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Classical solvability in dimension two of the second boundary-value problem associated with the Monge-Ampère operator

by

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Abstract. — Given two bounded strictly convex domains of $\mathbb{R}^n$ and a positive function on their product, all data being smooth, find a smooth strictly convex function whose gradient maps one domain onto the other with Jacobian determinant proportional to the given function. We solve this problem under the (technical) condition $n = 2$.

Key words : Strictly convex functions, prescribed gradient image, Monge-Ampère operator, continuity method, a priori estimates.

Résumé. — Soit deux domaines bornés strictement convexes de $\mathbb{R}^n$ et une fonction positive définie sur leur produit, ces données étant lisses, trouver une fonction lisse strictement convexe dont le gradient applique un domaine sur l’autre avec déterminant Jacobien proportionnel à la fonction donnée. Nous résolvons ce problème sous la condition (technique) $n = 2$.

Classification A.M.S. : 35 J 65, 35 B 45, 53 C 45.

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I. INTRODUCTION

Let $D$ and $D^*$ be bounded $C^\infty$ strictly convex domains of $\mathbb{R}^n$. We denote by $S(D, D^*)$ the subset of $C^\infty(D)$ consisting of strictly convex real functions $f$ whose gradient maps $D$ onto $D^*$. Given any $u \in C^\infty(D)$, we denote by $A(u)$ the Jacobian determinant of the gradient mapping $x \mapsto du(x)$. The nonlinear second order differential operator $A$ is called the Monge-Ampère operator on $D$. Basic features of $A$ restricted to $S(D, D^*)$ are listed in the preliminary

**PROPOSITION 1.** – $A$ sends $S(D, D^*)$ into

$$\Sigma := \{ f \in C^\infty(D), f > 0, \langle f \rangle = |D^*|/|D| \}$$

($\langle f \rangle$ denotes the average of $f$ over $D$ and $|D|$, the Lebesgue measure of $D$). On $S(D, D^*)$, $A$ is elliptic and its derivative is divergence-like. Given any defining function $h^*$ of $D^*$, the boundary operator $u \mapsto B(u) := h^*(du)|_{\partial D}$ is co-normal with respect to $A$ on $S(D, D^*)$. Furthermore, given any $u \in S(D, D^*)$ and any $x \in \partial D$, the co-normal direction at $x$ with respect to the derivative of $A$ at $u$ is nothing but the normal direction of $\partial D^*$ at $du(x)$.

We postpone the proof of proposition 1 till the end of this section. The second boundary-value problem consists in showing that $A : S(D, D^*) \to \Sigma$ is onto. More generally, we wish to solve in $S(D, D^*)$ two kinds of equations namely

$$\begin{align*}
\log A(u) &= f(x, du) + \langle u \rangle \\
\log A(u) &= F(x, du, u)
\end{align*}$$

where $f \in C^\infty(D \times D^*)$ and $F \in C^\infty(D \times D^* \times \mathbb{R})$, the latter being uniformly increasing in $u$. We aim at the following

**THEOREM.** – Equations (1) and (2) are uniquely solvable in $S(D, D^*)$ provided $n = 2$.

The second boundary-value problem was first posed and solved (with $n = 2$ but the methods, geometric in nature, extend to any dimension) in a generalized sense in [18] chapter V section 3 (see also [3] theorem 2, where the whole plane is taken in place of $D$). The elliptic Monge-Ampère operator with a quasilinear Neumann boundary condition is treated in [16], in any dimension, and it is further treated with a quasilinear oblique boundary condition in [21] provided $n = 2$. A general study of nonlinear oblique boundary-value problems for nonlinear second order uniformly

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(1) Here the meaning of “strictly convex” is restricted to having a positive-definite hessian matrix, which rules out e.g. the strictly convex function $u(x) = |x-y|^4$ near $y \in D$, as pointed out to us by Martin Zerner.
elliptic equations is performed in [15]. Quite recently, the following problem was solved [5]: existence and regularity on a given bounded domain $D$ of $\mathbb{R}^n$ (no convexity assumption, no restriction on $n$) of a diffeomorphism from $D$ to itself, reducing to the identity on $\partial D$, with prescribed positive Jacobian determinant (of average 1 on $D$).

**Remarks.**  1. The restriction $n=2$ is unsatisfactory but we could not draw second order boundary estimates without it. In May 1988, in Granada (Spain), Neil Trudinger informed us that Kai-Sing Tso had treated the problem in any dimension; however, from that time on, Tso’s preprint has not been available due to a serious gap in his proof, as he himself wrote us [20]. In June 1989, John Urbas visited us in Antibes and he kindly advised us to submit our own 2-dimensional result; it is a pleasure to thank him for his thorough reading of the present paper. This may be the right place to thank also the Referee for pointing out a mistake at the end of the original proof of proposition 2 below, and a few inaccuracies (particularly one in remark 6).

2. We do not assume the non-emptiness of $S(D, D^*)$ to prove the theorem; we thus obtain it (when $n=2$) as a by-product of our proof. In fact, we found no straightforward way of exhibiting any member of $S(D, D^*)$ except, of course, if $D=D^*$, although we can write down explicitly a $C^\infty(D)$ convex (but not strictly convex) function with gradient image $D^*$, constructed from any suitable support function for $D^*$. Provided non-emptiness, it is possible to prove that $S(D, D^*)$ is a locally closed Fréchet submanifold of the open subset of strictly convex functions in $C^\infty(D)$, as the fiber of a submersion.

