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Remarks on critical exponents for Hessian operators

by

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ABSTRACT. — The critical exponent for the k -th Hessian operator ($k=2, \dots, n$) is determined and the solvability of the associated Dirichlet problem with sub-critical nonhomogeneous term is discussed.

Key words : Hessian operators, Critical exponent, Associated Dirichlet problem.

RÉSUMÉ. — On détermine l'exposant critique pour les opérateurs de Hess, et l'on étudie le problème de Dirichlet associé, avec un terme non homogène sous-critique.

INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$. It is well-known that the Sobolev exponent $\frac{n+2}{n-2}$ plays a critical role concerning the solvability of the Dirichlet problem

$$\Delta u + u^p = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Namely, this problem admits no positive solutions in a star-shaped Ω when $p \geq \frac{n+2}{n-2}$ and it has a positive solution in any Ω when $\frac{n+2}{n-2} > p > 0$

and p is not equal to one. In a previous work [6] we have studied a corresponding problem for the Monge-Ampère operator on a convex Ω :

$$\det \nabla^2 u = (-u)^p \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

We found that in contrast to the semilinear case this problem admits non-zero convex solutions provided p is positive and is not equal to n . Since the Laplace operator and the Monge-Ampère operator are respectively the first and the last Hessian operators, we are led to the question of determining the "critical exponents" for the remaining Hessian operators.

Let $S_k(\nabla^2 u)$, $k=1, \dots, n$, be the k -th Hessian operator, *i. e.*, it is the k -th elementary symmetric function of the Hessian matrix of u . Consider

$$\left. \begin{aligned} S_k(\nabla^2 u) &= (-u)^p \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \right\} \quad (1)$$

In this note we shall establish the following result:

THEOREM. — Let Ω be a ball and let $\gamma(k) = \begin{cases} \frac{(n+2)k}{n-2k} & 1 \leq k < \frac{n}{2} \\ \infty & \frac{n}{2} \leq k \leq n \end{cases}$. Then

(i) (1) has no negative solution in $C^1(\bar{\Omega}) \cap C^4(\Omega)$ when $p \geq \gamma(k)$; (ii) It admits a negative solution which is radially symmetric and is in $C^2(\bar{\Omega})$ when $0 < p < \gamma(k)$, p is not equal to k .

The non-existence result actually holds in more general situation. See Proposition 1 in the next section. Part (ii) of this theorem is contained in Propositions 2 ($0 < p < k$) and 3 ($k < p < \gamma(k)$) in Section 2. When $p = k$ one should study an eigenvalue problem. However, we shall not consider it here.

NOTATION. — Subscripts like those in u_i, u_{ij}, F_z, \dots , stand for partial differentiations. Also, summation convention is always in effect.

SECTION 1. NON-EXISTENCE

Consider the following Dirichlet problem

$$\left. \begin{aligned} S_k(\nabla^2 u) &= f(x, -u) > 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \right\} \quad (2)$$

Many authors have studied the existence of negative solutions for this problem in the case $k = 1$. Negative solutions are sought because they are important in many applications. However, when k is greater than one, negative solutions will be sought not only as an analogue of the semilinear case but mainly because they are precisely the “admissible ones”. When k is greater than one a Hessian operator is nonlinear and so its type depends on the Hessian of u . According to Caffarelli-Nirenberg-Spruck [1] a C^2 -function u is called *admissible* (with respect to S_k , $k \geq 2$) if the eigenvalues of $\nabla^2 u(x)$, $x \in \Omega$, lie in the connected component Γ_k of the set $\{\lambda = (\lambda_1, \dots, \lambda_n)$: The k -th elementary symmetric function of λ is positive $\}$ which contains the cone $\{\lambda : \lambda_i > 0 \text{ for all } i\}$. It can be shown that for an admissible u $S_k(\nabla^2 u)$ is elliptic, *i. e.*, $\frac{\partial S_k(\nabla^2 u)}{\partial r_{ij}} \xi_i \xi_j > 0$, for all ξ , $|\xi| > 0$, and $x \in \Omega$. We claim that any solution of (2) which belongs to $C(\bar{\Omega}) \cap C^2(\Omega)$ is admissible if and only if it is negative. For, at a non-negative maximum of u , $S_k(\nabla^2 u)$ is non-positive. But since f is positive this is impossible. Conversely, if u is negative, it attains a minimum in Ω . At this minimum the eigenvalues of the Hessian of u lie in Γ_k . Hence by the continuity of $\nabla^2 u$ and the positivity of f u is also admissible.

