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Nonuniqueness for the heat flow of harmonic maps

by

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ABSTRACT. — We construct maps $u_0 : B^3 \to S^2$ such that the Cauchy problem « find $u : B^3 \times [0, +\infty) \to S^2$ such that $u(x, 0) = u_0(x)$ in B^3, $u_t - \Delta u = u \mid \nabla u \mid^2$, $u = u_0$ on $\partial B^3 \times [0, +\infty)$ » has infinitely many weak solutions.

Key-words: Heat flow; Harmonic maps; Uniqueness.

RÉSUMÉ. — On construit des applications u_0 de B^3 dans S^2 telles que le problème de Cauchy « trouver $u : B^3 \times [0, +\infty[\to S^2$ tel que $u_t - \Delta u = u \mid \nabla u \mid^2$, $u(x, 0) = u_0(x)$ dans B^3, $u = u_0$ sur $\partial B^3 \times [0, +\infty[$ » a une infinité de solutions faibles.

I. INTRODUCTION

Let M be a compact Riemannian manifold (with or without boundary) and let N be another compact Riemannian manifold without boundary. We will assume that N is isometrically embedded in $\mathbb{R}^k$. Let u be a map from M to N which belongs to $H^1(M; \mathbb{R}^k)$. We define the energy of u by

$$E(u) = \frac{1}{2} \int_M \mid \nabla u \mid^2 dM. \tag{1}$$

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The map u is harmonic if it is a critical point of E. The Euler-Lagrange equation satisfied by the harmonic maps is (see e. g. [*6*])

$$\tau(u) := \Delta u - \lambda\gamma(u) = 0 \tag{2}$$

where λ is a function from M into $\mathbb{R}$ and $\gamma(u)$ is a unit vector orthogonal to N at the point u.

The heat flow associated to (2) is

$$\frac{\partial u}{\partial t} - \tau(u) = 0 . \tag{3}$$

Let u_0 be a map from M into N. We consider the Cauchy problem: find u such that

$$(\mathrm{C}) \qquad \begin{cases} u : \mathrm{M} \times [0, +\infty) \to \mathrm{N} \\ u \textit{ is a solution of } (3) \\ u(x, 0) = u_0(x) \qquad \textit{for } x \textit{ in } M \\ u(x, t) = u_0(x) \qquad \textit{for } (x, t) \in \partial\mathrm{M} \times [0, +\infty). \end{cases}$$

In [*6*] Eells and Sampson have proved that if $\partial \mathrm{M} = \emptyset$ and $u_0 \in \mathrm{C}^\infty$ then (C) has a unique solution of class C^∞ if the dimension of M is 1 or if the Riemann curvature of N is nonpositive. In [*5*] it has been proved that if $\mathrm{M} = \mathrm{N} = \mathrm{S}^k$ with $k \geq 3$ then for some maps u_0 of class C^∞ problem (C) does not have smooth solutions. Hence it is natural to consider weak solutions of (C). Following Chen and Struwe (see [*3*]) we will say that $u : \mathrm{M} \times [0, +\infty[\to \mathrm{N}$ is a weak solution of (C) if

$$\frac{\partial u}{\partial t} \in \mathrm{L}^2(\mathrm{M} \times (0, +\infty)), \mathrm{E}(u(\cdot, t)) \leq \mathrm{E}(u(\cdot, 0)) \qquad \forall t \in [0, +\infty) \tag{5}$$

$$u \textit{ satisfies (3) in the sense of distributions in } \mathring{\mathrm{M}} \times (0, +\infty) \tag{6}$$

$$u(x, 0) = u_0(x) \textit{ in the trace sense} \tag{7}$$

$$u(x, t) = u_0(x) \textit{ on } \partial\mathrm{M} \times [0, +\infty) \textit{ in the trace sense.} \tag{8}$$

The existence of a weak solution to (C) has been proved independently by Chen in [*2*] and by Keller, Rubinstein and Sternberg in [*8*] when $\mathrm{N} = \mathrm{S}^n$. For general Riemannian manifold the existence of a weak solution is due to Chen and Struwe [*3*]; in [*3*] u_0 is assumed smooth and M without boundary but the proof can be easily extended to the cases where $u_0 \in \mathrm{H}^1(\mathrm{M}; \mathrm{N})$ and M has a boundary. Let us recall that, when the dimension of M is 2, Struwe has defined in [*11*] a notion of weak solution and has proved the existence and uniqueness of such a solution.

