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On the non-existence of energy stable minimal cones

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ABSTRACT. — We show that there are no non-trivial (potential) energy stable minimal cones in $\mathbb{R}^n \times \mathbb{R}^+$ with singularity at 0, if $2 \leqq n \leqq 5$. The sharpness of this result is demonstrated by proving that a certain six dimensional cone in $\mathbb{R}^7$ is stable. Moreover, we extend all results to the more general α-energy functional.

Key words : Stable cones.

RÉSUMÉ. — L'on démontre que, si $2 \leqq n \leqq 5$, il n'existe pas dans $\mathbb{R}^n \times \mathbb{R}^+$ de cônes minimaux stables en énergie. Ce résultat est optimal, car l'on exhibe dans $\mathbb{R}^7$ un cône de dimension 6 qui est stable. On étend également ces résultats à des fonctionnelles d'énergie plus générales.

A well known result due to J. Simons [SJ] states that there are no non-trivial n-dimensional stable minimal cones in $\mathbb{R}^{n+1}$ (with singularity at zero), provided $n \leqq 6$. One of the crucial ingrediences in his proof is an important identity for the Laplacian of the second fundamental form for minimal hypersurfaces. Using sharper estimates than had previously been

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realized, Schoen-Simon-Yau [SSY] gave a considerably simpler proof of Simons' result.

Simons also proved that the seven dimensional cone $x_1^2+\ldots+x_4^2=x_5^2+\ldots+x_8^2$ in $\mathbb{R}^8$ is stable and, in fact, it was proved by Bombieri-De Giorgi-Giusti [BDG] that it even minimizes area in $\mathbb{R}^8$. This result dashes the hope for general interior regularity of codimension one solutions to the problem of least area in $\mathbb{R}^8$.

In two papers [D 1] and [D 2] the author has investigated the cones

$$C_n^\alpha := \left\{ x=(x_1, \ldots, x_{n+1});\ 0 \leqq x_{n+1} \leqq \sqrt{\frac{\alpha}{n-1}}\,[x_1^2+\ldots+x_n^2]^{1/2} \right\} \subset \mathbb{R}^{n+1}$$

which have boundaries of least α-energy

$$\mathscr{E}_\alpha = \int x_{n+1}^\alpha \,|D\varphi_U| \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+,$$

provided one of the following conditions holds:

(i) $\alpha+p\geqq 6$, where $\alpha\geqq 2$ and $p:=n-1\geqq 2$,

or

(ii) $\alpha+p\geqq 7$, for $\alpha\geqq 1$ and $p\geqq 1$.

Here $\mathbb{R}^+=\{t\geqq 0\}$, $U\subset\mathbb{R}^n\times\mathbb{R}^+$, and $|D\varphi_U|$ is the n-dimensional Hausdorff measure restricted to the reduced boundary of U. Also a set $C\subset\mathbb{R}^n\times\mathbb{R}^+$ with characteristic function φ_C has a boundary of least α-energy in $\mathbb{R}^n\times\mathbb{R}^+$, if and only if for each $g\in L^1_{\text{loc}}(\mathbb{R}^n\times\mathbb{R}^+)$ with compact support $K\subset\mathbb{R}^n\times\mathbb{R}^+$ we have

$$\int_K x_{n+1}^\alpha \,|D\varphi_C| \leqq \int_K x_{n+1}^\alpha \,|D(\varphi_C+g)|.$$

Furthermore it could be shown in [D 2] that the six dimensional boundary of the cone C_6^1 does not minimize the energy $\mathscr{E}_1$ in $\mathbb{R}^6\times\mathbb{R}^+$. Similarly, the cone C_2^5 does not minimize $\mathscr{E}_5$ in $\mathbb{R}^2\times\mathbb{R}^+$.

We wish to emphasize the physical relevance of the problem. Namely if we regard the boundary $M=\partial U$ of U as a material surface of constant mass density, then $\mathscr{E}_1$ corresponds to the potential energy of M under gravitational forces. Here we have of course assumed that the gravitational force acts in the $-x_{n+1}$ direction. Therefore, we refer to $\mathscr{E}_\alpha$ as the α-energy, and, in particular, if $\alpha=1$ we shall simply omit the addition "α".

