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Closed orbits of fixed energy for a class of N-body problems (*)

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ABSTRACT. — We prove the existence of periodic solutions with prescribed energy for a class of N-body type problems.

Key words : Singular Hamiltonian systems, N-body problem, critical point theory.

RÉSUMÉ. — Nous démontrons l'existence de solutions périodiques à énergie fixée pour une classe de problèmes de type N-corps.

1. MAIN RESULTS

The aim of this paper is to prove the existence of periodic solutions with prescribed energy for a class of second order Hamiltonian systems, including the N-body problem. Precisely, we set $\Omega = \mathbf{R}^k \setminus \{0\}$ and consider

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a potential V of the form

$$V(x) = V(x_1, \dots, x_N) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}(x_i - x_j) \tag{1.1}$$

where $x_i \in \mathbf{R}^k$, $x = (x_1, \dots, x_N) \in \mathbf{R}^{Nk}$ and $V_{ij} \in C^1(\Omega, \mathbf{R})$ ($i, j = 1, \dots, N$). Given $m_i > 0$ ($i = 1, \dots, N$) and $h \in \mathbf{R}$, we seek for periodic solutions of

$$(\text{Ph}) \quad \begin{cases} m_i x_i'' + \nabla_{x_i} V(x_1, \dots, x_N) = 0 & (1 \leq i \leq N) & (\text{Ph. 1}) \\ \frac{1}{2} \sum_i m_i |x_i'(t)|^2 + V(x_1(t), \dots, x_N(t)) = h & & (\text{Ph. 2}) \end{cases}$$

Here ∇ (resp. ∇_{x_i}) denotes the gradient (resp. the gradient with respect x_i). We will use the notation $x \cdot y$, or simply xy (resp. $|x|$) to denote the Euclidean scalar product of any two vectors $x, y \in \mathbf{R}^m$ (resp. the Euclidean norm of x).

We assume $V(x)$ is in the form (1.1) with V_{ij} satisfying:

- (V1) $V_{ij}(\xi) = V_{ji}(\xi), \forall \xi \in \Omega$;
- (V2) $\exists \alpha \in [1, 2[$ such that $\nabla V_{ij}(\xi) \cdot \xi \geq -\alpha V_{ij}(\xi) > 0, \forall \xi \in \Omega$;
- (V3) $\exists \delta \in]0, 2[$ and $r > 0$ such that $\nabla V_{ij}(\xi) \cdot \xi \leq -\delta V_{ij}(\xi)$ for all $0 < |\xi| \leq r$;
- (V4) $V_{ij}(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$.

Remarks. – For future references let us note explicitly some consequences of the preceding assumptions. First of all, (V2)-(V3) imply, respectively:

$$V_{ij}(\xi) \leq -\frac{c_1}{|\xi|^\alpha}, \quad \forall |\xi| > 0 \tag{1.2}$$

$$V_{ij}(\xi) \geq -\frac{c_2}{|\xi|^\delta}, \quad \forall 0 < |\xi| \leq r \tag{1.3}$$

Here and always in the sequel c, c_1, c_2 , etc. denote positive constants.

Moreover, since $\nabla V(x) \cdot x = -\frac{1}{2} \sum_{i \neq j} \nabla V_{ij}(x_i - x_j)(x_i - x_j)$, then from (V2)-(V3) it follows:

$$\nabla V(x) \cdot x \geq -\alpha V(x) > 0, \quad \forall x = (x_1, \dots, x_N), \quad x_i \neq x_j; \tag{1.4}$$

$$\nabla V(x) \cdot x \leq -\delta V(x), \quad \forall x = (x_1, \dots, x_N), \quad 0 < |x_i - x_j| \leq r. \tag{1.5}$$

By a solution of (Ph) we mean an $x(t) = (x_i(t))_{1 \leq i \leq N}$ such that x is periodic with period $T > 0$ and for all $i, j = 1, \dots, N$ there results

- (i) $x_i \in H^{1,2}(0, T; \mathbf{R}^k)$;
- (ii) the set $\mathcal{C} = \{t \in [0, T] : x_i(t) = x_j(t)\}$ has measure zero;
- (iii) x_i is C^2 on $[0, T] \setminus \mathcal{C}$ and satisfies (Ph. 1)-(Ph. 2) therein.

A solution x such that $\mathcal{C} \neq \emptyset$ (resp. $= \emptyset$) is called a *collision* (resp. *non-collision*). We anticipate that our solutions are possibly collisions, found as limit of non-collisions.

