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On nonhomogeneous elliptic equations involving critical Sobolev exponent


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critical Sobolev exponent

by

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ABSTRACT. - Let \( p = \frac{2N}{N-2} \), \( N \geq 3 \) be the limiting Sobolev exponent and 
\( \Omega \subset \mathbb{R}^N \) open bounded set. 
We show that for \( f \in H^{-1} \) satisfying a suitable condition and \( f \neq 0 \), the 
Dirichlet problem:

\[
\begin{aligned}
-\Delta u &= |u|^{p-2} u + f & \text{on } \Omega \\
 u &= 0 & \text{on } \partial \Omega
\end{aligned}
\]

admits two solutions \( u_0 \) and \( u_1 \) in \( H^1_0(\Omega) \). 
Also \( u_0 \geq 0 \) and \( u_1 \geq 0 \) for \( f \geq 0 \). 
Notice that, in general, this is not the case if \( f = 0 \) (see [P]).

Key words : Semilinear elliptic equations, critical Sobolev exponent.

RÉSUMÉ. - Soit \( p = \frac{2N}{N-2} \) l'exposant de Sobolev critique et \( \Omega \subset \mathbb{R}^N \) un 
domaine borné.

Classification A.M.S. : 35 A 15, 35 J 20, 35 J 65.

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On montre que si \( f \in H^{-1}, f \neq 0 \) satisfait une certaine condition alors le problème de Dirichlet : \( \Delta u = |u|^{p-2}u + f \) dans \( \Omega \) et \( u = 0 \) dans \( \partial \Omega \), admet deux solutions \( u_0 \) et \( u_2 \) dans \( H_0^1(\Omega) \). De plus \( u_0 \geq 0 \) et \( u_1 \geq 0 \) si \( f \geq 0 \).

On remarque que ce n'est pas le cas, en général, si \( f = 0 \) (voir [P]).

1. INTRODUCTION AND MAIN RESULTS

In a recent paper Brezis-Nirenberg (B.N.1] have considered the following minimization problem:

\[
\inf_{u \in H, \|u\|_p = 1} \left( \int_{\Omega} (|\nabla u|^2 - fu) \right)
\]  \( (1.1) \)

where \( \Omega \subset \mathbb{R}^N \), is a bounded set, \( H = H_0^1(\Omega) \), \( f \in H^{-1} \) and \( p = \frac{2N}{N-2}, N \geq 3 \) is the limiting exponent in the Sobolev embedding.

It is well known that the infimum in \( (1.1) \) is never achieved if \( f = 0 \) (cf. [B]). In contrast, in [B.N.1] it is shown that for this infimum is always achieved. (See also [C.S.] for previous related results.)

Motivated by this result we consider the functional:

\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p} \int_{\Omega} |u|^p - \int_{\Omega} fu, \quad u \in H;
\]

whose critical points define weak solutions for the Dirichlet problem:

\[
\begin{align*}
-\Delta u &= |u|^{p-2}u + f \quad \text{on } \Omega \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]  \( (1.2) \)

We investigate suitable minimization and minimax principles of mountain pass-type (cf. [A.R.]), and show how, for suitable \( f \)'s, they produce critical values for \( I \) in spite of a possible failure of the Palais-Smale condition.

To start, notice that \( I \) is bounded from below in the manifold:

\[ \Lambda = \{ u \in H : \langle I'(u), u \rangle = 0 \} \]

[here \( \langle , \rangle \) denotes the usual scalar product in \( H = H_0^1(\Omega) \)]. Thus a natural question to ask is whether or not \( I \) achieves a minimum in \( \Lambda \).

We show that this is the case if \( f \) satisfies the following:

\[
\int_{\Omega} fu \leq c_N (\|\nabla u\|_2^{(N+2)/2}) \quad (\ast)_0
\]
\( \forall u \in H, \| u \|_p = 1 \), where \( c_N = \frac{4}{N-2} \left( \frac{N-2}{N+2} \right)^{(N+2)/4} \). More precisely we have:

**Theorem 1.** Let \( f \neq 0 \) satisfies (\( \ast \)). Then

\[
\inf_{\Lambda} I = c_0
\]

is achieved at a point \( u_0 \in \Lambda \) which is a critical point for \( I \) and \( u_0 \geq 0 \) for \( f \geq 0 \).

In addition if \( f \) satisfies the more restrictive assumption:

\[
\int_{\Omega} fu < c_N ( \| \nabla u \|_2^{(N+2)/2}
\]

\( \forall u \in H, \| u \|_p = 1 \), then \( u_0 \) is a local minimum for \( I \). 

