace on $\partial \Omega$ is $\psi$, we have that

$$F(u) = \int_{\Omega} |Du| + \int_{\partial \Omega} |\tilde{u}(x) - \psi(x)| \, d\mathcal{H}^n = \int_{\Omega'} |Dv_u| - \int_{\partial \Omega} |D\hat{\psi}|,$$

where the function $v_u$ is defined by $v_u(x) = u(x)$ for $x \in \Omega$, $v_u(x) = \hat{\psi}(x)$, for $x \in \Omega \setminus \Omega$. Since the first addendum of the right-hand side is lower semicontinuous with respect to $u$ in $L^1(\Omega)$, $F$ also is lower semicontinuous in $L^1(\Omega)$.

From now on, we let, for $t \geq 0$,

$$\varphi(t) = \int_0^t \mathcal{W}^{1/2}(s) \, ds, \quad (5)$$

$$\tilde{\sigma}(t) = \inf \{ \sigma(s) + 2 \left| \varphi(s) - \varphi(t) \right| : s \geq 0 \}, \quad (6)$$

and, for $u \in \text{BV}(\Omega)$,

$$\mathcal{E}_0(u) = 2 \int_{\Omega} |D(\varphi \circ u)| + \int_{\partial \Omega} \tilde{\sigma}(\tilde{u}(x)) \, d\mathcal{H}^{n-1}, \quad (7)$$

where, as above, $\tilde{u}$ denotes the trace of $u$ on $\partial \Omega$.

1.4. Proposition. — Let $(u_h)$ be a sequence of functions of class $C^1$ on $\Omega$. If $(u_h)$ converges in $L^1(\Omega)$ to a function $u_\infty$ and there exists a real constant $c$ such that

$$\int_{\Omega} |D(\varphi \circ u_h)| \, dx \leq c$$

for every $h \in \mathbb{N}$, then $\varphi \circ u_\infty \in \text{BV}(\Omega)$ and

$$\mathcal{E}_0(u_\infty) \leq \liminf_{h \to +\infty} \mathcal{E}_0(u_h).$$

Proof. — Let us denote $v_h(x) = \varphi(u_h(x))$ and fix an open subset $\Omega'$ of $\Omega$ such that $\overline{\Omega'} \subset \Omega$. If we consider the smooth function $\tilde{v}_h(x) = v_h(x) - \mathcal{I}_h$, where

$$\mathcal{I}_h = \int_{\Omega'} v_h \, dx,$$
Poincaré Inequality gives

\[ \int_{\Omega'} |\tilde{v}_h| \, dx \leq c_1(\Omega) \int_{\Omega'} |D\tilde{v}_h| \, dx \leq c_1(\Omega)c \]

for every \( h \in \mathbb{N} \) and for a real constant \( c_1(\Omega) \) depending on \( \Omega \) but independent of \( \Omega' \subseteq \Omega \). It follows that the sequence \((\tilde{v}_h)\) is bounded in \( BV(\Omega) \); hence, by Rellich's Theorem, there exists a subsequence \((\tilde{v}_{\sigma(h)})\) which converges in \( L^1(\Omega) \) to a function \( \tilde{v}_\infty \).

Since it is not restrictive to assume that \((\tilde{v}_{\sigma(h)})\) and \((v_{\sigma(h)})\) both converge almost everywhere in \( \Omega \), we infer that \((\vartheta_{\sigma(h)})\) converges in \( \mathbb{R} \) to \( \vartheta_\infty \), and finally that \((v_{\sigma(h)})\) converges in \( L^1(\Omega) \) to \( \tilde{v}_\infty + \vartheta_\infty \). We have of course \( \tilde{v}_\infty + \vartheta_\infty = \varphi \circ u_\infty \), so we conclude that the whole \((v_h)\) converges in \( L^1(\Omega) \) to \( v_\infty = \varphi \circ u_\infty \) and, by semicontinuity, that

\[ \int_{\Omega} |Dv_\infty| \leq \liminf_{h \to +\infty} \int_{\Omega} |Dv_h| \leq c < +\infty. \]

We now consider the inverse function \( \varphi^{-1} \) of \( \varphi \); note that \( \varphi^{-1} \) exists because \( \varphi'(t) = W(t) > 0 \) except for \( t = \alpha, \beta \). Denoting \( \tau(s) = \hat{\vartheta}(\varphi^{-1}(s)) \), we have that

\[ |\tau(s_1) - \tau(s_2)| \leq C |s_1 - s_2| \]

for every \( s_1, s_2 \) in the domain of \( \varphi^{-1} \); then Proposition 1.2 yields that

\[ \mathcal{E}_0(u_\infty) = 2 \int_{\Omega} |Dv_\infty| + \int_{\partial\Omega} \tau(\tilde{v}_\infty) \, d\mathcal{H}^1 \]

\[ \leq \liminf_{h \to +\infty} \left[ 2 \int_{\Omega} |Dv_h| \, dx + \int_{\partial\Omega} \tau(\tilde{v}_h) \, d\mathcal{H}^1 \right] = \liminf_{h \to +\infty} \mathcal{E}_0(u_h) \]

and Proposition 1.4 is proved. \( \blacksquare \)

We now turn to the liquid-drop problem (Po) by proving that the class of competing sets can be restricted to smooth sets.

1.5. **Proposition.** Suppose \( 0 < m_1 < |\Omega| \) and \( |\gamma| \leq 1 \). If \( \lambda \) is a fixed real number such that

\[ \lambda \leq P_\Omega(A) + \gamma \mathcal{H}^{n-1}(\partial(A \cap \Omega) \cap \partial\Omega) \]

for every open, bounded subset $A$ of $\mathbb{R}^n$ which has smooth boundary and satisfies $\mathcal{H}_{n-1}(\partial A \cap \partial \Omega) = 0$, $|A \cap \Omega| = m_1$, then

$$\lambda \leq \min \{ P_\Omega(E) + \gamma \mathcal{H}_{n-1}(\partial^* E \cap \partial \Omega) : E \subseteq \Omega, |E| = m_1 \}.$$

**Proof.** We omit the details because we closely follow the proof of the analogous result proved for the case $\gamma = 0$ in Lemmas 1 and 2 of [10].

