ANNALES DE L'I. H. P., SECTION C

J. B. HIRIART-URRUTY
PH. PLAZANET

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Annales de l'I. H. P., section C, tome S6 (1989), p. 325-338
<http://www.numdam.org/item?id=AIHPC_1989__S6__325_0>

MOREAU'S DECOMPOSITION THEOREM REVISITED

J.B. HIRIART-URRUTY
U.F.R. Mathématiques, Informatique, Gestion, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex, France

Ph. PLAZANET
Département de Mathématiques Appliquées, E.N.S.I.C.A., 49 Avenue Léon Blum, 31056 Toulouse Cédex, France

ABSTRACT

Given two convex functions g and h on a Hilbert space, verifying $g + h = \frac{1}{2}\|.\|^2$, we show there necessarily exists a lower-semicontinuous convex function F such that $g = F \,\square\, \frac{1}{2}\|.\|^2$ and $h = F^* \,\square\, \frac{1}{2}\|.\|^2$. An explicit formulation of F is given as a deconvolution of a convex function by another one. The approach taken here as well as the way of factorizing g and h shed a new light on what is known as Moreau's theorem in the literature on Convex Analysis.

1 - INTRODUCTION

The starting point of our study was the following question, which takes root in the regularization processes studied in [9]: *Let $(H, <.,.>)$ be a Hilbert space, let f be a function on H and $\alpha > 0$ such that*

(1.1) *both* $\frac{\alpha}{2}\|.\|^2 - f$ *and* $\frac{\alpha}{2}\|.\|^2 + f$ *are convex functions on* H *(Here* $\|.\|$ *denotes the norm on* H *associated with the inner product* $<.,\cdot>$*).*
How to show that f *is Gâteaux-differentiable on* H *with*

(1.2) $\|f'(x) - f'(y)\| \leqslant \alpha \|x-y\|$ *for all* x, y *in* H ?

The question of differentiability of f offers no difficulty since it readily comes from (1.1) that both $g := \frac{\alpha}{2}\|.\|^2 - f$ and $h := \frac{\alpha}{2}\|.\|^2 + f$ are finite convex functions on H, so that the directional derivative $f'(x,.)$ of f exists and satisfies:

(1.3) $f'(x,.) = \alpha < x, .> - g'(x,.) = h'(x,.) - \alpha < x, .>$
for all $x \in H$, whence $f'(x,.)$ is linear and continuous (since convex and concave) for all $x \in H$. The problem now is to prove that f' is Lipschitz on H, with Lipschitz constant α. It is clear, in view of (1.1), that α is the best Lipschitz constant one can expect on f'. Even if the problem can be reduced (by an argument of projection) to the same problem in a 2-dimensional context (cf.[6]), it is not simpler for all that. So, the question should be broached in a different way.
When reading (1.1), our first reaction is to observe that f is necessarily a d.c. function (i.e., a difference of convex functions) :

(1.4) $f = \frac{\alpha}{2}\|.\|^2 - g$ or $f = h - \frac{\alpha}{2}\|.\|^2$.

D.C. functions enjoy differentiability properties similar to those of convex functions, but to keep control of their derivatives is hopeless in general ([3, §II.2]). Things are however made easier since one of the functions involved in the decomposition of f is merely $\frac{\alpha}{2}\|.\|^2$. Referring back to (1.4), we see we are in the presence of two convex functions g and h such that

$$(1.5)\qquad g + h = \alpha\ \|.\|^2 .$$

We thus reformulate the question posed at the beginning in the following way : *Let g and h be convex functions on H and $\alpha > 0$ such that*

$$(1.6)\qquad g + h = \alpha\ \|.\|^2$$

Show that both g and h are Gâteaux-differentiable on H with

$$(1.7)\qquad < g'(x) - g'(y),\ h'(x) - h'(y) > \geqslant 0 \text{ for all } x,\ y \text{ in } H.$$

Let us prove that the two formulations are equivalent.
Suppose we have answered the question in its second formulation and wish to answer it in its first one. Then, posing $g = \frac{\alpha}{2}\|.\|^2 - f$ and $h = \frac{\alpha}{2}\|.\|^2 + f$, we get that f is differentiable and

$$(1.8)\qquad \begin{aligned} &< g'(x) - g'(y),\ h'(x) - h'(y) > \\ &= \alpha^2\ \|x-y\|^2 - \|f'(x) - f'(y)\|^2 \geqslant 0 \text{ for all } x,\ y \in H, \end{aligned}$$

which is (1.2) precisely.
Conversely, suppose we have answered the question in its original formulation and wish to answer it in its second one. Posing $f = \frac{\alpha}{2}\|.\|^2 - g = h - \frac{\alpha}{2}\|.\|^2$, we indeed have a function f such that both $\frac{\alpha}{2}\|.\|^2 + f$ and $\frac{\alpha}{2}\|.\|^2 + f$ are convex functions on H. Then, the differentiability of f induces that of g and h, and, in view of (1.8), the inequality (1.2) induces (1.7).