We prove in this paper that (C) may have infinitely many weak solutions. Let

$$\mathrm{B}^3 = \{ x \in \mathbb{R}^3 ; | x | < 1 \} .$$

We take $M = \overline{B^3}$ and $N = S^2 = \partial B^3$. The manifolds M and N are provided with the usual metrics. Equation (3) becomes

$$\frac{\partial u}{\partial t} - \Delta u = u |\nabla u|^2. \tag{9}$$

Our maps u_0 are defined in the following way. Let $p = (0, 0, 1)$ and let $\pi : S^2 \setminus \{ p \} \to \mathbb{R}^2 \times \{ 0 \}$ be the stereographic projection with pole p into the equatorial plane of B^3. We identify $\mathbb{R}^2 \times \{ 0 \}$ with $\mathbb{C}$ by $(x^1, x^2, 0) \sim x^1 + ix^2$. Let $g : \mathbb{C} \cup \{ \infty \} \to \mathbb{C} \cup \{ \infty \}$ be a rational function of z and let $f : B^3 \to S^2$ be defined by

$$f(x) = \pi^{-1}\left(g\left(\pi\left(\frac{x}{|x|}\right)\right)\right).$$

Let

$$q = \int_{S^2} |\nabla f|^2 x d\sigma(x) \in \mathbb{R}^3.$$

Our main result is

THEOREM 1. — *If $q \neq 0$, then the Cauchy problem* (C) *with $u_0 = f$ has infinitely many weak solutions.*

Our method to prove Theorem 1 is the following. We first notice that f is a (weakly) harmonic map and therefore

$$\bar{u}(x, t) := f(x)$$

is a weak solution of (C). Then we prove that the solution $\underline{u}$ constructed in [*2*] and [*8*] is different from $\bar{u}$. For this purpose we show using [*3*] (see also [*10*]) that $\underline{u}$ satisfies a monotonicity property which is not satisfied by $\bar{u}$. Therefore there exists at least two weak solutions of (C). Then one deduces easily the existence of infinitely many weak solutions. Indeed let, for τ in $[0, +\infty)$, w^τ be the map defined by

$$\begin{aligned} w^\tau &: M \times [0, +\infty[\to N \\ w^\tau(x, t) &= \bar{u}(x, t) \quad \text{if} \quad t \leq \tau \\ w^\tau(x, t) &= \underline{u}(x, t - \tau) \quad \text{if} \quad \tau \leq t. \end{aligned}$$

Then w^τ is a weak solution of (C).

REMARK 2. — *a*) It has been proved in [*1*] (p. 678) that if the degree of g is 1 then $q = 0$ if and only if $\pi^{-1} \circ g \circ \pi$ is a rotation. *b*) It has also been proved in [*1*] (remark 7.6) that if $q \neq 0$ then f is not a minimizing harmonic map. Note that it follows easily from the definition of a weak solution that if u_0 is a minimizing harmonic map then (C) has a unique solution (which is $u \equiv u_0$).

II. PROOF OF THEOREM 1

Let us first recall how a weak solution of (C) is constructed in [2] and in [8]. Let, for α in $(0, +\infty)$, E_α be the functional

$$E_\alpha(u) = \frac{1}{2}\int_{B^3} |\nabla u|^2 + \frac{\alpha}{2}(|u|^2 - 1)^2.$$

