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FORCED SECOND ORDER CONSERVATIVE SYSTEMS WITH PERIODIC NONLINEARITY

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1. Introduction

We consider the problem

$$(M(t)u')' + Au + D_uF(t,u) = h(t)$$
$$u(0) - u(T) = u'(0) - u'(T) = 0 \tag{1}$$

where $M : [0,T] \to S(\mathbf{R}^n, \mathbf{R}^n)$ is a continuous mapping in the space $S(\mathbf{R}^n, \mathbf{R}^n)$ of symmetric real (nxn)-matrices such that, for some $\mu > 0$ and all $(t,v) \in [0,T] \times \mathbf{R}^n$,

$$(M(t)v|v) \geqslant \mu|v|^2,$$

$A \in S(\mathbf{R}^n, \mathbf{R}^n)$, $F : [0,T] \times \mathbf{R}^n \to \mathbf{R}$ is continuous and bounded and $D_uF : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ exists, is continuous and is bounded, and $h \in L^1(0,T; \mathbf{R}^n)$.

The special case where $M(t) \equiv Id$ and $A = 0$ has been considered in [11,16] and was motivated by the study of the forced pendulum equation [10]. Our motivations for (1) are the equation describing the periodic motions of a satellite, with respect to its center of mass, which take place in the plane of its orbit around the direction of the radius vector

$$((1+e\cos t)^2u')' + (1+e\cos t)a\sin u = 4e\sin t(1+e\cos t) \quad (2)$$

$(|e| < 1)$ (see e.g. [1]), the equations of linearly coupled pendulum

$$\begin{aligned} u_1'' + a_1(u_1-u_2) + b\sin u_1 &= h_1(t) \\ u_2'' + a_2(u_2-u_1) + b\sin u_2 &= h_2(t) \end{aligned} \quad (3)$$

(see e.g. [7]), and the system of equations arising in the theory of Josephson multipoint junctions (see e.g. [6])

$$u'' + n^2Au + f(u) = h(t), \quad (4)$$

where

$$A = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & 0 & \\ & & & & \\ & 0 & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}$$

and $f(u) = (a_1 \sin u_1,\ldots,a_n \sin u_n)$.

The existence of the periodic solutions of (2) has been studied, using various methods (see e.g. the references in [8]) and the most recent contributions are those of Petryshyn-Yu [14], Mawhin [8] and Dang Dinh Hai [3]. The best conditions of existence are those of [8] and [3] which require only $|e| < 1$ and are obtained respectively by using the symmetry of the equation and Schauder fixed point theorem and by minimizing the associated action. Moreover, [8] proves the existence of a second periodic solution when $|e| < |a|/4$ using upper and lower solutions and degree techniques. The results of the present paper will provide the existence of two solutions for (2) when $|e| < 1$. The system (3) was studied by Marlin [7] when h satisfies some symmetry conditions and by Drabek-Invernizzi [4] for more general h. The same authors have also considered (4) and use topological degree type techniques which allow the presence of friction terms in (3) and (4) but provide existence conditions which are rather involved and require some smallness restrictions on h.

In this paper, we shall study the existence and multiplicity of the solutions of (1) using various methods of critical point theory. Of course, if the "linearized" problem

$$\begin{gathered}(M(t)u')' + Au = 0 \\ u(0) - u(T) = u'(0) - u'(T) = 0\end{gathered} \tag{5}$$

has only the trivial solution, then (1) has at least one solution as shown immediately by a direct application of Schauder's fixed point theorem (and the gradient structure of the nonlinearity is not necessary). The next case to consider in increasing order of complexity seems to be the situation where (5) admits only constant non-trivial solutions which will be necessary the elements of the null-space $N(A)$ of A. That was the case in the special case $M(t) = Id$ and $A = 0$ of [11] where the existence of at least two solutions was proved under a periodicity condition on F

$$F(t,u+T_je_j) = F(t,u) \qquad (1 \leqslant j \leqslant n) \tag{6}$$

for all $(t,u) \in [0,T] \times \mathbf{R}^n$ and some $T_j > 0$ $(1 \leqslant j \leqslant n)$, (the e_j are the elements of the canonical basis in $\mathbf{R}^n$) and a zero-mean value condition on h

$$\int_0^T h(t)dt = 0. \tag{7}$$

This is also the case in equation (2), which corresponds to $A = 0$ and to a forcing term having mean value zero, and this was assumed in [4] for (3) and (4) with the further restriction that $\dim N(A) = 1$.