3. From the proof below, it appears that, given any $\alpha \in (0, 1)$ $C^{2, \alpha}(\overline{D})$ solutions may be derived (by approximation) from the above theorem under the sole regularity assumptions: $D$ and $D^*$ are $C^{2, 1}$, $f$ and $F$ are $C^{1, 1}$. We did not study further 2-dimensional global regularity refinements as done in [19], [14] for the Dirichlet problem.

4. The uniqueness for (1) shows that, in general, the equation $\log A(u) = f(x, du)$ is not well-posed on $S(D, D^*)$. The idea of introducing in (1) the average term goes back to [6] and it proved to be useful in various contexts ([2], [8], [9], [10]). If $u \in S(D, D^*)$ solves (1), then $v = u + \text{Const.}$ solves in $S(D, D^*)$ the equation $\log A(v) = f(x, dv) + \langle u \rangle$, while the Legendre transform $v^*$ of $v$ solves in $S(D^*, D)$ the “dual” equation $\log A(v^*) = -f(dv^*, x) - \langle u \rangle$. In case $f(x, x^*) = f_1(x) - f_2(x^*)$, the value of $\langle u \rangle$ is a priori fixed by the constraint (due to the “Jacobian” structure of $A$)

$$\int_{D^*} e^{f_2(x^*)} dx^* = e^{\langle u \rangle} \int_D e^{f_1(x)} dx.$$  

The prescribed Gauss-curvature equation is an example of this type.
Proof of proposition 1. — By its very definition, as the Jacobian of the gradient mapping, \( A \) readily sends \( S(D, D^*) \) into the submanifold \( \Sigma \).

Let \( u \in S(D, D^*) \). In euclidean co-ordinates \((x^1, \ldots, x^n)\), \( A(u) \) reads
\[
A(u) = \det(u_{ij})
\]
and the derivative of \( A \) at \( u \) reads
\[
\delta u \in C^\infty(\bar{D}) \to dA(u)(\delta u) = A^{ij}(\delta u)_{ij}
\]
where
\[
A^{ij} = A(u)u^{ij}
\]
(indices denote partial derivatives, Einstein’s convention holds, \( (u^{ij}) \) is the matrix inverse of \( (u_{ij}) \) and \( (A^{ij}) \), its co-matrix). Since \( u \) is strictly convex, \( A \) is indeed elliptic at \( u \). Furthermore, one easily verifies the following identity: for any \( \delta u \in C^\infty(\bar{D}) \),
\[
A^{ij}(\delta u)_{ij} = [A^{ij}(\delta u)]_j.
\]
So, as asserted, \( dA(u) \) is divergence-like. The co-normal boundary operator associated with \( A \) at \( u \) is
\[
\delta u \in C^\infty(\bar{D}) \to \beta(\delta u) = A^{ij}N^i(\delta u)_{j} \in C^\infty(\partial D),
\]
\( N \) standing for the outward unit normal on \( \partial D \). Fix a defining function \( h^* \) for \( D^* \) (i.e. \( h^* \in C^\infty(\bar{D^*}) \)) is strictly convex and vanishes on \( \partial D^* \). Since \( u \in S(D, D^*) \), the function \( H := h^*(du) \in C^\infty(\bar{D}) \) is negative inside \( D \) and vanishes on \( \partial D \). Moreover, a straightforward computation yields in \( D \):
\[
u^{ij}H_{ij} - u^{ij}[\log A(u)]_{ij} = u_{ij}h_{ij}^* > 0.
\]
Hopf’s lemma [12] implies that \( H_{N} > 0 \) on \( \partial D \). Since
\[
H_{i} = u_{ij}h_{ij}^*
\]
the boundary operators satisfy
\[
A(u)dB(u) = H_{N}\beta.
\]
So \( B \) is indeed co-normal with respect to \( A \) at \( u \).

Last, the geometric interpretation of the co-normal direction \( \beta \) given at the end of proposition 1, simply follows from the fact that \( dB(u)(x) \) equals the derivative in the direction of \( dh^*[du(x)] \) which is precisely (outward) normal to \( \partial D^* \) at \( du(x) \). \( \square \)

II. THE CONTINUITY METHOD

Fix \((x_0, x_0^*) \in D \times D^* \) and \( \lambda \in (0, 1] \) such that the gradient of
\[
\nu_0 = \frac{\lambda}{2} |x - x_0|^2 + x_0^* \cdot x
\]
maps $\bar{D}$ into $D^*$ ($|\cdot|$ stands for the standard euclidean norm, $.$ for the euclidean scalar product). Set $u_0 := v_0 - \langle v_0 \rangle$, $D_0 := du_0(D)$. A routine verification shows that $D_0$ is $C^\infty$ strictly convex. Let $t \in [0, 1] \to D_t$ be a smooth path of bounded $C^\infty$ strictly convex domains connecting $D_0$ to $D_1 = D^*$, with $D_t \subset D_{t'}$ for $t < t'$; fix $t \to h_t$ a smooth path of corresponding defining functions. For each $t \in [0, 1]$, consider in $S(D, D_t)$ the two following equations:

\[
\begin{align*}
\log A(u) &= tf(x, du) + (1 - t)n \log \lambda + \langle u \rangle \\
\log A(u) &= tF(x, du, u) + (1 - t)(u - u_0 + n \log \lambda).
\end{align*}
\]

By construction $u_0$ solves both equations for $t = 0$, so (for $i = 1, 2$) the sets $T_i := \{ t \in [0, 1], (i, t) \text{ admits a solution in } S(D, D_t) \}$ are non-empty. Hereafter, we show that they are both relatively open and closed in $[0, 1]$: if so, by connectedness, they coincide with all of $[0, 1]$. The solutions for $t = 1$ are those announced in the theorem; their uniqueness is established at the end of this section.