The equation in (2) has a variational structure. To describe it we need to express S_k in terms of the Newtonian tensor. Recall that for $k = 0, \dots, n - 1$, the k -th Newtonian tensor is given by

$$T_k(\nabla^2 u)_{ij} = \frac{1}{k!} \delta_{i_1 \dots i_k}^{j_1 \dots j_k} u_{i_1 j_1} \dots u_{i_k j_k}.$$

Here $\delta_{i_1 \dots i_k}^{j_1 \dots j_k}$ is the generalised Kronecker delta: It is equal to 1 (resp. -1) if $i_1, \dots, i_k, j_1, \dots, j_k$ are distinct and $(j_1 \dots j_k)$ is an even (resp. odd) permutation; Otherwise it is equal to zero. A fundamental property of the Newtonian tensor is that it is of divergence free, *i. e.*,

$$\frac{\partial T_k(\nabla^2 u)_{ij}}{\partial x_j} = 0, \quad i = 1, \dots, n. \tag{3}$$

S_k is related to T_{k-1} via

$$\frac{\partial S_k(\nabla^2 u)}{\partial r_{ij}} = T_{k-1}(\nabla^2 u)_{ij} \tag{4}$$

and

$$S_k(\nabla^2 u) = \frac{1}{k} T_{k-1}(\nabla^2 u)_{ij} u_{ij}. \tag{5}$$

(5) follows from Euler’s identity. For proofs of (3) and (4) see Reilly [5].

Using (3)-(5) one readily verifies that the equation in (2) is the Euler-Lagrange equation for the functional

$$J_k(u) = \frac{-1}{k+1} \int_{\Omega} u S_k(\nabla^2 u) + \int_{\Omega} F(x, u)$$

where $F(x, z) = \int_0^z f(x, t) dt$.

Now, consider a general integrand $\mathcal{F}(x, z, r_{ij})$, $r_{ij} = r_{ji}$. Let u be a C^4 -solution of the Euler-Lagrange equation for \mathcal{F} . Then for any constant a the following identity holds:

$$\frac{\partial}{\partial x_i} \left[x_i \mathcal{F} + (x_k u_k + au) \frac{\partial \mathcal{F}}{\partial x_j} r_{ij} - \frac{\partial}{\partial x_j} (x_k u_k + au) \mathcal{F} r_{ij} \right] = n \mathcal{F} + x_i \mathcal{F}_{x_i} - au \mathcal{F}_z - (a+2) u_{,i} \mathcal{F} r_{ij}.$$

This identity, which can be verified directly, was first obtained in Pucci-Serrin [3]. Applying this identity to $\mathcal{F} = \frac{-z S_k(r_{ij})}{k+1} + F(x, z)$, we have, by (3)-(5),

$$\begin{aligned} & \frac{\partial}{\partial x_i} \left[x_i \left(\frac{-u S_k(\nabla^2 u)}{k+1} + F(x, u) \right) + (x_k u_k + au) \left(\frac{-u_j T_{k-1}(\nabla^2 u)_{ij}}{k+1} \right) \right. \\ & \quad \left. + \frac{\partial}{\partial x_j} (x_k u_k + au) \frac{u T_{k-1}(\nabla^2 u)_{ij}}{k+1} \right] \\ & = [k(a+2) + a - n] \frac{u S_k(\nabla^2 u)}{k+1} + n F - a u f + x_i F_{x_i}. \end{aligned} \tag{6}$$

Next we formulate the non-existence result. Recall that a bounded domain Ω is called star-shaped if for some x_0 , $\langle x - x_0, \nu \rangle \geq 0$ on $\partial\Omega$. (Here ν is the unit outer normal and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.)

PROPOSITION 1. — *Let Ω be a star-shaped C^2 -domain with x_0 being the origin. Suppose that f belongs to $C(\bar{\Omega} \times (-\infty, 0]) \cap C^1(\Omega \times (-\infty, 0))$, positive in $\Omega \times (-\infty, 0)$ and equal to zero on $\Omega \times \{0\}$. There are no negative solutions to (2) which belong to $C^1(\bar{\Omega}) \cap C^4(\Omega)$ if*

$$n F(x, z) - \frac{n-2k}{k+1} z f(x, z) + x_i F_{x_i}(x, z) > 0 \tag{7}$$

in $\Omega \times (-\infty, 0)$. The same conclusion holds under

$$nF(x, z) - \frac{n-2k}{k+1}zf(x, z) + x_i F_{x_i}(x, z) \geq 0 \tag{7}'$$

if $\langle x, v \rangle > 0$ on $\partial\Omega$.

