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Existence of geodesics for the Lorentz metric of a stationary gravitational field (*)

by

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ABSTRACT. - Let $g = g(z)$ ($z = (z_0, \ldots, z_3) \in \mathbb{R}^4$) be a Lorentz metric (with signature $+,-,-,-$) on the space-time manifold $\mathbb{R}^4$. Suppose that $g$ is stationary, i.e. $g$ does not depend on $z_0$. Then we prove, under some other mild assumptions on $g$, that for any two points $a, b \in \mathbb{R}^4$ there exists a geodesic, with respect to $g$, joining $a$ and $b$.

Key words: Lorentz metric, geodesic, critical point.

RÉSUMÉ. - Soit $g = g(z)$ ($z = (z_0, \ldots, z_3) \in \mathbb{R}^4$) une métrique de Lorentz (avec signature $+,-,-,-$) sur l'espace-temps $\mathbb{R}^4$. On suppose que $g$ soit stationnaire, c'est-à-dire indépendante de $z_0$. Nous démontrons, sous des autres convenables hypothèses sur $g$, l'existence d'arcs de géodésique joignant deux points $a, b$ arbitrairement donné dans $\mathbb{R}^4$.


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0. INTRODUCTION AND STATEMENT OF THE RESULTS

In General Relativity a gravitational field is described by a symmetric, second order tensor
\[ g \equiv g(z)[\ldots], \quad z = (z_0, \ldots, z_3) \in \mathbb{R}^4 \]
on the space-time manifold \( \mathbb{R}^4 \). The tensor \( g \) is assumed to have the signature +, −, −, −; namely for all \( z \in \mathbb{R}^4 \) the bilinear form \( g(z)[\ldots] \) possesses one positive and three negative eigenvalues. The "pseudometric" induced by \( g \) is called Lorentz-metric.

In this paper we study the existence of geodesics, with respect to \( g \), connecting two points \( a, b \in \mathbb{R}^4 \).

To this end we consider the "action" functional related to \( g \), i.e.
\[
f(z) = \frac{1}{2} \int_0^1 g(z(s)) [\dot{z}(s), \dot{z}(s)] ds = \frac{1}{2} \int_0^1 \sum_{i,j=0}^3 g_{ij}(z(s)) \dot{z}_i(s) \dot{z}_j(s) ds \quad (0.1)
\]
where \( g_{ij}(i,j=0,\ldots,3) \) denote the components of \( g \) and \( z = z(s) \) belongs to the Sobolev space
\[ H^1 \equiv H^1((0,1), \mathbb{R}^4) \]
of the curves \( z:(0,1) \to \mathbb{R}^4 \) which are square integrable with their first derivative \( \dot{z} = \frac{dz}{ds} \). If \( g \) is smooth, \( f \) defined in (0.1) is Fréchet differentiable in \( H^1 \). Let \( a, b \in \mathbb{R}^4 \), then a geodesic joining \( a \) and \( b \) is a critical point of \( f \) on the manifold
\[
M = \{ z \in H^1 \mid z(0) = a, z(1) = b \}. \quad (0.2)
\]
Due to the indefiniteness of the metric \( g \) it is easy to see that the functional (0.1) is unbounded both from below and from above even modulo submanifolds of finite dimension or codimension. Then the Morse index of a geodesic is \( +\infty \), in contrast with the situation for positive definite Riemannian spaces. This fact causes difficulties in the research of a geodesic connecting \( a \) and \( b \) and actually such a geodesic, in general, does not exist (cf. [3], § 5.2 or [5], remark 1.14).

However the above difficulties can be overcome if the events \( a, b \) are causally related, namely if \( a, b \) can be joined by a smooth curve \( z = z(s) \) such that
\[
g(z(s))[\dot{z}(s), \dot{z}(s)] \geq 0 \quad \text{for all} \ s \in (0,1). \quad (0.3)
\]
Such a curve is called causal.

In this case, under mild assumptions on \( g \), the existence of a causal geodesic joining \( a, b \) can be achieved just maximizing the functional

\[
f^*(z) = \int_0^1 \sqrt{g(z)(\dot{z}(s), \dot{z}(s))} \, ds
\]

over all the causal curves in \( M \) (cf. [1], [8] or [3], chapt. 6).