Let us show that $T_1$ is relatively open in $[0, 1]$; similar, more standard (due to the monotonicity assumption of $F$), reasonings hold for $T_2$. Fix $\alpha \in (0, 1)$ and denote by $U^{2, \alpha}$ the open subset of $C^{2, \alpha}(\bar{D})$ consisting of strictly convex functions. On $[0, 1] \times U^{2, \alpha}$, consider the smooth map $(M, B)$ defined by

\[
M(t, u) := \log A(u) - tf(x, du) - (1 - t)n \log \lambda - \langle u \rangle,
B(t, u) := h_t(du)|_{\partial D},
\]

and ranging in $C^{0, \alpha}(\bar{D}) \times C^{1, \alpha}(\partial D)$. Let $t_0 \in T_1$; there thus exists $u_0$ in $U^{2, \alpha}$ such that $(M, B)(t_0, u_0) = (0, 0)$. The proof is based on the Banach implicit function theorem applied to $(M, B)$ at $(t_0, u_0)$. We want to show that the map

\[
(m, b) := [M_u(t_0, u_0), B_u(t_0, u_0)] : C^{2, \alpha}(\bar{D}) \to C^{0, \alpha}(\bar{D}) \times C^{1, \alpha}(\partial D)
\]

is an isomorphism. Record the following expression of $(m, b)$ in euclidean co-ordinates:

\[
\begin{align*}
m(\delta u) &= u^{ij}_0(\delta u)_{ij} - t_0 f_u(x, du_0)(\delta u)_i - \langle \delta u \rangle, \\
b(\delta u) &= (h_t)_i(du_0)(\delta u)_i.
\end{align*}
\]

From proposition 1, we know that $b$ is oblique; so Hopf's maximum principle \cite{11} combined with Hopf's lemma \cite{12} imply that any $\delta u \in \text{Ker}(m, b)$ is constant, hence actually $\langle \delta u \rangle = 0$ and $\delta u \equiv 0$. Therefore $(m, b)$ is one-to-one.

Now we fix $(\delta M_0, \delta B_0) \in C^{0, \alpha}(\bar{D}) \times C^{1, \alpha}(\partial D)$ and we look for $\delta u$ in $C^{2, \alpha}(\bar{D})$ solving: $(m, b) (\delta u_0) = (\delta M_0, \delta B_0)$. Consider the auxiliary map

\[
(m', b') := \{ A(u_0)(m + \langle . \rangle), [A(u_0)/H_N] b \},
\]

where $H = h_t(du_0)$. It follows from proposition 1 that, given any $(\delta M', \delta B') \in C^{0, \alpha}(\bar{D}) \times C^{1, \alpha}(\partial D)$, the function $\delta u' \in C^{2, \alpha}(\bar{D})$ solves:

\[
(m', b')(\delta u') = (\delta M', \delta B'),
\]

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if and only if, for every $\delta v' \in W^{1,2}(D)$,

$$L(\delta u', \delta v') = \int_{\partial D} \delta B' \delta v \, da - \int_D \delta M' \delta v \, dx$$

$(da)$ is the measure induced on $\partial D$ by $dx$, where $L$ is the continuous bilinear form on $W^{1,2}(D)$ given by

$$L(\delta u', \delta v') = \int_D A(u_0)[u_0^j(\delta u')_i(\delta v')_j + t_0 f_u(x, du_0)(\delta u')_i(\delta v')] \, dx.$$ Let us argue on $(m', b')$ as in [6]. Combining the ellipticity of $m'$ and the obliqueness of $b'$ (asserted by proposition 1), with Hopf's maximum principle, Schauder's estimates and Fredholm's theory of compact operators, we know that the kernel of the adjoint of $(m', b')$ (formally obtained by varying the first argument of $L$ instead of the second, and by integrating by parts) is one-dimensional, let $\delta w \in C^{2,2}(D)$ span it, and that $(3)$ is solvable up to an additive constant if and only if

$$\int_{\partial D} \delta B' \delta w \, da - \int_D \delta M' \delta w \, dx = 0. \tag{4}$$

Observe that

$$\int_D A(u_0) \delta w \, dx \neq 0$$

since, otherwise, one could solve $(3)$ with $(\delta M', \delta B') = [A(u_0), 0]$ contradicting the maximum principle. We may thus normalize $\delta w$ by

$$\int_D A(u_0) \delta w \, dx = 1.$$ Then we can solve $(3)$ with right-hand side equals:

$$\left\{ A(u_0) \left[ \delta M_0 + \int_{\partial D} [A(u_0)/H_N] \delta B_0 \delta w \, da \right] - \int_D A(u_0) \delta M_0 \delta w \, dx \right\} [A(u_0)/H_N] \delta B_0$$

since the latter satisfies $(4)$. If $\delta u_0'$ is a solution, then

$$\delta u_0 = \delta u_0' - \langle \delta u_0' \rangle + \int_{\partial D} [A(u_0)/H_N] \delta B_0 \delta w \, da - \int_D A(u_0) \delta M_0 \delta w \, dx$$

solves the original equation

$$(m, b)(\delta u_0) = (\delta M_0, \delta B_0).$$
So \((m, b)\) is also onto. The implicit function theorem thus implies the existence of a real \(\delta > 0\) and of a smooth map
\[
t \in (t_0 - \delta, t_0 + \delta) \cap [0, 1] \rightarrow u_t \in U^{2, \alpha}
\]
such that \((M, B)(t, u) = (0, 0)\). By proposition 1 and standard elliptic regularity \([1]\), \(u_t \in S(D, D)_t\), hence \(T_1\) is relatively open.