One considers the Cauchy problem for the heat flow associated to E_α

$$(C_\alpha) \quad \begin{cases} i) \ \dfrac{\partial u}{\partial t} - \Delta u + \alpha(|u|^2 - 1)u = 0 \quad \textit{in } B^3 \times (0, +\infty) \\ ii) \ u(x, 0) = u_0(x) \quad \textit{for } x \textit{ in } B^3 \\ iii) \ u(x, t) = u_0(x) \quad \textit{for } (x, t) \textit{ in } \partial B^3 \times (0, +\infty). \end{cases}$$

where u_0 is given in $H^1(B^3; \mathbb{R}^3)$ with $|u_0| = 1$ a. e.. One easily sees that (C_α) has a unique solution u^α in $C^0([0, +\infty); H^1(B^3)) \cap C^\infty((0, +\infty) \times B^3; \overline{B^3})$. In [2] and [8] it is proved that there exists a sequence (α_i), $i \in \mathfrak{N}$ such that $\alpha_i \to +\infty$ as $i \to +\infty$ and if $u^i = u^{\alpha_i}$ then u^i tends weakly in $H^1_{loc}(\overline{B^3} \times [0, +\infty))$ to a map $\underline{u}$ which is a weak solution of (C).

We are going to prove that if $u_0 = f$ and $q \neq 0$ then $\bar{u} \neq \underline{u}$; Theorem 1 follows—see Introduction.

Let θ be a function from B^3 into $[0, 1]$ which is of class C^∞ with compact support and such that $\theta(x) = 1$ if $|x| \leq 1/2$. As in [3] (see also [10]), for a in B^3 with $|a| \leq 1/4$ and for $t_1 > 0$, we define maps $\varphi_i : [0, t_1^{1/2}] \to [0, +\infty)$ by

$$\varphi_i(r) = \frac{1}{r}\int_{t = t_1 - r^2} (|\nabla u^i|^2 + \frac{\alpha_i}{2}(|u^i|^2 - 1)^2)\theta^2 \exp -\frac{|x - a|^2}{4r^2}\, dx.$$

It follows from Lemma 4.2 in [3] (see also [10]) that there exists a constant C independent of u_0, a and t_1 such that if $0 < r_0 < r_1 \leq t_1^{1/2}$,

$$\varphi_i(r_0) \leq \varphi_i(r_1) + CE(u_0)(r_1 - r_0). \tag{10}$$

For convenience let us recall the proof of (10). Since the index i is fixed we will omit it. Let

$$v_r(y, \tau) = u(a + ry, t_1 - r^2\tau). \tag{11}$$

Then

$$\varphi(r) = \int_{\tau = 1} \left\{ |\nabla v_r|^2 + \frac{\alpha}{2} r^2(|v_r|^2 - 1)^2 \right\} \theta^2(a + ry) \exp -\frac{|y|^2}{4} dy. \tag{12}$$

We take the derivative of φ with respect to r; we get $\varphi'(r) = \mathrm{I}_0 + \mathrm{I}_1$ with

$$\mathrm{I}_0 = 2\int_{\tau=1}\left\{\nabla v_r \cdot \nabla \frac{\partial v_r}{\partial r} + \frac{\alpha r}{2}(|v_r|^2-1)^2 + \alpha r^2(|v_r|^2-1)v_r\frac{\partial v_r}{\partial r}\right\}\theta^2(a+ry)\exp -\frac{|y|^2}{4}\,dy \tag{13}$$

$$\mathrm{I}_1 = \frac{2}{r}\int_{t=t_1-r^2}\left\{|\nabla u|^2+\frac{\alpha}{2}(|u|^2-1)^2\right\}\left(\frac{x-a}{r}\cdot\nabla\theta\right)\theta\exp -\frac{|x-a|^2}{4r^2}\,dx. \tag{14}$$

We have

$$\frac{\partial v_r}{\partial r} = \frac{1}{r}\left(y\cdot\nabla v_r + 2\tau\frac{\partial v_r}{\partial \tau}\right) \tag{15}$$

and

$$\Delta v_r = -\frac{\partial v_r}{\partial \tau} + \alpha r^2 v_r(|v_r|^2-1). \tag{16}$$