Let $E_0$ be the set which realizes the minimum of $(P_\Omega)$. By a theorem of E. Gonzalez, U. Massari and I. Tamanini ([7], Th. 1), which was stated for $\gamma = 0$ but holds also in our situation because of its local character, we have that both $E_0$ and $\Omega \setminus E_0$ contain a non-empty open ball. Then, arguing as in Lemma 1 of [10], one can construct a sequence $(E_h)$ of open, bounded, smooth subsets of $\mathbb{R}^n$ such that $|E_h \cap \Omega| = m_1$, $\mathcal{H}_{n-1}(\partial E_h \cap \partial \Omega) = 0$ for every $h \in \mathbb{N}$, and

$$\lim_{h \to +\infty} |(E_h \cap \Omega) \Delta E_0| = 0, \quad (8)$$

$$\lim_{h \to +\infty} P_\Omega(E_h) = P_\Omega(E_0), \quad (9)$$

$$\lim_{h \to +\infty} \mathcal{H}_{n-1}(\partial (E_h \cap \Omega) \cap \partial \Omega) = \mathcal{H}_{n-1}(\partial^* E_0 \cap \partial \Omega). \quad (10)$$

The last assertion is not actually contained in Lemma 1 of [10] but it easily follows from (8) and from

$$\mathcal{H}_{n-1}(\partial (E_h \cap \Omega) \cap \partial \Omega) = \int_{\partial \Omega} \tilde{\chi}_{E_h} \cap \Omega d\mathcal{H}_{n-1}$$

$$\mathcal{H}_{n-1}(\partial^* E_0 \cap \partial \Omega) = \int_{\partial \Omega} \tilde{\chi}_{E_0} d\mathcal{H}_{n-1},$$

where $\tilde{\chi}_T$ denotes the trace on $\partial \Omega$ of the characteristic function of $T$ for $T = E_h \cap \Omega$ and $T = E_0$.

The proof of the proposition is now a straightforward consequence of (9) and (10). $\blacksquare$

The next result, stated here without proof, was proved in [10] (Lemma 4).

1.6. **Proposition.** Let $A$ be an open subset of $\mathbb{R}^n$ with smooth, non-empty, compact boundary $\partial A$ such that $\mathcal{H}_{n-1}(\partial A \cap \partial \Omega) = 0$. Define the function $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) = \text{dist}(x, \partial A)$ for $x \in A$, $h(x) = -\text{dist}(x, \partial A)$ for $x \notin A$. Then $h$ is Lipschitz continuous, $|D h(x)| = 1$ for almost all $x \in \mathbb{R}^n$.
and

$$\lim_{t \to 0} \mathcal{H}_{n-1}(S_t \cap \Omega) = \mathcal{H}_{n-1}(\partial A \cap \Omega)$$

where \(S_t = \{x \in \mathbb{R}^n : h(x) = t\}\).

2. THE MAIN RESULT

We recall that \(\Omega\) denotes an open, bounded subset of \(\mathbb{R}^n (n \geq 2)\) with smooth boundary, and \(W, \sigma : [0, +\infty[ \rightarrow \mathbb{R}\) denote two non-negative continuous functions. We assume also that \(W(t) = 0\) only for \(t = \alpha\) or \(t = \beta\) with \(0 < \alpha < \beta\).

2.1. THEOREM. — Fix \(m \in [\alpha|\Omega|, \beta|\Omega|]\) and, for every \(\varepsilon > 0\), let \(u_{\varepsilon}\) be a solution of the minimization problem \((P_e)\). If each \(u_{\varepsilon}\) is of class \(C^1\) and there exists a sequence \((\varepsilon_h)\) of positive numbers, converging to zero, such that \((u_{\varepsilon_h})\) converges in \(L^1(\Omega)\) to a function \(u_0\), then

(i) \(W(u_0(x)) = 0\) [i.e. \(u_0(x) = \alpha\) or \(u_0(x) = \beta\)] for almost all \(x \in \Omega\);

(ii) the set \(E_0 = \{x \in \Omega : u_0(x) = \alpha\}\) is a solution of the minimization problem \((P_0)\) with

\[
\gamma \cdot \frac{\hat{\sigma} - \hat{\sigma} (t)}{2c_0}, \quad m_1 = \beta |\Omega| - m / (\beta - \alpha),
\]

where [see (5) and (6)]

\[
\hat{\sigma}(t) = \inf \left\{ \sigma(s) + \frac{1}{2} \int_{\alpha}^{s} W^{1/2}(y) \, dy : s \geq 0 \right\}
\]

for \(t = \alpha, \beta, \) and

\[
c_0 = \int_{\alpha}^{\beta} W^{1/2}(y) \, dy;
\]

(iii) \(\lim_{h \to +\infty} \varepsilon_h^{-1} \mathcal{E}_{w_h}(u_{\varepsilon_h})\)

\[
= 2c_0 P_\Omega(E_0) + \hat{\sigma}(\alpha) \mathcal{H}_{n-1}(\partial^* E_0 \cap \partial \Omega) + \hat{\sigma}(\beta) \mathcal{H}_{n-1}(\partial \Omega \setminus \partial^* E_0).
\]
For some comments about this statement we refer to Remarks 2. 5. The proof of Theorem 2.1 is similar to that one of the result with \( \sigma = 0 \) given in [10]. Nevertheless the extension is not trivial, because in the asymptotic \( (\varepsilon = 0) \) boundary behavior, given by \( \hat{\sigma} \), both the boundary and the interior behavior for \( \varepsilon > 0 \), given by \( W \) and \( \sigma \), are involved.

