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ON THE SET-VALUED CALCULUS IN PROBLEMS OF VIABILITY AND CONTROL FOR DYNAMIC PROCESSES: THE EVOLUTION EQUATION

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1. Introduction

The topic of this paper is motivated by problems of evolution, estimation and control of uncertain dynamic processes described by differential inclusions. [1-6] One of the important proᵇˡᵉms for these systems is to specify the tube of all solutions to a differential inclusion that also satisfy a given state constraint (the "viability" property). [5,6]

It is known that the tube of all viable trajectories may be described by a new differential inclusion whose right-hand side is formed with the aid of a "tangent cone" to the multivalued map that gives the phase restriction [5,8]. Here however, we develop another approach to the problem that allows to avoid the procedure of constructing the cone-valued mappings mentioned above.

In the problem discussed here it occurs that the time-cross-sections of the set of viable trajectories represents the "state" of the uncertain system (in the phase vector for the standard control system). Then the problem of discovering the evolution law for the "states" of the uncertain process becomes relevant.

An evolutionary "funnel equation" for the tube of viable solutions is described in the paper in terms of set-valued calculus. For the linear-convex case the solution to this equation is given through set-valued Lagrangian techniques in the form of a set-valued "convolution integral". An application to the solution of a feedback control problem with state constraints is also introduced.

2. Statement of the Problem

Let R^n be the n-dimensional Euclidean space. For x , $y \in R^n$ let $x' y$ (or (x , y) denote the usual inner product of x and y with the prime as the transpose, $|| x || = (x' x)^{1/2}$, $S = \{x \in R^n : || x || \leq 1\}$. Also denote *conv* R^n to be the set of convex compact subsets of R^n and $h(A , B)$ to be the Hausdorff metric for $A , B \in conv\ R^n$.

Consider the following differential inclusion

$$\dot{x} \in F(t , x) \ (t_o \leq t \leq t_1) \tag{2.1}$$

where $x \in R^n$, F is a continuous map from $[t_o , t_1] \times R^n$ into $conv\ R^n$. We will assume the Lipshitz condition for F to be satisfied ($L > 0$):

$$h(F(t , x) , F(t , y)) \leq L||x-y|| , \forall\ x , y , \in R^n$$

Assuming set $X_o \in conv\ R^n$ to be given, denote $x[t] = x(t , t_o , x_o)$ $(t_o \leq t \leq t_1)$ to be the Carathéodory-type solution to (2.1) that starts at $x[t_o] = x_o \in X_o$. We further require all the solutions $x(t , t_o , x_o)$ $| \ x_o \in X_o\}$ to be extendable until the instant T [10].

Let $Y(t)$ be a continuous map from $[t_o , t_1]$ into $conv\ R^n$, $X_o \subseteq Y(t_o)$.

Definition 2.1 [2-5] A trajectory $x[t] = x(t , t_o , x_o)$ $(x_o \in X_o , t_o \leq t \leq t_1)$ of the differential inclusion (2.1) will be said to be ***viable*** on $[t_o , \tau]$ $(\tau \leq t_1)$ if

$$x[t] \in Y(t) \text{ for all } t \in [t_o , \tau] \tag{2.2}$$

For every $x_o \in X_o$ the set of all viable on $[t_o , \tau]$ trajectories $x(\cdot , t_o , x_o)$ will be denoted as $X(\cdot\ ; \tau , t_o , x_o)$, $X(\cdot\ ; \tau , t_o , X_o) = \bigcup\{X(\cdot\ ; \tau , t_o , x_o) \mid x_o \in X_o\}$, and its cross-section at instant τ as $X(\tau , t_o , x_o)$ and $X(\tau , t_o , X_o)$ respectively.

Let $X^*(\cdot , t_o , X_o)$ be the set of all solutions to the differential inclusion (2.1) that emerge from X^o (the "solution assembly" for X^o). Under our assumptions the set $\mathbf{Q} = \bigcup \{ X^*(t , t_o , X_o) \mid t_o \leq t \leq t_1\}$ of cross sections $X^*(t , t_o , X_o)$ is compact in R^n [9,10]. Let us denote the graph of the map $F(t , \cdot)$ as $gr_t F$ (t is fixed):

$$gr_t\ F = \{(x , y) \in R^n \times R^n : y \in F(t , x)\}$$

and the interior of $A \subseteq R^n$ as $int\ A$

Assumption A:

(1) For some $D \in conv\mathbf{R}^n$ such that $\mathbf{Q} \subset int\ D$, the set $D \bigcap gr_t\ F$ is convex for every $t \in [t_o , t_1]$.

(2) There exists a solution $x_*[\cdot]$ of inclusion (2.1) such that $x_*[t_o] \in X_o$ and $x_*[t] \in int\ Y(t)$, $\forall\ t \in [t_o , t_1]$.

Under assumption A the bundle $X(\cdot\ ; \tau , t_o , X_o)$ of viable trajectories is a convex compact subset of the space $C[t_o , t_1]$ of all continuous n-vector functions, and its τ-cross-section $X(\tau , t_o , X_o)$ is a convex compact subset of R^n.

It is known that sets $X(t, t_o, X_o)$ satisfy a *semigroup property*:

$$X(\tau, t_o, X_o) = X(\tau, s, X(s, t_o, X_o))$$

Therefore they define *a generalized dynamic system*. The construction of an adequate evolution equation describing this system is the first objective of this paper.

The situation will then be reduced to the linear case where it will be shown that the solution to the evolution equation derived here may be given in the form of a set-valued convolution integral.

3. The Evolution Equation

We will further demand that one of the following assumptions would be fulfilled.

Assumption B. The graph $gr\ Y \in conv\ \mathbf{R}^{n+1}$

Assumption C. For every $\ell \in R^n$ the support function $f(\ell, t) = \rho(\ell \mid Y(t)) = \max\{\ell' y \mid y \in Y(t)\}$ is differentiable in t and its derivative $\partial f(\ell, t) / \partial t$ is continuous in (ℓ, t).

The following basic theorem will be proved.

Theorem 3.1. Suppose assumption A is fulfilled and the map $Y(\cdot)$ satisfies either assumption B or assumption C. Then the τ-cross-section $X[\tau] = X(\tau, t_o, X_o)$ of the set $X(\cdot\,; \tau, t_o, X_o)$ of all viable trajectories to the differential inclusion (2.1) will satisfy the following evolution equation:

$$\lim_{\sigma \longrightarrow 0+} \sigma^{-1} h(X[\tau + \sigma], \bigcup_{x \in X[\tau]} (x + \sigma F(\tau, x)) \bigcap Y(\tau + \sigma)) = 0, \qquad (3.1)$$

$$X[t_o] = X_o, \ t_o \leq \tau \leq t_1 .$$

The proof of this theorem will follow from a number of lemmas given in the next section.

Concluding this paragraph we will remark that under the hypotheses of theorem 3.1 the set-valued map $X[\tau] = X(\tau, t_o, X_o)$ will be continuous in τ. However if one replaces assumption A (2) in theorem 3.1 by one which requires that $x_*[t] \in int\ Y(t)$ only for almost all $t \in [t_o, t_1]$, then the equation (3.1) for $X[\tau]$ will be fulfilled almost everywhere on $[t_o, t_1]$. In this case $X[\tau]$ may also be discontinuous on a set $\{\tau\}$ of a measure zero. (It is known that in general the function $X[\tau]$ is continuous from the left and upper semicontinuous from the right at every point $\tau \in [t_o, t_1]$ [6]).