Assuming \(n = 2\), we shall carry out a \(C^{2, \alpha}(\overline{D})\) a priori bound, independent of \(t \in [0, 1]\), on the solutions in \(S(D, D)_t\) of equations \((1. t)\) and \((2. t)\). Provided such a bound exists, the closedness of \(T_i (i = 1, 2)\) follows in a standard way from Ascoli’s theorem combined with proposition 1 and elliptic regularity \([1]\).

Last, let us prove that \((1)\) admits at most one solution in \(S(D, D^*)\); a similar argument holds for \((2)\). By contradiction, let \(u_0\) and \(u_1\) be two distinct solutions of \((1)\) in \(S(D, D^*)\). Then, for \(t \in [0, 1]\),
\[
\begin{align*}
u_t &= \frac{t}{1} u_t + \frac{(1 - t)}{1} u_0 \in S(D, D^*_t) \quad \text{and} \\
u &= u_1 - u_0 \quad \text{solves the linear boundary-value problem:}
\end{align*}
\]
which is elliptic inside \(D\) and oblique on \(\partial D\) by proposition 1. The maximum principle implies \(u \equiv 0\), contradicting the assumption.

III. PRELIMINARY A PRIORI ESTIMATES

In this section, we do not need yet the condition \(n = 2\). For any \(v \in S(D, D)_t\), \(dv \in D^*\), hence \(|dv|\) is bounded above by \(\rho(D^*_t) := \max_{x^* \in D^*} |x^*|\).

Set \(|f|_0 = \max_{D \times D^*} |f(x, x^*)|\), and let \(u \in S(D, D)_t\) solve \((1. t)\), then
\[
e^{-\int_0^1 A(u)} \leq e^{\int_0^1 \rho A(u)} \leq e^{\int_0^1 \rho(D^*_t)} \leq e^{\int_0^1 |D^*_t|} \leq e^{\int_0^1 |D^*_t|} \rho\Delta A(u).
\]

Integrating this over \(D\) yields for \(\langle u \rangle\) the pinching:
\[
\log |D_0| - |f|_0 \leq \langle u \rangle \leq \log |D^*_t| + |f|_0 + n |\log \lambda| \rho(D^*_t).
\]

Since \(du \leq \rho(D^*_t)\), \(u\) is a priori bounded in \(C^1(D)\) in terms of \(|D^*_t|\), \(\rho(D^*_t)\), \(|f|_0\), \(|D_0|\), \(\lambda\) and \(n\).

By assumption, there exists \(\mu \in (0, 1]\) such that \(F_t \geq \mu\) on \(\overline{D} \times \overline{D^*} \times \mathbb{R}\). The right-hand side of equation \((2. t)\), let us denote it by
\[
f(t, x, du, u),
\]
thus satisfies \( f_u \geq \mu \) as well, on \([0, 1] \times \bar{D} \times \bar{D}^* \times \mathbb{R} \). Let \( u \in S(D, D_t) \) solve (2. t). Set
\[
M := \max \limits_{\bar{D}} (u), \quad m := \min \limits_{\bar{D}} (u)
\]
\[
M_0 := \max \limits_{[0, 1] \times \bar{D} \times \bar{D}^*} [f(t, x, x^*, 0)], \quad m_0 := \min \limits_{[0, 1] \times \bar{D} \times \bar{D}^*} [f(t, x, x^*, 0)].
\]
From the mean value theorem, we know that
\[
\delta(D) \text{ standing for the diameter of } D. \text{ If } M \geq 0 \text{ and } m \leq 0, \text{ it implies } |u| \leq \mu \delta(D) \text{ and we are done. If not, say for instance } M < 0, \text{ then } A(u) = \exp \{f(t, x, du, u)\} \leq \exp[M_0 + \mu M]. \text{ Integrating this over } D \text{ yields: } \mu M \geq [\log(|D_0|/|D|) - M_0]/|D| \text{ and}
\]
\[
-m = \max \limits_{\bar{D}} |u| \leq \mu \delta(D) + [M_0 - \log(|D_0|/|D|)]/\mu.
\]
Similarly, \( m > 0 \) yields \( [\log(|D^*|/|D|) - m_0] > 0 \) and
\[
M = \max \limits_{\bar{D}} |u| \leq \mu \delta(D) + [\log(|D^*|/|D|) - m_0]/\mu.
\]
In any case, we obtain a \( C^1(D) \) \textit{a priori} bound on \( u \) in terms of \( |D^*|, \mu \delta(D) \), \( |D| \), \( \delta(D) \), \( |D_0| \), \( M_0, m_0 \) and \( \mu \).