In the language of \( \Gamma \)-convergence theory, the proof of Theorem 2.1 consists in verifying that \( (\varepsilon^{-1} \mathcal{E}_\varepsilon + I_m) \) converges as \( \varepsilon \to 0^+ \), in the sense of \( \Gamma (L^1(\Omega)) \)-convergence, to the functional \( \mathcal{E}_0 + I_m \), at the points \( u \in L^1(\Omega) \) such that \( W(u(x)) = 0 \) for almost all \( x \in \Omega \) (cf Section 3 in [10]). The functional \( \mathcal{E}_0 \) was defined in (7); \( I_m \) denotes here the \( 0/\infty \) characteristic function of the constraint \( \int_\Omega u(x) \, dx = m. \)

The main steps in the proof of Theorem 2.1 are the following propositions.

2.2. Proposition. — Suppose that \( (v_\varepsilon)_{\varepsilon > 0} \) is a family in \( \{ u \in C^1(\Omega) : u \geq 0 \} \) which converges in \( L^1(\Omega) \) as \( \varepsilon \to 0^+ \) to a function \( v_0 \). If
\[
\liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon (v_\varepsilon) < +\infty,
\]
then \( v_0 \in BV(\Omega), W(v_0(x)) = 0 \) for almost all \( x \in \Omega \), and
\[
\mathcal{E}_0(v_0) \leq \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon(v_\varepsilon). \tag{11}
\]

2.3. Proposition. — Let \( A \) be an open, bounded subset of \( \mathbb{R}^n \) with smooth boundary such that \( \mathcal{H}_{n-1}(\partial A \cap \partial \Omega) = 0 \). Define the function \( v_0 : \Omega \to \mathbb{R} \) by \( v_0(x) = \alpha \) for \( x \in A \cap \Omega \), \( v_0(x) = \beta \) for \( x \in \Omega \setminus A \). For every \( r > 0 \) denote
\[
U_r = \left\{ v \in H^1(\Omega) : v \geq 0, \| v - v_0 \|_{L^2(\Omega)} < r, \int_\Omega v \, dx = \int_\Omega v_0 \, dx \right\}.
\]
Then, for every \( r > 0 \), we have that
\[
\limsup_{\varepsilon \to 0^+} \inf_{v \in U_r} \varepsilon^{-1} \mathcal{E}_\varepsilon(v) \leq \mathcal{E}_0(v_0). \tag{12}
\]

2.4. Remark. — For the connection between (12) and the corresponding inequality in the usual definition of \( \Gamma \)-convergence, see Proposition 1.14 of [4].

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Proof of Proposition 2.2. — By the continuity of $W$ and by Fatou's Lemma we have that
\[
\int_{\Omega} W(v_0) \, dx \leq \liminf_{\varepsilon \to 0^+} \int_{\Omega} W(v_\varepsilon) \, dx \leq \liminf_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon(v_\varepsilon) = 0;
\]
since $W \geq 0$, we have at once proved that $W(v_0(x)) = 0$ for almost all $x \in \Omega$.

Now
\[
\int_{\Omega} |D(\varphi \circ v_\varepsilon)| = \int_{\Omega} |\varphi'(v_\varepsilon(x))| \cdot |Dv_\varepsilon(x)| \, dx
\]
\[
= \int_{\Omega} W(v_\varepsilon(x)) |Dv_\varepsilon(x)| \, dx
\]
\[
\leq \int_{\Omega} [\varepsilon |Dv_\varepsilon|^2 + \varepsilon^{-1} W(v_\varepsilon)] \, dx
\]
so Proposition 1.4 and $\hat{\sigma} \leq \sigma$ apply for obtaining
\[
\mathcal{E}_\varepsilon(v_0) \leq \liminf_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon(v_\varepsilon)
\]
\[
\leq \liminf_{\varepsilon \to 0^+} \left\{ \int_{\Omega} [\varepsilon |Dv_\varepsilon|^2 + \varepsilon^{-1} W(v_\varepsilon)] \, dx + \int_{\partial \Omega} \hat{\sigma}(v_\varepsilon) \, d\mathcal{H}_{n-1} \right\}
\]
\[
\leq \liminf_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon(v_\varepsilon).
\]
It remains to prove that $v_0 \in BV(\Omega)$. This is obvious because $v_0$ takes only the values $\alpha$ and $\beta$, and $\varphi \circ v_0 \in BV(\Omega)$; hence the proof of Proposition 2.2 is complete. □

Proof of Proposition 2.3. — Let us fix $r > 0$ and also, for further convenience, $L \geq 0$, $M \geq 0$ and $\delta > 0$. We shall not often indicate in the following the dependence on $r$, $L$, $M$, $\delta$ as well as on the other data $n$, $\Omega$, $W$, $\alpha$, $\beta$, $\sigma$, $A$; in particular we shall denote by $c_1$, $c_2$, $\ldots$ real positive constants depending on all such data.

The following lemma contains a purely technical part of the proof.

2.5. Lemma. — Consider, for every $\varepsilon > 0$, the first-order ordinary differential equation
\[
|y'| = \varepsilon^{-1} (\delta + W(y))^{1/2}.
\]
Then there exist three constants $c_1$, $c_2$, $c_3$, independent of $\varepsilon$, and a Lipschitz continuous function $\chi_\varepsilon(s, t)$, defined on the upper half-plane $\mathbb{R} \times [0, +\infty[$, satisfying the following properties:

$$
\begin{align*}
\chi_\varepsilon(s, t) &= \alpha \quad \text{for} \quad s \geq c_1 \varepsilon, \quad t \geq c_1 \varepsilon, \\
\chi_\varepsilon(s, t) &= \beta \quad \text{for} \quad s \leq 0, \quad t \geq c_1 \varepsilon, \\
\chi_\varepsilon(s, t) &= L \quad \text{for} \quad s \leq 0, \\
\chi_\varepsilon(s, t) &= M \quad \text{for} \quad s \geq c_1 \varepsilon; \\
0 \leq \chi_\varepsilon \leq c_2, \quad |D\chi_\varepsilon| \leq c_3/\varepsilon;
\end{align*}
$$

(14)

on the strip $\{s \leq 0, t \leq c_1 \varepsilon\}$ the function $\chi_\varepsilon(s, t)$ depends only on $t$ and fulfils the equation (13) in the set $\{\chi_\varepsilon(t) \neq \beta\}$; on the strip $\{s \geq c_1 \varepsilon, t \leq c_1 \varepsilon\}$ the function $\chi_\varepsilon(s, t)$ depends only on $t$ and fulfils (13) in the set $\{\chi_\varepsilon(t) \neq \alpha\}$; on the strip $\{0 \leq s \leq c_1 \varepsilon, t \geq c_1 \varepsilon\}$ the function $\chi_\varepsilon(s, t)$ depends only on $s$ and fulfils (13) in the set $\{\chi_\varepsilon(s) \neq \alpha\}$.