4. Proof of the Basic Theorem

Let $\tau \in [t_o , t_1]$ be fixed, $X[\tau] = X(\tau , t_o , X_o)$. First we have the following estimate

Lemma 4.1 Under Assumption A for every $\epsilon > 0$ *there exists a* $\sigma_* > 0$ *such that*

$$X[\tau + \sigma] \subseteq \bigcup_{x \in X[\tau]} (x + \sigma F(\tau , x)) \bigcap Y(\tau + \sigma) + \epsilon \sigma S \tag{4.1}$$

for every $\sigma \in [0 , \sigma_*]$

Since $X[\tau + \sigma] = X(\tau + \sigma , \tau , X[\tau])$ the definition of viable trajectories yields

$$X[\tau + \sigma] \subseteq X^*(\tau + \sigma , \tau , X[\tau]) \bigcap Y(\tau + \sigma) \tag{4.2}$$

Being the cross section at instant $\tau + \sigma$ of the solution assembly to the differential inclusion (1.1) that starts at $\{\tau , X[\tau]\}$, the set $X^*[\tau] = X^*(t , \tau , X[\tau])$ satisfies the "funnel equation", [9,10]

$$\lim_{\sigma \longrightarrow 0} \sigma^{-1} h(X^*[\tau + \sigma] , \bigcup_{x \in X^*[\tau]} (x + \sigma F(\tau , x))) = 0$$

Therefore

$$X^*(\tau + \sigma , \tau , X[\tau]) \subseteq \left[\bigcup_{x \in X[\tau]} (x + \sigma F(\tau , x)) \right] + o(\sigma)S \tag{4.3}$$

where $\sigma^{-1} o(\sigma) \longrightarrow 0$, with $\sigma \longrightarrow 0$.*

If P , Q , W are given subsets of $\mathbf{R}^n$ with $Q = -Q$, then it is possible to verify the inclusion

$$(P + Q) \bigcap W \subseteq P \bigcap (Q + W) + Q$$

From this inclusion and from (4.2), (4.3) it follows that

$$X[\tau + \sigma] \subseteq (\bigcup_{x \in X[\tau]} (x + \sigma F(\tau , x)) \bigcap (Y(\tau + \sigma) + o(\sigma) S) + o(\sigma)S \tag{4.4}$$

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* Here and in the sequel a function denoted by $o(\sigma)$ without or with any type of indices (e.g. $o^*(\sigma)$, o_i , (σ), etc.) will always be presumed to satisfy $\sigma^{-1} o(\sigma) \longrightarrow 0$ if $\sigma \longrightarrow +0$. *

Denoting $R(\sigma , \tau) = \{\bigcup (x + \sigma F(\tau , x)) \mid x \in X[\tau]\}$

we observe due to assumption A1 that the set $R(\sigma , \tau)$ is convex and compact for every value of $\sigma > 0$. We will now verify the following inclusion

$$R(\sigma , \tau) \bigcap (Y(\tau + \sigma) + o(\sigma)S) \subseteq (R(\sigma , \tau) \bigcap Y(\tau + \sigma)) + o_1(\sigma)S$$

for some function $o_1(\sigma)$.

From assumption A it follows that there exist vectors $x_* \in X[\tau]$, $v_* \in F(\tau , x_*)$ and numbers $r > 0$, $\sigma_* > 0$, $K > 0$ such that for every $\sigma \in [0 , \sigma_*]$ we have

$$x_* + \sigma v_* + r S \subseteq Y(\tau + \sigma) , x_* + \sigma v_* \in R(\sigma , \tau) , R(\sigma , \tau) \subseteq KS$$

$$0 < r^{-1} o(\sigma) < 1$$

Then however

$$R(\sigma , \tau) \bigcap (Y(\tau + \sigma) + o(\sigma)S) \subseteq R(\sigma , \tau) \bigcap Y(\tau + \sigma) + 2K S r^{-1} o(\sigma) \tag{4.5}$$

Indeed, suppose a number $\sigma \in [0 , \sigma_*]$ and a vector $z \in R(\sigma , \tau) \bigcap (Y(\tau + \sigma) + o(\sigma) S)$ are given. We will show that

$$z \in R(\sigma , \tau) \bigcap Y(\tau + \sigma) + 2K S r^{-1} o(\sigma) \tag{4.6}$$

Selecting vector

$$y = (1 - r^{-1} o(\sigma)) z + r^{-1} o(\sigma) (x_* + \sigma v_*)$$

we observe that $y \in R(\sigma , \tau)$, and

$$|| y - z || \leq 2 Kr^{-1} o(\sigma) \tag{4.7}$$

From the above we arrive at two inclusions

$$(1 - r^{-1} o(\sigma)) z \in (1 - r^{-1} o(\sigma)) (Y(\tau + \sigma) + o(\sigma) S) \subseteq$$

$$\subseteq (1 - r^{-1} o(\sigma)) Y(\tau + \sigma) + o(\sigma) S ,$$

$$r^{-1} o(\sigma)(x_* + \sigma v_* + r S) \subseteq r^{-1} o(\sigma) Y(\tau + \sigma)$$

where $Y(\tau+\sigma)$ is convex-valued. Taking the sums of the respective elements at the left and right hand parts of these relations we come to

$$y + o(\sigma)\, S \subseteq Y(\tau+\sigma) + o(\sigma)\, S$$

or otherwise, to the inclusion $y \in Y(\tau+\sigma)$ (since in this relation $o(\sigma)$ is a specific function of σ).

This immediately yields (4.6) and the inclusion (4.5) is therefore established. The result given in Lemma 4.1 now follows from relations (4.5), (4.6).

Consider the system

$$\dot{z} = v \,,\ v \in F(\tau\,,\, x_o) \tag{4.8}$$

$$z(t) \in Y(t) \,,\ z(\tau) = x_o \,,\ \tau \le t \le \tau + \sigma \,,$$

with $Z(\tau+\sigma\,,\,\tau\,,\,x_o)$ being the cross section of the tube of viable solutions to this system.

Denote

$$Z(\sigma\,,\,\tau) = \bigcup \{Z(\tau+\sigma\,,\,\tau\,,\,x_o) \mid x_o \in X[\tau]\}$$

Lemma 4.2 Under assumption A for every $\epsilon > 0$ *there exists a* $\sigma_* > 0$ *such that for all* $\sigma \in [0\,,\,\sigma_*]$ *the following inclusions are true*

$$X[\tau+\sigma] \subseteq Z(\sigma\,,\,\tau) + \epsilon\,\sigma\, S \tag{4.9}$$

$$Z(\sigma\,,\,\tau) \subseteq \left(\bigcup_{x \in X[\tau]} (x + \sigma\, F(\tau\,,\,x))\right) \bigcap Y(\tau+\sigma) + \epsilon\,\sigma\, S \tag{4.10}$$

Lemma 4.2 is a detailed version of Lemma 4.1. It is proved through a similar scheme.