Proof. - Let $\Omega_\varepsilon = \{x \in \Omega : u(x) < -\varepsilon\}$. By a strong maximum principle (see Lemma below) Ω_ε has C^4 -boundary for small $\varepsilon > 0$. Choose $a = \frac{n-2k}{k+1}$ so that $k(a+2) + a - n = 0$ in (6) and then integrate both side of (6) over Ω_ε . By divergence theorem we have

$$\begin{aligned} \int_{\partial\Omega_\varepsilon} \langle x, v \rangle & \left(\frac{\varepsilon S_k(\nabla^2 u)}{k+1} + F(x, -\varepsilon) \right) - \frac{1}{k+1} \int_{\partial\Omega_\varepsilon} x_k u_k u_j T_{k-1}(\nabla^2 u)_{ij} v_i \\ & - \varepsilon \int_{\partial\Omega_\varepsilon} \frac{u_j T_{k-1}(\nabla^2 u)_{ij} v_i}{k+1} - \varepsilon \int_{\partial\Omega_\varepsilon} \frac{x_k u_k u_j T_{k-1}(\nabla^2 u)_{ij} v_i}{k+1} \\ & = \int_{\Omega_\varepsilon} \left(nF(x, u) - \frac{n-2k}{k+1}uf(x, u) + x_i F_{x_i}(x, u) \right). \end{aligned}$$

A repeated application of (3)-(5) gives

$$\int_{\partial\Omega_\varepsilon} u_j T_{k-1}(\nabla^2 u)_{ij} v_i = k \int_{\Omega_\varepsilon} f(x, u) \tag{8}$$

and

$$\begin{aligned} \int_{\partial\Omega_\varepsilon} x_k u_k u_j T_{k-1}(\nabla^2 u)_{ij} v_i & = k \int_{\Omega_\varepsilon} f(x, u) + \int_{\Omega_\varepsilon} x_k \frac{\partial}{\partial x_k} S_k(\nabla^2 u) \\ & = (k-n) \int_{\Omega} f(x, u). \end{aligned}$$

Consequently,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{\partial\Omega_\varepsilon} \langle x, v \rangle |\nabla u|^2 T_{k-1}(\nabla^2 u)_{ij} v_i v_j \\ = -(k+1) \int_{\Omega} \left(nF - \frac{n-2k}{k+1}uf + x_i F_{x_i} \right). \end{aligned}$$

If u is a negative solution of (2), $T_{k-1}(\nabla^2 u)_{ij} v_i v_j > 0$ by ellipticity. Hence the left hand side of the above identity is non-negative. However, this contradicts with (7). On the other hand, if $\langle x, v \rangle \geq \delta$ for some positive δ on $\partial\Omega$, by (8) and the lemma below the left hand side of the above

identity is positive. But then this contradicts with (7)'. Proposition 1 is thus proved.

LEMMA. — $\frac{\partial u}{\partial \nu} > 0$ on $\partial\Omega$.

Proof. — Let $x \in \partial\Omega$ and let B be a ball which is inscribed inside Ω and touches $\partial\Omega$ at x . For a non-negative radially symmetric function g with $g(|x|) \leq f(x, u(x))$ let w be the solution for

$$S_k(\nabla^2 w) = g \quad \text{in } B, \quad w = 0 \quad \text{on } \partial B,$$

w can be solved explicitly in terms of g ; see (10) in the next section. By the maximum principle $w \geq u$ in B . Hence $\frac{\partial w}{\partial \nu}(x) \geq \frac{\partial u}{\partial \nu}(x) > 0$. ■

SECTION 3. EXISTENCE

Throughout this section Ω denotes a ball of radius R centered at the origin and $f(x, z)$ in (2) is equal to $g(r, z)$, $r = |x|$, where

$$g \in C([0, R] \times (-\infty, 0]), \quad g > 0 \quad \text{in } [0, R] \times (-\infty, 0]. \quad (9)$$

We shall look for radially symmetric solutions for (2). Denote such a solution by $y(r)$. y satisfies

$$\begin{aligned} C_{k-1}^{n-1} \left(\frac{y'}{r}\right)^{k-1} y'' + C_k^{n-1} \left(\frac{y'}{r}\right)^k &= g(r, y), \\ y(R) &= 0, \quad y'(0) = 0. \end{aligned} \quad (10)$$

Here C_{k-1}^{n-1} and C_k^{n-1} are combinatorial constants. It is easy to see that whenever y solves (10), $u(x) = y(|x|)$ is a solution for (3).