Proof. — We have to determine $c_1$, $c_2$, $c_3$ and to complete the definition of $\chi_\varepsilon$ on the strips

$$
S_1 = \{s \leq 0, t \leq c_1 \varepsilon\}, \quad S_2 = \{s \geq c_1 \varepsilon, t \leq c_1 \varepsilon\}, \quad S_3 = \{0 \leq s \leq c_1 \varepsilon, t \geq c_1 \varepsilon\},
$$

and on the square $Q = [0, c_1 \varepsilon] \times [0, c_1 \varepsilon]$.

Let us begin by $S_1$, where we have the prescribed boundary values $\chi_\varepsilon(s, c_1 \varepsilon) = \beta$, $\chi_\varepsilon(s, 0) = L$. If $\beta = L$, we define $\chi_\varepsilon(t) = \beta$; if $\beta > L$, we solve the Cauchy problem

$$
y'(t) = \varepsilon^{-1} (\delta + W(y(t)))^{1/2}, \quad y(0) = L,
$$

and we define $\chi_\varepsilon(t) = \min \{\beta, y(t)\}$; if $\beta < L$, we solve the same Cauchy problem with $-y'$ instead of $y'$ and we define $\chi_\varepsilon(t) = \max \{\beta, y(t)\}$. Since

$$
|\chi_\varepsilon'(t)| = \varepsilon^{-1} (\delta + W(\chi_\varepsilon(t)))^{1/2} \geq \varepsilon^{-1} \delta^{1/2}
$$

provided that $\chi_\varepsilon(t) \neq \beta$, we have $\chi_\varepsilon(t) = \beta$ for $t \geq \varepsilon |\beta - L|/\delta$; then, in order that $\chi_\varepsilon$ takes the prescribed boundary values $\chi_\varepsilon(s, c_1 \varepsilon) = \beta$, we need $c_1 \geq |\beta - L|/\delta$. The same holds on $S_2$ and $S_3$, so we are led to define

$$
c_1 = \max \{|\beta - L|/\delta, |\alpha - \beta|/\delta, |\alpha - M|/\delta\}.
$$
Define also $c_2 = \max \{\alpha, \beta, L, M\}$, so that

$$0 \leq \chi_{\varepsilon} \leq c_2$$

and

$$|D\chi_{\varepsilon}| \leq \varepsilon^{-1} (\delta + \max \{W(s): 0 \leq s \leq c_2\})^{1/2}$$

on $(\mathbb{R} \times [0, +\infty]) \setminus Q$. Finally, as we know $\chi_{\varepsilon}$ on three sides of the square $Q$, we can extend $\chi_{\varepsilon}$ on $Q$ in such a way that $\chi_{\varepsilon}$ becomes Lipschitz continuous on the whole upper half-plane and (15) is satisfied with

$$c_3 = 3c_1 (\delta + \max \{W(s): 0 \leq s \leq c_2\})^{1/2}.$$ 

The proof of Lemma 2.5 is now complete. ■

Let us return to the proof of Proposition 2.3. The first part of the proof consists in constructing a family $(v_{\varepsilon})$ in $U$, such that $v_{\varepsilon}$ converges to $v_0$ as $\varepsilon \to 0^+$, and

$$\inf_{v \in U} \mathcal{E}_\varepsilon(v)$$

is approximatively equal to $\mathcal{E}_\varepsilon(v_{\varepsilon})$.

Define

[Diagram: \begin{itemize}
  \item $v_{\varepsilon} = \alpha$
  \item $v_{\varepsilon} = \beta$
  \item $v_{\varepsilon} = L$
  \item $v_{\varepsilon} = M$
  \item $C_{\varepsilon}$
  \item $B_{\varepsilon}$
  \item $\Omega$
\end{itemize}]

Fig. 1.
and let $\chi_\varepsilon$ be the function constructed in Lemma 2.5. Let, for $x \in \Omega$,

$$v'_\varepsilon(x) = \chi_\varepsilon (d_A(x), d_\Omega(x)).$$

Look at Figure 1 for understanding the meaning of our construction.

Denoting

- $S_s = \{x \in A \cap \Omega : d_A(x) = s\}$,
- $\Sigma^s = \{x \in \Omega \cap A : d_\Omega(x) = t\}$,
- $\Sigma^b = \{x \in \Omega \setminus A : d_\Omega(x) = t\}$,

Federer’s coarea formula and $|Dd_\Omega| = |Dd_A| = 1$ (see Proposition 1.6) yield

$$\int_\Omega |v'_\varepsilon - v_0| \, dx$$

$$\leq c_4 \left| \{x \in \Omega : d_\Omega(x) \leq c_1 \varepsilon \} \right| + \left| \{x \in A \cap \Omega : d_A(x) \leq c_1 \varepsilon \} \right|$$

$$= c_4 \int_0^{c_1 \varepsilon} \left[ \mathcal{H}^{n-1}(\Sigma^s \cup \Sigma^b) + \mathcal{H}^{n-1}(S_t) \right] \, dt;$$

hence, as $\partial A$ and $\partial \Omega$ are smooth, Proposition 1.6 implies

$$\int_\Omega |v'_\varepsilon - v_0| \, dx \leq c_5 \varepsilon$$

for $\varepsilon$ small enough. It follows that $v'_\varepsilon$ converges to $v_0$ in $L^1(\Omega)$ as $\varepsilon \to 0^+$ and, defining

$$\eta_\varepsilon = \int_\Omega v'_\varepsilon \, dx - \int_\Omega v_0 \, dx,$$

we have that

$$|\eta_\varepsilon| \leq c_5 \varepsilon$$

for $\varepsilon$ small enough.