The main results of this paper are:

THEOREM A. — *Suppose (V1)-(V4) hold. Then for all $h < 0$ problem (Ph) has a periodic solution.*

THEOREM B. — *Suppose V satisfies (V1), (V3), (V4) and (V2') $\exists \alpha \in]0, 2[$ such that $\nabla V_{ij}(\xi) \xi \geq -\alpha V_{ij}(\xi) > 0, \forall \xi \in \Omega;$
(V5) $V_{ij} \in C^2(\Omega, \mathbf{R})$ and $3 \nabla V_{ij}(\xi) \xi + V''_{ij}(\xi) \xi \cdot \xi > 0.$*

Then for all $h < 0$ (Ph) has a periodic solution.

It is worth pointing out that Theorems A and B above cover the case of the N-body problem, namely when $V_{ij}(\xi) = -\frac{m_i m_j}{|\xi|}, x \in \mathbf{R}^3$, and (Ph. 1) is nothing but the equation of motion of N bodies in \mathbf{R}^3 of position x_1, \dots, x_N and masses m_1, \dots, m_N subjected to their mutual gravitational attraction. In fact, it is immediate to verify that the potentials $V_{ij}(\xi) = -\frac{m_i m_j}{|\xi|}$ satisfy both the assumptions (V1)-(V4) with $\alpha = \delta = 1$, as well as (V5).

Theorems A and B must be related with the results of [1] where problem (Ph) has been studied for potentials of the form $V(x) \cong -\frac{1}{|x|^\alpha}, \alpha > 0$. Actually, Theorem B extends Theorem 4.12 of [1] to problems of the N-body type under quite similar assumptions, in particular (V2') and (V5). On the contrary, in Theorem A we eliminate (V5) but require that (V2) holds for $\alpha \geq 1$.

Both the proofs of theorem A and B are based upon critical point theory. In the latter we employ the same techniques of [1]: roughly, (V5) allows us to find solutions of (Ph) looking for critical points of a functional f constrained on a suitable manifold M , where the Palais-Smale condition (PS) holds true.

The proof of Theorem A is more direct and relies on an application of the Mountain-Pass theorem to f . Actually, when (V2') is substituted by the stronger (V2) it is possible to prove that (PS) holds for f without constraints. An example shows that indeed the lack of (PS) arises when $V_{ij}(\xi) = -|\xi|^{-\alpha}$ with $\alpha < 1$.

Existence of periodic solutions with prescribed period for some classes of N-body problems has been proved in [3], [4], [5]. On the contrary, we do not know any result *in the large* concerning the existence of trajectories with prescribed energy.

2. APPROXIMATE PROBLEMS

Let us introduce the following notation:

$$\begin{aligned}
 H &= H^{1,2}(\mathbb{S}^1, \mathbb{R}^k) \\
 H_{\#} &= \left\{ u \in H : u\left(t + \frac{1}{2}\right) = -u(t) \right\} \\
 E &= \left\{ u = (u_1, \dots, u_N) : u_i \in H_{\#} \ (i=1, \dots, N) \right\} \\
 \Lambda_0 &= \left\{ u \in E : u_i(t) \neq u_j(t), \forall t, i \neq j \right\} \\
 (u|v) &= \int u'v', \quad \|u\|^2 = \int |u'|^2 \quad (u, v \in H_{\#}).
 \end{aligned}$$

Here and always in the sequel \int stands for $\int_0^1 dt$. It is well known that $\|u_i\|$ is a norm on $H_{\#}$ equivalent to the usual one and one has:

$$\|u_i\| \geq 4 \|u_i\|_{\infty}$$

As an immediate consequence, for all $u = (u_1, \dots, u_N) \in E$ setting

$$\|u\|_E^2 = \sum_i m_i \|u_i\|^2$$

there results

$$\|u\|_E \geq c |u(t)|, \quad \forall t \tag{2.1}$$

Define the following functionals on Λ_0 :

$$f(u) = \frac{1}{2} \|u\|_E^2 \cdot \int [h - V(u)]$$

Formally, it is known (cf. [1], see also Lemma 2 below) that critical points of f on Λ_0 give rise, after a rescaling of time, to periodic solutions of (Ph). Actually, since Λ_0 is an open subset of E , critical point theory cannot be employed directly. A device to overcome this problem has been used in [1] (see also [3], [5]) and consists in substituting V with

$$V_{\varepsilon}(x) = V(x) - \varepsilon W(x), \quad W(x) = \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^2} \quad (\varepsilon > 0)$$

Note that from (1.4) it follows:

$$\nabla V_{\varepsilon}(x) \cdot x = \nabla V(x) \cdot x + 2\varepsilon W(x) > 0 \tag{2.2}$$

Let us set $f_{\varepsilon}(u) = \frac{1}{2} \|u\|_E^2 \cdot \int [h - V_{\varepsilon}(u)]$. Since $h - V_{\varepsilon}(u) > h + \varepsilon W(u)$, one can show (see, for ex. [5]) that f_{ε} is suitable for the critical point theory because there results

$$u_n \rightarrow u, \text{ weakly in } E, \quad \text{and} \quad u \in \partial\Lambda_0 \Rightarrow \int V_{\varepsilon}(u_n) \rightarrow -\infty \tag{2.3}$$

The procedure to find solutions of (Ph) will consists in two steps: first, critical points of f_ε are found, giving rise to solutions x_ε of corresponding approximate problems; second, we show that x_ε converge, as $\varepsilon \rightarrow 0$, to a solution of (Ph).