If $N(A) \neq \{0\}$ and $N(A) = \text{span}(\alpha_1,\ldots,\alpha_m)$ for some $1 \leqslant m \leqslant n$, we shall study (1) under the generalized condition (6)

$$F(t,u+T_j\alpha_j) = F(t,u) \qquad (1 \leqslant j \leqslant m) \tag{8}$$

for all $(t,u) \in [0,T] \times \mathbb{R}^n$ and some $T_j > 0$ $(1 \leqslant j \leqslant m)$ and the generalized condition (7)

$$\int_0^T (h(t)|\alpha_j)dt = 0 \qquad (1 \leqslant j \leqslant m). \tag{9}$$

The existence of at least one solution will be proved in Theorem 1 and the existence of more solutions will be obtained in Theorems 2, 3 and 4 under further conditions. In a further paper, we shall use more sophisticated algebraic topological tools to avoid those further conditions.

2. The assumptions and the Palais-Smale condition

Let $M : [0,T] \to S(\mathbb{R}^n, \mathbb{R}^n)$ be a continuous mapping in the space $S(\mathbb{R}^n, \mathbb{R}^n)$ of symmetric (nxn)-real matrices such that

$$(M(t)v|v) \geqslant \mu|v|^2 \tag{10}$$

for some $\mu > 0$ and all $(t,v) \in [0,T] \times \mathbb{R}^n$, with $(v|w)$ the usual inner product of v and w in $\mathbb{R}^n$ and $|v|$ the corresponding norm. Let $A \in S(\mathbb{R}^n, \mathbb{R}^n)$ and let us assume that the following condition holds

(H_1) dim $N(A) = m \geqslant 1$ and M and A are such that

$$\begin{gathered}(M(t)u')' + Au = 0\\ u(0) - u(T) = u'(0) - u'(T) = 0\end{gathered} \tag{11}$$

if and only if u is constant and $u \in N(A)$.

An easy consequence of (H_1) is that the non homogeneous corresponding problem

$$(M(t)u')' + Au = h(t)$$
$$u(0) - u(T) = u'(0) - u'(T) = 0 \qquad (12)$$

has a solution if and only if h satisfies the following condition

(H_2) $$\int_0^T (h(t)|v)dt = 0$$

for each $v \in N(A)$.

Let $H_T^1 = \{u : [0,T] \to \mathbb{R}^n \mid u$ is absolutely continuous on $[0,T]$, $u(0) = u(T)$ and $u' \in L^2(0,T;\mathbb{R}^n)\}$

equipped with the inner product

$$\langle u|v\rangle = \int_0^T [(M(t)u'(t)|v'(t)) + (u(t)|v(t))]dt$$

and the corresponding norm

$$\|u\| = (\int_0^T [(M(t)u'(t)|u'(t)) + |u(t)|^2]dt)^{1/2}.$$

By (10), $\|u\|$ is equivalent to the classical norm

$$(\int_0^T (|u'(t)|^2 + |u(t)|^2)dt)^{1/2}$$

and H_T^1 is a Hilbert space. Now, the quadratic form q defined on H_T^1 by

$$q(u) = \int_0^T (1/2)[(M(t)u'(t)|u'(t)) - (Au(t)|u(t))]dt$$

is such that

$$q(u) = (1/2)\|u\|^2 - \int_0^T (1/2)((A+I)u(t)|u(t))dt =$$
$$= (1/2)[\|u\|^2 - \langle Ku|u\rangle] = (1/2)\langle (I-K)u|u\rangle$$

where the linear self-adjoint operator $K : H^1_T \to H^1_T$ defined via Riesz representation theorem by

$$\int_0^T ((A+I)u(t)|v(t))dt = \langle Ku|v\rangle \ , \ (u,v \in H^1_T)$$

is compact because of the compact embedding of H^1_T into $C([0,T], \mathbb{R}^n)$. On the other hand, the critical points of q on H^1_T coincide with the elements of $N(I-K)$ and with the solutions of (11), so that

$$N(I-K) = N(A)$$

(if we identify constant functions with their value).
By classical spectral theory, we can decompose H^1_T into the orthogonal direct sum of invariant subspaces for $I-K$,

$$H^1_T = H^- \oplus \overset{\circ}{H} \oplus H^+$$

where $\overset{\circ}{H} = N(I-K) = N(A)$, H^- is finite dimensional (as K has only finitely many eigenvalues λ_i with $\lambda_i > 1$) and there exists $\delta > 0$ such that

$$q(u) \leqslant -(\delta/2) \|u\|^2 \quad \text{for } u \in H^-, \tag{13}$$

and

$$q(u) \geqslant (\delta/2) \|u\|^2 \quad \text{for } u \in H^+. \tag{14}$$

If $u \in H^1_T$, we shall write correspondingly $u = u^- + \overset{\circ}{u} + u^+$ with $u^- \in H^-$, $\overset{\circ}{u} \in \overset{\circ}{H}$, $u^+ \in H^+$.