Using (13), (15), and (16) one gets with an integration by parts

$$\mathrm{I}_0 = \frac{1}{r}\int_{\tau=1}\left\{\left|y\cdot\nabla v_r + 2\tau\frac{\partial v_r}{\partial\tau}\right|^2 + \alpha r^2(|v_r|^2-1)^2\right\}\theta^2(a+ry)\exp -\frac{|y|^2}{4}\,dy - \mathrm{I}_2 \tag{17}$$

with

$$\mathrm{I}_2 = 4\int_{\tau=1}((\theta\nabla\theta)(a+ry)\cdot\nabla v_r)\left(y\cdot\nabla v_r + 2\tau\frac{\partial v_r}{\partial\tau}\right)\exp -\frac{|y|^2}{4}\,dy. \tag{18}$$

We will denote by C various constants independent of u_0, a, t_1 and r. We have

$$|\mathrm{I}_2| \leq \frac{1}{r}\int_{\tau=1}\left|y\cdot\nabla v_r + 2\tau\frac{\partial v_r}{\partial\tau}\right|^2\theta^2(a+ry)\exp -\frac{|y|^2}{4}\,dy + \mathrm{C}\int_{\tau=1} r\,|\nabla v_r|^2\exp -\frac{|y|^2}{4}\,dy. \tag{19}$$

The last integral in (19) is equal to

$$\int_{t=t_1-r^2}|\nabla u|^2\exp -\frac{|x-a|^2}{4r^2}\,dx$$

and therefore is not larger than $\mathrm{E}(u_0)$ (note that $\mathrm{E}_\alpha(u(t))\leq\mathrm{E}_\alpha(u(0))=\mathrm{E}(u_0)$). Hence it follows from (17) and (19) that

$$\mathrm{I}_0 \geq -\mathrm{CE}(u_0). \tag{20}$$

Since $\nabla\theta(x) = 0$ if $|x - a| \leq 1/4$ we have

$$|\mathrm{I}_1| \leq \mathrm{CE}_\alpha(u(t_1 - r^2)) \leq \mathrm{CE}(u_0)\,. \tag{21}$$

Finally (10) follows from (20), (21) and $\varphi'(r) = \mathrm{I}_0 + \mathrm{I}_1$.

We now use (10) with $r_1 = t_1^{1/2}$ and $r_0 = t_0^{1/2} < t_1^{1/2}$; we get

$$\frac{1}{t_0^{1/2}}\int_{B^3} \theta^2 |\nabla u^i|^2(x, t_1 - t_0) \exp - \frac{|x - a|^2}{4t_0}$$

$$\leq \mathrm{CE}(u_0)\left(t_1^{1/2} - t_0^{1/2}\right) + \frac{1}{t_1^{1/2}}\int_{B^3} \theta^2 |\nabla u_0|^2 \exp - \frac{|x - a|^2}{4t_1}. \tag{22}$$

We now let i go to ∞ and get for a. e. t_0 and a. e. t_1 with $0 < t_0 < t_1$

$$\frac{1}{t_0^{1/2}}\int_{B^3} |\nabla \underline{u}|^2(x, t_1 - t_0) \exp - \frac{|x - a|^2}{4t_0}$$

$$\leq \mathrm{CE}(u_0)\left(t_1^{1/2} - t_0^{1/2}\right) + \frac{1}{t_1^{1/2}}\int_{B^3} \theta^2 |\nabla u_0|^2 \exp - \frac{|x - a|^2}{4t_1}. \tag{23}$$

Let us assume that $\underline{u} = \bar{u}$; then (23) leads to (with $u_0 = f$)

$$\frac{1}{t_0^{1/2}}\int_{B^3} \theta^2 |\nabla u_0|^2 \exp \frac{|x - a|^2}{4t_0}$$

$$\leq \mathrm{CE}(u_0)\left(t_1^{1/2} - t_0^{1/2}\right) + \frac{1}{t_1^{1/2}}\int_{B^3} \theta^2 |\nabla u_0|^2 \exp - \frac{|x - a|^2}{4t_1}. \tag{24}$$

We define a function φ from $(0, +\infty)$ into $(0, +\infty)$ by

$$\varphi(\ell) = \frac{1}{\ell}\int_{B^3} \theta^2 |\nabla u_0|^2 \exp - \frac{|x - a|^2}{4\ell^2}.$$