Let us start with:

LEMMA 1. — For any $\varepsilon > 0$, let $u_\varepsilon \in \Lambda_0$ be such that $f'_\varepsilon(u_\varepsilon) = 0$ and $\|u_\varepsilon\| > 0$ and set

$$\omega_\varepsilon^2 = \frac{\int \nabla V_\varepsilon(u_\varepsilon) u_\varepsilon}{\|u_\varepsilon\|_E^2} > 0. \tag{2.4}$$

Then $x_\varepsilon(t) := u_\varepsilon(\omega_\varepsilon t)$ is a non-collision solution of

$$m_i x_i'' + \nabla_{x_i} V_\varepsilon(x_1, \dots, x_N) = 0 \tag{Ph. 1 \varepsilon}$$

$$\frac{1}{2} \sum_i m_i |x_i'(t)|^2 + V_\varepsilon(x_1(t), \dots, x_N(t)) = h \tag{Ph. 2 \varepsilon}$$

Proof. — The proof is similar to that of Lemma 2.3 of [1] and therefore we will be sketchy. From $f'_\varepsilon(u_\varepsilon) = 0$ it follows:

$$\|u_\varepsilon\|_E^2 \int [h - V_\varepsilon(u_\varepsilon)] - \frac{1}{2} \|u_\varepsilon\|_E^2 \int \nabla V_\varepsilon(u_\varepsilon) u_\varepsilon = 0$$

and hence [cf. (2.2)]:

$$\int [h - V_\varepsilon(u_\varepsilon)] = \frac{1}{2} \int \nabla V_\varepsilon(u_\varepsilon) u_\varepsilon > 0 \tag{2.5}$$

Moreover $u_\varepsilon = (u_{\varepsilon,i})_{1 \leq i \leq N}$ satisfies:

$$\sum_i m_i \int u'_{\varepsilon,i} v_i \cdot \int [h - V_\varepsilon(u_\varepsilon)] - \frac{1}{2} \|u_\varepsilon\|_E^2 \int \nabla V_\varepsilon(u_\varepsilon) v = 0$$

$$\forall v = (v_1, \dots, v_N) \in E$$

and hence, dividing by $\frac{1}{2} \|u_\varepsilon\|_E^2$ and using (2.5):

$$\omega_\varepsilon^2 \sum_i m_i \int u'_{\varepsilon,i} v_i - \int \nabla V_\varepsilon(u_\varepsilon) v = 0, \quad \forall v = (v_1, \dots, v_N) \in E \tag{2.6}$$

Next, since $V_{ij}(x) = V_{ji}(x)$, one shows as in [5], Thm. 1.1, that (2.6) holds not only for all $v \in E$ but also for all $v \in H^N = H \times H \times \dots \times H$ (N-times). Thus u_ε satisfies

$$\omega_\varepsilon^2 m_i u'_{\varepsilon,i} + \nabla_{x_i} V_\varepsilon(u_\varepsilon) = 0 \tag{2.7}$$

Rescaling the time, one finds that $x_\varepsilon(t) = u_\varepsilon(\omega_\varepsilon t)$ satisfies (Ph. 1 \varepsilon). Integrating (2.7) the conservation of the energy (Ph. 2 \varepsilon) holds, too. ■

3. EXISTENCE OF CRITICAL POINTS OF f_ε

Critical points of f_ε on Λ_0 will be found by means of the Mountain-Pass Theorem. Let us begin proving:

LEMMA 2. — *There exist $\rho, \beta > 0$ such that*

- (i) $f_\varepsilon(u) \geq \beta$ for all $\varepsilon > 0$ and all $u \in \Lambda_0, \|u\|_E = \rho$;
- (ii) there exist $\varepsilon_0 > 0, u_0, u_1 \in \Lambda_0$ with $\|u_0\|_E < \rho < \|u_1\|_E$, such that $f_\varepsilon(u_0), f_\varepsilon(u_1) < \beta, \forall 0 < \varepsilon \leq \varepsilon_0$.