Let us choose a point $x_0 \in \Omega \setminus \partial A$ and, for fixing the ideas, assume that $x_0 \in \Omega \cap A$. In the case $\Omega \cap A = \emptyset$ or $x_0 \in \Omega \setminus A$ the changes in the proof...
are trivial. Note that the closed ball $B_{\epsilon} = B(x_0, \epsilon^{1/n})$ is contained, for $\epsilon$ small enough, in the set $\{v'_{\epsilon} = \alpha\}$; then the function $v_{\epsilon}$, defined on $\Omega$ by $v_{\epsilon} = v'_{\epsilon}$ for $x \notin B_{\epsilon}$, and by

$$v_{\epsilon}(x) = \alpha + h_{\epsilon} \left(1 - \epsilon^{-1/n} |x - x_0|\right),$$

for $x \in B_{\epsilon}$, is Lipschitz continuous whenever $h_{\epsilon} \in \mathbb{R}$.

We now choose

$$h_{\epsilon} = -n \omega_{n-1} \eta_{\epsilon} \epsilon^{(1-n)/n},$$

with $\omega_{n-1}$ equal to the volume of the unit ball in $\mathbb{R}^{n-1}$, so that

$$\int_{B_{\epsilon}} (v_{\epsilon} - v'_0) \, dx = \int_{B_{\epsilon}} h_{\epsilon} \left(1 - \epsilon^{-1/n} |x - x_0|\right) \, dx = -\eta_{\epsilon},$$

and, by the definition of $\eta_{\epsilon}$ and $v_{\epsilon}$,

$$\int_{B_{\epsilon}} v_{\epsilon} \, dx = \int_{B_{\epsilon}} v'_0 \, dx$$

(18)

for $\epsilon$ small enough. Since, by (17),

$$|h_{\epsilon}| \leq c_\epsilon \epsilon^{1/n},$$

(19)

we have, for $\epsilon$ small enough,

$$0 \leq v_{\epsilon} \leq c_\gamma,$$

(20)

and

$$\lim_{\epsilon \to 0^+} \int_{\Omega} |v_{\epsilon} - v'_0|^2 \, dx = 0;$$

(21)

hence

$$\lim_{\epsilon \to 0^+} \inf_{v \in \mathcal{V}_\epsilon} \epsilon^{-1} \mathcal{E}_\epsilon(v) \leq \lim_{\epsilon \to 0^+} \sup_{v \in \mathcal{V}_\epsilon} \epsilon^{-1} \mathcal{E}_\epsilon(v_{\epsilon}).$$

(22)

The second part of the proof consists in a sharp estimate of the right-hand side of such inequality. For the sake of simplicity, let

$$\epsilon^{-1} \mathcal{E}_\epsilon(v_{\epsilon}) = \mathcal{E}'_\epsilon(v_{\epsilon}; \Omega) + \mathcal{E}''_\epsilon(v_{\epsilon})$$

with
\[ \delta''(v_\varepsilon; C) = \int_C [\varepsilon |Dv_\varepsilon|^2 + \varepsilon^{-1} W(v_\varepsilon)] \, dx \quad (C \subseteq \Omega), \]

and
\[ \delta''(v_\varepsilon) = \int_{\partial \Omega} \sigma(\tilde{v}_\varepsilon) \, d\mathcal{H}^{n-1}. \]

By (20) and (21), and by the continuity of $\sigma$ and of the trace operator, we at once obtain
\[ \limsup_{\varepsilon \to 0^+} \delta''(v_\varepsilon) \leq \int_{\partial \Omega} \sigma(\tilde{v}_0) \, d\mathcal{H}^{n-1} = \sigma(L) \mathcal{H}^{n-1}(\partial \Omega \setminus A) + \sigma(M) \mathcal{H}^{n-1}(\partial \Omega \cap A). \quad (23) \]

The evaluation of $\delta''(v_\varepsilon; \Omega)$ is more complicated. Let us divide $\Omega$ in seven parts, corresponding to the construction of $\chi_\varepsilon$ in Lemma 2.5 and of $v_\varepsilon$ (see Fig. 1):

\[ B_\varepsilon = B(x_0, \varepsilon^{1/n}), \]
\[ \Omega^x = \{ x \in \Omega : d_\varepsilon(x) > c_1 \varepsilon, \quad d_\varepsilon(x) > c_1 \varepsilon, \quad x \notin B_\varepsilon \}, \]
\[ \Omega^p = \{ x \in \Omega : d_\varepsilon(x) \leq 0; \quad d_\varepsilon(x) > c_1 \varepsilon \}, \]
\[ \Omega^{x_p} = \{ x \in \Omega : 0 < d_\varepsilon(x) \leq c_1 \varepsilon, \quad d_\varepsilon(x) > c_1 \varepsilon \}, \]
\[ \Omega^{x_L} = \{ x \in \Omega : d_\varepsilon(x) \leq 0, \quad d_\varepsilon(x) \leq c_1 \varepsilon \}, \]
\[ \Omega^{x_M} = \{ x \in \Omega : d_\varepsilon(x) > c_1 \varepsilon, \quad d_\varepsilon(x) \leq c_1 \varepsilon \}, \]
\[ \Omega^0 = \{ x \in \Omega : 0 < d_\varepsilon(x) \leq c_1 \varepsilon, \quad d_\varepsilon(x) \leq c_1 \varepsilon \}. \]

On $B_\varepsilon$ we have, by (19),
\[ \delta''(v_\varepsilon; B_\varepsilon) \]
\[ = \varepsilon |h_\varepsilon|^2 \varepsilon^{-2/n} |B_\varepsilon| + \varepsilon^{-1} \int_{B_\varepsilon} W(\alpha + h_\varepsilon(1 - \varepsilon^{-1/n}|x - x_0|)) \, dx \]
\[ \leq c_7 \left[ \varepsilon^2 + \int_0^1 W(\alpha + h_\varepsilon(1 - r)) r^{n-1} \, dr \right]. \]
hence

\[
\limsup_{\varepsilon \to 0^+} \mathcal{E}'(v_{\varepsilon} ; B_{\varepsilon}) = 0. \tag{24}
\]