In this paper we will employ the method of Schoen-Simon-Yau [SSY] to obtain a result on the non-existence of non-trivial α-stable minimal cones in $\mathbb{R}^n\times\mathbb{R}^+$ *i.e.*, cones which are stable with respect to the α-energy $\mathscr{E}_\alpha$. We in fact prove (Theorem 2) that such a result holds true provided that

$$\alpha+p<3+\sqrt{8}, \qquad p=n-1.$$

On the other hand we show in Theorem 1 that the cones $x_{n+1}=+\sqrt{\frac{\alpha}{p}}\left[x_1^2+\ldots+x_n^2\right]^{1/2}$ are α-stable, if

$$\alpha+p\geqq 3+\sqrt{8}.$$

Note that this in particular implies stability, if $\alpha=1$, $n=6$, or $\alpha=5$, $n=2$, but because of [D 2] the boundaries of the set C_6^1 or C_2^5 do not minimize the corresponding α-energy in $\mathbb{R}^n\times\mathbb{R}^+$. In fact, we might even obtain a field of α-stable and non-minimizing minimal cones, e. g. the two-dimensional cones $x_3=\sqrt{\alpha}\,[x_1^2+x_2^2]^{1/2}$ in $\mathbb{R}^3$ where $2+\sqrt{8}\leqq\alpha\leqq 5$.

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1. NOTATIONS AND RESULTS

In this section we set up our terminology and, in particular, we give simple expressions for the first and second variation of the α-energy. Finally, we formulate our main results.

Let M be an n-dimensional submanifold of class C^2 contained in the open half-space $\mathbb{R}^n+\mathring{\mathbb{R}}^+\subset\mathbb{R}^{n+1}$, $\mathring{\mathbb{R}}^+=\{t>0\}$, and let $U\subset\mathbb{R}^n\times\mathring{\mathbb{R}}^+$ be open with $U\cap M\neq\varnothing$, $(\operatorname{clos} M-M)\cap U=\varnothing$, $\mathscr{H}_n(M\cap K)<\infty$ for each compact set $K\subset U$; here $\mathscr{H}_t$, $t\geqq 0$, denotes t-dimensional Hausdorff measure. We consider one parameter families $\{\Phi_t\}$, $-1\leqq t\leqq 1$, of diffeomorphisms from U into U, with the following properties:

$$\Phi(t, x)=:\Phi_t(x)\in C^2((-1,1)\times U, U), \tag{1}$$

$$\Phi_0(x)=x \quad \text{for all } x\in U, \tag{2}$$

$$\Phi_t(x)=x \quad \text{for all } t\in(-1,1) \text{ and all } x\in U-K \text{ for some compact set } K\subset U. \tag{3}$$

Put

$$X(x):=\frac{\partial\Phi}{\partial t}(t, x)\Big|_{t=0},$$

and

$$Z(x):=\frac{\partial^2\Phi}{\partial t^2}(t, x)\Big|_{t=0},$$

to denote the initial velocity and acceleration vectors of Φ_t respectively. Then, because of (3), X and Z have compact support $K\subset U$, and furthermore

$$\Phi_t(x)=x+t\,X(x)+\frac{t^2}{2}Z(x)+o(t^2).$$

Let $M_t := \Phi_t(M)$ denote the image of $M = M_0$ under Φ_t; then we are interested in the first and second variation of the α-energy functional

$$\mathscr{E}_\alpha(M) = \int_M x_{n+1}^\alpha \, d\mathscr{H}_n,$$

where $x = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^+$, $\alpha > 0$, *i.e.* we want to compute

$$\delta\mathscr{E}_\alpha(M, X) = \frac{d}{dt} \int_{M_t} x_{n+1}^\alpha \, d\mathscr{H}_n \Big|_{t=0},$$

and

$$\delta^2 \mathscr{E}_\alpha(M, X, Z) = \frac{d^2}{dt^2} \int_{M_t} x_{n+1}^\alpha \, d\mathscr{H}_n \Big|_{t=0}.$$

Choose a local field of orthonormal frames $\tau^1, \ldots, \tau^n, \nu$ such that $\tau^1, \ldots, \tau^n \in T_x M$ are tangent to M. For a given vectorfield Y on M (not necessarily tangential) we denote by $D_{\tau^i} Y$ the directional derivative of Y in the direction τ^i. Also

$$\operatorname{div} Y = \sum_{i=1}^n (D_{\tau^i} Y) \tau^i$$

stands for the divergence on M, and

$$\nabla f = \sum_{i=1}^n (D_{\tau^i} f) \tau^i$$

denotes the gradient of the function $f \in C^1(M, \mathbb{R})$ respectively. We shall also employ the symbol Δ to denote the Laplacian on M, *i.e.* $\Delta = \nabla_i \nabla_i$ where $\nabla_i = D_{\tau^i}$, and $Y^\perp = Y - \sum_{j=1}^n (Y \cdot \tau^j) \tau^j$ stands for the normal part of Y.