Notice that assumption (\( \ast \)) certainly holds if

\[
\| f \|_{H^{-1}} \leq c_N S^{N/4}
\]

where \( S \) is the best Sobolev constant (cf. [T]).

Also if \( f = 0 \) Theorem 1 remains valid and gives the trivial solution \( u_0 = 0 \).

Moreover in the situation where \( u_0 \) is a local minimum for \( I \), necessarily:

\[
\| \nabla u_0 \|_p^2 - (p-1) \| u_0 \|_p^p \geq 0
\]

This suggests to look at the following splitting for \( \Lambda \):

\[
\Lambda^+ = \{ u \in \Lambda : \| \nabla u \|_2^2 - (p-1) \| u \|_p^p > 0 \}
\]

\[
\Lambda_0 = \{ u \in \Lambda : \| \nabla u \|_2^2 - (p-1) \| u \|_p^p = 0 \}
\]

\[
\Lambda^- = \{ u \in \Lambda : \| \nabla u \|_2^2 - (p-1) \| u \|_p^p < 0 \}
\]

It turns out that assumption (\( \ast \)) implies \( \Lambda_0 = \{ 0 \} \) (see Lemma 2.3 below). Therefore for \( f \neq 0 \) and (1.4) we obtain \( u_0 \in \Lambda^+ \) and consequently

\[
c_0 = \inf_{\Lambda} I = \inf_{\Lambda^+} I.
\]

So we are led to investigate a second minimization problem. Namely:

\[
\inf_{\Lambda^-} I = c_1.
\]

In this direction we have:

**Theorem 2.** Let \( f \neq 0 \) satisfies (\( \ast \)). Then \( c_1 > c_0 \) and the infimum in (1.5) is achieved at a point \( u_1 \in \Lambda^- \) which define a critical point for \( I \).

Furthermore \( u_1 \geq 0 \) for \( f \geq 0 \).
Notice that the assumption $f \neq 0$ is necessary in Theorem 2. In fact for $f = 0$ we have:

$$\inf I = \inf_{u \neq 0} \frac{1}{N} \left[ \frac{1}{\| u \|_2^2} \right]^{N/2} \inf_{\| u \|_1 = 1} \| \nabla u \|_2^{2N/2}$$

and the infimum in the right hand side is never achieved.

The proofs of Theorem 1 and Theorem 2 rely on the Ekeland's variational principle (cf. [A.E.]) and careful estimates inspired by these in [B.N.1].

As an immediate consequence of Theorems 1 and 2 we have the following for the Dirichlet problem (1.2).

**Theorem 3.** - Problem (1.2) admits at least two weak solutions $u_0, u_1 \in H^1_0(\Omega)$ for $f \neq 0$ satisfying (*); and at least one weak solution for $f$ satisfying ($\ast$)

Moreover $u_0 \geq 0, u_1 \geq 0$ for $f \geq 0$. □

This result for $f \geq 0$ was also pointed out by Brezis-Nirenberg in [B.N.1]. Their approach however uses in an essential way the fact that $f$ does not change sign. It relies on a result of Crandall-Rabinowitz [C.R.] and techniques developed in [B.N.2].

Furthermore for $f \geq 0$ it is known that (1.2) cannot admit positive solution when $\| f \|_{H^{-1}}$ is too large (see [C.R.], [M.] and [Z]). So our approach necessarily breaks down when $\| f \|_{H^{-1}}$ is large. In fact we suspect that assumptions ($\ast$)_0 and ($\ast$) on $f$ are not only sufficient but also necessary to guarantee the statements of Theorems 1 and 2.

By a result of Brezis-Kato [B-K] we know that Theorem 3 gives classical solutions if $f$ is sufficiently regular and $\partial \Omega$ is smooth; and for $f \geq 0$, via the strong maximum principle, such solutions are strictly positive in $\Omega$.

Obviously an equivalent of Theorem 3 holds for the subcritical case where one replaces the power $p = \frac{2N}{N-2}$ in (1.2) by $q \in \left( 2, \frac{2N}{N-2} \right)$. In such a case more standard compactness arguments apply, and the proof can be consistently simplified. The details are left to the interested reader.

Finally going back to the functional $I$, if $f$ satisfies ($\ast$) then Theorem 1 suggests a mountain-pass procedure; which will be carried out as follows.