The following result characterizes the A for which $\dim H^- = 0$.

PROPOSITION 1. *If* $M : [0,T] \to S(\mathbb{R}^n, \mathbb{R}^n)$ *is continuous and positive and if* $A \in S(\mathbb{R}^n, \mathbb{R}^n)$, *then* $\dim H^- = 0$ *if and only if* A *is semi-negative definite.*

Proof. Necessity. If $\dim H^- = 0$, then $q(u) \geqslant 0$ for each $u \in H^1_T$ and in particular for each constant c; thus

$$0 \leqslant q(c) = -\int_0^T \tfrac{1}{2}(Ac|c)dt = -\tfrac{T}{2}(Ac|c)$$

for all $c \in \mathbb{R}^n$ and A is semi-negative definite.

Sufficiency. If A is semi-negative definite, then

$$q(u) = (1/2) \int_0^T (M(t)u'(t)|u'(t))dt - \frac{1}{2} \int_0^T (Au(t)|u(t))dt \geqslant (\mu/2)\|u'\|^2 \geqslant 0.$$

When M is constant, the *Morse index of* q, i.e. dim H^-, can be easily obtained through the properties of M and A. M and A being symmetric and M positive definite, they can be simultanously diagonalized by a unitary matrice. Thus, without loss of generality, we can assume

$$M = \mathrm{diag}(m_1, \ldots, m_n),\ A = \mathrm{diag}(a_1, \ldots, a_n)$$

with $m_i > 0$ $(1 \leqslant i \leqslant n)$ and m of the a_i are equal to zero. Therefore, writing, for $u \in H^1_T$, $u_j = \sum_{k \in \mathbb{Z}} c_{j,k} e^{ik\omega t}$, $(1 \leqslant j \leqslant n)$, we find

$$q(u) = (1/2) \left[\sum_{j=1}^{n} \sum_{k \neq 0} (m_j k^2 \omega^2 - a_j)|c_{j,k}|^2 - \sum_{j=1}^{n} a_j |c_{j,0}|^2 \right]$$

and hence

$$\dim H^- = \# \{j | a_j > 0\} + 2 \sum_{j=1}^{n} \# \{k \in \mathbb{N}^* | k^2\omega^2 < a_j/m_j\}.$$

In particular, dim H^- is the number of positive eigenvalues of A if $\omega^2 \geqslant a_j/m_j$ for all $1 \leqslant j \leqslant n$.

Now let $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous bounded function such that $D_uF : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ exists and is continuous and bounded. We shall assume that F satisfies the following condition.

(H_3) There exist $\alpha_j \in \mathbb{R}^n$ and $T_j > 0$ $(1 \leqslant j \leqslant m)$ such that $N(A) = \mathrm{span}\,(\alpha_1, \ldots, \alpha_m)$ and

$$F(t,u+T_j\alpha_j) = F(t,u) \quad (1 \leqslant j \leqslant m)$$

for all $(t,u) \in [0,T] \times \mathbb{R}^n$.

Now, let $h \in L^1(0,T; \mathbb{R}^n)$ satisfying (13). Then, the function r, defined over H^1_T by

$$r(u) = \int_0^T [F(t,u(t))-(h(t)|u(t))]\, dt$$

is of class C^1, with gradient given by

$$<\nabla r(u)|v> = \int_0^T (D_uF(t,u(t))-h(t)|v(t))dt.$$

The compact imbedding of H_T^1 into $C([0,T], \mathbb{R}^n)$ implies that

$$r(u_n) \to r(u) \text{ if } u_n \rightharpoonup u \text{ in } H_T^1.$$

Finally, a classical result of the calculus of variations (see e.g. [12]) implies that the solutions of (1) are given by the critical points of the function f defined on H_T^1 by

$$f(u) = q(u)-r(u) = \int_0^T [(1/2)(M(t)u'(t)|u'(t))-(1/2)(Au(t)|u(t))-$$

$$-F(t,u(t))+(h(t)|u(t))]dt.$$

We shall show that f satisfies a weak form of a Palais-Smale condition.