Proof. — First of all let us remark that from (1.2) it follows

$$-V(x) = -\frac{1}{2} \sum_{i \neq j} V_{ij}(x_i - x_j) \geq \frac{c_1}{2} \sum_{i \neq j} \frac{1}{|x_i - x_j|^\alpha} \geq \frac{c_2}{|x|^\alpha}, \left. \begin{array}{l} \forall 0 < |x_i - x_j| \leq r \end{array} \right\} \quad (3.1)$$

Using (3.1) jointly with (2.1) one deduces:

$$f_\varepsilon(u) \geq \frac{1}{2} \|u\|_E^2 \int [h - V(u)] \geq \frac{1}{2} \|u\|_E^2 \int \left[h + \frac{c_2}{|u|^\alpha} \right] \geq \frac{h}{2} \|u\|_E^2 + c_3 \|u\|_E^{2-\alpha}$$

proving (i).

To complete the proof we take $u = (u_1(t), \dots, u_N(t))$, with

$$u_i(t) = \xi \cos\left(2\pi\left(t + \frac{i}{N}\right)\right) + \eta \sin\left(2\pi\left(t + \frac{i}{N}\right)\right) \quad (i = 1, \dots, N)$$

where $\xi, \eta \in \mathbf{R}^k$ satisfy: $|\xi| = |\eta| = 1, \xi\eta = 0$.

For $R > 0$ we consider

$$f_\varepsilon(Ru) = \frac{1}{2} R^2 \|u\|_E^2 \int [h - V_\varepsilon(Ru)]$$

Note that $|u_i(t) - u_j(t)| = a_{ij}$ is independent on t and hence

$$\sum_{i < j} \frac{1}{|u_i(t) - u_j(t)|^2} = c_4.$$

From this it follows:

$$f_\varepsilon(Ru) = \frac{1}{2} R^2 \|u\|_E^2 \int \left[h - V(Ru) + \frac{c_4 \varepsilon}{R^2} \right]$$

Since $|Ru_i(t) - Ru_j(t)| = R a_{ij}$, then $h < 0$ and (V4) imply

$$\limsup \left[h - \int V(Ru) \right] < 0$$

and hence $f_\varepsilon(Ru) \rightarrow -\infty$ as $R \rightarrow \infty$, proving the existence of $u_i \in \Lambda_0$, such that $\|u_1\| > \rho$ and $f_\varepsilon(u_1) < \beta$.

Lastly, let $R > 0$ be small enough and recall that $|u_i(t) - u_j(t)| = a_{ij}$ is constant. Then using (1.3) one finds

$$-V(Ru(t)) \leq \frac{c_5}{R^\delta} \sum_{i < j} a_{ij}^{-\delta} \leq \frac{c_6}{R^\delta}$$

Hence

$$f_\varepsilon(Ru) \leq \frac{h}{2} R^2 \|u\|_E^2 + c_7 R^{2-\delta} + c_8 \varepsilon$$

Since $0 < \delta < 2$, then the existence of $\varepsilon_0 > 0$ and u_0 satisfying (ii) follows. ■

Next, we investigate the Palais-Smale [in short (PS)] condition. For this, some lemmas are in order.

LEMMA 3. — Let $u_n \in \Lambda_0$ be such that

$$(*) \quad \begin{cases} f_\varepsilon(u_n) \leq c \\ f'_\varepsilon(u_n) \rightarrow 0. \end{cases}$$

Then $\|u_n\|_E \leq c'$.

Proof. — Since $f(u) \leq f_\varepsilon(u)$, from $f_\varepsilon(u_n) \leq c$ we infer

$$-\frac{1}{2} \|u_n\|_E^2 \int V(u_n) \leq c - \frac{1}{2} h \|u_n\|_E^2 \tag{3.2}$$

Setting $\sigma_{\varepsilon, n} = \sigma_n = (f'_\varepsilon(u_n) | u_n)$ one has:

$$\sigma_n = \|u_n\|_E^2 \int \left[h - V_\varepsilon(u_n) - \frac{1}{2} \nabla V_\varepsilon(u_n) u_n \right]$$

Using (1.4) we deduce:

$$\begin{aligned} \sigma_n &= \|u_n\|_E^2 \int \left[h - V(u_n) - \frac{1}{2} \nabla V(u_n) u_n \right] \\ &\leq \|u_n\|_E^2 \int \left[h - \left(1 - \frac{\alpha}{2}\right) V(u_n) \right]. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) it follows

$$\sigma_n \leq h \|u_n\|_E^2 + \left(1 - \frac{\alpha}{2}\right) (2c - h \|u_n\|_E^2) = \frac{\alpha}{2} h \|u_n\|_E^2 + c_1$$

and thus

$$-\frac{\alpha}{2} h \|u_n\|_E^2 \leq c_2 + \|f'_\varepsilon(u_n)\| \|u_n\|_E.$$

Since h is negative we infer $\|u_n\|_E \leq c'$. ■

LEMMA 4. — Let u_n be a sequence satisfying (*). If $|u_n|_\infty \rightarrow 0$ then $\limsup f_\varepsilon(u_n) \leq 0$.