LEMMA 1. – *Let* M, $\Phi_t : U \to U$ *and*

$$X(x) = (X_1(x), \ldots, X_{n+1}(x)),$$
$$Z(x) = (Z_1(x), \ldots, Z_{n+1}(x))$$

be defined as above. Then

$$\delta\mathscr{E}_\alpha(M, X) = \int_M \{ x_{n+1}^\alpha \operatorname{div} X + \alpha x_{n+1}^{\alpha-1} X_{n+1} \} \, d\mathscr{H}_n \tag{4}$$

and

$$\delta^2 \mathscr{E}_\alpha(\mathrm{M}, \mathrm{X}, \mathrm{Z}) = \int_{\mathrm{M}} \{ \alpha(\alpha-1) x_{n+1}^{\alpha-2} \mathrm{X}_{n+1}^2 + \alpha x_{n+1}^{\alpha-1} \mathrm{Z}_{n+1} + 2\alpha x_{n+1}^{\alpha-1} \mathrm{X}_{n+1} \operatorname{div} \mathrm{X} + x_{n+1}^{\alpha} [\operatorname{div} \mathrm{Z} + (\operatorname{div} \mathrm{X})^2 + \sum_{i=1}^{n} |(\mathrm{D}_{\tau^i} \mathrm{X})^{\perp}|^2 - \sum_{i,\, j=1}^{n} (\tau^i \mathrm{D}_{\tau^j} \mathrm{X})(\tau^j \mathrm{D}_{\tau^i} \mathrm{X})] \} \, d\mathscr{H}_n. \quad (5)$$

Proof. — From the general area formula (*see* e. g. [SL], § 8, or [FH] 3.2.20 Cor.), we infer that

$$\mathscr{E}_\alpha(\Phi_t(\mathrm{M} \cap \mathrm{K})) = \int_{\mathrm{M} \cap \mathrm{K}} (\psi_t)_{n+1}^{\alpha} \, \mathrm{J} \psi_t \, d\mathscr{H}_n,$$

where $\psi_t = \Phi_t|_{\mathrm{M} \cap \mathrm{U}}$, $\mathrm{J}\psi_t$ denotes the Jacobian of ψ_t and $(\psi_t)_{n+1}^{\alpha}$ is the $(n+1)$-th component of ψ_t to the power α. The Jacobian $\mathrm{J}\psi_t$ may be computed as in [SL], p. 50,

$$\mathrm{J}\psi_t = 1 + t \,.\operatorname{div} \mathrm{X} + \frac{t^2}{2} \left\{ \operatorname{div} \mathrm{Z} + (\operatorname{div} \mathrm{X})^2 + \sum_{i=1}^{n} |(\mathrm{D}_{\tau^i} \mathrm{X})^{\perp}|^2 - \sum_{i,\, j=1}^{n} (\tau^i \mathrm{D}_{\tau^j} \mathrm{X})(\tau^j \mathrm{D}_{\tau^i} \mathrm{X}) \right\} + o(t^2).$$

Similarly we find

$$(\psi_t)_{n+1}^{\alpha} = x_{n+1}^{\alpha} + t \alpha x_{n+1}^{\alpha-1} \mathrm{X}_{n+1} + \frac{t^2}{2} [\alpha(\alpha-1) x_{n+1}^{\alpha-2} \mathrm{X}_{n+1}^2 + \alpha x_{n+1}^{\alpha-1} \mathrm{Z}_{n+1}] + o(t^2).$$

Now the result follows immediately by computing the coefficients of t and $\frac{t^2}{2}$ in the product $(\psi_t)_{n+1}^{\alpha} \,.\, \mathrm{J}\psi_t$. □

Lemma 1 motivates the following definition.

DEFINITION 1. — A C^1-submanifold $\mathrm{M} \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ is called α-stationary in $\mathrm{U} \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$, if $\mathscr{E}_\alpha(\mathrm{M} \cap \mathrm{K}) < \infty$ for all compact sets $\mathrm{K} \subset \mathrm{U}$, and

$$\int_{\mathrm{M}} \{ x_{n+1}^{\alpha} \operatorname{div} \mathrm{X} + \alpha x_{n+1}^{\alpha-1} \mathrm{X}_{n+1} \} \, d\mathscr{H}_n = 0, \quad (6)$$

for all vector fields $\mathrm{X} \in C_c^1(\mathrm{U}, \mathbb{R}^{n+1})$.