For simplicity, let us give a unified treatment of higher order \( a \text{ priori} \) estimates for equations (1. t) and (2. t) by rewriting these equations into a single general form
\[
\log A(u) = \Gamma(t, x, du, u, \langle u \rangle).
\]
Let \( u \in S(D, D_t) \) solve (*). In this section, a constant will be said \textit{under control} provided it depends only on the following quantities: \( |u|_1 \), \textit{i.e.} the \( C^1(D) \)-norm of \( u \), on the \( C^2 \)-norm of \( \Gamma \) on
\[
K := [0, 1] \times \bar{D} \times \bar{D}^* \times 1 \times 1,
\]
where \( I = [-|u|_1, |u|_1] \), on the \( C^0([0, 1], C^2) \)-norm of \( t \to h_t \) (the fixed path of defining functions, \textit{cf. supra}), and on the \textit{positive} real
\[
\sigma := \min \limits_{t \in [0, 1]} \sigma(t)
\]
where \( \sigma(t) \) is the smallest eigenvalue of \([h_{ij}] \) over \( D_t \).

Since \( u \) is convex, a \( C^2(D) \) bound on \( u \) follows from a bound on
\[
M_2 := \max \limits_{(x, \theta) \in \bar{D} \times S} [u_{00}(x)]
\]
S standing for the unit sphere of \( \mathbb{R}^n \). Set \( H := h_t(du) \) and consider
\[
Q : (c, \theta, x) \in (0, \infty) \times S \times \bar{D} \to Q(c, \theta, x) = \log [u_{00}(x)] + c H(x).
\]
Proposition 2. — There exists $C \in (0, \infty)$ under control such that, if
\[
\max_{(\theta, x) \in S \times D} [Q(\theta, x)] \text{ occurs at } (z, x_0) \in S \times D \text{ with } x_0 \text{ interior to } D, \text{ then } \mu_2 \text{ is under control.}
\]

This proposition does not refer to any boundary condition and constitutes by no means an interior estimate (it is rather the type of argument suited on a compact manifold). A similar proposition (with $\Delta u$ and $|du|^2$, respectively in place of $u_{\theta \theta}$ and $H$) is lemma 2 of [13], later (and independently) reproved in [7] (p. 694); a similar argument is used in [4] (p. 398). Here proposition 2 may serve for the higher dimensional theorem, due to the special form of $Q$; so for completeness, we provide a detailed proof of it.

Proof. — Fix $(c, \theta) \in (0, \infty) \times S$ and consider $Q$ as a function of $x$ only. Let us record some auxiliary formulae: differentiating twice equation (*) in the $\theta$-direction yields,
\[
\begin{align*}
\frac{\partial^2 u}{\partial \theta^2} &= (\Gamma)_\theta \equiv \Gamma_{\theta \theta} + \Gamma_u u_{\theta \theta} + \Gamma_u u_{\theta i} u_{\theta j} \\
\frac{\partial^2 u}{\partial \theta^2} &= (\Gamma)_{\theta \theta} = (\Gamma)_{\theta \theta} + u_{\theta k} u_{\theta m} u_{\theta i} u_{\theta j}
\end{align*}
\]
with
\[
(\Gamma)_{\theta \theta} = \Gamma_{\theta i} u_{\theta i} + \Gamma_{\theta j} u_{\theta j} + 2(\Gamma_{u i} u_{\theta i} + \Gamma_{u j} u_{\theta j}) + \Gamma_{\theta i} u_{\theta i} + \Gamma_{\theta j} u_{\theta j} + \Gamma_{\theta i} u_{\theta j} + [\Gamma_{\theta \theta} + 2 \Gamma_{u i} + \Gamma_{\theta i} (u_\theta)^2].
\]

Differentiating twice $H$ yields (with the subscript $t$, of $h$, dropped),
\[
\begin{align*}
H_i &= h_{ik} u_{ik} \\
H_{ij} &= h_{ik} u_{ij} + h_{km} u_{ik} u_{jm}
\end{align*}
\]
and similarly for $Q$,
\[
Q_i = (u_{\theta \theta} / u_{\theta \theta}) + c H_i \\
Q_{ij} = (u_{\theta \theta} / u_{\theta \theta}) - [u_{\theta i} u_{\theta j} (u_{\theta \theta})^2] + c H_{ij}.
\]

Combining (8) with (5) and (7), we get
\[
\frac{\partial^2 u}{\partial \theta^2} = h_i (\Gamma_i + \Gamma_u u_i) + \Gamma_{ui} H_i + h_{ij} u_{ij}
\]
while from (6) we get,
\[
\frac{\partial^2 u}{\partial \theta^2} = [(\Gamma)_{\theta \theta} / u_{\theta \theta}] + (1 / u_{\theta \theta}) [u_{ij} u_{ij} u_{\theta 0 k m} - (1 / u_{\theta \theta}) u_{ij} u_{\theta 0 i} u_{\theta 0 j}] + cu_{ij} H_{ij}.
\]

Expanding the square
\[
(u_{\theta} u_{\theta i} - u_{\theta i} u_{\theta 0}) (u_{\theta} u_{\theta k m} - u_{\theta k} u_{\theta 0 m}) u_{ij} u_{im}
\]
one immediately verifies the identity:
\[
u_{ij} u_{\theta i} u_{\theta j} u_{\theta 0 k m} \geq (1 / u_{\theta \theta}) u_{ij} u_{\theta 0 i} u_{\theta 0 j}.
\]

So,
\[
u_{ij} Q_{ij} \geq [(\Gamma)_{\theta \theta} / u_{\theta \theta}] + cu_{ij} H_{ij}.
\]
Combining the expression of $(\Gamma)_{\theta \theta}$ with that of $Q_i$ and (9) yields,

$$u^j Q_{ij} - \Gamma_{ui} Q_i \geq c h_{ij} u_{ij} + (1/u_{\theta \theta}) \Gamma_{\theta \theta u_i u_\theta} u_{\theta i} u_{\theta j}$$

$$+ (2/u_{\theta \theta}) (\Gamma_{\theta i} + u_0 \Gamma_{u i}) u_{\theta i} + \Gamma_{ui} + c h_{ij} (\Gamma_{i} + \Gamma_{u} u_i)$$

$$+ (1/u_{\theta \theta}) [\Gamma_{\theta \theta} + 2 u_0 \Gamma_{\theta u} + \Gamma_{uu} (u_{\theta})^2]. \quad (10)$$