Proof. – Let us set

$$r_n = \min \{ |u_n(t)| : 0 \leq t \leq 1 \}, \quad R_n = \max \{ |u_n(t)| : 0 \leq t \leq 1 \}.$$

We claim that $R_n/r_n \leq c_1$. To see this we argue by contradiction. Suppose that (without relabeling) $\frac{R_n}{r_n} \rightarrow \infty$, and let t_n and s_n be such that $R_n = |u_n(t_n)|$ and $r_n = |u_n(s_n)|$. One has

$$\begin{aligned} \log \frac{R_n}{r_n} &= \log \frac{|u_n(t_n)|}{|u_n(s_n)|} = \int_{s_n}^{t_n} \frac{d}{d\tau} \log |u_n(\tau)| \leq \int_{s_n}^{t_n} \frac{|u'_n|}{|u_n|} \\ &\leq \left[\int |u'_n|^2 \right]^{1/2} \left[\int \frac{1}{|u_n|^2} \right]^{1/2} \leq c_2 \|u_n\|_E \left[\int \frac{1}{|u_n|^2} \right]^{1/2}. \end{aligned}$$

Since $\log \frac{R_n}{r_n} \rightarrow \infty$, then

$$\|u_n\|_E \left[\int \frac{1}{|u_n|^2} \right]^{1/2} \rightarrow \infty \tag{3.4}$$

Furthermore, from $|u_n|_\infty \rightarrow 0$ and (3.1) it follows $\int h - V(u_n) \rightarrow \infty$. In particular, $\int [h - V(u_n)] > 0$ for n large and hence, using (3.4) we infer

$$f_\varepsilon(u_n) = \frac{1}{2} \|u_n\|_E^2 \cdot \int [h - V(u_n) + \varepsilon W(u_n)] \geq \frac{\varepsilon}{2} \|u_n\|_E^2 \int \frac{1}{|u_n|^2} \rightarrow \infty,$$

a contradiction with $f_\varepsilon(u_n) \leq c$, proving the claim.

Next, let us set

$$\begin{aligned} \gamma_n &= - \int V(u_n) \\ A_n &= \frac{1}{2} \|u_n\|_E^2 [h + \gamma_n] \\ B_n &= \|u_n\|_E^2 \int W(u_n) \end{aligned}$$

From [see (3.2)]

$$\sigma_n = \|u_n\|_E^2 \left[h + \gamma_n - \frac{1}{2} \int \nabla V(u_n) u_n \right] \tag{3.5}$$

it follows that

$$A_n = \frac{1}{2} \frac{\sigma_n}{\left[h + \gamma_n - (1/2) \int \nabla V(u_n) u_n \right]} [h + \gamma_n].$$

Using (1.5) one has $\int \nabla V(u_n) u_n \leq \delta \gamma_n$ and hence

$$A_n \leq \frac{1}{2} \frac{\sigma_n [h + \gamma_n]}{[h + (1 - (\delta/2)) \gamma_n]}$$

Since $\sigma_n \rightarrow 0$ and $\gamma_n \rightarrow \infty$ then $\limsup A_n \leq 0$.

To estimate B_n we use again (3.1) and (3.5) yielding, respectively:

$$\begin{aligned} \left[h + \gamma_n - \frac{1}{2} \int \nabla V(u_n) u_n \right] &> h + \left(1 - \frac{\delta}{2} \right) \gamma_n > h + c_3 \int |u_n|^{-\alpha} \quad (> 0) \\ \|u_n\|_E \left[h + \gamma_n - \frac{1}{2} \int \nabla V(u_n) u_n \right] &\leq \|f'_\varepsilon(u_n)\| \end{aligned}$$

These two inequalities imply

$$\|u_n\|_E \leq \frac{\|f'_\varepsilon(u_n)\|}{h + c_3 \int |u_n|^{-\alpha}}$$

and hence

$$B_n \leq \|f'_\varepsilon(u_n)\|^2 \frac{c_4 \int |u_n|^{-2}}{\left(h + c_3 \int |u_n|^{-\alpha} \right)^2}$$

From $r_n \leq |u_n(t)| \leq R_n$ we deduce

$$B_n \leq \|f'_\varepsilon(u_n)\|^2 \frac{c_4 r_n^{-2}}{(h + c_3 R_n^{-\alpha})^2}$$

Since $R_n/r_n \leq c_1$, $\alpha \geq 1$ and $\|f'_\varepsilon(u_n)\| \rightarrow 0$, it follows that $B_n \rightarrow 0$. Finally, from

$$f_\varepsilon(u_n) = A_n + \frac{\varepsilon}{2} B_n$$

we infer that $\limsup f_\varepsilon(u_n) \leq 0$. This completes the proof of the lemma. ■

We are now in position to prove:

LEMMA 5. — *The functional f_ε satisfies:*

(PS⁺) *If $u_n \in \Lambda_0$ is such that $0 < \beta \leq f_\varepsilon(u_n) \leq c$, and $f'_\varepsilon(u_n) \rightarrow 0$, then (up to a subsequence) $u_n \rightarrow u^* \in \Lambda_0$.*

Proof. — From lemma 3 it follows that $\|u_n\|_E \leq c'$ and $\exists u^* \in E$ such that (up to a subsequence) $u_n \rightarrow u^*$, weakly and uniformly in $[0, 1]$. From lemma 4 we infer that $u^* \neq 0$, otherwise $\limsup f_\varepsilon(u_n) \leq 0$, in contradiction with $f_\varepsilon(u_n) \geq \beta > 0$. If $u^* \in \partial \Lambda_0$, then (2.3) implies $h - \int V_\varepsilon(u_n) \rightarrow +\infty$. This

and (3.6) would contradict $f_\varepsilon(u_n) \leq c$, proving that $u^* \in \Lambda_0$. Hence:

$$\liminf \|u_n\|_E \geq \|u^*\|_E > 0 \tag{3.6}$$

as well as

$$V(u_n) \rightarrow V(u^*), \quad W(u_n) \rightarrow W(u^*), \quad \nabla V(u_n)u_n \rightarrow \nabla V(u^*)u^* \tag{3.7}$$

Moreover from

$$\sigma_n = \|u_n\|_E^2 \int \left[h - V_\varepsilon(u_n) - \frac{1}{2} \nabla V_\varepsilon(u_n)u_n \right]$$

we infer

$$\int [h - V_\varepsilon(u_n)] = \frac{1}{2} \int \nabla V_\varepsilon(u_n)u_n + \frac{\sigma_n}{\|u_n\|_E^2} \tag{3.8}$$

Taking into account (3.6), (3.7) and since $\sigma_n \rightarrow 0$ we can pass to the limit into (3.8) yielding

$$\int [h - V_\varepsilon(u_n)] \rightarrow \frac{1}{2} \int \nabla V_\varepsilon(u^*)u^* > 0 \tag{3.9}$$

Finally, from $f'_\varepsilon(u_n) \rightarrow 0$ it follows:

$$(u_n | v) \int [h - V_\varepsilon(u_n)] - \frac{1}{2} \|u_n\|_E^2 \int \nabla V_\varepsilon(u_n)v \rightarrow 0, \quad \forall v \in H^N$$

Then (3.9) and $\int \nabla V_\varepsilon(u_n)v \rightarrow \int \nabla V_\varepsilon(u^*)v$ imply that $u_n \rightarrow u^*$ strongly in E. ■

LEMMA 6. — *Let (V1)-(V4) hold. Then $\exists \varepsilon_0 > 0$ such that $\forall 0 < \varepsilon \leq \varepsilon_0$ there is $u_\varepsilon \in \Lambda_0$ such that $f'_\varepsilon(u_\varepsilon) = 0$. Moreover $\exists a, b > 0$ such that $0 < a \leq \|u_\varepsilon\|_E \leq b$, $\forall 0 < \varepsilon \leq \varepsilon_0$.*

Proof. — Lemmas 2 and 5 allow us to apply the Mountain-Pass Theorem [2] yielding a critical point $u_\varepsilon \in \Lambda_0$ of f_ε . From the min-max characterization of $f_\varepsilon(u_\varepsilon)$ it follows:

$$f_\varepsilon(u_\varepsilon) \leq \max_{R > 0} f_\varepsilon(Ru) \leq \max_{R > 0} f_{\varepsilon_0}(Ru) \equiv c. \tag{3.10}$$

Since $f'_\varepsilon(u_\varepsilon) = 0$, then the arguments of lemma 3 imply the existence of $b > 0$ such that $\|u_\varepsilon\|_E \leq b$. Furthermore from (2.5) we infer readily

$$h = \int \left[V_\varepsilon(u_\varepsilon) + \frac{1}{2} \nabla V_\varepsilon(u_\varepsilon)u_\varepsilon \right] = \int \left[V(u_\varepsilon) + \frac{1}{2} \nabla V(u_\varepsilon)u_\varepsilon \right].$$

If $\|u_\varepsilon\|_E \rightarrow 0$ as $\varepsilon \rightarrow 0$, then $\|u_\varepsilon\|_\infty \rightarrow 0$ and (1.4) implies

$$h \leq \left(1 - \frac{\delta}{2}\right) \int V(u_\varepsilon), \tag{3.11}$$

while (2.7) yields $\int V(u_\varepsilon) \rightarrow -\infty$. This and (3.11) led to a contradiction, proving the lemma. ■

4. PROOF OF THEOREM A

Let u_ε the Mountain-Pass critical point of f_ε given by Lemma 6. Since $\|u_\varepsilon\|_E > 0$, from Lemma 1 it follows that $x_\varepsilon(t) = u_\varepsilon(\omega_\varepsilon t)$ is a solution of (Ph.1 ε)-(Ph.2 ε). Furthermore, again from lemma 6 one has that $\|u_\varepsilon\|_E \leq b$, and $u_\varepsilon \rightarrow u$ ($\varepsilon \rightarrow 0$) uniformly in $[0, 1]$. In order to show that $u = (u_i)_{1 \leq i \leq N}$ gives rise to a solution of (Ph) we follow the same procedure as in [1]. For completeness we outline these arguments referring to [1] for more details. First, one proves that

(i) $\exists t : V(u(t)) \neq h$.