On \( \Omega^e_a \) and \( \Omega^e_\beta \) the function \( v_{\varepsilon} \) equals respectively \( \alpha \) and \( \beta \), so that

\[
\mathcal{E}'(v_{\varepsilon} ; \Omega^e_a) + \mathcal{E}'(v_{\varepsilon} ; \Omega^e_\beta) = 0. \tag{25}
\]

On \( \Omega_{a\beta}^e \) we have \( v_{\varepsilon}(x) = \chi_{\varepsilon}(d_\Lambda(x), d_\Omega(x)) \); moreover, by (16), \( \chi_{\varepsilon}(s, t) = \chi_{\varepsilon}(s) \) depends only on the first variable and satisfies the equation

\[ -\chi_{\varepsilon}'(s) = \varepsilon^{-1} (\delta + W(\chi_{\varepsilon}(s)))^{1/2} \]

on an interval \( ]0, \tau_{\varepsilon}[ \), with \( 0 < \tau_{\varepsilon} < c_1 \varepsilon \), while \( \chi_{\varepsilon}(s) = \alpha \) for \( s \geq \tau_{\varepsilon} \). Then, applying Federer’s coarea formula and \( \chi_{\varepsilon}(0) = \beta \), we obtain that

\[
\begin{align*}
\mathcal{E}'(v_{\varepsilon} ; \Omega_{a\beta}^e) &= \int_0^{\tau_{\varepsilon}} \left[ \varepsilon \chi_{\varepsilon}'(s)^2 + \varepsilon^{-1} W(\chi_{\varepsilon}(s)) \right] \mathcal{H}_{n-1}(S_s) \, ds \\
&\leq \left( \sup_{0 \leq s \leq \tau_{\varepsilon}} \mathcal{H}_{n-1}(S_s) \right) \int_0^{\tau_{\varepsilon}} 2 (\delta + W(\chi_{\varepsilon}))^{1/2} \, ds \\
&= \left( \sup_{0 \leq s \leq \tau_{\varepsilon}} \mathcal{H}_{n-1}(S_s) \right) \left( 2 \int_\alpha^\beta (\delta + W(t))^{1/2} \, dt \right),
\end{align*}
\]

and therefore, by Proposition 1.6,

\[
\limsup_{\varepsilon \to 0^+} \mathcal{E}'(v_{\varepsilon} ; \Omega_{a\beta}^e) \leq 2 \mathcal{H}_{n-1}(\partial A \cap \Omega) \int_\alpha^\beta (\delta + W(t))^{1/2} \, dt. \tag{26}
\]

The same argument leads to

\[
\limsup_{\varepsilon \to 0^+} \mathcal{E}'(v_{\varepsilon} ; \Omega_{\beta L}^e) \leq 2 \mathcal{H}_{n-1}(\partial A \cap A) \left| \int_\alpha^\beta (\delta + W(t))^{1/2} \, dt \right|. \tag{27}
\]

and to

\[
\limsup_{\varepsilon \to 0^+} \mathcal{E}'(v_{\varepsilon} ; \Omega_{aM}^e) \leq 2 \mathcal{H}_{n-1}(\partial A \cap A) \left| \int_\alpha^\beta (\delta + W(t))^{1/2} \, dt \right|. \tag{28}
\]

Finally, on \( \Omega^e_0 \) we have, by (15),

\[
\mathcal{E}'(v_{\varepsilon} ; \Omega^e_0) \leq c_8 \varepsilon^{-1} |\Omega^e_0|.
\]
Note that, again by coarea formula,
\[
\left| \Omega_0^\varepsilon \right| = \int_0^{c_1^\varepsilon} \mathcal{H}^{n-1}_\varepsilon \left( \left\{ x \in \Omega : d_A(x) = s, \ d_\Omega(x) \leq c_1^\varepsilon \right\} \right) ds \\
\leq c_1 \left( \sup_{0 \leq s \leq c_1^\varepsilon} \mathcal{H}^{n-1}_\varepsilon (S_\varepsilon \setminus \Omega_0^\varepsilon) \right),
\]
where \( \Omega_0^\varepsilon \) denotes here the set \( \left\{ x \in \Omega : d_\Omega(x) > \rho \right\} \). Since we have \( \mathcal{H}^{n-1}_\varepsilon (\partial A \cap \partial \Omega_0^\varepsilon) = 0 \) for almost all \( \rho > 0 \), Proposition 1.6 gives
\[
\limsup_{\varepsilon \to 0^+} \left( \sup_{0 \leq s \leq c_1^\varepsilon} \mathcal{H}^{n-1}_\varepsilon (S_\varepsilon \setminus \Omega_0^\varepsilon) \right)
\leq \limsup_{\varepsilon \to 0^+} \left( \sup_{0 \leq s \leq c_1^\varepsilon} \mathcal{H}^{n-1}_\varepsilon (S_\varepsilon \setminus \Omega_0^\varepsilon) \right) = \mathcal{H}^{n-1}_\varepsilon (\partial A \cap \partial (\Omega \setminus \Omega_0^\varepsilon))
\]
for almost all \( \rho > 0 \); by taking the infimum for \( \rho > 0 \), we conclude that
\[
\lim_{\varepsilon \to 0^+} \varepsilon \varepsilon^\prime (v, \Omega_0^\varepsilon) = 0.
\]
(29)