Take:

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}} \quad \varepsilon > 0, \quad x \in \mathbb{R}^N$$

be an extremal function for the Sobolev inequality in $\mathbb{R}^N$.

For $a \in \Omega$ let $u_{\varepsilon, a}(x) = u_\varepsilon(x-a)$, and

$$\xi_a \in C_0^\infty(\Omega) \quad \text{with} \quad \xi_a \geq 0 \quad \text{and} \quad \xi_a = 1 \text{ near } a.$$
Set

\[ \mathcal{F} = \left\{ h : [0, 1] \to H \text{ continuous, } h(0) = u_0, \right\} \]

\[ h(1) = R_0 \xi_{\varepsilon} u_{\varepsilon, \alpha} \]

R_0 > 0 fixed.

We have:

**THEOREM 4.** For a suitable choice of R_0 > 0, a \in \Omega and \varepsilon > 0 the value

\[ c = \inf_{h \in \mathcal{F}} \max_{t \in [0, 1]} I(h, (t)) \]

defines a critical value for I, and c \geq c_1. \quad \Box

It is not clear whether or not c = c_1. So no additional multiplicity can be claimed for (1.2). However, in case c = c_1 then it is possible to claim a critical point of mountain-pass type (cf. [H]) for I in \Lambda^{-}. This follows by a refined version of the mountain-pass lemma (see [A-R]) obtained by Ghoussoub-Preiss and the fact that \Lambda^{-} cannot contain local minima for I (see [G.P., theorem (ter) part d]).

The referee has brought to our attention a paper of O. Rey (See [R.]) where, by a different approach, a result similar to that of Theorem 3 is established when f \neq 0, f \geq 0 and \| f \|_{H^{-1}} is sufficiently small.

2. THE PROOF OF THEOREM 1

To obtain the proof of Theorem 1 several preliminary results are in order.

We start with a lemma which clarifies the purpose of assumption (\ast).

**LEMMA 2.1.** Let f \neq 0 satisfy (\ast). For every u \in H, u \neq 0 there exists a unique \( t^{+} = t^{+}(u) > 0 \) such that \[ I(t^{+}u) = \max_{t \in [0, 1]} I(tu) \]

and \[ I(t^{+}u) = \max_{t \geq t_{\max}} I(tu) \]

Moreover, if \[ \int_{\Omega} fu > 0 \], then there exists a unique \( t^{-} = t^{-}(u) > 0 \) such that \( t^{-}u \in \Lambda^{+} \).

In particular,

\[ t^{-} = \left[ \frac{\| \nabla u \|_{L^2}^2}{(p-1) \| u \|_{L^p}^p} \right]^{1/(p-2)} \]

and \( I(t^{-}u) \leq I(tu), \forall t \in [0, t^{+}] \).
Set $\varphi(t) = t \| u \|_p^{\frac{2}{p}} - t^{p-1} \| u \|_p^p$. Easy computations show that $\varphi$ is concave and achieves its maximum at

$$t_{\text{max}} = \left\| \frac{\| \nabla u \|_2^2}{(p-1) \| u \|_p^p} \right\|^{1/(p-2)}.$$ 

Also

$$\varphi(t_{\text{max}}) = \left[ \frac{1}{p-1} \right]^{(p-1)/(p-2)} (p-2) \left[ \| \nabla u \|_2^2 \| u \|_p^{p-1} \right]^{1/(p-2)},$$

that is

$$\varphi(t_{\text{max}}) = c_N \frac{\| \nabla u \|_2^{(N+2)/2}}{\| u \|_p^{N/2}}.$$ 

Therefore if $\int_{\Omega} fu \leq 0$ then there exists a unique $t^+ > t_{\text{max}}$ such that:

$$\varphi(t^+) = \int_{\Omega} fu \quad \text{and} \quad \varphi'(t^+) < 0.$$ 

Equivalently $t^+ u \in \Lambda^-$ and $I(t^+ u) \geq I(tu) \forall t \geq t_{\text{max}}.$

In case $\int_{\Omega} fu > 0$, by assumption (*) we have that necessarily

$$\int_{\Omega} fu < c_N \frac{\| \nabla u \|_2^{(N+2)/2}}{\| u \|_p^{N/2}} = \varphi(t_{\text{max}}).$$

Consequently, in this case, we have unique $0 < t^- < t_{\text{max}} < t^+$ such that

$$\varphi(t^+) = \int_{\Omega} fu = \varphi(t^-)$$

and

$$\varphi'(t^-) > 0 > \varphi'(t^+).$$

Equivalently $t^+ u \in \Lambda^-$ and $t^- u \in \Lambda^+$.