From (24) we get

$$\varphi'(\ell) \geq -\mathrm{CE}(u_0)\,. \tag{25}$$

We have

$$\varphi'(\ell) = \int_{B^3} \beta\theta^2 |\nabla u_0|^2\left(-\frac{1}{\ell^2} + \frac{|x - a|^2}{2\ell^4}\right) \tag{26}$$

with

$$\beta(x) = \exp - \frac{|x - a|^2}{4\ell^2}$$

Let

$$\mathrm{I} = \int_{B^3} \beta\theta^2 |\nabla u_0|^2. \tag{27}$$

In the following for any function $h : (0, +\infty) \to (0, +\infty)$ we will denote by $\mathrm{O}(h(\ell))$ various functions such that

$$|\mathrm{O}(h(\ell))| \leq \mathrm{C}\,|h(\ell)| \qquad \forall \ell \in (0, +\infty)$$

for some constant C which does not depend on a and ℓ. We have, with usual conventions of notation,

$$\mathrm{I} = \frac{1}{3}\int_{\mathrm{B}^3} (\beta\theta^2(x-a)^k)_k u^i_{0j} u^i_{0j} + \frac{1}{6\ell^2}\int_{\mathrm{B}^3} \beta\theta^2 \,|\, x-a \,|^2 u^i_{0j} u^i_{0j} + \mathrm{O}(\ell^2). \tag{28}$$

Let, for k in $\{ 1, 2, 3 \}$,

$$\mu^k = (u^i_{0j} u^i_{0j})_k - 2(u^i_{0k} u^i_{0j})_j. \tag{29}$$

We recall (see e. g. [9] p. 146) that u_0 is stationary if and only if

$$\mu^k = 0 \qquad \forall k \in \{ 1, 2, 3 \}. \tag{30}$$

From (28) and (29) we get

$$\mathrm{I} = \frac{2}{3}\int_{\mathrm{B}^3} (\beta\theta^2(x-a)^k)_j u^i_{0j} u^i_{0k} - \frac{1}{3}\langle \mu^k, \beta\theta^2(x-a)^k \rangle + \frac{1}{6\ell^2}\int_{\mathrm{B}^3} \beta\theta^2 \,|\, x-a \,|^2 u^i_{0j} u^i_{0j} + \mathrm{O}(\ell^2)$$

where $\langle\ ,\ \rangle$ denotes the duality between distributions and smooth functions with compact support in B^3.

$$\mathrm{I} = -\frac{1}{3\ell^2}\int_{\mathrm{B}^3} \beta\theta^2 \,|\, (x-a)\cdot\nabla u_0 \,|^2 + \frac{2}{3}\mathrm{I} + \frac{1}{6\ell^2}\int_{\mathrm{B}^3} \beta\theta^2 \,|\, x-a \,|^2 \,|\, \nabla u_0 \,|^2 - \frac{2\ell^2}{3}\langle \mu^k_k, \beta\theta^2 \rangle + \mathrm{O}(\ell^2). \tag{31}$$

From (26), (27) and (31) we get

$$\varphi'(\ell) = \frac{1}{\ell^4}\int_{\mathrm{B}^3} \beta\theta^2 \,|\, (x-a)\cdot\nabla u_0 \,|^2 + 2\langle \mu^k_k, \beta\theta^2 \rangle + \mathrm{O}(1). \tag{32}$$

A straightforward computation leads to

$$\mu = q\delta_0 \qquad (\in \mathcal{D}'(\mathrm{B}^3)^3), \tag{33}$$

where δ_0 is the Dirac mass at the origin. From (32) and (33) we obtain, for $\ell \leqslant 1$,

$$\varphi'(\ell) = \frac{1}{\ell^4}\int_{\mathrm{B}^3} \beta\theta^2 \,|\, a\cdot\nabla u_0 \,|^2 - \frac{\theta^2(0)\beta(0)}{\ell^2} a\cdot q + \mathrm{O}(1). \tag{34}$$