LEMMA 2. — *Suppose* M *is* α-*stationary in* U *and of class* C^2. *Then the mean curvature* H *of* M *with respect to the unit normal* $\nu = (\nu_1, \ldots, \nu_{n+1})$ *is given by*

$$\mathrm{H}(x) = \alpha x_{n+1}^{-1} \nu_{n+1} \quad \textit{for all } x \in \mathrm{M} \cap \mathrm{U}.$$

Proof. — Take some arbitrary function $\xi \in C_c^1(M, \mathbb{R})$ with compact support in U and put $X = \xi . \nu$. Then we infer from (6)

$$\begin{aligned} 0 &= \int_M \{ x_{n+1}^\alpha \operatorname{div}(\xi . \nu) + \alpha x_{n+1}^{\alpha-1} \xi . \nu_{n+1} \} \, d\mathcal{H}_n \\ &= \int_M \{ \operatorname{div}(x_{n+1}^\alpha \xi . \nu) + \alpha x_{n+1}^{\alpha-1} \xi . \nu_{n+1} \} \, d\mathcal{H}_n, \\ &= -\int_M \{ x_{n+1}^\alpha \xi . \nu . \underline{\underline{H}} - \alpha x_{n+1}^{\alpha-1} \xi . \nu_{n+1} \} \, d\mathcal{H}_n, \end{aligned}$$

where $\underline{\underline{H}} = \nu H$ is the mean curvature vector of M. The lemma follows by applying the fundamental lemma in the calculus of variations. □

We take again the special variation $X = \xi . \nu$, $\xi \in C_c^1(M, \mathbb{R})$ and find successively,

$$\operatorname{div} X = -X . \underline{\underline{H}} = -\alpha \nu_{n+1} x_{n+1}^{-1} \xi$$

$$\sum_{i=1}^n |(D_{\tau^i} X)^\perp|^2 = \sum_{i=1}^n |\nu D_{\tau^i} \xi|^2 = |\nabla \xi|^2,$$

and

$$\sum_{i,\, j=1}^n (\tau^i D_{\tau^j} X)(\tau^j D_{\tau^i} X) = \xi^2 |A|^2,$$

where $|A|$ denotes the length of the second fundamental form $A = h_{ij} \tau^i \otimes \tau^j$, *i.e.*

$$|A|^2 = \sum_{i,\, j=1}^n h_{ij}^2.$$

Thus we have proved

LEMMA 3. — *Suppose* $M \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ *is a submanifold of class* C^2 *which is* α*-stationary in* $U \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$, (clos $M - M) \cap U = \varnothing$. *If* $X = \xi . \nu$ *for some function* $\xi \in C_c^1(M, \mathbb{R})$ *with compact support in* U, *then the second variation is given by*

$$\delta^2 \mathscr{E}(M, \xi) = \int_M \{ x_{n+1}^\alpha |\nabla \xi|^2 - \alpha x_{n+1}^{\alpha-2} \nu_{n+1}^2 \xi^2 - x_{n+1}^\alpha |A|^2 \xi^2 \} \, d\mathcal{H}_n. \quad \square$$

Hence it is reasonable to define stability as follows.

DEFINITION 2. — Suppose $M \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ is a n-dimensional submanifold of class C^2 which is α-stationary in $U \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$, (clos $M - M) \cap U = \varnothing$. Then M is called α-stable in U, if

$$\int_M \{ x_{n+1}^\alpha |\nabla \xi|^2 - \alpha x_{n+1}^{\alpha-2} \nu_{n+1}^2 \xi^2 - x_{n+1}^\alpha |A|^2 \xi^2 \} \, d\mathcal{H}_n \geqq 0 \tag{7}$$

for each $\xi \in C_c^1(M, \mathbb{R})$ with compact support in U. In particular, if $\mathscr{C} = \operatorname{clos} M$ is a cone in $\mathbb{R}^n \times \mathbb{R}^+$ with singularity at $\{0\}$, and if $M = \mathscr{C} - \{0\} \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ is α-stationary in $\mathbb{R}^n \times \mathring{\mathbb{R}}^+$, then $\mathscr{C}$ is called α-stable if (7) holds for all $\xi \in C_c^1(M, \mathbb{R})$.

Put $c_n^\alpha(y) = \sqrt{\frac{\alpha}{p}}[y_1^2 + \ldots + y_n^2]^{1/2}$, $y \in \mathbb{R}^n$, $\alpha > 0$, $p = n-1$ and define the cones

$$\mathscr{C}_n^\alpha = \{(y, c_n^\alpha(y)) : y \in \mathbb{R}^n\},$$

then we have

THEOREM 1. — *The cones $\mathscr{C}_n^\alpha$ are α-stable, if $\alpha + p \geqq 3 + \sqrt{8}$.*

Observe that the critical number $3 + \sqrt{8}$ also enters the discussion of the ordinary differential system [11] in [D 1]. Here, it appears as a necessary, though not sufficient condition for the construction of a minimal foliation about the cone $\mathscr{C}_n^\alpha$.