Now, by collecting (22) to (29), we have that
\[
\limsup_{\varepsilon \to 0^+} \inf_{v \in \mathcal{U}_r} \varepsilon^{-1} \varepsilon (v) \leq 2 \mathcal{H}^{n-1}_\varepsilon (\partial A \cap \Omega) \int_\beta (\delta + W(t))^{1/2} dt \\
+ \mathcal{H}^{n-1}_\varepsilon (\partial \Omega \cap A) \left( 2 \left\| \int_\gamma (\delta + W(t))^{1/2} dt \right\| + \sigma(M) \right) \\
+ \mathcal{H}^{n-1}_\varepsilon (\partial \Omega \cap A) \left( 2 \left\| \int_\beta (\delta + W(t))^{1/2} dt \right\| + \sigma(L) \right).
\]
The left-hand side does not depend on \( \delta \), \( L \), and \( M \), so, by taking first the infimum for \( \delta > 0 \), and then the infima for \( M \geq 0 \) and for \( L \geq 0 \) of the right-hand side, we obtain, by the definition of \( \widehat{\sigma} \) and \( c_0 \), that
\[
\limsup_{\varepsilon \to 0^+} \inf_{v \in \mathcal{U}_r} \varepsilon^{-1} \varepsilon (v) \leq 2 c_0 \mathcal{H}^{n-1}_\varepsilon (\partial A \cap \Omega) + \widehat{\sigma} (\alpha) \mathcal{H}^{n-1}_\varepsilon (\partial \Omega \cap A) \\
+ \widehat{\sigma} (\beta) \mathcal{H}^{n-1}_\varepsilon (\partial \Omega \setminus A) \\
= 2 c_0 \mathcal{H}^{n-1}_\varepsilon (\partial A \cap \Omega) + \int_{\delta \Omega} \widehat{\sigma} (\bar{\tau}) d\mathcal{H}^{n-1}_\varepsilon. \quad (30)
\]
Remarking that the Fleming-Rishel formula yields

\[ 2 \int_\Omega |D(\phi \circ v_0)| = 2 \int_\Omega P_\Omega(\{ x \in \Omega : \phi(v_0(x)) > t \}) dt \]

\[ = 2 \int_{\phi(t)} P_\Omega(A \cap \Omega) dt = 2c_0 \mathcal{H}_{n-1}(\partial A \cap \Omega), \tag{31} \]

the right-hand side of (30) agrees with \( \mathcal{E}_0(v_0) \) and the proof of Proposition 2.3 is complete. \( \blacksquare \)

Now, we can prove Theorem 2.1.

**Proof of Theorem 2.1.** — Assume for simplicity that all \((u_\varepsilon)\) converges, as \(\varepsilon \to 0^+\), to \(u_0\). By constructing, as in the proof of Theorem I of [10], a suitable family of comparison piecewise affine functions, we first obtain that

\[ \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon(u_\varepsilon) < +\infty; \tag{32} \]

hence Proposition 2.2 gives \( W(u_0(x)) = 0 \) and

\[ \mathcal{E}_0(u_0) \leq \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon(u_\varepsilon). \]

Now, let \( \mathcal{A} \) be the class of all open, bounded subsets \( A \) of \( \mathbb{R}^n \), with smooth boundary, such that \( \mathcal{H}_{n-1}(\partial A \cap \Omega) = 0 \) and \( |A \cap \Omega| = |E_0| = m_1 \). For every \( A \in \mathcal{A} \), we define \( v_0^A(x) = \alpha \) for \( x \in A \cap \Omega \), \( v_0^A(x) = \beta \) for \( x \in \Omega \setminus A \); applying Proposition 2.3 with \( r = 1 \), we infer that

\[ \limsup_{\varepsilon \to 0^+} \inf_{v \in U} \varepsilon^{-1} \mathcal{E}_\varepsilon(v) \leq \mathcal{E}_0(v_0^A), \]

where

\[ U = \left\{ v \in H^1(\Omega) : v \geq 0, \int_\Omega |v - v_0^A|^2 dx < 1, \int_\Omega v dx = \int_\Omega v_0^A dx \right\} \]

Since

\[ \int_\Omega v_0^A dx = m, \]

we have, by the minimality of \( u_\varepsilon \), that
\[
\mathcal{E}_\varepsilon(u_\varepsilon) \leq \mathcal{E}_\varepsilon(v), \quad \forall v \in U,
\]
and we conclude that
\[
\mathcal{E}_0(u_0) \leq \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \mathcal{E}_0(v_0^A) \tag{33}
\]
for every \( A \in \mathcal{A} \). Arguing as for (30) and (31), we obtain
\[
\mathcal{E}_0(u_0) = 2c_0 \mathcal{P}_\Omega(E_0) + \hat{\sigma}(\alpha) \mathcal{H}_{n-1}(\partial^* E_0 \cap \partial\Omega)
+ \hat{\sigma}(\beta) \mathcal{H}_{n-1}(\partial\Omega \setminus \partial^* E_0) \tag{34}
\]
and
\[
\mathcal{E}_0(v_0^A) = 2c_0 \mathcal{P}_\Omega(A) + \hat{\sigma}(\alpha) \mathcal{H}_{n-1}(\partial\Omega \cap A) + \hat{\sigma}(\beta) \mathcal{H}_{n-1}(\partial\Omega \setminus A),
\]
so that
\[
\mathcal{P}_\Omega(E_0) + \gamma \mathcal{H}_{n-1}(\partial^* E_0 \cap \partial\Omega) \leq \mathcal{P}_\Omega(A) + \gamma \mathcal{H}_{n-1}(\partial(A \cap \Omega) \cap \partial\Omega)
\]
for every \( A \in \mathcal{A} \). Then the required minimality property (ii) of \( E_0 \) follows from Proposition 1.5. Finally, by employing again (33) and Proposition 1.5, with
\[
\lambda = \limsup_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon(u_\varepsilon),
\]
we have that
\[
\mathcal{E}_0(u_0) = \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon(u_\varepsilon);
\]
therefore the result (iii) follows from (34) and this concludes the proof of Theorem 2.1. ■

2.5. Remarks. — (a) The assumption that \( \partial\Omega \) is smooth in Theorem 2.1 cannot be easily replaced by \( \partial\Omega \) Lipschitz continuous, except for \( \sigma = 0 \) (cf. [10]). In fact, as we already observed in Remark 1.3, the liquid-drop problem \( (\text{P}_0) \) in bounded domains with angles requires a particular treatment.