Here we are interested to find sufficient conditions on the metric tensor \( g \) which guarantee the existence of geodesics connecting \emph{any} two given points \( a, b \in \mathbb{R}^4 \).

We shall prove the following result.

**Theorem 0.1.** - Let \( g_{ij}(i,j=0,\ldots,3) \) denote the components of the metric tensor \( g \). We assume that:

- \( g_{ij} \in C^1(\mathbb{R}^4, \mathbb{R}) \) \((i,j=0,\ldots,3)\).
- \( g_{00}(z) \geq \nu > 0 \) for all \( z \in \mathbb{R}^4 \).
- There exists \( \mu > 0 \) s. t.

\[
- \sum_{i,j=1}^3 g_{ij}(z) \xi_i \xi_j \geq \mu \| \xi \|^2 \quad \text{for all } z \in \mathbb{R}^4
\]

and all

\[
\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.
\]

\( g_4 \) The functions \( g_{0i} (i=0,\ldots,3) \) are bounded.

\( g_5 \) \( \frac{\partial g_{ij}}{\partial z_0}(z) = 0 \) for all \( z \in \mathbb{R}^4 \).

Then for any two points \( a, b \in \mathbb{R}^4 \) there exists a geodesic, with respect to the metric \( g \), joining \( a \) and \( b \).

The assumptions \( g_1 \), \ldots, \( g_4 \) are reasonably mild.

The most restrictive assumption is \( g_5 \) which means that the gravitational field is stationary (cf. [4], §88). The proof of theorem 0.1 is attained by using some minimax arguments which have been recently developed in the study of nonlinear differential equations having a variational structure (cf. e.g. [7] for a review on these topics).

### 1. PROOF OF THEOREM 0.1

The manifold \( M \) in \( H^1 \) defined in (0.2) can be written as follows

\[
M = \tilde{z} + H_0^1
\]

where

\[
\tilde{z} = a + (b - a) s, \quad s \in (0,1)
\]
In order to prove theorem 0.1 we shall first carry out a finite dimensional approximation.

Let \( n \in \mathbb{N} \) and set

\[
M_n = \bar{z} + H_n
\] 

where

\[ H_n = \text{span} \{ \varphi_j \sin \pi l \theta : j = 0, \ldots, 3; l = 1, \ldots, n \} \]

\( \varphi_j(j=0,\ldots,3) \) being the canonical base in \( \mathbb{R}^4 \).

Moreover we set

\[
\begin{align*}
V_n &= \text{span} \{ \varphi_0 \sin \pi l \theta : l = 1, \ldots, n \} \\
W_n &= \text{span} \{ \varphi_j \sin \pi l \theta : j = 1, 2, 3; l = 1, \ldots, n \} \\
S_n &= \bar{z} + V_n \\
Q_n(R) &= \bar{z} + W_n \cap B_R
\end{align*}
\]

where

\[
B_R = \{ z \in H^1_0 \mid \| z \| \leq R \}, \quad R > 0
\]

and \( \| \cdot \| \) denotes the standard norm in the Sobolev space \( H^1 \). Finally we set

\[
f_n = f|_{M_n}
\]

where \( f \) denotes the functional defined in (0.1). First we prove the existence of a critical point of \( f_n \), that is to say of a point \( z_n \in M_n \) such that

\[
\langle f'(z_n), \zeta \rangle = 0 \quad \text{for all} \quad \zeta \in H_n
\]

where \( f' \) is the Fréchet-differential of \( f \) and \( \langle \ldots, \rangle \) denotes the pairing between \( H^1 \) and its dual. More precisely the following theorem holds.

**Theorem 1.1.** Suppose that \( g \) satisfies the assumptions of theorem 0.1. Then there exists a critical point \( z_n \in M_n \) of \( f_n \) such that

\[
c' \leq f(z_n) \leq c''
\]

where \( c' \) and \( c'' \) are two constants independent on \( n \).

The proof of theorem 1.1 is based on a variant of the "saddle point theorem" of P. H. Rabinowitz [cf. [6] or propositions 2.1 and 2.2 in [2]]. We need some lemmas.