THEOREM 2. — *Suppose $\mathscr{C} \subset \mathbb{R}^n \times \mathbb{R}^+$ is an α-stable n-dimensional cone with singularity at $\{0\}$. If $\alpha + p < 3 + \sqrt{8}$ then $\mathscr{C}$ is a hyperplane $\mathscr{P}$. Furthermore, $\mathscr{P}$ must be perpendicular to the plane $\{x_{n+1} = 0\}$.*

COROLLARY. — *In particular, if $2 \leqq n \leqq 5$ there are no non-trivial (potential-) energy stable cones in $\mathbb{R}^n \times \mathbb{R}^+$ with singularity at $\{0\}$.*

2. PROOFS

Let $\xi \in C_c^1(\mathscr{C}_n^\alpha - \{0\}, \mathbb{R})$ be arbitrary and put $X(x) = x.|x|^{-2}\xi^2$ for $x \in \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ where $|x|^2 = (x_1^2 + \ldots + x_{n+1}^2)$. A standard calculation yields (*see* [SL], § 17)

$$\operatorname{div} X = \sum_{i=1}^{n} (D_{\tau^i} X)\,\tau^i = 2\,|x|^{-2}(x\nabla\xi)\,\xi$$
$$+ (n-2)\,\xi^2\,|x|^{-2} + 2\,|x|^{-2}\xi^2\,|(D\,|x|)^\perp|^2.$$

Since $\mathscr{C}_n^\alpha - \{0\}$ is α-stationary in $\mathbb{R}^n \times \mathring{\mathbb{R}}^+$, we conclude from (6) that

$$\int_{\mathscr{C}_n^\alpha - \{0\}} x_{n+1}^\alpha \{2\,|x|^{-2}(x\nabla\xi)\,\xi + (n-2+\alpha)\,|x|^{-2}\,\xi^2\}\,d\mathscr{H}_n \leqq 0.$$

We apply Schwarz inequality and obtain

$$\left(\frac{n-2+\alpha}{2}\right)^2 \int_{\mathscr{C}_n^\alpha - \{0\}} x_{n+1}^\alpha\,|x|^{-2}\,\xi^2\,d\mathscr{H}_n \leqq \int_{\mathscr{C}_n^\alpha - \{0\}} x_{n+1}^\alpha\,|\nabla\xi|^2\,d\mathscr{H}_n.$$

Therefore $\mathscr{C}_n^\alpha$ is α-stable, if

$$\left(\frac{n-2+\alpha}{2}\right)^2 \geqq |x|^2\,|A|^2+\alpha x_{n+1}^{-2}\,|x|^2\,\nu_{n+1}^2. \tag{8}$$

An elementary calculation shows that for the cone $\mathscr{C}_n^\alpha$ the length of the second fundamental form is given by

$$|A|^2=\frac{\alpha p}{\alpha+p}\,r^{-2}=\alpha\,|x|^{-2} \quad \text{for all } x\in\mathscr{C}_n^\alpha-\{0\},$$

where we have put $r^2=(x_1^2+\ldots+x_n^2)$. Then along $\mathscr{C}_n^\alpha$, $x_{n+1}=\sqrt{\frac{\alpha}{p}}\,r$ and we infer from (8) that $\mathscr{C}_n^\alpha$ is stable, if

$$\left(\frac{n-2+\alpha}{2}\right)^2 \geqq \alpha+\alpha x_{n+1}^{-2}\,|x|^2\,\nu_{n+1}^2$$
$$=\alpha+\frac{\alpha p}{\alpha+p}\left[1+\frac{r^2}{x_{n+1}^2}\right]=\alpha+p.$$

This is true in case that $\alpha+p\geqq 3+\sqrt{8}$. Theorem 1 follows.

Proof of Theorem 2. – In the following we shall always assume that $M=\mathscr{C}-\{0\}$ is an α-stable cone in $\mathbb{R}^n\times\mathring{\mathbb{R}}^+$, so that in particular (7) holds true. Replacing ξ by $|A|\,\xi$ in (7) we get

$$\int_M \{x_{n+1}^\alpha\,|A|^4\,\xi^2+\alpha x_{n+1}^{\alpha-2}\,\nu_{n+1}^2\,|A|^2\,\xi^2\}\,d\mathscr{H}_n$$
$$\leqq \int_M x_{n+1}^\alpha\,\{|A|^2\,|\nabla\xi|^2+|\nabla|A||^2\xi^2+2\,\xi\,|A|\,(\nabla\xi\nabla|A|)\}\,d\mathscr{H}_n. \tag{9}$$