Lemmma 4.3 With assumption A fulfilled it is possible for any $\epsilon > 0$ *to indicate a* $\sigma_* > 0$ *such that for every* $\sigma \in (0\,,\,\sigma_*]$ *we have*

$$Z(\sigma\,,\,\tau) \subseteq X[\tau+\sigma] + \epsilon\,\sigma\, S \tag{4.11}$$

Inclusion (4.11) gives us the next step, relative to (4.9), to prove the Basic Theorem.

In order to verify the assertion of Lemma 4.3 assume $z^* \in Z(\sigma\,,\,\tau)$. Then there exists a pair

$$x_o \in X[\tau] \,,\ v(t) \in F(\tau\,,\,x_o) \,,\ \tau \le t \le \tau + \sigma \,,$$

such that the respective solution $z[t] = z(t\ ,\ \tau\ ,\ x_o)$ to equation (4.8) satisfies the conditions

$$z[\tau + \sigma] = z(\tau + \sigma\ ,\ \tau\ ,\ x_o) = z^*\ ;\ z[t] \in Y(t)\ ,\ t \in [\tau\ ,\ \tau + \sigma]$$

Therefore

$$\dot{z}\ [t] \in F(\tau\ ,\ x_o)\ ,$$

and

$$F(\tau, x_o) = F(\tau, z[t] - \int_\tau^t v(\xi)\, d\xi) \subseteq F(\tau, z[t]) + L\, K\, S\, \sigma \subseteq$$

$$\subseteq F(t\ ,\ z[t]) + (0(\sigma) + L\ K\ \sigma) S$$

where

$$K = \max\ \{\ ||\ z\ ||\ \mid z \in F(t\ ,\ x)\ ,\ t \in [t_o\ ,\ t_1]\ ,\ x \in \mathbf{Q}\}$$

The last relations are derived due to the earlier assumptions that $F(\tau\ ,\ z)$ is Lipschitz in (with constant L) and continuous in t uniformly in $z \in M$. Here the function $O(\sigma) \longrightarrow 0$ wi $\sigma \longrightarrow +0$.

If we now introduce the differential inclusion

$$\dot{y} \in F(t\ ,\ y)\ ,\ y(\tau) = x_o = z(\tau) \tag{4.1}$$

then from the Gronwall lemma for differential inclusions [5] it follows that there exists a soluti $y(t)$ to (4.12) that satisfies

$$||\ y(t) - z(t)\ || \leq \int_\tau^t h\,(F(t, y(t)), F(t, z(t)))\, dt + (O(\sigma) + L\, N\, \sigma)\,(t - \tau)$$

$$\leq \int_\tau^t L\, ||\ y(\xi) - z(\xi)\ ||\ d\xi + (O(\sigma) + L\, N\, \sigma)\,(t - \tau)$$

and therefore yields

$$||\ y(t)\ -\ z(t)\ ||\ \leq (\exp\ L\ \sigma)\ (O(\sigma)\ +\ L\ N\ \sigma)\ \sigma = o^*(\sigma)\ ,\ t \in [\tau\ ,\ \tau + \sigma] \tag{4.1}$$

Hence

$$y(t) \in Y(t) + o^*(\sigma)\, S \;,\; t \in [\tau \,,\, \tau + \sigma]$$

Due to assumption A the sets $X[t] \in conv\, \mathbf{R}^m$. Following the scheme for Lemma 4.1, it is possible to construct a function $w(t)$ that satisfies

$$\dot{w} \in F(t \,,\, w) \;,\; w(t_o) \in X[\tau] \;,\; w(t) \in Y(t)$$

$$||\; y(t) - w(t) \;|| \leq o_1^*(\sigma) \;,$$

for a certain function $o_1^*(\sigma)$.

Therefore

$$y(\tau + \sigma) \in X[\tau + \sigma] + o_1^*(\sigma) \tag{4.14}$$

and in view of (4.13), (4.14) we have

$$z^* \in X[\tau + \sigma] + o^*(\sigma) + o_1^*(\sigma) = X[\tau + \sigma] + o_2^*(\sigma)$$

where the function $o_2^*(\sigma)$ does not depend upon the vector $z^* \in Z(\sigma \,,\, \tau)$.

The last Lemma leads to

Corollary 4.1 Under Assumption A we have

$$\lim_{\sigma \longrightarrow +0} \sigma^{-1}\, h(X[\tau + \sigma] \,,\, Z(\tau \,,\, \sigma)) = 0 \tag{4.15}$$

Assertion (4.15) follows from (4.11), (4.9).

In order to finalize the proof of the basic theorem we will have to establish an inclusion opposite to either (4.10) or (4.1). This however will require some additional assumptions in the form of either B or C in § 2.

Lemma 4.4 Under Assumptions A, B for any $\sigma > 0$ we have

$$R(\sigma \,,\, \tau) \bigcap Y\,(\tau + \sigma) \subseteq Z(\sigma \,,\, \tau) \tag{4.16}$$

Consider the set $Z'(\tau+\sigma\ ,\ \tau\ ,\ x_o)$ of viable solutions to (4.8) in the class of constant functions v $(v(t) \equiv const)$

Denote

$$Z'\ (\sigma\ ,\ \tau) = \bigcup \{Z'(\tau+\sigma\ ,\ \tau\ ,\ x_o)\ \mid\ x_o \in X[\tau]\}$$

Clearly $Z'(\sigma\ ,\ \tau) \subseteq Z(\sigma\ ,\ \tau)$. If we now assume $z \in R(\sigma\ ,\ \tau) \bigcap Y(\tau+\sigma)$ then there exists a pair of vectors $x \in X[\tau]$, $v \in F(\tau\ ,\ x)$ such that

$$x + \sigma\ v \in Y(\tau+\sigma)\ ,\ x \in Y(\tau)$$

Since $gr\ Y \in conv\ \mathbf{R}^{n\ +\ 1}$ we have

$$x + s\ v = (1 - s\ \sigma^{-1})\ x + (s\ \sigma^{-1})\ (x + \sigma\ v) \in Y\ (s)$$

for any $s \in [0\ ,\ \sigma]$

Therefore $z \in Z'(\sigma\ ,\ \tau)$ and (4.16) is proved.

Relations (4.16), (4.10) yield

Corollary 4.2 Under assumptions A, B we have

$$\lim_{\sigma \longrightarrow +0} \sigma^{-1}\ h\ (Z(\sigma\ ,\ \tau)\ ,\ R(\sigma\ ,\ \tau) \bigcap\ Y\ (\tau+\sigma)) = 0$$

Combining the latter equality with (4.15) we arrive at the proof of the basic theorem under Assumptions A, B.

We will now prove the same theorem under Assumptions A, C. Having already found (4.1), we will only need to establish an opposite inclusion. However prior to that we will prove an additional assertion.

Let us introduce some auxiliary constructions. Define for an arbitrary closed set $P \subseteq R^n$ *a contingent cone* $T_P(x)$ $(x \in P)$:

$$T_P(x) = \{v \in R^n : \liminf_{\sigma \longrightarrow 0+} \sigma^{-1}\ d(x + \sigma v\ ,\ P) = 0\}$$

and for a multivalued mapping $Y(\cdot)$ a ***contingent derivative*** [5,8]

$$DY(t\ ,\ y)(\alpha) = \{v \in R^n : (\alpha\ ,\ v) \in T_{grY}\ (t\ ,\ y)\}$$

(here $d(x, p) = \min\{ || x - P || : p \in P\}$, $\alpha \in R^1$, $(t, y) \in grY$).