Introducing the constant $\sigma$ (defined above) we get

$$(1/u_{\theta \theta}) \Gamma_{\theta \theta u_i u_\theta} u_{\theta i} u_{\theta j} + \frac{1}{3} c h_{ij} u_{ij}$$

$$\geq (1/u_{\theta \theta}) u_{ik} u_{jm} \left( \theta^k \theta^m \Gamma_{u_{ij}} + \frac{1}{3} c \sigma u_{\theta \theta} \delta_{ij} u^{k m} \right)$$

$$\geq (1/u_{\theta \theta}) u_{\theta i} u_{\theta j} \left( \Gamma_{u_{ij}} + \frac{1}{3} c \sigma \delta_{ij} \right)$$

this last inequality being obtained by noting that, identically for $u$ strictly convex, $u_{\theta \theta} u^{k m} \geq \theta^k \theta^m$. Hence our first requirement on $c$ is:

$$\left( \Gamma_{u_{ij}} + \frac{1}{3} c \sigma \delta_{ij} \right) \geq 0$$

in the sense of symmetric matrices, over $K$. To express our second requirement on $c$, we first note that the inequality between the arithmetic and the geometric means of $n$ positive numbers applied to the eigenvalues of $(u_{ij})$ and combined with ($*$), yields on $D$:

$$\Delta u \geq n \exp \left( \frac{1}{n} \min_{\kappa} \Gamma \right) = : \gamma.$$

Then we take $c$ such that

$$2 \min \left[ \Gamma_{z \gamma} (r) + u_{\gamma} (x) \Gamma_{u u_{\gamma}} (r) \right] + \frac{1}{3} c \sigma \gamma \geq 0$$

the minimum being taken on $(r, x, y) \in K \times \bar{D} \times S$. From now on, $c$ has a fixed value under control, $C$, meeting both requirements and we take $(\theta, x) = (z, x_0)$ as defined in proposition 2. In particular, $u_{zz} (x_0)$ is now the maximum eigenvalue of $[u_{ij} (x_0)]$; diagonalizing the latter and using the second requirement on $C$, we obtain at $x_0$:

$$\frac{1}{3} C h_{ij} u_{ij} + (2/u_{zz}) (\Gamma_{zu} + u_z \Gamma_{uu}) u_{zi} \geq \frac{1}{3} C \sigma \Delta u + 2 (\Gamma_{zu} + u_z \Gamma_{uu}) \geq 0.$$

Now (10) yields for $x \rightarrow Q = Q (C, z, x)$ at $x_0$,

$$u^j Q_{ij} - \Gamma_{ui} Q_i \geq C \left( \frac{1}{3} \sigma u_{zz} - C' \right) - C'' (1/u_{zz}), \quad (11)$$

for some positive constants under control $C', C''$. Since $Q (C, z, \cdot)$ assumes its maximum at $x_0 \in D$, (11) implies a controlled bound from above on
hence also on \((\theta, x) \to Q(C, \theta, x)\) and on \((\theta, x) \to u_{00}(x)\). Therefore \(M_2\) is under control. \(\square\)

According to proposition 2, we may assume, without loss of generality, that the point \(x_0\) above lies on \(\partial D\), hence a \(C^2(\overline{D})\) \(a\ priori\) bound on \(u\) follows from an \(a\ priori\) bound on \(u_{zz}(x_0)\) which, in turn, coincides with \(\max_{(\theta, x) \in S \times \partial D} [u_{00}(x)]\).

### IV. A PRIORI ESTIMATES OF SECOND DERIVATIVES ON THE BOUNDARY \((n = 2)\)

In this section we fix a defining function of \(D\), denoted by \(k\), and we include in the definition of constants \(under control\) the possible dependence on \(|k|_A\), on \(\tau := \min k_N > 0\) and on the minimum over \(\overline{D}\) of the smallest \(\tau\) eigenvalue of \((k_{ij})\), denoted by \(s > 0\).

We still let \(u \in \mathcal{S}(D, D_t)\) solve equation \((\ast)\). According to proposition 1 \(H = h_t (du)\) which vanishes on \(\partial D\), satisfies there \(H_N > 0\); moreover, \((7)\) implies on \(\partial D\) (dropping the subscript \(t\) of \(h\)):

\[
h_i[du(x)] = H_N u^{ij} N^j(x). \quad (12)
\]

In particular, the function on \(\partial D\)

\[\varphi(x) := N^i(x) h_i[dv(x)]\]

is \(positive\). Fix an arbitrary point \(x_0 \in \partial D\) and a direct system of euclidean co-ordinates \((O, x^1, x^2)\) satisfying \(N(x_0) = \partial/\partial x^2\). Then \((12)\) reads at \(x_0\),

\[
\begin{align*}
u_{11}(x_0) &= (e^T/H_N) \varphi(x_0) \\
u_{12}(x_0) &= -(e^T/H_N)(x_0) h_1[du(x_0)]
\end{align*} \quad (13)
\]

while equation \((\ast)\) itself provides for \(u_{22}(x_0) = u_{NN}(x_0)\),

\[
\varphi u_{22}(x_0) = H_N(x) + (e^T/H_N)(x_0) \{ h_1[du(x_0)] \}^2. \quad (14)
\]

We thus need positive lower bounds under control on \(H_N(x_0)\) and \(\varphi(x_0)\), as well as a controlled upper bound on \(H_N(x_0)\).