Now

$$2\int_M x_{n+1}^\alpha\,|A|\,\xi\,(\nabla\xi\nabla|A|)\,d\mathscr{H}_n=\int_M x_{n+1}^\alpha\,(\nabla\xi^2)\,\nabla\left(\frac{1}{2}|A|^2\right)d\mathscr{H}_n$$
$$=-\int_M x_{n+1}^\alpha\,\xi^2\,\Delta\left(\frac{1}{2}|A|^2\right)d\mathscr{H}_n-\int_M \xi^2\,(\nabla x_{n+1}^\alpha)\left(\nabla\frac{1}{2}|A|^2\right)d\mathscr{H}_n. \tag{10}$$

In order to conclude further we need a sharp estimate for the Laplacian of $|A|^2$. This will be provided by the following

LEMMA 4 ([SSY], [SL, appendix B]). – *If* $M=\mathscr{C}-\{0\}$ *is a cone, then*

$$-\frac{1}{2}\Delta\,|A|^2\leqq|A|^4-2\,|x|^{-2}\,|A|^2-|\nabla|A||^2-h_{ij}H_{,ij}-H\,h_{mi}h_{mj}h_{ij} \quad [1]$$

[1] The summation convention is used freely here!

Here $H_{,ij}$ denote the second covariant derivatives of the mean curvature H with respect to τ^i and τ^j, and, as above, h_{ij} are the coefficients of A.

Proof of Lemma 4. – B.8 Lemma and B.9 Lemma in [SL] yield the relations

$$\Delta\left(\frac{1}{2}|A|^2\right)=\sum_{i,j,k} h_{ij,k}^2-|A|^4+h_{ij}H_{,ij}+H\,h_{mi}h_{mj}h_{ij},$$

here $H=h_{kk}=\text{trace } A$ and $h_{ij,k}$ denotes the covariant derivative of A with respect to τ^k; also

$$\sum_{i,j,k} h_{ij,k}^2-|\nabla|A||^2\geqq 2|x|^{-2}|A|^2 \quad \text{for all } x\in M.$$

Both relations imply Lemma 4. □

From (9), (10) and Lemma 4 we conclude that

$$\int_M \xi^2\left\{2x_{n+1}^\alpha |x|^{-2}|A|^2+\alpha x_{n+1}^{\alpha-2}|A|^2 v_{n+1}^2 + \nabla(x_{n+1}^\alpha)\nabla\left(\frac{1}{2}|A|^2\right)+x_{n+1}^\alpha h_{ij}H_{,ij}+x_{n+1}^\alpha H\,h_{mi}h_{mj}h_{ij}\right\}d\mathscr{H}_n \leqq \int_M x_{n+1}^\alpha |A|^2|\nabla\xi|^2 d\mathscr{H}_n. \quad (11)$$

Relation (11) will be of crucial importance in what follows.

To begin, select an orthonormal frame $\tau^1,\dots,\tau^n\in T_xM$ so that $\tau^n=\dfrac{x}{|x|}$ and $\tau^1,\dots,\tau^n$ are constant along the ray through x. Also we can assume that $\tau_{n+1}^1=\tau_{n+1}^2=\dots=\tau_{n+1}^{n-1}=0$. Then $h_{in}=h_{ni}=0$ for $i\in\{1,\dots,n\}$ and, since $h_{ij}(\lambda x)=\lambda^{-1}h_{ij}(x)$, $\lambda>0$, we have $h_{ij,n}=-|x|^{-1}h_{ij}$.

We first compute the expression

$$(\nabla x_{n+1}^\alpha)\left(\nabla\frac{1}{2}|A|^2\right)=\alpha x_{n+1}^{\alpha-1}(D_{\tau^k}x_{n+1})\left(D_{\tau^k}\left(\frac{1}{2}|A|^2\right)\right) =\alpha x_{n+1}^{\alpha-1}h_{ij}h_{ij,k}\tau_{n+1}^k=-\alpha x_{n+1}^\alpha|x|^{-2}|A|^2, \quad (12)$$