For a negative radially symmetric u , the functional $J_k(u)$ is equal to

$$\alpha \int_0^R (y')^{k+1} r^{n-k} + \tau \int_0^R G(r, y) r^{n-1}$$

where $G(r, z) = \int_0^z g(r, t) dt$, $\alpha = \frac{\tau}{k(k+1)} C_{k-1}^{n-1}$ and τ is the volume of the unit sphere in \mathbf{R}^n . It is more convenient to let J_k act on the whole space rather than on negative functions. To this end we extend g to be an even

function in z (maintaining the same notation) and consider the functional

$$j_k(y) = \alpha \int_0^R |y'|^{k+1} r^{n-k} + \tau \int_0^R G(r, y) r^{n-1}.$$

LEMMA 1. — Let $E = \{y \in C^1([0, R]) : y(R) = 0\}$ (i) For $1 \leq k \leq \frac{n}{2}$ and $0 < \delta < \gamma(k)$, there exists a constant $C = C(\delta, k, R, n)$ such that

$$\left(\int_0^R |y|^{\delta+1} r^{n-1} \right)^{1/(\delta+1)} \leq C \left(\int_0^R |y'|^{k+1} r^{n-k} \right)^{1/(k+1)}$$

for all $y \in E$; (ii) For $n \geq k > \frac{n}{2}$, there exists a constant $C = C(k, R, n)$ such that

$$\|y\|_{C^{0,\sigma}([0, R])} \leq C \left(\int_0^R |y'|^{k+1} r^{n-k} \right)^{1/(k+1)}, \quad \sigma = \frac{2k-n}{k+1}.$$

Proof. — Applying Hölder inequality to $y(r) = \int_R^r y'(s) ds$ we have

$$|y(r)| \leq C r^{-(n-2k)/(k+1)} \left(\int_0^R |y'|^{k+1} r^{n-k} \right)^{1/(k+1)}.$$

Raising both side to their $(\delta + 1)$ -th power, multiplying by r^{n-1} and then integrating from 0 to R we obtain (i). (ii) can be proved in a similar way. ■

Remark. — In fact, a sharper result holds: There exists a constant $C = C(k, n)$ such that

$$C \left(\int_0^R |y'|^{k+1} r^{n-k} \right)^{1/(k+1)} \cong \begin{cases} \left(\int_0^R |y|^{\gamma(k)+1} r^{n-1} \right)^{1/(\gamma(k)+1)} & \text{if } k < \frac{n}{2} \\ \|y\|_{C([0, R])} & \text{if } k = \frac{n}{2} \\ \|y\|_{C^{0,\sigma}([0, R])}, \quad \sigma = \frac{2k-n}{k+1}, & \text{if } k > \frac{n}{2} \end{cases}$$

for all y in E . For a proof see Lin [2].

Let W_k be the Banach space obtained by completing E under the norm $\left(\int_0^R |y'|^{k+1} r^{n-k}\right)^{1/(k+1)}$. Using Lemma 1 one can easily show that the following lemma holds:

LEMMA 2. — (i) W_k is continuously embedded in $\{y \in C^{0, 1-(1/(k+1))}([r, R]) : y(R) = 0\}$ for $r > 0$; (ii) W_k is compactly embedded in $L^{p+1}(r^{n-1} dr)$ for $p < \gamma(k)$ when $k \leq \frac{n}{2}$ and in $C([0, R])$ when $k > \frac{n}{2}$.

We also have

LEMMA 3. — Suppose g satisfies

$$|g(r, z)| \leq C(1 + |z|^p), \quad 0 \leq r \leq R \tag{11}$$

for some p , $0 < p < \gamma(k)$. Then j_k belongs to $C^1(W_k; R)$. Moreover, $y \rightarrow \int_0^R G(r, y) r^{n-1}$ is weakly continuous on W_k .

The proof of this lemma is parallel to the proof for the corresponding result in the semilinear case; See Rabinowitz [4], Appendix B, for details. Any critical point of j_k will be called a *generalised solution* of (10). In other words, y is a generalised solution of (10) if and only if

$$(k+1)\alpha \int_0^R |y'|^{k-1} y' \varphi' r^{n-k} + \tau \int_0^R g(r, y) \varphi r^{n-1} = 0 \tag{12}$$

for all φ in W_k . The regularity of such solutions is given in

LEMMA 4. — Under (9) and (11), any generalised solution of (10) is in $C^2([0, R])$, and solves (10) in classical sense. Moreover, it is negative in $[0, R]$ unless it vanishes identically.