Determine $V(t, y) = DY(t, y)(1)$ for $(t, y) \in gr\ Y$. Under assumption C for all $(t, y) \in gr\ Y$ the set $V(t, y)$ is *closed and convex* in R^n [5].

Following [12] consider a *local approximation* $Y_\sigma(\tau)$ for the set-valued map $Y(\cdot)$ in the neighbourhood of a fixed point τ:

$$Y_\tau(\sigma) = \bigcap_{y \in Y(\tau)} (y + \sigma\ V(t, y)),\ Y_\tau(0) = Y(\tau)$$

Lemma 4.5 [12].

1. *Under Assumption C the following equality is true for all* $\sigma > 0$

$$Y_\tau(\sigma) = \{z \in R^n : \ell' z \le \rho\,(\ell \mid Y(\tau)) + \sigma\,\partial f(\ell, \tau)/\partial\tau,\ \forall\, \ell \in R^n\}$$

2. *Under Assumptions A(2), C for every* $\epsilon > 0$ *there exists a* $\sigma_* > 0$ *such that for all* $\sigma \in (0, \sigma_*]$

$$Y(\tau + \sigma) \subseteq Y_\tau(\sigma) + \epsilon\,\sigma\,S \tag{4.17}$$

$$Y_\tau(\sigma) \subseteq Y(\tau + \sigma) + \epsilon\,\sigma\,S \tag{4.18}$$

As a function of σ the graph of the map $Y_\tau(\sigma)$ is convex. This allows to establish

Theorem 4.1 Under Assumptions A, C the set-valued map $X[\tau]$ *is a solution to the equation*

$$\lim_{\sigma \longrightarrow +0} \sigma^{-1} h(X[\tau + \sigma], \left(\bigcup_{x \in X[\tau]} (x + \sigma F(\tau, x)) \right) \bigcap Y_\tau(\sigma)) = 0$$

From (4.17) and from the scheme for proving Lemma 4.1 (since Assumption A(2) remains true for $Y_\tau(\sigma)$) it follows that there is an upper bound for $X[\tau + \sigma]$, namely

$$X\,[\tau + \sigma] \subseteq \left(\bigcup_{x \in X[\tau]} (x + \sigma F(\tau, x) \right) \bigcap Y_\tau(\sigma)) + \bar{o}(\sigma)S$$

In order to prove the opposite relation

$$\bigcup_{x \in X[\tau]} (x + \sigma F(\tau, x)) \bigcap Y_\tau(\sigma) \subseteq X[\tau + \sigma] + o(\sigma)S \tag{4.19}$$

for some $o(\sigma)$ assume

$$z \in \bigcup_{x \in X[\tau]} (x + \sigma F(\tau , x)) \bigcap Y_\tau(\sigma)$$

Then $z = x + \sigma v \in Y_\tau(\sigma)$ for some $x \in X[\tau]$, $v \in F(\tau , x)$. Since $gr\ Y_\tau \in conv\ \mathbf{R}^n$ we will have $x + sv \in Y_\tau(s)$ for all $s \in [0 , \sigma]$.

As in (4.12), (4.13) it is possible to establish the existence of a solution $y(t)$ to the inclusion (4.12)

$$\dot{y} \in F(t , y) , y(\tau) = x , t \in [\tau , \tau - \quad]$$

that satisfies the inequality

$$|| x + sv - y(\tau + s) || \leq \tilde{o} (\sigma) , s \in [0 , +\sigma]$$

for a certain $\tilde{o}(\sigma)$ and therefore yields

$$y(\tau + s) \in Y_\tau(s) + \tilde{o} (\sigma)S , s \in [0 , \sigma] \tag{4.20}$$

$$|| y(\tau + \sigma) - z || \leq \tilde{o}(\sigma) \tag{4.21}$$

Due to Lemma 4.5 (2) we may substitute (4.20) for $y(\tau + s) \in Y(\tau + s) + \tilde{o}_1(\sigma)$.

Then, following the schemes of Lemma 4.1, we may find due to Assumption A a solution $Y_*(t)$ to (4.12) that satisfies relations

$$y_*(\tau) \in X [\tau] , y_*(\tau + s) \in Y (\tau + s) , 0 \leq s \leq \sigma$$

$$|| y_*(\tau + \sigma) - y(\tau + \sigma) || \leq \tilde{o}_2(\sigma)$$

The latter inequality together with (4.21) leads to (4.19). Theorem 4.1 is therefore proved.

Now we may come to the proof of the inclusion opposite to (4.1). From Assumptions A, C and from Lemma 4.5 (2) we observe that

$$R(\sigma , \tau) \bigcap Y(\tau + \sigma) \subseteq R(\sigma , \tau) \bigcap (Y_\tau(\sigma) + o(\sigma)) \tag{4.22}$$

and that Assumption A(2) remains true for $Y_\tau(\sigma)$. Then following the reasoning of Lemma 4.1 we will have

$$R(\sigma , \tau) \bigcap (Y_\tau + o(\sigma)) \subseteq R(\sigma , \tau) \bigcap Y_\tau(\sigma) + o(\sigma) S \qquad (4.23)$$

From theorem 4.1 and from (4.22), (4.23) we come to the inclusion

$$R(\sigma , \tau) \bigcap Y(\tau + \sigma) \subseteq X(\tau , \sigma) + o_*(\sigma) S \qquad (4.24)$$

for a certain function $o_* (\sigma)$.

This finalizes the proof of the basic theorem under Assumption A, ☺, since (4.1), (4.24) yield (3.1).

5. The Linear System

Consider the following system

$$\dot{x} \in A(t)x + P(t) \; (t_o \leq t \leq T) \qquad (5.1)$$

where $x \in R^n$, $A(t)$ is a continuous $n \times n$-matrix function, $P(t)$ is a continuous map from $[t_o , t_1]$ into *conv* R^n and therefore $F(t , x) = A(t) \, x + P(t)$

Here assumption A (1) will be fulfilled automatically Hence to retain assumption A(2) we will introduce

Assumption A*. There exists solution $x_*[\cdot]$ of (5.1) such that

$$x_*[t] \in int \; Y(t) \; , \forall \; t \in [t_o , T]$$

The following result is a direct consequence of theorem 3.1 (it also generalizes theorem 4.1 of paper [3]).

Theorem 5.1 Assume assumption A^* to be fulfilled. If the map $Y(\cdot)$ satisfies either assumption B or assumption C then the set-valued function $X[\tau] = X(\tau , t_o , X_o)$ is the solution to the evolution equation

$$\lim_{\sigma \longrightarrow 0+} \sigma^{-1} \, h(X[\tau + \sigma] \, , ((E + \sigma \, A(\tau)) \, X[\tau] + \qquad (5.2)$$

$$+ \sigma P(\tau)) \bigcap Y(\tau + \sigma)) = 0 \; , \; X[t_o] = X_0 \; ,$$

$t_0 \le \tau \le t_1$ *(here E is the identity $n \times n$-matrix).*

A separate question is how to solve equation (5.1). We will further demonstrate that this solution may be given by a certain multivalued "convolution integral".