Let us start with \(H_N(x_0)\). Aside from \((9)\), \(H\) also satisfies in \(D\) [still by combining \((8), (5), (7)\)],

\[
u^{ij} H_{ij} - u^{ij}(\Gamma_i H_j) = h_{ij} u_{ij}. \quad (15)
\]

Set \(T = u_{11} + u_{22}, T^* = u^{11} + u^{22}\), and note the identity: \(T^* = A(u) T\). It implies the existence of positive constants under control, \(\alpha, \beta\), such that

\[
\alpha T^* \leq T \leq \beta T^*, \quad (16)
\]

which we simply denote by: \(T \simeq T^*\). Consider the function

\[(c, x) \in (0, \infty) \times \overline{D} \to w(c, x) = H(x) - c k(x).\]
From (9) and $T \geq \gamma$ (cf. supra), we infer
\[
u^{ij}[w(\cdot, \cdot)]_{ij} \leq - T \left[ \frac{1}{2} c (s/\beta) - (u_{ij}/T) (h_{ij} + \Gamma_{ui} h_j) \right] - \left[ \frac{1}{2} \gamma c (s/\beta) - h_i (\Gamma_i + \Gamma_u u_i) \right],
\]
and there readily exists $c = C > 1$, under control, such that the latter right-hand side is non-positive. Similarly (15) (16) yield:
\[
u^{ij}[w(\cdot, \cdot)]_{ij} - u^{ij}(\Gamma)_{i}[w(\cdot, \cdot)]_{ij} \geq \frac{1}{2} \sigma T
\]
\[- T \max \left\{ 0, (c/\alpha) u^{ij}/T^* \right\} [k_{ij} - k_i (\Gamma_j + \Gamma_u u_j)] + \left[ \frac{1}{2} \sigma\gamma + ck_i \Gamma_u \right],
\]
($\sigma$ was defined at the beginning of section III) and there exists $c \in (0, 1)$ under control such that the right-hand side is nonnegative. Since $w$ identically vanishes on $(0, \infty) \times \partial D$, Hopf’s maximum principle [11] implies the following pinching under control on $\partial D$:
\[
\text{c} \tau \leq c k_N \leq H_N \leq C k_N \leq C \left| k \right|_1.
\]
Combined with (13), it implies a controlled upper bound on $|u_{11}(x_0)| + |u_{12}(x_0)|$. Furthermore, combined with (14), it implies also (the notation $\approx$ is defined at (16))
\[
u^{ij}_{22}(x_0) \approx 1/\varphi(x_0).
\]
We now turn to a lower bound on $\varphi(x_0)$ and consider the function
\[
(c, x) \in (0, \infty) \times \bar{D} \rightarrow P(c, x) = \psi - ck,
\]
where
\[
\psi(x) = k_i(x) h_i [du(x)].
\]
A routine computation using (5) yields in $D$:
\[
u^{ij}\psi_{ij} = k_i h_i (\Gamma_j + \Gamma_u u_j + \Gamma_{um} u_{jm}) + 2 k_i h_{ij} + k_i h_{ijm} u_{jm} + u^{ij} k_{ijm} h_m.
\]
It implies the existence of a constant $c_1$ under control such that, in $D$,
\[
u^{ij} P_{ij} \leq c_1 (1 + T) - c (s/\beta) T = - \left[ \frac{1}{2} c \gamma (s/\beta) - c_1 \right] - T \left[ \frac{1}{2} c (s/\beta) - c_1 \right];
\]
let us choose $c = C_0 = 2 c_1 \beta / s \min (1, \gamma)$, so that $u^{ij} [P(C_0, \cdot)]_{ij} \leq 0$ in $D$. By Hopf’s maximum principle [11], $P(C_0, \cdot)$ necessarily assumes its minimum over $\bar{D}$ at a boundary point $y_0$ where
\[
\psi_N \leq C_0 k_N.
\]
Pick a euclidean system of co-ordinates $(O, y^1, y^2)$ such that $N(y_0) = \partial / \partial y^2$. Then $dk(y_0) = k_N \partial / \partial y^2$, while, using (13) (17):
\[
\left| u_{12}(y_0) \right| \leq C_1 := e^{1 + T} |h|_{1/\varepsilon}
\]

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is under control, and (19) reads:
\[ u_{22}(y_0) k_N(y_0) h_{22}[du(y_0)] \leq C_0 k_N(y_0) - k_{21}(y_0) h_1[du(y_0)] - k_N(y_0) u_{12}(y_0) h_{12}[du(y_0)]. \]

It implies
\[ \sigma \gamma u_{22}(y_0) \leq C_0 |k_1| + |h_1| |k_2| + C_1 |k_1| |h_2| \]
i.e. a controlled bound from above on \( u_{22}(y_0) \). Recalling (18), it means a
controlled positive bound from below, \( \lambda \), on \( \varphi(y_0) \). Since on \( \partial D \), \( P(C_0, .) \equiv k_N \varphi \), and since \( P(C_0, .) \) assumes its minimum at \( y_0 \), we infer
on \( \partial D \):
\[ \varphi(x) \geq \lambda k_N(y_0)/k_N(x) \geq \lambda \tau / |k_1|. \]

Using (18) again, we obtain a controlled upper bound on \( u_{22}(x_0) \). The
second derivatives of \( u \) are thus a priori bounded on \( \partial D \).