**Lemma 1.2.** Fix \( n \in \mathbb{N} \) and \( R > 0 \). Then \( S_n \) and the boundary \( \partial Q_n(R) \) link, namely for any continuous map \( h : M_n \to M_n \) s.t. \( h(\bar{z}) = z \) for all \( z \in \partial Q_n(R) \), we have

\[
h(Q_n(R)) \cap S_n \neq \emptyset
\]
Proof. Let \( h: M_n \to M_n \) s. t. \( h(z) = z \) for all \( z \in \partial Q_n(R) \) and define
\[
\tilde{h}: H_n \to H_n \quad \text{s. t.} \quad \forall \gamma \in H_n: \quad \tilde{h}(\gamma) = h(\gamma + \tilde{z}) - \tilde{z}
\]
It is easy to see that
\[
\tilde{h}(\gamma) = \gamma, \quad \forall \gamma \in \partial (B_R \cap W_n)
\]
Then by using the Brower degree (cf. [2], prop. 2.1 or [6]) it can be shown that there exists \( \gamma \in \tilde{h}(W_n \cap B_R) \cap V_n \) and therefore \( \tilde{z} + \gamma \in h(Q_n(R)) \cap \mathbb{S}_n \). \( \square \)

We denote by \( f'|_{M_n} \) the Fréchet differential of \( f \) on the manifold \( M_n \) and by \( \| . \| \) the standard norm in \( H^1 \). Moreover we set
\[
t = z_0 \quad \text{and} \quad x = (z_1, z_2, z_3)
\]
Now we prove that \( f|_{M_n} \) satisfies the Palais-Smale condition. More precisely the following lemma holds.

**Lemma 1.3.** Let \( g \) satisfy the assumptions of Theorem 0.1. Let \( \{ z_k \} \) be a sequence in \( M_n \) such that
\[
f'|_{M_n}(z_k) \to 0 \quad \text{as} \quad k \to \infty \quad \text{(1.5)}
\]
and
\[
\{ f(z_k) \} \text{ is bounded} \quad \text{(1.6)}
\]
Then \( \{ z_k \} \) is bounded in the \( H^1 \) norm and consequently it is precompact.

Proof. Since \( z_k \in M_n \), we can set
\[
z_k = (t_k, x_k) = \tilde{z} + (\tau_k, \xi_k)
\]
with \( \tau_k \in V_n \) and \( \xi_k \in W_n \) [cf. (1.1), (1.2)].

By (1.5) we deduce that
\[
\langle f'(z_k), \zeta \rangle = \epsilon_k \| \zeta \| \quad \text{for all} \quad \zeta \in H_n \quad \text{(1.7)}
\]
where \( \epsilon_k \to 0 \) as \( k \to \infty \).

Then for all \( \zeta = (\tau, \xi) \), with \( \tau \in V_n \) and \( \xi = (\xi_1, \xi_2, \xi_3) \in W_n \), we have
\[
\int_0^1 g(x_k)[\dot{z}_k, \xi] \, ds + \frac{1}{2} \int_0^1 \sum_{i,j=0}^3 \sum_{i=1}^3 \frac{\partial g_{ij}}{\partial x_i}(x_k) \cdot (\dot{\xi}_i)(\dot{\xi}_j) \, ds = \epsilon_k \| \zeta \|. \quad \text{(1.8)}
\]
And, if we take \( \zeta = (\tau_k, 0) = \tau_k \), we get
\[
\int_0^1 g(x_k)[\dot{z}_k, \dot{\tau}_k] \, ds = \epsilon_k \| \tau_k \| \quad \text{(1.9)}
\]
Now set
\[
T = \begin{pmatrix}
+1 & 0 \\
-1 & -1 \\
0 & -1
\end{pmatrix}
\] (1.10)

Then
\[
\tau_k = \frac{1}{2} [z_k - \bar{z} + T(z_k - \bar{z})]
\]

and from (1.9) we get
\[
\frac{1}{2} \int_0^1 g(x_k)[\dot{z}_k, z_k - \bar{z}] ds - \varepsilon_k \| \tau_k \| = -\frac{1}{2} \int_0^1 g(x_k)[\dot{z}_k, T(z_k - \bar{z})] ds. \tag{1.11}
\]

By (1.6) there exists \( c_1 > 0 \) such that for all \( k \in \mathbb{N} \)
\[
|f(z_k)| = \frac{1}{2} \int_0^1 g(x_k)[\dot{z}_k, \dot{z}_k] ds \leq c_1.
\]