Proof. — We shall first show that any generalised solution is bounded. When k is greater than $\frac{n}{2}$ this follows from Lemma 1. So we assume k is less than or equal to $\frac{n}{2}$.

Let y be a generalised solution. For $s \geq 1$ and $N \geq 1$ we define a function Φ in $C^1[1, \infty)$ by setting $\Phi(z) = z^s - 1$ for $z \in [1, N]$ and taking it to be linear for $z \geq N$. Select $\varphi = \int_1^{y^+ + 1} |\Phi'(t)|^{k+1} dt$ where $y^+ = \max(y, 0)$

as a test function. Substituting φ into (12) we have

$$\int \left| \frac{d}{dr} \Phi(y^+ + 1) \right|^{k+1} r^{n-k} \leq C \int g(r, y) |\Phi'(y^+ + 1)|^{k+1} y^+ r^{n-1}$$

after using $\varphi \leq |\Phi'(y^+ + 1)|^{k+1} y^+$. It follows from Lemma 1, Hölder inequality and (11) that for a fixed β , $\max(p, k) < \beta < \gamma(k)$,

$$\left(\int \Phi(y^+ + 1)^{\beta+1} r^{n-1} \right)^{1/(\beta+1)} \leq C \left(\int (\Phi'(y^+ + 1)(y^+ + 1))^{q+1} r^{n-1} \right)^{1/(q+1)}$$

where $q=p$ if $p > k$ and $q=k$ if $p \leq k$ and C is a constant depending on $\|y\|_{w_k}$. Letting $N \rightarrow \infty$, we see that $y^+ \in L^{s(q+1)}(r^{n-1} dr)$ implies the stronger inclusion $y^+ \in L^{s(\beta+1)}(r^{n-1} dr)$. Also, we have

$$\|y^+ + 1\|_{L^{s(\beta+1)}(r^{n-1} dr)} \leq (Cs)^{1/s} \|y^+ + 1\|_{L^{s(q+1)}(r^{n-1} dr)}$$

where $\kappa = \frac{1+\beta}{1+q} > 1$. Setting $s = \kappa^m, m \geq 1$, an iteration yields

$$\sup y^+ \leq C(1 + \|y\|_{L^{q+1}(r^{n-1} dr)}).$$

Similarly one can show that the above estimate holds for $y^- = -\min(y, 0)$ in place of y^+ . Thus y is bounded.

Take $\varphi = \int_r^R \eta(t) dt$ where η is smooth as a test function in (12). After an integration by parts, we have

$$(k+1)\alpha \int_0^R |y'|^{k-1} y' \eta r^{n-k} dr = \tau \int_0^R \int_0^r g(t, y) t^{n-1} dt \eta dr.$$

Henceforth

$$(k+1)\alpha r^{n-k} |y'|^{k-1} y' = \tau \int_0^r g(t, -y) t^{n-1} dt \quad \text{a. e.}$$

Since g is non-negative, y is increasing (hence is negative) unless it vanishes identically. From this the desired conclusion of Lemma 4 can be drawn. ■

Now we can state the existence results.

PROPOSITION 2. — Under (9) and

$$\lim_{z \rightarrow \infty} \frac{g(r, z)}{z^p} = 0, \quad p < \gamma(k), \quad \lim_{z \rightarrow 0} \frac{g(r, z)}{z^k} = \infty \quad (13)$$

uniformly on $[0, R]$, (2) admits a negative radially symmetric solution in $C^2(\Omega)$.

This solution is obtained by minimizing j_k over W_k . (13) ensures that j_k is bounded from below and is negative somewhere. It follows from the direct method, which works in view of Lemma 3, that a non-zero absolute minimum exists.

PROPOSITION 3. — *In addition to (9) and (11), suppose g satisfies*

$$\lim_{z \rightarrow 0} \frac{g(r, z)}{z^k} = 0 \text{ uniformly in } [0, R], \text{ and, further, there exist } \theta \in (0, 1) \text{ and a}$$

positive constant M such that

$$G(r, z) \leq \frac{\theta}{k+1} z g(r, z), \quad 0 \leq r \leq R, \quad |z| \geq M. \quad (14)$$

Then (2) admits a negative radially symmetric solution which belongs to $C^2(\bar{\Omega})$.

(14) implies that j_k fulfills the Palais-Smale condition. Proposition 3 follows from the mountain pass lemma. For details see the corresponding proof for the semilinear case in [4], Chapter 2.

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