Also $I(t^+ u) \geq I(tu), \forall t \geq t^-$ and $I(t^- u) \leq I(tu), \forall t \in [0, t^+]$.

**Lemma 2.2.** For $f \neq 0$

$$\inf_{\| u \|_p = 1} \left( c_N \| \nabla u \|_2^{(N+2)/2} - \int_{\Omega} fu \right) = \mu_0 \quad (2.1)$$

is achieved. In particular if $f$ satisfies (*), then $\mu_0 > 0$.

The proof of Lemma 2.2 is technical and a straightforward adaptation of that given in [B.N.1] for an analogous minimization problem.

It will be given in the appendix for the reader’s convenience.
Next, for \( u \neq 0 \) set
\[
\psi(u) = c_N \frac{\| \nabla u \|_2^{(N+2)/2} - \int_\Omega fu}{\| u \|_p^{N/2}}.
\]

Since for \( t > 0, \| u \|_p = 1 \) we have:
\[
\psi(tu) = t \left[ c_N \| \nabla u \|_2^{(N+2)/2} - \int_\Omega fu \right];
\]
given \( \gamma > 0 \), from Lemma 2.2 we derive that
\[
\inf_{\| u \| \geq \gamma} \psi(u) \geq \gamma \mu_0. \tag{2.2}
\]

In particular if \( f \) satisfies (*) then the infimum (2.2) is bounded away from zero.
This remark is crucial for the following:

**Lemma 2.3.** Let \( f \) satisfy (*). For every \( u \in \Lambda, u \neq 0 \) we have
\[
\| \nabla u \|_2^2 - (p-1) \| u \|_p^p \neq 0
\]
(i.e. \( \Lambda_0 = \{ 0 \} \)).

**Proof.** Although the result also holds for \( f = 0 \), we shall only be concerned with the case \( f \neq 0 \).

Arguing by contradiction assume that for some \( u \in \Lambda, u \neq 0 \) we have
\[
\| \nabla u \|_2^2 - (p-1) \| u \|_p^p = 0 \tag{2.3}
\]

Thus
\[
0 = \| \nabla u \|_2^2 - \| u \|_p^p - \int_\Omega fu = (p-2) \| u \|_p^p - \int_\Omega fu. \tag{2.4}
\]

Condition (2.3) implies
\[
\| u \|_p \geq \left( \frac{S}{p-1} \right)^{1/(p-2)} := \gamma,
\]
and from (2.2) and (2.4) we obtain:

\[
0 < \mu_0 \gamma \leq \psi(u) = \left[ \frac{1}{p-1} \right]^{(p-1)/(p-2)} (p-2) \left[ \frac{\| \nabla u \|_2^2 (p-1)}{\| u \|_p^p} \right]^{1/(p-2)} - \int_{\Omega} fu
\]

\[
= (p-2) \left[ \frac{1}{p-1} \right]^{(p-1)/(p-2)} \left[ \frac{\| \nabla u \|_2^2 (p-1)}{\| u \|_p^p} \right]^{1/(p-2)} - \| u \|_p^p
\]

\[
= (p-2) \| u \|_p^p \left( \frac{\| \nabla u \|_2^2 (p-1)}{(p-1) \| u \|_p^p} - 1 \right) = 0
\]

which yields to a contradiction. \( \square \)

As a consequence of Lemma 2.3 we have:

**Lemma 2.4.** Let \( f \neq 0 \) satisfy (\(*\)). Given \( u \in \Lambda, u \neq 0 \) there exist \( \varepsilon > 0 \) and a differentiable function \( t = t(w) > 0, w \in H \| w \| < \varepsilon \) satisfying the following:

\[
t(0) = 1, \quad t(w)(u - w) \in \Lambda, \quad \text{for } \| w \| < \varepsilon,
\]

and

\[
\langle t'(0), w \rangle = \frac{2 \int_{\Omega} \nabla u \cdot \nabla w - p \int_{\Omega} |u|^{p-2} u w \int_{\Omega} f w}{\| \nabla u \|_2^2 - (p-1) \| u \|_p^p}.
\]

**Proof.** Define \( F : \mathbb{R} \times H \rightarrow \mathbb{R} \) as follows:

\[
F(t, w) = t \| \nabla (u - w) \|_2^2 - t^{p-1} \| u - w \|_p^p - \int_{\Omega} f(u - w).
\]