From (1.11) we get
\[
\frac{1}{2} \int_0^1 g(x_k)[\dot{z}_k, T\dot{z}_k] ds
\]
\[
\leq c_1 + \frac{1}{2} \int_0^1 g(x_k)[\dot{z}_k, \bar{z}] ds + \frac{1}{2} \int_0^1 g(x_k)[\dot{z}_k, T\bar{z}] ds + \varepsilon_k \| \tau_k \|
\]
\[
= c_1 + \int_0^1 (g_{00}(x_k) \dot{z}_k + \sum_{i=1}^3 g_{0i}(x_k) (\dot{x}_k)_i T \dot{z}_k) ds + \varepsilon_k \| \tau_k \| \tag{1.12}
\]

where \((\bar{t}, \bar{x}) = \bar{z}\).

Since \( g_{0i} (i=0,1,2,3) \) are bounded, from (1.12) we easily get
\[
\int_0^1 g(x_k)[\dot{z}_k, T\dot{z}_k] ds \leq 2c_1 + c_2 \| z_k \| + 2\varepsilon_k \| \tau_k \| \tag{1.13}
\]

where \( c_2 \) is a positive constant depending on \( \bar{t} \) and \( g_{0i} (i=0, \ldots, 3) \).

Now it can be easily verified that
\[
g(x_k)[\dot{z}_k, T\dot{z}_k] = g_{00}(x_k) \dot{z}_k^2 - \sum_{i,j=1}^3 g_{ij}(x_k) (\dot{x}_k)_i (\dot{x}_k)_j. \tag{1.14}
\]

From (1.13) and (1.14) and by using \((g_2), (g_3)\) we get
\[
c_3 \| z_k \|^2 \leq 2c_1 + c_2 \| z_k \| + 2\varepsilon_k \| \tau_k \| \tag{1.15}
\]

where \( c_3 \) is a positive constant.
From (1.15) we deduce that

\{ z_k \} is bounded in \( H^1 \). \( \square \)

**Proof of Theorem 1.1.** — Set

\[
W = \sum_{n \in \mathbb{N}} W_n, \quad V = \sum_{n \in \mathbb{N}} V_n
\]

(the closures are taken in the \( H^1_0 \)-norm)

\[
S = z + V, \quad Q = Q(R) = z + W \cap B_R.
\]

It is easy to see that

\[
f(z) \to -\infty \quad \text{as} \quad \| z \| \to \infty, \quad z \in z + W
\]

and

\[
\inf f(S) > -\infty.
\]

Then if R is large enough we get

\[\sup f(\partial Q(R)) < \inf f(S).\]

Let \( n \in \mathbb{N} \) and set

\[
c_n = \inf_{h \in \mathcal{H}_n} \sup f(h(Q_n)) \quad (1.16)
\]

where

\[
\mathcal{H}_n = \{ h: M_n \to M_n, h \text{ continuous and s. t. } h(u) = u, \forall u \in \partial Q_n \}
\]

and \( Q_n \) is defined in (1.2).

By Lemma 1.2 \( c_n \) is well defined and

\[
c' = \inf f(S) \leq c_n \leq \sup f(Q) = c''.
\]

Moreover by lemma 1.3 \( f|_{M_n} \) satisfies the Palais-Smale condition; then, by the saddle point theorem (cf. [6] or Theorem 2.3 in [2]), \( c_n \) defined by (1.16) is a critical value of \( f|_{M_n} \). \( \square \)

We are now ready to prove Theorem 0.1.

**Proof of Theorem 0.1.** — Consider the sequence \( \{ z_n \} \) of the critical points of \( f|_{M_n} \) found in Theorem 1.1.

The same arguments used in proving lemma 1.3 show that \( \{ z_n \} \) is bounded in \( H^1 \), then there exists a subsequence, which we continue to call \( \{ z_n \} \) such that

\[
z_n \to z^* \quad \text{weakly in } H^1. \quad (1.17)
\]

We shall prove that

\[
z_n \to z^* \quad \text{strongly in } H^1. \quad (1.18)
\]

We set

\[
z_n = z + \zeta_n, \quad \zeta_n = (\tau_n, \xi_n)
\]
\[ z^* = \tilde{z} + \zeta^*, \quad \zeta^* = (\tau^*, \xi^*) \]

and \((\tau_n^*, \xi_n^*) = \zeta_n^* = P_n \zeta^* \)

\(P_n\) being the projection on \(H_n\).