PROPOSITION 2. *Assume that* M *and* A *satisfy* (H_1), *that* D_uF *is bounded and that* $h \in L^1(0,T; \mathbb{R}^n)$. *Then each sequence* (u_k) *in* H_T^1 *such that* $(\overset{\circ}{u}_k)$ *is bounded and*

$$\nabla f(u_k) \to 0$$

contains a convergent subsequence.

Proof. Let (u_k) be such a sequence; then $(\nabla f(u_k))$ is bounded and hence there exists $C_1 > 0$ with $\|\nabla f(u_k)\| \leq C_1$ for all $k \in \mathbb{N}$. Then, using (13) and (14), we have

$$C_1(\|u_k^+\|^2+\|u_k^-\|^2)^{1/2} = C_1\|u_k^+-u_k^-\| \geq$$

$$\geq <\nabla f(u_k)|u_k^+-u_k^-> = <(I-K)u_k|u_k^+-u_k^-> - <\nabla r(u_k)|u_k^+-u_k^->$$

$$= <(I-K)u_k^+|u_k^+> - <(I-K)u_k^-|u_k^-> - <\nabla r(u_k)|u_k^+-u_k^-> \geq$$

$$\geq \delta(\|u_k^+\|^2 + \|u_k^-\|^2) - C_2(\|u_k^+\|^2+\|u_k^-\|^2)^{1/2},$$

where C_2 depends only on the bound on D_uF and of $\|h\|_{L^1}$.

Thus, there exists $C_3 > 0$ such that

$$(\|u_k^+\|^2 + \|u_k^-\|^2)^{1/2} \leqslant C_3$$

and hence (u_k) is bounded. Going if necessary to a subsequence, we can assume that $u_k \rightharpoonup u$ in H_T^1 and $u_k \to u$ in $C([0,T], \mathbb{R}^n)$.

Now

$$\langle \nabla f(u_k) - f(u) | u_k - u \rangle = \|u_k - u\|^2 - \langle K(u_k - u) | u_k - u \rangle -$$

$$- \int_0^T (D_u F(t, u_k(t)) - D_u F(t, u(t)) \cdot | u_k(t) - u(t)) dt$$

and the left-hand member as well as the two last terms of the right-hand member tend to zero if $k \to \infty$. Consequently $\|u_k - u\| \to 0$ and the proof is complete.

PROPOSITION 3. *Assume that* M *and* A *satisfy* (H_1), *that* h *satisfies* (H_2) *and that* F *satisfies* (H_3). *Then each* $c \in \mathbb{R}$ *for which a sequence* (u_k) *exists with*

$$f(u_k) \to c, \quad \nabla f(u_k) \to 0 \tag{15}$$

as $k \to \infty$, *is a critical value of* f.

__Proof__. Let c and (u_k) satisfy (15); if we write

$$\overset{\circ}{u}_k = \sum_{j=1}^{m} c_j \alpha_j,$$

there will exist $k_j \in \mathbb{Z}$ and $\hat{c}_j \in [0, T_j[$ such that

$$c_j = \hat{c}_j + k_j T_j \quad (1 \leqslant j \leqslant m).$$

Set $\hat{u}_k = u_k^- + \sum_{j=1}^{m} \hat{c}_j \alpha_j + u_k^+$, so that $\hat{u}_k^- = u_k^-$, $\hat{u}_k^+ = u_k^+$ and $\overset{\circ}{\hat{u}}_k = \sum_{j=1}^{m} \hat{c}_j \alpha_j$ is bounded. Now, as $\hat{u}_k - u_k \in N(I-K)$, we have

$$q(\hat{u}_k) = q(u_k), \; \nabla q(\hat{u}_k) = \nabla q(u_k).$$

On the other hand, by (H_2) and (H_3),

$$r(\hat{u}_k) = \int_0^T [F(t,u_k^-(t)+u_k^+(t)+\overset{\circ}{u}_k - \sum_{j=1}^m k_j T_j \alpha_j) - (h(t)|u_k^-(t)+u_k^+(t))]\, dt =$$

$$= \int_0^T [F(t,u_k(t)) - (h(t)|u_k(t))]\, dt = r(u_k)$$

and similary,

$$\nabla r(\hat{u}_k) = \nabla r(u_k), \quad k \in \mathbb{N}.$$

Thus

$$\nabla f(\hat{u}_k) \to 0$$

as $k \to \infty$ and, by Proposition 2, going if necessary to a subsequence, we have

$$\hat{u}_k \to u \quad \text{in } H_T^1.$$

Therefore,

$$\nabla f(u) = 0 \quad \text{and } f(u) = \lim_{k\to\infty} f(\hat{u}_k) = \lim_{k\to\infty} f(u_k) = c,$$

i.e. c is a critical value for f.