6. The Linear System. A Direct Solution

We will now pursue a direct calculation of the support function $\rho(\ell \mid X[\tau])$ based on the techniques of convex analysis and the set-valued analogies of Lagrangian techniques.

Denote $C^n(\mathbf{T})$ $(C_k^n(\mathbf{T}))$ to be the set of all n-vector-valued continuous functions defined on T (respectively the set of k times continuously differentiable functions with values in R^n, defined on T). Let $\mathbf{M}^n(\mathbf{T})$ stand for the set of all n-vector-valued polynomials of any finite degree, defined on $\mathbf{T}$. Obviously $g(\cdot) \in \mathbf{M}^n(\mathbf{T}_\tau)$ if

$$g(s) = \sum_{i=1}^{k} l^{(i)} s^i , \quad s \in \mathbf{T}_\tau , \quad l^{(i)} \in \mathbf{R}^n$$

and $\mathbf{M}^n(\mathbf{T}) \subseteq C_\infty^n(\mathbf{T})$.

Applying the duality concepts of infinite dimensional convex analysis [8] as given in the form presented in [6] we come to the following relations. For any $l \in \mathbf{R}^n$, $\lambda(\cdot) \in C^n(\mathbf{T})$ denote

$$\Phi_\tau(l,\lambda(\cdot)) = \rho(l'S(t_0,\tau) - \int_{t_0}^{\tau} \lambda'(\xi)\mathrm{S}(t_0,\xi)d\xi \,|\, X^0) + \tag{6.1}$$

$$\int_{t_0}^{\tau} \rho(l'\mathrm{S}(\xi,\tau) - \int_{\xi}^{\tau} \lambda'(s)\mathrm{S}(\xi,s)ds \,|\, \mathbf{P}(\xi))\, d\xi + \int_{t_0}^{\tau} \rho(\lambda(\xi) \,|\, Y(\xi))d\xi$$

Here, in the first variable the function $\mathrm{S}(t,\tau)$ is the matrix solution for the equation

$$\dot{s} = -sA(t) , \quad \mathrm{S}(\tau,\tau) = E , \quad t \le \tau ,$$

the second and third members of the sum (2.1) are Lebesgue-type integrals of multivalued maps $\mathrm{P}(\xi)$, $Y(\xi)$ respectively (see, for example, [5-7]).

In [6], § 6, it was proved that

$$\max \{(l,x) \,|\, x \in \mathbf{X}[\tau]\} = \rho(l|\mathbf{X}[\tau]) = \inf \{\Phi_\tau(l,\lambda(\cdot)) \,|\, \lambda(\cdot) \in C^n[\mathbf{T}_\tau]\} . \tag{6.2}$$

A slight modification of the respective proof shows that the class of functions $C^n(\mathbf{T}_\tau)$ in the last formula may be substituted by either $C^n_\infty(\mathbf{T}_\tau)$ or even $\mathbf{M}^n(\mathbf{T}_\tau)$. Hence

$$inf\{\Phi_\tau(l,\lambda(\cdot))\,|\,\lambda(\cdot) \in C^n(\mathbf{T}_\tau)\} = \tag{6.3}$$

$$inf\,\{\Phi_\tau(l,\lambda(\cdot))\,|\,\lambda(\cdot) \in C^n_\infty(\mathbf{T}_\tau)\} = inf\,\{\Phi_\tau(l,\lambda(\cdot))\,|\,\lambda(\cdot) \in \mathbf{M}^n(\mathbf{T}_\tau)\}$$

From relations (2.2) it is possible to derive the following assertion

Lemma 6.1 The following equality is true

$$\mathbf{X}[\tau] = \bigcap\,\{R(\tau,M(\cdot))\,|\,M(\cdot) \in C^{n\times n}(\mathbf{T}_\tau)\} = \tag{6.4}$$

$$= \bigcap\,\{R(\tau,M(\cdot))\,|\,M(\cdot) \in C^{n\times n}_\infty(\mathbf{T}_\tau)\} = \bigcap\,\{R(\tau,M(\cdot))\,|\,M(\cdot) \in \mathbf{M}^{n\times n}(\mathbf{T}_\tau)\}\ .$$

where

$$R(\tau,M(\cdot)) = (\mathbf{S}(t_0,\tau) - \int_{t_0}^{\tau} M(\xi)\mathbf{S}(t_0,\xi)d\xi)X^0 +$$

$$+ \int_{t_0}^{\tau} (\mathbf{S}(\tau,\xi) - \int_{\xi}^{\tau} M(s)\ S(\xi\ ,\ s)\ ds)\ \mathbf{P}(\xi)d\xi + \int_{t_0}^{\tau} M(s)\,Y(s)\,ds$$

and $C^{n\times n}_k(\mathbf{T})$, $(0 \le k \le \infty)$, $\mathbf{M}^{n\times n}(\mathbf{T})$ *stand for the respective spaces of* $(n\times n)$*-matrix-valued functions defined on* $\mathbf{T}$.

The proof of Lemma 6.1 follows immediately from (6.2), (6.3) after a substitution $\lambda'(\cdot) = l'M(\cdot)$ for $l \neq 0$. The infimum over $\lambda(\cdot)$ in (6.2) is then substituted by an infimum over $M(\cdot)$. Hence for every $l \neq 0$ we have

$$\rho(l\,|\,\mathbf{X}[\tau]) \le \Phi_\tau(l,M'(\cdot)l) \tag{6.5}$$

for any $M(\cdot) \in C^{n\times n}(\mathbf{T}_\tau)$ (or $C^{n\times n}_\infty(\mathbf{T}_\tau)$ or $\mathbf{M}^{n\times n}(\mathbf{T}_\tau)$). From (6.1) - (6.5) it now follows that $\mathbf{X}[\tau] \subseteq R(\tau,M(\cdot))$ for any $M(\cdot)$.

Hence

$$\mathbf{X}[\tau] \subseteq \bigcap\,\{R(\tau,M(\cdot))\,|\,M(\cdot) \in C^{n\times n}(\mathbf{T}_\tau)\} \tag{6.6}$$

(or over $C_{\infty}^{n\times n}(\mathbf{T}_\tau)$ or $\mathbf{M}^{n\times n}(\mathbf{T}_\tau)$).

Equalities (6.4) now follow from (6.6) and (6.2), (6.3).

Lemma 6.1 acquires a specific form when $X^o = \mathbf{R}^n$. In this case there are no initial restrictions on $x^o = x(t_o)$.

Corollary 6.1 Assume $X^o = \mathbf{R}^n$. *Then*

$$X[\tau] = \bigcap \{J(\tau, M(\cdot)\} = J[\tau], \tag{6.7}$$

$$J(\tau, M) = \int_{t_o}^{\tau} (S(\tau, \xi) - \int_{\xi}^{\tau} M(s)\, S(\xi, s) ds)\, \mathbf{P}(\xi)\, d\xi + \int_{t_o}^{\tau} M(s)\, Y(s) ds$$

over all $M(\cdot) \in C^{n\times n}(T_\tau)$ *that satisfy the equation*

$$\int_{t_o}^{\tau} M(\xi)\, S(\tau, \xi) d\xi = E \tag{6.8}$$

Relations (6.7), (6.8) are the direct analogies of the *convolution integral* introduced for single-valued functions, for example, in [13]. Following the conventional term we will therefore refer to $J[\tau]$ as the *set-valued convolution integral.* We will also extend this term to the right-hand part of (6.4).