Since \(z_n\) are critical points of \(f|_{M_n}\) we have

\[ \langle f'(z_n), T(\xi_n - \zeta_n^*) \rangle = \int_0^1 g(x_n)[\dot{x}_n, T(\xi_n - \zeta_n^*)] \, ds \]

\[ - \frac{1}{2} \int_0^1 \sum_{i, j=0}^3 \frac{\partial g_{ij}}{\partial x_l}(x_n)(\xi_n - \zeta_n^*)_l(\xi_n)_i(\zeta_n)_j \, ds = 0 \quad (1.19) \]

where \(T\) is defined in (1.10) and \((t_n, x_n) = z_n\).

\(H^1\) is compactly embedded into \(L^\infty\), then by (1.17), \(\xi_n \to \xi^*\) in \(L^\infty\) and \(\{ z_n \}\) is bounded in \(L^\infty\). Therefore

\[ \frac{\partial g_{ij}}{\partial x_l}(x_n)(\xi_n - \zeta_n^*)_l \to 0 \quad \text{in} \quad L^\infty \]

\[(i, j, 0, \ldots, 3 \text{ and } l = 1, 2, 3) \quad (1.20)\]

Then from (1.19), (1.20), (1.17) we deduce that

\[ \int_0^1 g(x_n)[\dot{z}_n, T(\xi_n - \zeta_n^*)] \, ds = O(1). \quad (1.21) \]

In (1.21) and in the sequel \(O(1)\) denotes a sequence converging to zero.

Since \(z_n = \tilde{z} + \xi_n\) we have

\[ \int_0^1 g(x_n)[\dot{z}_n, T(\xi_n - \zeta_n^*)] \, ds + \int_0^1 g(x_n)[\dot{\xi}_n, T(\zeta_n - \xi_n^*)] \, ds = O(1) \]

Then, since

\[ T(\xi_n - \zeta_n^*) \to 0 \quad \text{weakly in} \quad L^2 \quad (1.22) \]

and \(g_{ij}(x_n)(\dot{z}_n)_l\) converges (strongly) in \(L^\infty\), we get

\[ \int_0^1 g(x_n)[\dot{\xi}_n, T(\zeta_n - \xi_n^*)] \, ds = O(1) \quad (1.23) \]

which can also be written as

\[ \int_0^1 g(x_n)[(\xi_n - \zeta_n^*), T(\zeta_n - \xi_n^*)] \, ds + \int_0^1 g(x_n)[\xi_n^*, T(\xi_n - \zeta_n^*)] \, ds = O(1) \]

by (1.22) and since \(g_{ij}(x_n)(\xi_n^*)_l\) converges in \(L^2\), we get

\[ \int_0^1 g(x_n)[(\xi_n^* - \zeta_n^*), T(\xi_n - \zeta_n^*)] \, ds = O(1). \quad (1.24) \]
On the other hand,
\[ \int_0^1 g(x_n) [(\xi_n - \xi_n^*), T(\xi_n - \xi_n^*)] ds \geq \text{const.} \| \xi_n - \xi_n^* \|^2. \]  
(1.25)

From (1.24) and (1.125) and since \( \phi_n \to \phi^* \) in \( H_0^1 \) we get
\[ z_n \to z^* \quad \text{in} \quad H^1. \]  
(1.26)

Let us finally show that \( z^* \) is a critical point of \( f_{| M} \). By (1.26) we have
\[ \forall \xi \in H_0^1, \quad \langle f'(z_n), \xi \rangle \to \langle f'(z^*), \xi \rangle \quad \text{as} \quad n \to \infty. \]  
(1.27)

On the other hand
\[ \langle f'(z_n), \xi \rangle = \langle f'(z_n), \xi_n \rangle + \langle f'(z_n), \xi - \xi_n \rangle \]  
(1.28)

where \( \xi_n = P_n \xi \).

Since \( z_n \) is a critical point of \( f_{| M_n} \) and \( \xi - \xi_n \to 0 \) as \( n \to \infty \), from (1.28) we deduce that
\[ \langle f'(z_n), \xi \rangle = O(1). \]  
(1.29)

Finally from (1.27) and (1.29) we deduce that
\[ \forall \xi \in H_0^1, \quad \langle f'(z^*), \xi \rangle = 0 \]
and therefore \( z^* \) is a critical point of \( f_{| M} \).

\[ \square \]

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