3. The existence of critical points for f

The following geometrical properties of f will be useful.

PROPOSITION 4. *If* M *and* A *satisfy* (H_1), h *satisfies* (H_2) *and* F *is bounded, then* f *is bounded below on* $H^\circ \oplus H^+$.

Proof. If $u = u^\circ + u^+ \in H^\circ \oplus H^+$, then

$$f(u) = (1/2)\langle Ku^+|u^+\rangle - \int_0^T F(t,u(t))dt + \int_0^T (h(t)|u^+(t))dt$$

$$\geq (\delta/2)\|u^+\|^2 - c_1 - \|h\|_{L^1}\|u^+\|_{L^\infty} \geq$$

$$\geq (\delta/2)\|u^+\|^2 - c_1 - c_2\|u^+\| \geq -(c_2^2/2\delta) - c_1.$$

PROPOSITION 5. *If* M *and* A *satisfy* (H_1), F *is bounded and* $\dim H^- \geq 1$, *then*

$$f(u) \to -\infty \text{ as } \|u\| \to \infty \text{ in } H^- .$$

Proof. If $u \in H^-$, then

$$f(u) \leq -(\delta/2)\|u\|^2 + C_1 + C_2\|u\|$$

and the result follows.

We can now prove our first existence theorem for (1).

THEOREM 1. *Assume that* M *and* A *satisfies* (H_1), h *satisfies* (H_2) *and* h *satisfies* (H_3). *Then* (1) *has at least one solution* u_1 *with critical value* $c_1 = f(u_1)$ *characterized as follows.*

a) if A *is semi-negative definite,* $c_1 = \inf\limits_{H^1_T} f$

b) if A *is not semi-negative definite, then*

$$c_1 = \inf_{\sigma \in \Sigma_R} \max_{s \in B^-_R} f(\sigma(s)), \tag{16}$$

where $R > 0$ *is such that*

$$\max_{S^-_R} f < \inf_{H^\circ \oplus H^+} f,$$

$B^-_R = \{u \in H^- : \|u\| \leq R\}$, $S^-_R = \{u \in H^- : \|u\| = R\}$ *and*

$$\Sigma_R = \{\sigma \in C(B^-_R, H^1_T)\ \sigma\big|_{S^-_R} = Id\}.$$

Proof. If A is semi-negative definite, then, by Proposition 1, $\dim H^- = 0$, and hence f is bounded below on H by Proposition 4. Consequently, the result follows from Ekeland's variational principle [5] and Proposition 3 with $c = \inf\limits_H f$. If A is not semi-negative definite, then $\dim H^- \geq 1$ and the result follows from the version of Rabinowitz saddle point theorem [15] given in [9] (see also [12]) and Proposition 3 with c given by (16).

When $\dim H^- = 0$, we can extend to the above situation the result of Mawhin-Willem [11] about the existence of a second *geometrically distinct* solution u_2 (i.e. a solution distinct from the "equivalent" solutions $u_1 + \sum_{j=1}^{m} k_j T_j \alpha_j$, $k_j \in \mathbb{Z}$) by adapting the argument of [11] based on the mountain pass lemma. The details are left to the reader.

THEOREM 2. *Assume that* M *and* A *satisfy* (H_1), A *is semi-negative definite,* h *satisfies* (H_2) *and* F *satisfies* (H_3). *Then* (1) *has a solution* u_2 *such that*

$$u_2 \neq u_1 + kT_i\alpha_i,\ k \in \mathbb{Z},\ i \leqslant i \leqslant m.$$

The result however can be improved using Lusternik-Schnirelmann category [13] . Since $f(u + T_j\alpha_j) = f(u)$ $(u \in H^1_T,\ 1 \leqslant j \leqslant m)$, it is natural to define f on the Riemannian manifold

$$\mathcal{T} = T^m \times (H^- \oplus H^+),$$

where T^m is the m-dimensional torus, by the relation

$$f(c_1, \ldots, c_m, u^-, u^+) = f(\sum_{i=1}^{m} c_i\alpha_i + u^- + u^+)$$

with $u° = \sum_{i=1}^{m} c_i\alpha_i$. Of course, distinct critical points of f on $\mathcal{T}$ will correspond to geometrically distinct solutions of (1).