7. A Generalized "Lagrangian" Formulation

The assertions of the above yield the "standard" duality formulations for calculating $\gamma_o(l) = \rho(l \mid X[\tau])$, (see [6, 14, 15]).

Denoting

$$P(\cdot) = \{p(\cdot) : p(t) \in P(t), t \in \mathbf{T}_\tau\}$$

we come to the following "standard"

Primary Problem

$$\text{maximize}(\ell, x[\tau]) \tag{7.1}$$

over all

$$u(\cdot) \in P(\cdot)\ ,\ x^0 \in X_0 \tag{7.2}$$

where $x[t]$ is the solution to the equation

$$\dot{x}[t] = A(t)x[t] + u(t)\ ,\ x[t_0] = x^0 \tag{7.3}$$

In other words

$$\gamma_o(\ell) = \max\{\Psi(x^0\ ,\ u(\cdot) \mid x^0 \in \mathbf{R}^n\ ,\ u(\cdot) \in \mathbf{L}_2^n\,(T_\tau)\} \tag{7.4}$$

under restriction (7.2) where

$$\Psi(x^0\ ,\ u(\cdot)) = (\ell\ ,\ x[\tau]) + \delta(x^o \mid X^0) +$$

$$+ \int_{t_o}^{\tau} (\delta(x[t] \mid Y(t)) + \delta(u(t) \mid P(t)))\ dt$$

Here

$$\delta(x \mid Y) = \begin{cases} 0 & \text{if } x \in Y \\ +\infty & \text{if } x \bar{\in} Y \end{cases}$$

The primary problem generates a corresponding "standard"

Dual Problem:

Determine

$$\gamma^0(\ell) = \inf\,\{\Phi_\tau(\ell\ ,\ \lambda(\cdot)) \mid \lambda(\cdot) \in C^n\,(\mathbf{T}_\tau)\} \tag{7.5}$$

along the solutions $s[t]$ to the equation

$$s[t] = -\ s[t]A(t) + \lambda(t)\ ,\ s[\tau] = \ell \tag{7.6}$$

Here $\Phi_\tau(\ell\ ,\ \lambda(\cdot))$ may be rewritten as

$$\Phi_\tau(\ell\ ,\ \lambda(\cdot)) = \rho(s[t_0] \mid X^0) +$$

$$+ \int_{t_o}^{\tau} (\rho(s[t] \mid P(t)) + \rho(\lambda(t) \mid Y(t))\ dt$$

Relations (2.2), (2.3) indicate that $\gamma_o(\ell) = \gamma^0(\ell)$ and that $\lambda(\cdot)$ in (6.5) may be selected from $C^n_\infty(\mathbf{T}_\tau)$ or even from $M^n(\mathbf{T}_\tau)$.

A "standard" Lagrangian formulation is also possible here.

Lemma 7.1 The value $\gamma_0(\ell) = \gamma(\ell)$ may be achieved as the solution to the problem

$$\gamma(\ell) = \inf_{\lambda(\cdot)} \max_{u(\cdot)\,,\,x^o} L(\lambda(\cdot)\,,\,u(\cdot)\,,\,x^0) \tag{7.7}$$

where

$$L(\lambda(\cdot)\,,\,u(\cdot)\,,\,x^0) = (s[t_0]\,,\,x^0) + \int_{t_0}^{\tau} ((s[t]\,,\,u(t)) + \rho(\lambda(t) \mid Y(t)))\,dt \tag{7.8}$$

and

$$\lambda(\cdot) \in C^n(\mathbf{T}_\tau)\,,\; u(\cdot) \in P(\cdot)\,,\; x^0 \in X^0\,.$$

The passage from (6.2), (6.3) to (6.4) yields another form of presenting $X[\tau]$. Namely, denote $S[t]$ to be the solution to the matrix differential equation

$$\dot{S}[t] = -S[t]\,A(t) + M(t)\,,\; S[\tau] = E$$

Also denote

$$\wedge\,(x^0\,,\,u(\cdot)\,,\,M(\cdot)) =$$

$$S[t_0]x^0 + \int_{t_0}^{\tau} (S[t]u(t) + M(t)\;Y(t))\,dt$$

Obviously

$$R(\tau\,M(\cdot)) = \bigcup \{\wedge(x^0\,,\,u(\cdot)\,,\,M(\cdot)) \mid x^0 \in X^0\,,\,u(\cdot) \in P(\cdot)\} =$$

$$= S[t_0]X^0 + \int_{t_0}^{\tau} (S[t]\;P(t) + M(t)\;Y(t))\,dt$$

Lemma 6.1 may now be reformulated as

Lemma 7.2 The set $X[\tau]$ *may be determined as*

$$X[\tau] = \bigcap_{M(\cdot)} \bigcup_{x^0 , u(\cdot)} \wedge(x^0 , u(\cdot) , M(\cdot))$$

over all

$$M(\cdot) \in C^{n\times n}(\mathbf{T}_\tau) , x^0 \in X^0 , u(\cdot) \in \mathbf{P} .$$

This result may be treated as a *generalization of the standard Lagrangian formulation.* However here one deals with set $X[\tau]$ as a whole rather than with its projections $\rho(l \mid X[\tau])$ on the elements $l \in \mathbf{R}^n$. The results of the above indicate that the description of set $X[\tau]$ may be "decoupled" into the specification of sets $R(\tau , M(\cdot))$, the variety of which describes the generalized dynamic system $X(t , t_0 , X^0)$.

However it should be clear that the mapping $R(\tau , M(\cdot))$ may not always be an adequate element for the decoupling procedure, especially for the description of the evolution of $X(t , t_0 , X^0)$ in t. The reasons for this are the following.

Assuming function $M(\cdot)$ to be fixed, redenote $R(\tau , M(\cdot))$ as $\mathbf{R}_M(\tau , t_0 , X^0)$. Then, in general, for any fixed M, we have

$$\mathbf{R}_M(\tau , t_0 , X^0) \neq \mathbf{R}_M(\tau , s, \ \mathbf{R}_M(s , t_0 , X^0)) .$$

Therefore the map $\mathbf{R}_M(\tau , t_0 , X^0)$ does not generate a semigroup of transformations that may define a generalized dynamic system. The necessary properties may be however achieved for an alternative variety of mappings, each of the elements of which will possess both the property of type (2.4) and the "semigroup" property, [4].

8. An Alternative Presentation of X $[\tau]$

Denote $C^{n\times n}(\mathbf{T}_\tau)$ to be the subclass of $C^{n\times n}(\mathbf{T})$ that consists of all continuous matrix functions $M(\cdot)$ that satisfy

Assumption 8.1 *For any* $\varsigma \in \mathbf{T}_\tau$ *we have*

$$\det \left(S(\varsigma , \tau) - \int_{\varsigma}^{\tau} M(s) \ S(\varsigma , s) \ ds\right) \neq 0$$

In other words, if $K[t]$ is the solution to the equation

$$\dot{K}(t) = -K(t)\, A(t) + M(t)\, ,\ K(\tau) = E\, ,\ (t_o \le t \le \tau) \tag{8.1}$$

then $M(t)$ must be such that det $K[t] \neq 0$ for all $t \in [t_o\, ,\, \tau]$.