Starting from convex functions g and h such that $g + h = \alpha \|x\|^2$, we actually can prove more about g and h, namely that g and h can be *factorized* in the following form : $g = 2\alpha(F \,\square\, \frac{1}{2} \|.\|^2)$ and $h = 2\alpha \, (F^* \,\square\, \frac{1}{2} \|.\|^2)$ for some lower-semicontinuous convex function F. As a result, g and h will appear as Moreau-Yosida regularized versions of F and F^* respectively, so that all the announced properties on g and h follow.

2 - MOREAU'S DECOMPOSITION THEOREM REVISITED

2.1 - Let $\Gamma_o(H)$ denote the set of convex functions F from H into $(-\infty, +\infty]$ which are lower-semicontinuous and not identically equal to $+\infty$. What is known as Moreau's theorem in the context of Convex Analysis asserts the following : *for any* $F \in \Gamma_o(H)$

$$(2.1) \qquad F \,\square\, \frac{1}{2} \|.\|^2 + F^* \,\square\, \frac{1}{2} \|.\|^2 = \frac{1}{2} \|.\|^2. \qquad ([9])$$

By choosing F as the indicator function of a closed convex cone K of H, F^* is the indicator function of the polar cone K^o to K, $F \,\square\, \|.\|^2$ is the square of the distance function to K, so that (2.1) reads as a kind of Pythagore's theorem :

$$(2.2) \qquad d_K^2 + d_{K^o}^2 = \|.\|^2. \qquad ([7,9])$$

Such a decomposition has proved useful in all areas involving a Hilbertian structure (Euclidean spaces of matrices in Statistics, Sobolev spaces in Nonlinear Analysis [7,11], etc).

Our goal now is to prove a sort of *converse* to Moreau's theorem : starting with convex functions g and h such that $g + h = \frac{1}{2} \|.\|^2$, we want to factorize g and h in the form $F \,\square\, \frac{1}{2} \|.\|^2$ and $F^* \,\square\, \frac{1}{2} \|.\|^2$ respectively, by providing also an *explicit* formulation for F.

THEOREM (of factorization)

Let g and h be convex functions on H such that $g + h = \frac{1}{2} \|.\|^2$.
There then exists $F \in \Gamma_o(H)$ such that

(2.3) $\quad g = F \boxminus \frac{1}{2} \|.\|^2 \quad$ *and* $\quad h = F^* \boxminus \frac{1}{2} \|.\|^2$.

Moreover

(2.4) $\quad g'(x) \in \partial F\ (h'(x))$ *and* $h'(x) \in \partial F^*(g'(x))$ *for all* $x \in H$.

Before going into the details of the proof, we need to recall some facts about an operation on convex functions which has been recently introduced ([4]), and which bears the name of *deconvolution of a function by another one.*
Given φ and ψ in $\Gamma_o(H)$, the deconvolution of φ by ψ is the function denoted $\varphi \boxminus \psi$ and defined as:

$$\forall\, x \in H,\ (\varphi \boxminus \psi)\ (x) = \sup_{\psi(u) < +\infty} \{\varphi(x+u) - \psi(u)\}.$$

The two main properties to be noticed are : $\varphi \boxminus \psi \in \Gamma_o(H)$ (or possibly identically equal to $+\infty$) and
$(\varphi \boxminus \psi)^* = (\varphi^* - \psi^*)^{**}$ (see [5] and the references therein).

Proof of Theorem 1

We set $F = g \boxminus \frac{1}{2} \|.\|^2$, that is :

$$\forall\ x \in H,\ F(x) = \sup_{u \in H} \left\{ g(x+u) - \frac{1}{2} \|u\|^2 \right\}.$$

Since $g + h = \frac{1}{2} \|.\|^2$, we also have :

$$\forall\ x \in H,\ F(x) = \sup_{w \in H} \left\{ g(v) - \frac{1}{2} \|x-v\|^2 \right\}$$

$$= \sup_{v \in H} \left\{ \frac{1}{2} \|v\|^2 - h(v) - \frac{1}{2} \|x-v\|^2 \right\}$$

$$= \sup_{v \in H} \left\{ < x, v> -h(v) - \frac{1}{2} \|x\|^2 \right\}$$

$$= h^*(x) - \frac{1}{2} \|x\|^2 .$$

Whence

$$(2.5) \qquad F = g \boxminus \frac{1}{2} \|.\|^2 = h^* - \frac{1}{2} \|.\|^2 \qquad (\in \Gamma_o(H)).$$