Since $|\, \nabla u_0(x) \,| \leqslant \mathrm{C}/|\, x \,|$ there exists a constant C_0 independent of a and ℓ such that if $|\, a \,| \leq \ell$

$$\int_{\mathrm{B}^3} \frac{\beta\theta^2}{\ell^4} \,|\, a\cdot\nabla u_0 \,|^2 \leq \mathrm{C}_0 \frac{|\, a \,|^2}{\ell^3}. \tag{35}$$

Using (34) and (35) we have for some constant C_1 independent of a and ℓ

$$\varphi'(\ell) \leq -\frac{1}{\ell}\left(-C_0\frac{|a|^2}{\ell^2} + \frac{a}{\ell}\cdot q \exp -\frac{|a|^2}{4\ell^2}\right) + C_1. \quad (36)$$

We now assume that $q \neq 0$. Let e be a unit vector such that $e \cdot q > 0$. We choose $a = \nu\ell e$ with $\nu \in (0, 1)$. We take ν small enough in such a way that

$$\nu(e\cdot q) \exp -\frac{\nu^2}{4} > C_0\nu^2 \quad (37)$$

and finally let ℓ go to 0 to see that (36) and (25) are not compatible.

REMARK 3. — It follows from (32) that if u_0 is any (weakly) harmonic map which is stationary (and therefore satisfies (30)) then (24) holds. It would be interesting to know if the converse is true.

III. CONCLUSION

We have proved that for some initial data u_0 problem (C) has infinitely many weak solutions. Since we do not have always uniqueness of weak solutions it is tempting to add an extra condition to get it, as for example for hyperbolic semilinear equations one adds the entropy condition in order to have a unique solution to the Cauchy problem. Such a condition could be (if $M = B^3$ for example) that for any θ in $C_0^\infty(B^3)$, any compact K of the interior of $\{\theta = 1\}$ there exists a constant C such that for a. e. t_1, t_2 with $t_1 < t_2$ and for any a in K

$$t_1^{-1/2}\int_{B^3} \theta^2 |\nabla u|^2(x, t_2 - t_1) \exp -\frac{|x-a|^2}{4t_1}$$
$$\leq C\left(t_2^{1/2} - t_1^{1/2}\right) + t_2^{-1/2}\int_{B^3} \theta^2 |\nabla u_0|^2 \exp -\frac{|x-a|^2}{4t_2}. \quad (38)$$

The maps w^τ for $\tau > 0$ and $q \neq 0$ do not satisfy (38); the map $\underline{u}$ satisfies (38) whatever the initial data u_0 is. Hence there exists a weak solution of (C) which satisfies (38) and our method does not allow us to produce an initial data such that (C) has at least two weak solutions satisfying (38).

On the other hand it seems natural to conjecture from [*1*] section VII (see also [*4*] and [*7*]) that, with the notations of the introduction, if g has a degree larger than 1 then, even if $q = 0$, $\underline{u} \neq \bar{u}$. Indeed one could expect that for any $t > 0$, $\underline{u}(t)$ has d point singularities of degree 1 instead of a unique point singularity of degree d as $\bar{u}(t)$.

Moreover condition (38) is not natural since it does not respect the semi-group structure of the Cauchy problem. In other words (38) and the semi-group structure of the Cauchy problem would imply for a. e. $t_0 < t_1 < t_2$

$$
\begin{aligned}
(t_1 - t_0)^{-1/2} \int_{B^3} \theta^2 \, |\nabla u|^2(x, t_2 - t_1 + t_0) \exp - \frac{|x-a|^2}{4(t_1 - t_0)} \\
\leq C((t_2 - t_0)^{1/2} - (t_1 - t_0)^{1/2}) \\
+ (t_2 - t_0)^{-1/2} \int_{B^3} \theta^2 \, |\nabla u|^2(x, t_0) \exp - \frac{|x-a|^2}{4(t_2 - t_0)} \quad (39)
\end{aligned}
$$

where C does not depend on a, t_1 and t_2. However, it is not clear if there exists always a weak solution which satisfies (39).

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