We will further denote $K[t] = K(t\, ,\, \tau\, ;\, M(\cdot))$ for a given function $M(\cdot)$ in (7.1).

Consider the equation

$$\dot{Z} = (A(t) - L(t))\, Z\, , \qquad t_o \le t \le \tau \tag{8.2}$$

whose matrix solution $Z[t]$ $(Z[\tau] = E)$ will be also denoted as $Z[t] = Z(t\, ,\, \tau\, ;\, L(\cdot))$ $(Z'(t\, ,\, \tau\, ,\, \{0\}) \equiv S(\tau\, ,\, t))$

Under Assumption 8.1 there exists a function $L(\cdot) \in C^{n \times n}(\mathbf{T}_\tau)$ such that

$$K[t] \equiv Z(\tau\, ,\, t\, ,\, L(\cdot))\, ,\ \forall\, t \in \mathbf{T}_\tau\, , \tag{8.3}$$

Indeed, if for $t \in \mathbf{T}_\tau$ we select $L(t)$ according to the equation

$$L(t) = A(t) - K^{-1}(t)\, \dot{K}(t) = \tag{8.4}$$

$$A(t) - K^{-1}(t)\, (-K(t)\, A(t) + M(t)) = K^{-1}(t)\, M(t)$$

then, obviously, equation (8.3) will be satisfied. From (8.4), (8.3), (8.4) it now follows $(M(\cdot) \in C^{n \times n}(\mathbf{T}_\tau))$

$$R(\tau\, ,\, M(\cdot)) = Z(\tau\, ,\, t_0\, ;\, L(\cdot))\, X^0 + \tag{8.5}$$

$$\int_{t_0}^{\tau} Z(\tau\, ,\, t\, ;\, L(\cdot))\, (P(t) + L(t)\ Y(t))\, dt$$

However it is not difficult to observe that the right-hand part of (8.5) is $X_{L(\cdot)}(\tau\, ,\, t_0\, ,\, X^0) = X\, [\tau \mid L(\cdot)\,]$ which is the cross-section at instant τ of the set $X_{L(\cdot)}\, (\cdot\, ,\, t_o\ X^0) = X\, [\cdot \mid L(\cdot)]$ of all solutions to the differential inclusion

$$\dot{x}(t) \in (A(t) - L(t))x + \mathbf{P} + L(t)\ Y(t) \tag{8.6}$$

$$x(t_0) \in X^o\, ,\ t \in \mathbf{T}_\tau\, ,$$

Since the class of all functions $L(\cdot) \in C^{n\times n}(\mathbf{T}_\tau)$ generates a subclass of functions $M(\cdot) \in C^{n\times n}(\mathbf{T}_\tau)$ we now come to the following assertion in view of (6.3), (8.5), (8.6).

Lemma 8.1 The following inclusion is true

$$\mathbf{X}[\tau] \subseteq \bigcap \{X[\tau \mid L(\cdot)] \mid L(\cdot) \in C^{n\times n}(\mathbf{T}_\tau)\} \tag{8.7}$$

Therefore $\mathbf{X}[\tau]$ is contained in the ***attainability domains*** at instant τ for the inclusion (8.6), whatever is the function $L(t)$.

However the main point is that (8.7) actually turns to be an equality. In order to prove this one has to establish an inclusion opposite to (8.7) which is a rather long procedure already presented in [4]. The result is given by

Theorem 8.1 The following equality is true

$$X[\tau] = \bigcap\{X[\tau \mid L(\cdot)] \mid L(\cdot) \in C^{n\times n}(T_\tau)\} \tag{8.8}$$

Since each of the multivalued functions $X_L[\tau] = X[\tau\, ,\, L(\cdot)]$ is a solution to differential inclusion (8.6) it may be also considered as a solution to the funnel equation $(X[t_o] = X^o$

$$\lim_{\sigma \longrightarrow 0} \sigma^{-1}\, h(X_L\, [\tau + \sigma]\, ,\, (E + A(\tau)\sigma)\, X_L[\tau] + \mathbb{P}(\tau)\, \sigma) = 0 \tag{8.9}$$

Combining Theorem 5.1 with (8.8), (8.9) we arrive at

Theorem 8.2 Under assumptions A^*, B or A^*, C the solution $X[\tau]$ to the "generalized" funnel equation (5.2) may be decoupled into the variety $\{X_L[\tau]\}$ of solutions to the "ordinary" funnel equation (8.9) so that equality (8.8) will be fulfilled.

The results of this paper may be applied to the solution of feedback control problems under state constraints. One of the possible schemes is to solve the problem in the class of set-valued control strategies this requires the solution of a problem inverse to those of the above.

9. The Inverse Problem

Consider system (5.1), (2.2) for $t \in [s,t_1]$, with set $\mathbf{M} \in comp\ \mathbf{R}^n$.

Definition 9.1. The *viable domain* for system (5.1), (2.2) at time s is the set $W(s,t_1)$ that consists of all vectors $w \in \mathbf{R}^n$ such that

$$x(t,s,w) \subseteq Y(t)\ , \quad s \leq t \leq t_1\ , \tag{9.1}$$

$$x(t_1,s,w) \subseteq \mathbf{M} . \tag{9.2}$$

Using the duality relations of convex analysis as given in [6] it is possible to observe that

$$W(s,t_1) \subseteq \mathbf{R}_-(s,M(\cdot)) \ , \ \forall \, M(\cdot) \in C^{n\times n}\,[T^s] \ ,$$

where

$$. \ T^s = \{t : s \le t \le t_1\} \ .$$

$$\mathbf{R}_-(S,M(\cdot) = (\mathbf{S}(\theta,s) - \int_{t_1}^{s} M(t)\mathbf{S}(\theta,t)\,dt)\,\mathbf{M} +$$

$$+ \int_{t_1}^{s} (\mathbf{S}(\xi,S) - \int_{\xi}^{s} M(t)\mathbf{S}(t,\xi)\,dt)\mathbf{P}(\xi)\,d\xi + \int_{t_1}^{s} \rho(M(t)\,|\,Y(t))\,dt \ .$$

Similar to § 6 we come to

Lemma 9.1. *The set $W(s,t_1)$ may be determined as*

$$W[s] = W(s,t_1) = \bigcap \{\mathbf{R}_-(s,M(\cdot))\,|\,M(\cdot) \in C^{n\times n}[T^S]\} \ .$$

Under assumptions $\mathbf{A}^*$, *B or* $\mathbf{A}^*$, *C it also satisfies the funnel equation*

$$\lim_{\sigma \longrightarrow 0} \sigma^{-1} h(W[\tau-\sigma], \Big((E - A(\tau)\sigma)\, W[\tau] - \mathbf{P}[\tau]\Big) \bigcap Y(\tau-\sigma)\Big) = 0, \ W[t_1] = \mathbf{M} \tag{9.3}$$

An important technical element is the directional derivative in t of the support function $\rho(\ell \mid W[\tau])$.