By inverting the role of g and h, we get in a same way :

$$(2.6) \qquad h \boxminus \frac{1}{2} \|.\|^2 = g^* - \frac{1}{2} \|.\|^2 \quad (\in \Gamma_o(H)).$$

But the formula giving the conjugate function of $g \boxminus \frac{1}{2} \|.\|^2$ (as aforesaid) yields that

$$\left(g \boxminus \frac{1}{2} \|.\|^2 \right)^* = \left(g^* - \frac{1}{2} \|.\|^2 \right)^{**} = g^* - \frac{1}{2} \|.\|^2 .$$

Thus, the function defined in (2.6) is nothing else than F^*. Consequently, the usual calculus rules on conjugate functions, applied to
$h^* = F + \frac{1}{2} \|.\|^2$ and $g^* = F^* + \frac{1}{2} \|.\|^2$, induce that
$g = F \square \frac{1}{2} \|.\|$ and $h = F^* \square \frac{1}{2} \|.\|^2$.

Now, calculus rules on subdifferentials, applied to
$h^* = F + \frac{1}{2} \|.\|^2$ for example, yield that

$$\partial h^* (h'(x)) = \partial F(h'(x)) + \{h'(x)\} \text{ for all } x \in H.$$

But $x \in \partial h^*(h'(x))$ for all $x \in H$, whence

$g'(x) \in \partial F\,(h'(x))$ for all $x \in H$.

Remark 1 The factorization of g and h in the form $F \,\square\, \frac{1}{2}\,\|\cdot\|^2$ and $F^* \,\square\, \frac{1}{2}\,\|\cdot\|^2$ respectively, with $F \in \Gamma_0(H)$, is *unique* : indeed, if $\phi \in \Gamma_0(H)$ verifies $\phi \,\square\, \frac{1}{2}\,\|\cdot\|^2 = g$ and $\phi^* \,\square\, \frac{1}{2}\,\|\cdot\|^2 = h$, we get that

$$\phi = h^* - \frac{1}{2}\,\|\cdot\|^2 = \left(g^* - \frac{1}{2}\,\|\cdot\|^2\right)^*, \tag{2.7}$$

that is $\phi = g \,\boxminus\, \frac{1}{2}\,\|\cdot\|^2$.

Remark 2. The dual formulation of the theorem of factorization is as follows : *If k, $\ell \in \Gamma_0(H)$ satisfy $k \,\square\, \ell = \frac{1}{2}\,\|\cdot\|^2$, there then exists an unique $K \in \Gamma_0\,(H)$ such that*

$$k = K + \frac{1}{2}\,\|\cdot\|^2 \textit{ and } \ell = K^* + \frac{1}{2}\,\|\cdot\|^2.$$

Example. Let S be a nonempty closed convex set of H. We have that

$$\underbrace{\frac{1}{2}\,d_S^2}_{g} + \underbrace{\frac{1}{2}\left(\|\cdot\|^2 - d_S^2\right)}_{h} = \frac{1}{2}\,\|\cdot\|^2.$$

It is known that $h = \frac{1}{2}\left(\|\cdot\|^2 - d_S^2\right)$ is convex ([1]) (*). Then the only solution F yielded by the factorization theorem is $F = \psi_S$ (the indicator function of S). Note incidentally the pairing result :

$$\frac{1}{2}\left(\|\cdot\|^2 - d_S^2\right) = \psi_S^* \,\square\, \frac{1}{2}\,\|\cdot\|^2, \tag{2.8}$$

which also can be obtained from direct calculations or as an example of Moreau's theorem (cf. (2.1)).

2.2. *Applications*

2.2.1. As a first application of the factorization theorem, we look back at the question posed in the Introduction and which motivated our study.

Consider two convex functions g and h on H, $\alpha > 0$, such that $g + h = \alpha \, \|.\|^2$. According to the factorization theorem, there exists a unique $F \in \Gamma_0(H)$ such that :

$$g/2\alpha = F \,\square\, \frac{1}{2} \, \|.\|^2 \text{ and } h/2\alpha = F^* \,\square\, \frac{1}{2} \, \|.\|^2,$$

$$g'(x) \in \partial F(h'(x)) \text{ for all } x \in H.$$

Due to the monotonicity property of ∂F, the second relation above induces that

$$< g'(x) - g'(y), \; h'(x) - h'(y) > \; \geqslant 0 \text{ for all } x \in H,$$

which is the relation (1.7) required.