THEOREM 3. *Under the assumptions of Theorem 2,* (1) *has at least* $m+1$ *geometrically distinct solutions.*

__Proof__. Proposition 2 implies that f satisfies the usual Palais-Smale condition on $\mathcal{T}$ and Proposition 4 implies that f is bounded below on $\mathcal{T}$. Then, by a classical result [13], f has at least cat $\mathcal{T}$ critical points, where cat $\mathcal{T}$ denotes the Lusternik-Schnirelmann category of $\mathcal{T}$ (see e.g. [13] for the definitions). Now, the following equalities are easily verified

$$\operatorname{cat} \mathcal{T} = \operatorname{cat}_{\mathcal{T}} (T^m \times \{0\}) = \operatorname{cat} (T^m \times \{0\}) = \operatorname{cat} T^m,$$

and the result follows from the classical result

$$\operatorname{cat} T^m = m+1.$$

A better estimate can be obtained, using Morse theory, under the assumptions that all the critical points are non-degenerate. The result is modelled on the one given in [12] (in the case where $M(t) = I$ and $A = 0$) to which we refer, together with [2] , for the terminology and tools of Morse theory.

THEOREM 4. *Under the assumptions of Theorem 2, if* $D_u^2 F$ *exists and is continuous on* $[0,T] \times \mathbb{R}^n$ *and if the set* S *of solutions of* (1) *is finite, namely* $S = \{u_1, u_2, \dots, u_j\}$, *then there exists a polynomial* $Q(t)$ *with nonnegative integer coefficients such that*

$$\sum_{k=0}^{\infty} \left(\sum_{i=1}^{j} \dim C_k(f,u_i)\right) t^k = \sum_{k=0}^{m} \binom{m}{k} t^k + (1+t)\, Q(t),$$

where $C_k(f,u_i)$ *denotes the* k^{th} *critical group of* u_i. *Moreover, if all the solutions of* (1) *are non-degenerate,* (1) *has at least* 2^m *solutions.*

<u>Proof</u>. By a classical result of algebraic topology the Poincaré polynomial of $\mathcal{T}$

$$P(t,\mathcal{T},\phi) = \sum_{k=0}^{\infty} \dim H_k(\mathcal{T}) t^k$$

(where $H_k(\mathcal{T})$ denotes the k^{th} singular homology group of $\mathcal{T}$) is given by

$$P(t,\mathcal{T},\phi) = P(t,T^m,\phi) = \sum_{k=0}^{m} \binom{m}{k} t^k.$$

On the other hand, Proposition 2 shows that f satisfies the Palais-Smale condition on $\mathcal{T}$. Therefore, by Morse theory [2, 12] , there will exist a polynomial $Q(t)$ with integer nonnegative coefficients such that the Morse polynomial

$$M(t,\mathcal{T},\phi) = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{j} \dim C_k(f,u_i)\right) t^k$$

satisfies the relation $M(t,\mathcal{T},\phi) = P(t,\mathcal{T},\phi)+(1+t)Q(t)$, which proves the first part of the theorem.

The second part is trivial if f has infinitely many critical points so that we can assume that it has only a finite number, namely $u_1, \ldots, u_j$. Now, if all the u_i are nondegenerate, we have

$$\dim C_k(f,u_i) = \delta_{k,M_i}$$

where M_i is the Morse index of $D^2f(u_i)$. But then

$$j = \sum_{k=0}^{\infty} \# \{u_i \mid 1 \leq i \leq j,\ M_i = k\} = \sum_{k=0}^{\infty} \sum_{i=1}^{j} \delta_{k,M_i} =$$

$$= \sum_{k=0}^{\infty} \sum_{i=1}^{j} \dim C_k(f,u_i) \geq \sum_{k=0}^{m} \binom{m}{k} = 2^m,$$

and the proof is complete.

Remark. Theorems 3 and 4 still hold without the assumption that A is semi-negative definite, but the corresponding proofs require more delicate algebraic topological arguments and will be given in a subsequent paper.