and then

$$\begin{aligned}
\frac{1}{\alpha}\mathrm{H}_{,ij} &= \frac{1}{\alpha}\nabla_i\nabla_j\mathrm{H} = \nabla_i\nabla_j\left(\frac{\nu_{n+1}}{x_{n+1}}\right)\\
&= \nabla_i\{-x_{n+1}^{-2}(\nabla_j x_{n+1})\,\nu_{n+1} + x_{n+1}^{-1}\nabla_j\nu_{n+1}\}\\
&= 2\,x_{n+1}^{-3}\nabla_i x_{n+1}\nabla_j x_{n+1}\,\nu_{n+1} - x_{n+1}^{-2}(\nabla_i\nabla_j x_{n+1})\,\nu_{n+1}\\
&\quad - x_{n+1}^{-2}\nabla_j x_{n+1}\nabla_i\nu_{n+1}\\
&\quad - x_{n+1}^{-2}\nabla_i x_{n+1}\nabla_j\nu_{n+1} + x_{n+1}^{-1}\nabla_i\nabla_j\nu_{n+1}\\
&= 2\,x_{n+1}^{-3}\,\tau_{n+1}^i\,\tau_{n+1}^j\,\nu_{n+1}\\
&\quad - x_{n+1}^{-2}\,\nabla_i\,\tau_{n+1}^j\,\nu_{n+1} - x_{n+1}^{-2}\,\tau_{n+1}^j\,\nabla_i\,\nu_{n+1}\\
&\quad - x_{n+1}^{-2}\,\tau_{n+1}^i\,\nabla_j\,\nu_{n+1} + x_{n+1}^{-1}\,\nabla_i\,\nabla_j\,\nu_{n+1}.
\end{aligned}$$

By virtue of

$$\nabla_i\,\nu = -h_{il}\,\tau^l \qquad \text{and} \qquad \nabla_i\,\tau^j = h_{ij}\,\nu$$

we obtain

$$\begin{aligned}
\frac{1}{\alpha}\mathrm{H}_{,ij} = 2\,x_{n+1}^{-3}\,\tau_{n+1}^i\,\tau_{n+1}^j\,\nu_{n+1}&\\
- x_{n+1}^{-2}\,h_{ij}\,\nu_{n+1}^2 + x_{n+1}^{-2}\,\tau_{n+1}^j\,h_{il}\,\tau_{n+1}^l&\\
+ x_{n+1}^{-2}\,\tau_{n+1}^i\,h_{jl}\,\tau_{n+1}^l - x_{n+1}^{-1}\,\nabla_i\,[h_{jl}\,\tau_{n+1}^l]&.
\end{aligned}$$

Using the Codazzi equations we conclude

$$\begin{aligned}
\nabla_i\,[h_{jl}\,\tau_{n+1}^l] &= h_{jl,\,i}\,\tau_{n+1}^l + h_{jl}\,\nabla_i\,\tau_{n+1}^l\\
&= h_{ij,\,l}\,\tau_{n+1}^l + h_{jl}\,h_{il}\,\nu_{n+1},
\end{aligned}$$

whence

$$\begin{aligned}
\frac{1}{\alpha}h_{ij}\,\mathrm{H}_{,ij} = 2\,x_{n+1}^{-3}\,\tau_{n+1}^i\,\tau_{n+1}^j\,h_{ij}\,\nu_{n+1} - x_{n+1}^{-2}\,|\mathrm{A}|^2\,\nu_{n+1}^2&\\
+ x_{n+1}^{-2}\,h_{ij}\,h_{il}\,\tau_{n+1}^j\,\tau_{n+1}^l + x_{n+1}^{-2}\,h_{ij}\,h_{jl}\,\tau_{n+1}^i\,\tau_{n+1}^l&\\
- x_{n+1}^{-1}\,h_{ij}\,h_{ij,\,l}\,\tau_{n+1}^l - x_{n+1}^{-1}\,h_{ij}\,h_{jl}\,h_{il}\,\nu_{n+1}&.
\end{aligned}$$

Thus

$$\frac{1}{\alpha}h_{ij}\,\mathrm{H}_{,ij} = -x_{n+1}^{-2}\,|\mathrm{A}|^2\,\nu_{n+1}^2 + |x|^{-2}|\mathrm{A}|^2 - x_{n+1}^{-1}\,h_{ij}\,h_{jl}\,h_{il}\,\nu_{n+1},$$

and finally

$$h_{ij}\,\mathrm{H}_{,ij} = -\alpha\,x_{n+1}^{-2}\,|\mathrm{A}|^2\,\nu_{n+1}^2 + \alpha\,|x|^{-2}\,|\mathrm{A}|^2 - \mathrm{H}\,h_{ij}\,h_{jl}\,h_{il}. \tag{13}$$

(12), (13), and (11) yield the relation

$$2\int_{M} x_{n+1}^{\alpha}\,|x|^{-2}\,|A|^{2}\,\xi^{2}\,d\mathcal{H}_{n} \leqq \int_{M} x_{n+1}^{\alpha}\,|A|^{2}\,|\nabla\xi|^{2}\,d\mathcal{H}_{n} \tag{14}$$

for all $\xi \in C_c^1(M, \mathbb{R})$.