2.2.2. A second application of the factorization theorem is the following result.

COROLLARY 2. *Let $f : H \longrightarrow \mathbb{R}$ be a Gâteaux-differentiable function and $\alpha > 0$. Then the next statements are equivalent* :

(2.9) $\quad |< f'(x) - f'(y), \; x-y >| \leqslant \alpha \, \|x-y\|^2$ for all $x, y \in H$;

(2.10) $\quad \| f'(x) - f'(y) \| \leqslant \alpha \, \|x-y\|$ for all $x, y \in H$.

Although it was known for C^2 - functions, this equivalence is rather surprising ; clearly, (2.9) which involves f on line segments is easier to check.

(*) Actually, h is convex whatever S be. But to ensure the convexity of g also, we need the convexity of S.

To prove that (2.9) implies (2.10), it suffices to observe that both $\frac{\alpha}{2}\|.\|^2 - f$ and $\frac{\alpha}{2}\|.\|^2 + f$ are convex functions on H ; (2.10) then follows from the equivalence properties stated in the Introduction.
Corollary 2 answers a question the first author alluded to in [3, p. 48 bottom] concerning the comparison between (globally) $C^{1,1}$ functions f and those satisfying an inequality like (2.9).

2.2.3. A third application of the factorization theorem is a characterization of the so-called α-strongly convex functions. We recall that, given $\alpha > 0$, $f \in \Gamma_0(H)$ is said to be *α-strongly convex* (or strongly convex with modulus α) if

$$f(tx + (1-t)\,x') \leqslant t\,f(x) + (1-t)\,f(x') - \frac{\alpha}{2}\,t(1-t)\,\|x-x'\|^2$$

for all x, x' in H and $t \in]0,1[$. In other words, that means that $f - \frac{\alpha}{2}\|.\|^2$ is still a convex function ($\in \Gamma_0(H)$). The next characterization of α-strongly convex functions has also been observed by Volle ([10]) who, furthermore, introduced a new conjugacy mapping for such functions by substituting the "coupling functional"
$(x, y) \longmapsto \frac{\alpha}{2}\|x-y\|^2$ for the usual bilinear functional
$(x, y) \longmapsto \langle x, y \rangle$.

COROLLARY 3. Let $f \in \Gamma_o(H)$. The following are equivalent :
(2.11) *f is α-strongly convex ;*
(2.12) $\frac{1}{2\alpha}\|.\|^2 - f^* \in \Gamma_o(H)$;
(2.13) *There exists $\varphi \in \Gamma_o(H)$ such that $f \,\square\, \varphi = \frac{\alpha}{2}\|.\|^2$.*

Condition (2.12) actually says more than what is stated : since f^* is itself in $\Gamma_0(H)$, condition (2.12) implies that

f^* is finite on H ; in fact we will see in the course of the proof that f^* is a $C^{1,1}$ function (*).

Likewise, a consequence of (2.13) is that $\varphi^* = \frac{1}{2\alpha}\|.\|^2 - f^*$, whence the exhibited function φ is α-strongly convex ; indeed,

$$(2.14)\qquad \varphi = \left(\frac{1}{2\alpha}\|.\|^2 - f^*\right)^* = \frac{\alpha}{2}\|.\|^2 \boxminus f,$$

$$(2.15)\qquad f = \left(\frac{1}{2\alpha}\|.\|^2 - \varphi^*\right)^* = \frac{\alpha}{2}\|.\|^2 \boxminus \varphi.$$

Proof. *(2.12)* $\Longrightarrow$ *(2.11)*. Let g denote the convex function $\frac{1}{2\alpha}\|.\|^2 - f^*$. Since $\alpha\, g + \alpha\, f^* = \frac{1}{2}\|.\|^2$, the theorem of factorization yields that there exists $F \in \Gamma_0(H)$ such that $\alpha\, f^* = F \,\square\, \frac{1}{2}\|.\|^2$. Consequently, f assigns $\frac{1}{\alpha} F^*(\alpha x) + \frac{\alpha}{2}\|x\|^2$ to $x \in H$, so that $f - \frac{\alpha}{2}\|.\|^2$ is still a convex function. We thus have proved f is α-strongly convex.