(b) Well-known growth conditions at infinity on \( W \) guarantee that the minimizers \( u_\varepsilon \) are of class \( C^1 \). Of course, if \( u_\varepsilon \in L^{\infty}(\Omega) \), then \( u_\varepsilon \) is smooth.
(c) The (relative) compactness of \( \{u_k\} \) in \( L^1(\Omega) \) may be studied as in Proposition 4 of [10]. It is ensured either by equiboundedness of \( \{u_k\} \) (cf. [9]), or again by a growth condition at infinity on \( W \).

3. A DISCUSSION ABOUT CRITICAL POINT WETTING

We make here more precise some statements of Introduction, about the connection between Theorem 2.1 and the critical point wetting theory by J. W. Cahn [2].

According to this author, and looking in particular at page 3668 and Figure 4 of [2], we assume that the contact energy \( \sigma \) is a non-negative, convex, decreasing function of class \( C^1 \). Moreover, we denote by \( W_T \) the Gibbs free energy at the temperature \( T \) (recall that we are concerned with isothermal phenomena), by \( \alpha_T \) and \( \beta_T \) the corresponding zeros, by \( M_T \) the maximum height of the hump between \( \alpha_T \) and \( \beta_T \). We assume that \( W_T(t) \) increases for \( t \geq \beta_T \). By thermodynamic and experimental reasons (cf. [2], page 3669), we assume also that \( \beta_T \) and \( M_T \) are decreasing in \( T \), \( \alpha_T \) is increasing in \( T \) and \( (\beta_T - \alpha_T) \to 0 \), \( M_T \to 0 \) when \( T \) increases towards a critical temperature \( T_0 \) (critical point of a binary system). The \( \varphi \) and \( \hat{\sigma} \) corresponding to \( \sigma \) and \( W_T \) will be denoted by \( \varphi_T \) and \( \hat{\sigma}_T \).

Let us compute now \( \varphi_T(t) \) for \( t \geq \alpha_T \). Since \( \sigma \) is decreasing and

\[
\lim_{t \to +\infty} \varphi_T(t) = +\infty,
\]

we obtain that the minimum of \( s \mapsto \sigma(s) + 2|\varphi_T(t) - \varphi_T(s)| \) is attained at a point \( s = \lambda_{t,T} \geq t \). Moreover, either \( \lambda_{t,T} = t \), or

\[
-\sigma'(\lambda_{t,T}) = 2\varphi'(\lambda_{t,T}) = 2W^{1/2}(\lambda_{t,T}).
\]

For \( T_0 - T \) small enough, that is for a temperature \( T \) below and close to the critical one, the hump in the graph of \( 2W^{1/2}_T \) between \( \alpha_T \) and \( \beta_T \) does not intersect the graph of \( -\sigma' \) in the same interval; on the other hand, since \( \sigma \) is convex, the decreasing function \( -\sigma' \) does intersect the increasing function \( 2W^{1/2}_T \) at a single point \( \lambda_T \geq \beta_T \) (see Fig. 2).
It is easy to check that $\lambda_T$ (independent of $t$) is actually the minimum point of $s \mapsto \sigma(s) + 2|\varphi_T(t) - \varphi_T(s)|$; hence we conclude that

$$\hat{\sigma}_T(t) = \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(t)), \quad \forall t \geq \lambda_T,$$

and hence

$$\gamma_T = \frac{\hat{\sigma}_T(\alpha_T) - \hat{\sigma}_T(\beta_T)}{2(\varphi_T(\beta_T) - \varphi_T(\alpha_T))} = 1$$

in correspondence with the phenomenon of the perfectly wetting phase $\beta$ quoted in Introduction. If one prefers not to consider the modified energy $\tilde{\sigma}_T$, it could be alternatively thought that a very thin layer of a third phase of the fluid, with density $\lambda_T > \beta_T$, appears on the whole boundary of the container.

When the temperature $T$ is much more below $T_0$, a possible relative behavior of $-\sigma'$ and $2W^{1/2}$ is shown in Figure 3, with both $\mu_T$ and $\lambda_T$ relative minima of

$$s \mapsto \sigma(s) + 2|\varphi_T(t) - \varphi_T(s)|$$

for every $t \geq \alpha_T.$
Note that
\[ \hat{\sigma}_T(\beta_T) = \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(\beta_T)), \]
while the value of \( \sigma_T(\alpha_T) \) depends on the areas A and B. Indeed, if \( A \leq B \), then
\[ \hat{\sigma}_T(\alpha_T) = \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(\alpha_T)) \]
and \( \gamma_T = 1 \) as above. On the contrary, if \( A > B \), then
\[ \hat{\sigma}_T(\alpha_T) = \sigma(\mu_T) + 2(\varphi_T(\mu_T) - \varphi_T(\alpha_T)) < \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(\alpha_T)) \]
and \( \gamma_T < 1 \); since we have analogously \( \gamma_T > -1 \), this means that both the fluid phases wet the container walls. Or, alternatively, two thin layers of fluid, with densities \( \mu_T \) and \( \lambda_T \), are interposed between the phases \( \alpha_T \) and \( \beta_T \) and the container.

Finally, we want to remark that the equation \( \hat{\sigma} = \sigma \) is equivalent to the inequality
\[ |\sigma(s_1) - \sigma(s_2)| \leq 2|\varphi(s_1) - \varphi(s_2)|, \quad \forall 0 \leq s_1 \leq s_2, \tag{35} \]
which gives in particular
\[ \sigma'(\alpha) \geq \varphi'(\alpha) = W^{1/2}(\alpha) = 0 \]
and analogously $\sigma'(\beta) \geq 0$; hence (35) cannot be satisfied in the case $\sigma' < 0$. It would be interesting to know whether the inequality (35), and then the equality $\sigma = \hat{\sigma}$, are verified in some other thermodynamic situation, different from the phenomenon studied in [2] by Cahn.

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