In fact, otherwise, $V(u(t)) \equiv h$, hence $u \in \Lambda_0$ and $V(u_\varepsilon) \rightarrow V(u)$, $\nabla V(u_\varepsilon)u_\varepsilon \rightarrow \nabla V(u)u$, uniformly in $[0, 1]$. Then

$$h = \int V(u_\varepsilon) + \frac{1}{2} \nabla V(u_\varepsilon)u_\varepsilon \rightarrow \int V(u) + \frac{1}{2} \nabla V(u)u = h + \frac{1}{2} \int \nabla V(u)u$$

implies $\int \nabla V(u)u = 0$, a contradiction because $\nabla V(x)x > 0$.

Next, one shows:

(ii) $\exists t : u_i(t) \neq u_j(t)$ for some $i \neq j$.

Otherwise, the components $u_{\varepsilon, i}$ of u_ε are such that $|u_{\varepsilon, i} - u_{\varepsilon, j}| \rightarrow 0$ uniformly in $[0, 1]$ for all i, j and (1.2) implies $\int V(u_\varepsilon) \rightarrow -\infty$. On the other side, using (1.5) one finds

$$h = \int V(u_\varepsilon) + \frac{1}{2} \nabla V(u_\varepsilon)u_\varepsilon \leq \left(1 - \frac{\delta}{2}\right) \int V(u_\varepsilon),$$

a contradiction.

Next, we claim that for the ω_ε given by (2.4) the following estimate holds:

(iii) $\exists 0 < \Omega_0 < \Omega_1$ such that $\Omega_0 \leq \omega_\varepsilon \leq \Omega_1$.

To prove this fact, let us take a closed interval $I \subset [0, 1]$, with measure $|I| > 0$, such that $u_i(t) \neq u_j(t)$, $V(u(t)) \neq h$, $\forall t \in I$. Such an interval exists because of (i) and (ii) above. Since $h - V_\varepsilon(u_\varepsilon) = \frac{1}{2} \nabla V_\varepsilon(u_\varepsilon)u_\varepsilon > 0$ and

$\|u_\varepsilon\|_E \leq b$, it follows

$$\omega_\varepsilon^2 = \frac{\int \nabla V_\varepsilon(u_\varepsilon) u_\varepsilon}{\|u_\varepsilon\|_E^2} = \frac{2 \int h - V_\varepsilon(u_\varepsilon)}{\|u_\varepsilon\|_E^2} \geq \frac{2 \int_I h - V_\varepsilon(u_\varepsilon)}{b^2} \tag{4.1}$$

Furthermore, from $V_\varepsilon(u_\varepsilon(t)) \rightarrow V(u(t))$ (uniformly on I), $h - V_\varepsilon(u_\varepsilon) > 0$ and (i) it follows that $h - V(u) > 0$ on I . Then, taking also into account that $|I| > 0$, we infer:

$$\omega_\varepsilon^2 \geq \frac{2 \int_I [h - V_\varepsilon(u_\varepsilon)]}{b^2} \rightarrow \frac{2 \int_I [h - V(u)]}{b^2} > 0 \tag{4.2}$$

From (4.1) and (4.2) it follows immediately that $\omega_\varepsilon \geq \Omega_0 > 0$.

In a similar way, using lemma 6 and (3.10) we find:

$$\omega_\varepsilon^2 = \frac{2 \int h - V_\varepsilon(u_\varepsilon)}{\|u_\varepsilon\|_E^2} = \frac{4 f_\varepsilon(u_\varepsilon)}{\|u_\varepsilon\|_E^4} \leq \frac{4c}{a^4} \equiv \Omega_1^2.$$

As a consequence of (iii) one has that $\omega_\varepsilon \rightarrow \omega$. Letting $x(t) = u(\omega t)$, a standard argument shows that x solves (Ph) (see the proof of theorem 4.12 of [1] and [5]). This completes the proof of the theorem A. ■

5. PROOF OF THEOREM B

The proof of Theorem B requires different arguments, because when (V2) is replaced by the weaker (V2') the (PS⁺) condition can fail (see Example below). The difficulty can be overcome, as in [1], by looking for critical points of f_ε constrained on a suitable manifold.