If ξ does not have compact support in $M = \mathscr{C} - \{0\}$ then (14) continues to hold, if only

$$\int_{M} x_{n+1}^{\alpha}\,|x|^{-2}\,|A|^{2}\,\xi^{2}\,d\mathcal{H}_{n} < \infty. \tag{15}$$

In fact, replace ξ by $\xi.\gamma_\varepsilon$ where γ_ε is a suitable cut off function with

$$\gamma_\varepsilon = \begin{cases} 1 & \text{for } |x| \in (\varepsilon, \varepsilon^{-1}) \\ 0 & \text{for } |x| < \dfrac{\varepsilon}{2} \text{ or } |x| > 2\,\varepsilon^{-1} \end{cases}$$

and $0 \leqq \gamma_\varepsilon \leqq 1$, $|\nabla\gamma_\varepsilon(x)| \leqq 3\,|x|^{-1}$ in all of $\mathbb{R}^n \times \mathbb{R}^+$. Then $\xi.\gamma_\varepsilon$ is admissible in (14) and the assertion follows by letting $\varepsilon \to 0^+$ and using (15).

Note that (15) is satisfied, if

$$\int_{M} |x|^{\alpha-2}\,|A|^{2}\,\xi^{2}\,d\mathcal{H}_{n} < \infty. \tag{16}$$

From the coarea formula we infer that

$$\int_{M} \varphi(x)\,d\mathcal{H}_{n}(x) = \int_{0}^{\infty} r^{n-1} \int_{\Sigma} \varphi(r\omega)\,d\mathcal{H}_{n-1}\,dr \tag{17}$$

for all non-negative $\varphi \in C^0(M)$, where $\Sigma = M \cap S^n$, and $S^n \subset \mathbb{R}^{n+1}$ denotes the unit n-phere. Also, since M is a cone, we find

$$|A(x)|^2 = |x|^{-2}\,|A(x/|x|)|^2 \quad \text{for all } x \in M.$$

Hence, we readily infer from (17) and (16) that

$$\xi = |x|^{1+\varepsilon-\alpha}.\,|x|_1^{1+\alpha-(n/2)-2\,\varepsilon},$$

where

$$|x|_1 = \max(1, |x|),$$

is admissible in (14), if $\varepsilon > \dfrac{\alpha}{2}$ (where we have of course assumed that $n \geqq 2$).

Furthermore we find

$$|\nabla\xi|^2 \leqq \begin{cases} (1+\varepsilon-\alpha)^2\,|x|^{2\,\varepsilon-2\,\alpha} & \text{in } M \cap B_1(0), \qquad B_1(0) = \{|x| < 1\}, \\ \left(2-\dfrac{n}{2}-\varepsilon\right)^2 |x|^{2-n-2\,\varepsilon} & \text{in } (\mathbb{R}^{n+1} - B_1(0)) \cap M \end{cases}$$

and (14) implies

$$2\int_{M\cap B_1} x_{n+1}^{\alpha}|A|^2|x|^{2\varepsilon-2\alpha}\,d\mathscr{H}_n + 2\int_{M\cap(\mathbb{R}^{n+1}-B_1)} x_{n+1}^{\alpha}|A|^2|x|^{2-n-2\varepsilon}\,d\mathscr{H}_n \leqq (1+\varepsilon-\alpha)^2\int_{M\cap B_1} x_{n+1}^{\alpha}|A|^2|x|^{2\varepsilon-2\alpha}\,d\mathscr{H}_n + \left(2-\frac{n}{2}-\varepsilon\right)^2\int_{M\cap(\mathbb{R}^{n+1}-B_1)} x_{n+1}^{\alpha}|A|^2|x|^{2-n-2\varepsilon}\,d\mathscr{H}_n.$$

We would like to choose n, ε, α so that

$$\varepsilon>\frac{\alpha}{2},\qquad (1+\varepsilon-\alpha)^2<2 \qquad\text{and}\qquad \left(\frac{n}{2}+\varepsilon-2\right)^2<2. \tag{18}$$

(18) is equivalent to

$$-1-\sqrt{2}+\alpha<\varepsilon<\sqrt{2}+\alpha-1 \qquad\text{and}\qquad \frac{\alpha}{2}<\varepsilon<2+\sqrt{2}-\frac{n}{2}. \tag{19}$$

If $\alpha+n<4+2\sqrt{2}$ then a suitable choice of ε is

$$\varepsilon=\frac{\alpha}{2}+\delta,$$

where

$$\delta=N^{-1}\left[2+\sqrt{2}-\frac{n}{2}-\frac{\alpha}{2}\right]>0$$

with $N\in\mathbb{N}$ large. Thus we conclude that $|A|^2\equiv 0$ *i.e.* M is a hyperplane $\mathscr{P}$. Because of $0=H=\alpha\,\dfrac{\nu_{n+1}}{x_{n+1}}$ we must have $\nu_{n+1}=0$ as required.

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