(2.11) $\Longrightarrow$ *(2.13)*. Let χ denote the convex function $\frac{f}{\alpha} - \frac{1}{2}\|.\|^2$; we set $\varphi = \alpha\, \chi^* + \frac{\alpha}{2}\|.\|^2$. Starting from the relation $\frac{f}{\alpha} = \chi + \frac{1}{2}\|.\|^2$,
we get successively

(*) The equivalence of (2.11) and (2.12) appears also as a by-product of more general results on the duality relations between uniformly convex functions and uniformly smooth convex functions ([2]).

$$\left(\frac{f}{\alpha}\right)^{*} = \chi^{*} \,\square\, \frac{1}{2}\|.\|^{2}$$

(2.16)

$$= \frac{1}{2}\|.\|^{2} - \left(\chi \,\square\, \frac{1}{2}\|.\|^{2}\right)$$ by Moreau's theorem.

Let us calculate $g = \left(\frac{f}{\alpha}\right) \square \left(\frac{\varphi}{\alpha}\right)$. Since $g^{*} = \left(\frac{f}{\alpha}\right)^{*} + \left(\frac{\varphi}{\alpha}\right)^{*}$, we infer from the definition of φ and (2.16) :

$$g^{*} = \frac{1}{2}\|.\|^{2} - \left(\chi \,\square\, \frac{1}{2}\|.\|^{2}\right) + \chi \,\square\, \frac{1}{2}\|.\|^{2} = \frac{1}{2}\|.\|^{2}.$$

Whence $g = \frac{1}{2}\|.\|^{2}$ and (2.13) is secured.

(2.13) $\Longrightarrow$ *(2.12)* From $f \,\square\, \varphi = \frac{\alpha}{2}\|.\|^{2}$ we derive $f^{*} + \varphi^{*} = \frac{1}{2\alpha}\|.\|^{2}$, so that $\frac{1}{2\alpha}\|.\|^{2} - f^{*} = \varphi^{*} \in \Gamma_{0}(H)$. ∎

3 - COMPARISON WITH MOREAU'S APPROACH

In his seminal 1965 paper ([8]), Moreau extensively studied the functions of the form $F \,\square\, \frac{1}{2}\|.\|^{2}$, $F \in \Gamma_{0}(H)$, and defined the so-called *proximal mapping* prox_F which assigns to $x \in H$ the unique point where the infimum of $u \longmapsto F(u) + \frac{1}{2}\|x - u\|^{2}$ is achieved. Among other properties, he proved that prox_F is a Lipschitz mapping (with Lipschitz constant 1) and that prox_F is actually a gradient mapping (i.e., there is a differentiable function Φ, called primitive function of prox_F, such that $\Phi'(x) = \mathrm{prox}_F(x)$ for all $x \in H$).

In a much less read section ([8, §9]), Moreau introduced a binary relation between convex functions by defining what he meant by "a convex function g less convex than a convex function f". More interesting is the characterization of such a relationship when f is $\frac{1}{2}\|.\|^2$ precisely, which now allows us to make connections with our approach.

According to Moreau ([8, définition 9.b]), a convex function g is less convex than a convex function f (or f is more convex than g) if there exists a convex function h such that f = g + h. He then proved the equivalence of the following properties ([8, Proposition 9.b and Proposition 10.b] :

(3.1) *$g \in \Gamma_o(H)$ is less convex than* $\frac{1}{2}\|.\|^2$;

(3.2) *The conjugate function of $g \in \Gamma_o(H)$ is more convex than* $\frac{1}{2}\|.\|^2$;

(3.3) *g is the primitive function of a proximal mapping* ;

(3.4) *$g \in \Gamma_o(H)$ is differentiable and g'is Lipschitz on H with a Lipschitz constant 1.*

(3.1) expresses the existence of a convex function h such that $g + h = \frac{1}{2}\|.\|^2$, which is precisely the situation we have considered here. According to (3.4), such a g is differentiable and $\|g'(x) - g'(y)\| \leqslant \|x-y\|$ for all x, y $\in$ H ; the property we were looking for from the beginning is stronger, namely : $\|g'(x) - g'(y) - \frac{x-y}{2}\| \leqslant \frac{1}{2}\|x-y\|$ (cf. Introduction).

Moreover, the factorization of g (and h) does not appear explicitly and a characterization like (3.3) uses heavily the properties of the proximal mapping.

Our approach, based on the deconvolution operation, allowed us to get at an explicit formulation of F in the factorization theorem (Theorem 1), thereby shedding a new light on Moreau's theorem.

Acknowledgment. We are indebted to J-P. Crouzeix for having posed a question which eventually gave rise to the present work, and to J-J Moreau for his constructive criticism on the first fruits of our reflections.

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