Since \( F(1, 0) = 0 \) and \( F_t(1, 0) = \| \nabla u \|_2^2 - (p-1) \| u \|_p^p \neq 0 \) (by Lemma 2.3), we can apply the implicit function theorem at the point \((1, 0)\) and get the result. \( \square \)

We are now ready to give:

**The Proof of Theorem 1**

We start by showing that \( I \) is bounded from below in \( \Lambda \). Indeed for \( u \in \Lambda \) we have:

\[
\int_{\Omega} |\nabla u|^2 - \int_{\Omega} |u|^p - \int_{\Omega} fu = 0.
\]

Thus:

\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p} \int_{\Omega} |u|^p - \int_{\Omega} fu = \frac{1}{N} \int_{\Omega} |\nabla u|^2 - \left( 1 - \frac{1}{p} \right) \int_{\Omega} fu
\]

\[
\geq \frac{1}{N} \| \nabla u \|_2^2 - \frac{N+2}{2N} \| f \|_{H^{-1}} \| \nabla u \|_2 \geq - \frac{1}{16N} \| f \|_{H^{-1}}^2.
\]
In particular
\[ c_0 \geq -\frac{1}{16N} [(N+2) \| f \|_{H^{-1}}]^2. \] (2.6)

We first obtain our result for \( f \) satisfying \((*)\). The more general situation where \( f \) satisfies \((*)_0\) will be subsequently derived by a limiting argument.

So from now on we assume that \( f \) satisfy \((*)\).

In order to obtain an upper bound for \( c_0 \), let \( v \in H \) be the unique solutions for \(-\Delta u = f\). So for \( f \neq 0 \)
\[ \int_{\Omega} f v = \| \nabla v \|_2^2 > 0. \]

Set \( t_0 = t^- (v) > 0 \) as defined by Lemma 2.1.

Hence \( t_0 v \in A^+ \) and consequently:
\[
I(t_0 v) = \frac{t_0^2}{2} \| \nabla v \|_2^2 - \frac{t_0^p}{p} \| v \|_p^p - t_0 \| \nabla v \|_2^2 \\
= -\frac{t_0^2}{2} \| \nabla v \|_2^2 + \frac{p-1}{p} \frac{t_0^p}{p} \| v \|_p^p < -\frac{t_0^2}{N} \| \nabla v \|_2^2 = -\frac{t_0^2}{N} \| f \|_{H^{-1}}^2
\]

This yields,
\[ c_0 < -\frac{t_0^2}{N} \| f \|_{H^{-1}}^2 < 0. \] (2.7)

Clearly Ekeland’s variational principle (see [A.E.], Corollary 5.3.2) applies to the minimization problem (1.3). It gives a minimizing sequence \( \{ u_n \} \subset \Lambda \) with the following properties:

(i) \( I(u_n) < c_0 + \frac{1}{n} \).

(ii) \( I(w) \geq I(u_n) - \frac{1}{n} \| \nabla (w - u_n) \|_2, \forall w \in \Lambda. \)

By taking \( n \) large, from (2.7) we have:
\[ I(u_n) = \frac{1}{N} \int_{\Omega} |\nabla u_n|^2 - \frac{N+2}{2N} \int_{\Omega} f u_n < c_0 + \frac{1}{n} < -\frac{t_0^2}{N} \| f \|_{H^{-1}}^2 \] (2.8)

This implies
\[ \int_{\Omega} f u_n \geq \frac{2}{N+2} \frac{t_0^2}{N} \| f \|_{H^{-1}}^2 > 0. \] (2.9)

Consequently \( u_n \neq 0 \), and putting together (2.8) and (2.9) we derive:
\[ \frac{2}{N+2} \| f \|_{H^{-1}}^2 \leq \| \nabla u_n \|_2 \leq \frac{N+2}{2} \| f \|_{H^{-1}}. \] (2.10)
Our goal is to obtain \( \| I'(u_n) \| \to 0 \) as \( n \to +\infty \).
Hence let us assume \( \| I'(u_n) \| > 0 \) for \( n \) large (otherwise we are done).