Remarks. — 5. Proposition 1 and (12) show that the lower bound \( \varphi \geq \lambda \)
ensures a priori the uniform obliqueness of the boundary operator at \( u \).
Geometrically, it implies another positive lower bound on the scalar
product of the outward unit normals, to \( \partial D \) at \( x \) and to \( \partial D_t \) at \( du(x) \).

6. Let (T, N) and (T*, N*) be direct orthonormal moving frames on
\( \partial D \) and on \( \partial D_t \), respectively (N* stands for the outward unit normal on
\( \partial D_t \)) and let \( z_0 \) be a critical point of: \( x \in \partial D \rightarrow N(x) \cdot N*[du(x)] \). Denote
by \( J du \) the Jacobian (or differential) of the gradient mapping \( du \). With
the help of Frénet’s formulae, one verifies that
\[ |J du[T(z_0)]| = (R^*_0/R_0), \]
\( R_0 \) (resp. \( R^*_0 \)) standing for the curvature radius of \( \partial D \) at \( z_0 \) [resp. of \( \partial D_t \)
\( \partial du(z_0) \)], i.e. if we let \( R_0 \) go to infinity and \( R^*_0 \)
remain bounded ? From (20), \( |J du[T(z_0)]| \) goes to zero hence \( |J du[N(z_0)]| \)
go to infinity. In a direct system of euclidean co-ordinates \( (0, x^1, x^2) \)
such that \( N(z_0) = \partial / \partial x^2 \), it implies that \( |u_{11}(z_0)| + |u_{12}(z_0)| \) goes to zero
while \( |u_{22}(z_0)| \) blows up like \( R_0 \) i.e. the control on \( u_{NN}(z_0) \) is lost.

V. HIGHER ORDER A PRIORI ESTIMATES

Let \( u \in S(D, D_t) \) solve equation (*) . Fix a generic point \( x \in \bar{D} \) and choose
a euclidean co-ordinates system which puts \( [u_{ij}(x)] \) into a diagonal form.
Observe that for each \( i \in \{1, \ldots, n\} \),
\[
    u_{ii}(x) = \frac{A(u)}{\prod_{j \neq i} u_{jj}(x)} \geq \gamma/|u|_2^{n-1}.
\]  

(21)

In case \( n = 2 \), the \( C^2(\overline{D}) \) a priori estimate drawn on \( u \) in the two preceding sections thus implies the controlled uniform ellipticity of \( d[\log A(u)] \) on \( \overline{D} \). Given \( \alpha \in (0, 1) \), a \( C^{2, \alpha}(\overline{D}) \) a priori bound on \( u \) now follows from the general theory of [15] (section 6); however, this bound is so straightforward for \( n = 2 \) that we include it for completeness.

First of all, given any interior subdomain \( D' \) of \( D \) and any \( z \in S \), the 2-dimensional regularity theory of [17] applied to \( u_z \), which satisfies (5) in \( D' \), yields a \( C^{1, \alpha}(\overline{D}') \) a priori bound under control on \( u_z \), hence, since \( z \) is arbitrary, a controlled \( C^{2, \alpha}(\overline{D}') \) a priori bound on \( u \). The theory of [17] also applies to \( H \) which satisfies (9) in \( D \) and vanishes on \( \partial D \): it yields a \( C^{1, \alpha}(\overline{D}) \) a priori bound under control on \( H \). Solving for \( u_{11}, u_{12} \) and \( u_{22} \), the \( 3 \times 3 \) system given by (7) and equation (\(*\)), we get (dropping the subscript \( t \) of \( h \)):
\[
    \begin{align*}
        u_{11} &= (H_1^2 + (h_2)^2 e_1^2)/\Delta \\
        u_{12} &= (H_1 H_2 - h_1 h_2 e_1^2)/\Delta \\
        u_{22} &= (H_2^2 + (h_1)^2 e_1^2)/\Delta,
    \end{align*}
\]  

(22)

where
\[
    \Delta(x) = H_I(x) h_I[du(x)].
\]

Note that (7) and (21) imply
\[
    \Delta(x) \geq (\gamma/|u|_2) |dH[du(x)]|^2.
\]  

(23)

Given any small enough \( \delta \in (0, 1) \), let
\[
    D_\delta := \{ x \in D, \text{dist}(x, \partial D) < \delta \}.
\]

From the \( C^2(\overline{D}) \) a priori estimate precedingly drawn on \( u \), it follows that the gradient image \( du(D_\delta) \) is contained in \( (D_\delta)_{C_6} \) for some positive constant \( C \) under control. If \( \tau^* := \min_{t \in [0, 1]} (\min_{[\partial D]} |dh_t|) \),
then there readily exists \( \delta_0 \in (0, 1) \) under control such that, for any \( x \in D_{\delta_0} \), \( |dh_t[du(x)]| \geq \tau^*/2 \). Therefore (22) and (23) imply a \( C^\alpha(D_{\delta_0}) \) a priori bound under control on the second derivatives of \( u \). A \( C^{2, \alpha}(\overline{D}) \) a priori bound on \( u \) follows.

Actually, a straightforward "bootstrap" argument now provides \( C^{k, \alpha}(\overline{D}) \) a priori bounds on \( u \) for each integer \( k > 2 \).

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