Referring to [1] for more details, let us outline the proof.

Set $g(u) := \int \left[V(u) + \frac{1}{2} \nabla V(u) u \right]$ and note that

$$(f'_\varepsilon(u) | u) = \|u\|_E^2 \int \left[h - V(u) - \frac{1}{2} \nabla V(u) u \right] = \|u\|_E^2 (h - g(u))$$

Hence, if u is any possible critical point of f_ε , then $g(u) = h$. Setting $M_h = \{u \in \Lambda_0 : g(u) = h\}$, it turns out that, under assumptions (V1), (V2'), (V3), (V4), $M_h \neq \emptyset, \forall h < 0$. Furthermore, (V5) implies that $(g'(u) | u) \neq 0, \forall u \in M_h$ and hence M_h is a (smooth) manifold of codimension 1 in E . Moreover, if u is a critical point of f_ε on M_h there results $f'_\varepsilon(u) = \lambda g'(u)$ for some $\lambda \in \mathbf{R}$. From this it follows:

$$(f'_\varepsilon(u) | u) = \lambda (g'(u) | u)$$

Since $(f'_\varepsilon(u)|u) = 0$ for $u \in M_h$ while $(g'(u)|u) \neq 0$, then $\lambda = 0$ and $f'_\varepsilon(u) = 0$. Noticing that $\forall u \in M_h$ there results $\|u\|_E > 0$, then Lemma 2 implies $x_\varepsilon(t) := u(\omega_\varepsilon t)$ solves (Ph. 1 ε)-(Ph. 2 ε), with ω_ε given by (2.4). To find critical points of f_ε on M_h we first note that for all $u \in M_h$ there results

$$f_\varepsilon(u) = \frac{1}{4} \|u\|_E^2 \int \nabla V_\varepsilon(u) u > 0. \text{ Moreover, repeating the arguments of}$$

Lemmas 4.5-6 of [1] [the fact that now the potential V has the form (1.1) requires minor changes, already indicated in the preceding section] one shows that f_ε satisfies (PS) on M_h . As a consequence f_ε achieves the minimum on M_h . Let us remark explicitly that here we do not need to use min-max arguments, because, in view of the symmetry assumption (V1), we are working in Λ_0 . Lemmas 4.9-10-11 of [1] enable us to show that $u_\varepsilon \rightarrow u$ and $\omega_\varepsilon \rightarrow \omega$ as $\varepsilon \rightarrow 0$, yielding a solution $x(t) := u(\omega t)$ of (Ph). ■

The following example shows that the (PS) condition can fail when $V(2)$ is replaced by $(V2')$. For simplicity we take a potential $V(x) = -|x|^{-\alpha}$ and not in the form (1.1).

Example. – Let us consider

$$f_\varepsilon(u) = \frac{1}{2} \|u\|_E^2 \cdot \int \left[h + \frac{1}{|u|^\alpha} + \frac{\varepsilon}{|u|^2} \right] \quad (0 < \alpha < 1)$$

We claim that for all $k \in \mathbb{N}$ there exists a sequence $u_n = u_{n,k}$ such that

- (i) $f_\varepsilon(u_n) \rightarrow 2k^2 \pi^2 \varepsilon$;
- (ii) $f'_\varepsilon(u_n) \rightarrow 0$.

To see this, we take a sequence $r_n \rightarrow 0$ and set (using complex notation) $u_n(t) = r_n e^{i 2 \pi k t}$.

Since $\alpha < 1$ there results:

$$f_\varepsilon(u_n) = 2k^2 \pi^2 r_n^2 (h + r_n^{-\alpha} + \varepsilon r_n^{-2}) \rightarrow 2k^2 \pi^2 \varepsilon,$$

proving (i).

Furthermore one has readily:

$$(f'_\varepsilon(u_n)|v) = (h + r_n^{-\alpha} + \varepsilon r_n^{-2}) \int u'_n v' + 2k^2 \pi^2 r_n^2 \left(-\frac{\alpha}{r_n^{\alpha+2}} \int u_n v - 2\frac{\varepsilon}{r_n^4} \int u_n v \right).$$

Letting $v = \sum v_k e^{i 2 \pi k t}$ it follows:

$$\begin{aligned} (f'_\varepsilon(u_n)|v) &= 4k^2 \pi^2 r_n v_k (h + r_n^{-\alpha} + \varepsilon r_n^{-2}) - 4k^2 \pi^2 r_n^2 v_k \left(\frac{\alpha}{2} r_n^{-\alpha-1} + \varepsilon r_n^{-3} \right) \\ &= 4k^2 \pi^2 r_n v_k \left(h + \left(1 - \frac{\alpha}{2} \right) r_n^{-\alpha} \right) \rightarrow 0, \end{aligned}$$

and (ii) follows.

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