4. The satellite-type equation

If $m : [0,T] \to \mathbb{R}$ is a continuous and positive function, $h \in L^1(0,T)$ and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, $D_uF : [0,T] \times \mathbb{R} \to \mathbb{R}$ exists and is continuous, and if

$$F(t,u+T_1) = F(t,u) \tag{18}$$

for all $(t,u) \in [0,T] \times \mathbb{R}$ and some $T_1 > 0$, let us consider the T-periodic problem for the satellite-type system

$$\begin{gathered} (m(t)u') + D_uF(t,u) = h(t) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{gathered} \tag{19}$$

Condition (18) implies that F and D_uF are bounded on $[0,T] \times \mathbb{R}$ and hence condition (H_1) is satisfied for $M = m$ and $A = 0$. Condition (H_2) becomes

$$\int_0^T h(t)dt = 0 \tag{20}$$

and (18) implies condition (H_3) with $\alpha_1 = 1$. Now,

$$q(u) = \int_0^T (1/2)\ m(t)\ (u'(t))^2 dt \geqslant 0$$

for all $u \in H_T^1$, and hence $\dim H^- = 0$. We deduce then from Theorem 1 and 2 the following

COROLLARY 1. *If* F *satisfies* (18), *then the problem* (19) *has at least two solutions not differing from a multiple of* T_1 *for every* $h \in L^1(0,T)$ *satisfying* (20).

This will be the case in particular for equation (2), with $T = 2\pi$, under the only condition that $|e| < 1$. This improves the result of [8] which requires in addition that

$$|e| < \max\ (|a|/4,\ e_0)$$

where $e_0 = 0{,}2982...$ is the positive root of the equation

$$9e^4 + (12 + (2 + \frac{3}{2^3})^2)e^2 + 4 - (2 + \frac{3}{2^3})^2 = 0.$$

Let us notice that, in the special case of equation (2), the existence of a second solution follows easily from the existence of an odd 2π-periodic solution given in [8] and the symmetry of the equation, by the following argument of C. Fabry (personal communication). Equation (2) has the form

$$c_e(t)u''(t) + 2c_e'(t)u'(t) + a \sin u(t) = 2c_e'(t), \tag{21,a}$$

where $c_e(t) = 1 + e \cos t$ is such that $c_e(t) = c_{-e}(t+\pi)$.
Let v_0 be the odd 2π-periodic solution of (21,-a) (so that

$$v_0(k\pi) = 0,\ k \in \mathbb{Z}\ ,)$$

whose existence is proved in [8], and let

$$u_3(t) = \pi + v_0(t)$$

so that u_3 is 2π-periodic and does not differ by $2k\pi$ from the solution u_0 of (21, a) as $u_0(k\pi) = 0$ and $u_3(k\pi) = \pi$. Now, for each $t \in \mathbb{R}$,

$$c_e(t)u_3''(t) + 2c_e'(t)u_3'(t) + a \sin u_3(t) - 2c_e'(t) =$$

$$= c_e(t)v_0''(t) + 2c_e'(t)v_0'(t) - a \sin v_0(t) - 2c_e'(t) = 0,$$

and u_3 is a 2π-periodic solution of (21).

Of course, Corollary 1 will conclude to the existence of at least two distinct T-periodic solutions for the equation

$$c(t)u''(t) + 2c'(t)u'(t) + g(u(t)) = 2c'(t)$$

for every T-periodic and positive C^1 function c and every continuous T_1-periodic g with mean value zero.

5. The linearly coupled pendulums equation

We consider now the forced linearly coupled pendulums equation, which can be written (see [7])

$$\begin{aligned} m_1u_1'' + u_1 - u_2 + a_1 \sin u_1 &= e_1(t) \\ m_2u_2'' + u_2 - u_1 + a_2 \sin u_2 &= e_2(t) \end{aligned} \tag{22}$$

where $m_i > 0$ and $e_i \in L^1(0,T)$ $(i = 1, 2)$. Hence

$$M(t) = \text{diag}\,(m_1, m_2)\ ,\ A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},\ F(t, u) = -a_1 \cos u_1 - a_2 \cos u_2.$$