10. A Directional Derivative

Let us calculate the left derivative $\partial_- \, \rho(\ell \mid W[\tau]) \,/\, \partial t$ for a given direction $\ell \in \mathbf{R}^n$. Since

$$h(P \ , \ Q) = \max \{ \ | \ \rho(\ell \mid P) - \rho(\ell \mid Q) \ | \ \Big| \ || \ \ell \ || = 1\}$$

we observe that the increments

$$\Delta_1(\sigma) = \sigma^{-1}(\rho(\ell \mid W[t - s]) - \rho(\ell \mid W[t])$$

and

$$\Delta_2(\sigma) = \sigma^{-1}(\rho(\ell \mid ((E - \sigma A(t))\ W[t] - \sigma\,\mathbf{P}(t)) \bigcap Y(t-\sigma)) - \rho(\ell \mid W[t])$$

are such that

$$\lim_{\sigma \longrightarrow 0} \mid \Delta_1(\sigma) - \Delta_2(\sigma) \mid = 0$$

Therefore it suffices to calculate the left derivative

$$d_-\ g(\sigma)\ /\ d\sigma \ |_{\sigma=0} \tag{10.1}$$

for the function

$$g(\sigma) = \rho(\ell \mid ((E - \sigma A(t))\ W[t] - \sigma\,\mathbf{P}(t)) \bigcap Y(t-\sigma)) =$$

$$= \min\{\rho(p \mid (E - \sigma A(t))\ W[t] - \sigma\,\mathbf{P}(t)) + \rho(\ell - p \mid Y\,(t-\sigma)) \mid p \in \mathbf{R}^n\}$$

The calculation of (10.1) then follows the techniques of [16]. The results are given by

Lemma 10.1 Under the assumptions of theorem 5.1 the directional derivative $\partial_-\,\rho(\ell \mid W[t])\ /\ \partial t$ *exists for every* $\ell \in \mathbf{R}^n$ *and almost all* $t \in T$. *It is given by formula*

$$\partial_-\,\rho(\ell \mid W[t])\ /\ \partial t =$$

$$= \min\{\rho(-p'\,A(t) \mid \partial_\ell\,k(t\,,\ell)) + \rho(-p \mid \mathbf{P}(t)) -$$

$$- \frac{\partial}{\partial t}\,(\rho(\ell - p \mid Y(t))) \mid p \in \mathbf{P}(t\,,\ell)\}$$

where $\partial_\ell\,k(t\,,\ell)$ *is the subdifferential in the variable* ℓ *of the function*

$$k(t\,,\ell) = \rho(\ell \mid W[t])\,,\ \mathbf{P}(t\,,\ell) = \{p \in \mathbf{R}^n : k(t\,,\ell) - k(t\,,p) =$$

$$= \rho(\ell - p \mid Y(t))\}$$

The formula of the above may be used for proving the existence of a feedback solution strategy in a control problem with state constraints. We will pursue this solution following the "extemal aiming" rule of [1] and the schemes of [6].

11. A Feedback Control Problem

Consider the system

$$\dot{x} \in A(t)x + u \tag{11.1}$$

with control

$$u \in \mathbf{P}(t)$$

and constraints

$$x(t) \in Y(t)\ , \qquad t_0 \leq t \leq t_1$$

$$x(t_1) \in \mathbf{M}$$

The set-valued functions $\mathbf{P}(t)$, $Y(t)$ are similar to § § 8-10, $\mathbf{M} \in conv\ \mathbf{R}^n$.

Problem 11.1 Devise a *feedback strategy* in the form of a set-valued function

$$u = U(t\ ,\ x)\ ,\ U(t\ ,\ x) \subseteq \mathbf{P}(t)$$

that would ensure for a certain range $W_s = \{(s\ ,\ w)\}$ of *positions* $(s\ ,\ w)$ $(s \in \mathbf{R}\ ,\ w \in \mathbf{R}^n)$ that restrictions (9.1), (9.2) would be fulfilled.

The admissible class of multivalued strategies $U(t\ ,\ x)$ will consist of those that ensure the existence of a solution to the inclusion

$$\dot{x} \in A(t)x + U(t\ ,\ x) \tag{11.2}$$

Lemma 11.1 Assuming instant τ is given, the set W_τ positions for which there exists a solution to problem 11.1 may be defined as

$$W_\tau = U\{(\tau\ ,\ w) \mid w \in \mathrm{W}(\tau\ ,\ t_1)\}$$

Assuming that the set $\mathrm{W}(\tau\ ,\ t_1)$ of § 9 is already specified, the solution to problem 11.1 is given by

Theorem 11.1 The solution $U(t\ ,\ x)$ to problem 11.1 may be given by the set-valued function

$$U^*(t\ ,\ x) = \begin{cases} \mathbf{P}(t) & \text{if } x \in \mathrm{W}(t\ ,\ t_1) \\ \partial\rho(-\ell \mid P(t)) & ,\ \ell \in \partial\, d(x\ ,\ \mathrm{W}(t\ ,\ t_1)) \\ & \text{if } x \,\bar{\in}\, W(t\ ,\ t_1) \end{cases}$$

Here $\partial f(\ell)$ is the subdifferential of function f at point ℓ. A standard proof indicates that $U^*(t, x)$ is an admissible strategy [6].

In order to prove theorem 11.1 it suffices to show that the derivative

$$d_+ (x(t), W[t]) / dt \leq 0 \tag{11.3}$$

if calculated along the solutions of (11.1) with $u = U^*(t, x)$, for any $x(t) \bar{\in} W(t, t_1)$.

Without loss of generality we may assume $A(t) \equiv 0$. (Since by substituting $\dot{z} = S(t, t_1)x$ the equation (11.1) may be reduced to $\dot{z} = S(t, t_1)u$)

Therefore we ought to differentiate the function

$$d(x, W(t, t_1)) = \max \left\{ 0, \max \{ (\ell, x) - \rho (\ell \mid W(t, t_1)) \;\Big|\; || \ell || = 1 \} \right\}$$

in t. If $\bar{\in} W(t, t_1)$ then

$$d(x, W(t, t_1)) = (\ell^o, x) - \rho(\ell^o \mid W(t, t_1))$$

where $\ell^o = \ell^o(t, x)$.

Using the result of Lemma 10.1 and the formula for differentiating a function of the "maximum" type [16] we have

$$d_- \, d(x, W(t, t_1) / dt = (\ell^o, u) - \partial_- \rho(\ell^o \mid W(t, t_1) / \partial t =$$

$$= (-\ell^o, u) - min \{+ \rho(-p \mid P(t)) - \frac{\partial_-}{\partial t} \rho(\ell^o - p \mid Y(t)) \;\Big|\; p \in P(t, \ell)\} \geq$$

$$\geq (-\ell^o, u) - \rho(-\ell^o \mid P(t)) = 0 \tag{11.4}$$

The last inequality is true if $u \in U^*(t, x)$. It follows from the definition of $U^*(t, x)$

Since $\rho(\ell \mid W(t, t_1))$ is differentiable both from the left and the right, inequality (11.4) proves (11.3). The latter in turn ensures that (9.1), (9.2) would be fulfilled. (Otherwise if $x(t)$ would belong to the boundary of $W(t, t_1)$ and $x(t + \sigma) \bar{\in} W(t + \sigma, t_1)$ for some $\sigma > 0$, then there would exist an instant $t + \sigma', \sigma' < \sigma$ such that $x(t + \sigma') \bar{\in} W(t + \sigma, t_1)$ and $d \, d \, W(t + \sigma', t_1) / dt > 0$. This contradicts with (11.3)).