Thus,

$$N(A) = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = v_2\}$$

and $(Av|v) = (v_1-v_2)^2 \geqslant 0$, so that we always have $\dim H^- \geqslant 1$. Condition (H_1) will be satisfied if and only if, with $\omega = 2\pi/T$,

$$\text{dét} \begin{pmatrix} 1-k^2\omega^2m_1 & -1 \\ -1 & 1-k^2\omega^2m_2 \end{pmatrix} \neq 0$$

for all $k \in \mathbb{N} \setminus \{0\}$, i.e. if and only if

$$(m_1^{-1} + m_2^{-1}) \neq k^2\omega^2 \tag{23}$$

for all $k \in \mathbb{N} \setminus \{0\}$. Condition (H_2) is here

$$\int_0^T [\, e_1(t) + e_2(t) \,] \, dt = 0 \quad .$$

and condition (H_3) holds with $\alpha_1 = (1, 1)$ and $T_1 = 2\pi$. We have therefore the following

COROLLARY 2. *If condition* (23) *holds, the problem* (22) *has at least one solution for each* $e \in L^1(0,T;\, \mathbb{R}^2)$ *such that* (24) *is satisfied.*

If e_1 and e_2 are odd T-periodic functions such that $e_i(0) = e_i(T/2) = 0$ and if condition (23) holds, it is easy to show, by Schauder's fixed point theorem, that (22) has at least one odd T-periodic solution $\hat{u}(t)$ for each $(a_1, a_2) \in \mathbb{R}^2$, which vanishes at $kT/2$, $k \in \mathbb{Z}$. Denote by $\hat{v}(t)$ the corresponding solution of

$$m_1 u_1'' + (u_1 - u_2) - a_1 \sin u_1 = e_1(t)$$
$$m_2 u_2'' + (u_2 - u_1) - a_2 \sin u_2 = e_2(t)$$

and set

$$\tilde{u}(t) = \pi(1,1) + \hat{v}(t).$$

Then

$$m_1 \tilde{u}_1'' + (\tilde{u}_1 - \tilde{u}_2) + a_1 \sin \tilde{u}_1 - e_1(t) =$$

$$= m_1 \hat{v}_1'' + (-v_1 - \hat{v}_2) - a_1 \sin \hat{v}_1 - e_1(t) = 0$$

and similarly for the second equation. Thus $\tilde{u}$ is a second T-periodic solution of (22) which do not differ by a multiple of 2π.

Corollary 2 improves substantially the result of Drabek-Invernizzi [4] who require, besides (24) that other conditions like restrictions on $|a_i|$, and $\|e\|_\infty$ are satisfied. Notice however that their result is also valid in the presence of friction terms, and that [4] deals also with some situations where (24) does not hold.

6. The Josephson multipoint system

As another application, let us consider the problem

$$\begin{aligned} & u'' + n^2 Du + g(u) = e(t) \\ & u(0) - u(T) = u'(0) - u'(T) = 0 \end{aligned} \tag{25}$$

where D is the symmetric (n x n) matrix

$$D = \begin{pmatrix} 1 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \\ & & \ddots & & & \\ & & & -1 & 2 & -1 \\ & 0 & & & -1 & 1 \end{pmatrix}$$

$e \in L^1(0,T; \mathbb{R}^n)$, $g(u) = (a_1 \sin u_1, \ldots, a_n \sin u_n)$, which occurs in the theory of multipoint Josephson functions or in the space discretization of some boundary value problems for the sine-Gordon equation

$$u_{tt} - u_{xx} + a \sin u = 0$$

(see e.g. [6]). Equation (25) is a special case of (1) with $M = I$, $A = n^2 D$, $F(t,u) = - \sum_{j=1}^{n} a_j \cos u_j$ and as

$$(Dv|v) = \sum_{j=1}^{n-1} (v_j - v_{j+1})^2$$

we see that $\dim N(A) = 1$ and $N(A) = \{v \in \mathbb{R}^n : v_1 = v_2 = \ldots = v_n\} =$ $=$ span α_1 with $\alpha_1 = (1, 1, \ldots, 1)$. Also A is positive semi-definite and condition (H_1) will be satisfied if and only if

$$\det (k^2 \omega^2 I - n^2 D) \neq 0 \tag{26}$$

for all $k \in \mathbb{N} \setminus \{0\}$. Condition (H_2) is equivalent to

$$\sum_{j=1}^{n} \int_0^T e_j(t) dt = 0$$

and condition (H_3) is verified with $T_1 = 2\pi$.
We have therefore the following result for (25).

COROLLARY 3. *If condition* (26) *holds, then* (25) *has at least one solution for each* $e \in L^1(0,T; \mathbb{R}^n)$ *verifying* (27).

Corollary 3 significantly generalizes, in the variational situation, the results of Drabek-Invernizzi [4] .

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