Applying Lemma 2.4 with \( u = u_n \) and \( w = \delta \frac{I'(u_n)}{\| I'(u_n) \|} \), \( \delta > 0 \) small, we find, \( t_n(\delta) := t \left[ \delta \frac{I'(u_n)}{\| I'(u_n) \|} \right] \)
such that

\[
  w_\delta = t_n(\delta) \left[ u_n - \delta \frac{I'(u_n)}{\| I'(u_n) \|} \right] \in \Lambda.
\]

From condition (ii) we have:

\[
  \frac{1}{n} \left\| \nabla (w_\delta - u_n) \right\|_2 \geq I(u_n) - I(w_\delta) = (1 - t_n(\delta)) \left< I'(w_\delta), u_n \right>
  + \delta t_n(\delta) \left< I'(w_\delta), \frac{I'(u_n)}{\| I'(u_n) \|} \right> + o(\delta).
\]

Dividing by \( \delta > 0 \) and passing to the limit as \( \delta \to 0 \) we derive:

\[
  \frac{1}{n} (1 + \left| t'_n(0) \right| \left\| \nabla u_n \right\|_2) \geq -t'_n(0) \left< I'(u_n), u_n \right> + \left\| I'(u_n) \right\| = \left\| I'(u_n) \right\|
\]

where we have set \( t'_n(0) = \left< I'(0), \frac{I'(u_n)}{\| I'(u_n) \|} \right> \).

Thus from (2.10) we conclude:

\[
  \left\| I'(u_n) \right\| \leq \frac{C}{n} (1 + \left| t'_n(0) \right|)
\]

for a suitable positive constant \( C \).

We are done once we show that \( \left| t'_n(0) \right| \) is bounded uniformly on \( n \).

From (2.5) and the estimate (2.10) we get:

\[
  \left| t'_n(0) \right| \leq \frac{C_1}{\left\| \nabla u_n \right\|_2^2 - (p - 1) \left\| u_n \right\|_p^p}
\]

\( C_1 > 0 \) suitable constant.

Hence we need to show that \( \left\| \nabla u_n \right\|_2^2 - (p - 1) \left\| u_n \right\|_p^p \) is bounded away from zero.

Arguing by contradiction, assume that for a subsequence, which we still call \( u_n \), we have:

\[
  \left\| \nabla u_n \right\|_2^2 - (p - 1) \left\| u_n \right\|_p^p = o(1). \tag{2.11}
\]

From the estimate (2.10) and (2.11) we derive:

\[
  \left\| u_n \right\|_p \geq \gamma \quad (\gamma > 0 \text{ suitable constant})
\]
and
\[ \left( \frac{\| \nabla u_n \|_2^2}{p-1} \right)^{(p-1)/(p-2)} - \left( \| u_n \|_p \right)^{(p-1)/(p-2)} = o(1). \]

In addition (2.11), and the fact that \( u_n \in \Lambda \) also give:
\[ \int_\Omega f u_n = (p-2) \| u_n \|_p^p + o(1). \]

This, together with (2.2) implies:
\[ 0 < \mu \gamma^{(N+2)/2} \leq \| u_n \|_p^{p/(p-2)} \psi(u_n) \]
\[ = (p-2) \left[ \left( \frac{\| \nabla u_n \|_2^2}{p-1} \right)^{(p-1)/(p-2)} - \left( \| u_n \|_p \right)^{(p-1)/(p-2)} \right] = o(1). \]

which is clearly impossible.

In conclusion:
\[ \| I'(u_n) \| \to 0 \quad \text{as} \quad n \to +\infty. \quad (2.12) \]

Let \( u_0 \in H \) be the weak limit in \( H_0^1(\Omega) \) of (a subsequence of) \( u_n \).

From (2.9) we derive that:
\[ \int_\Omega f u_0 > 0 \]

and from (2.12) that
\[ \langle I'(u_0), w \rangle = 0, \quad \forall w \in H, \]

i.e. \( u_0 \) is a weak solution for (1.2).

In particular, \( u_0 \in \Lambda \).

Therefore:
\[ c_0 \leq I(u_0) = \frac{1}{N} \| \nabla u_0 \|_2^2 - \int_\Omega f u_0 \leq \lim_{n \to +\infty} I(u_n) = c_0. \]

Consequently \( u_n \to u_0 \) strongly in \( H \) and \( I(u_0) = c_0 = \inf_\Lambda I \). Also from Lemma 2.1 and (2.12) follows that necessarily \( u_0 \in \Lambda^+ \).

To conclude that \( u_0 \) is a local minimum for \( I \), notice that for every \( u \in H \) with \( \int_\Omega f u > 0 \) we have:
\[ I(su) \geq I(t-u) \]
for every \( 0 < s < \left( \frac{\| \nabla u \|_2^2}{(p-1)\| u \|_p^p} \right)^{(1/(p-2))} \). \quad (2.13)

(see Lemma 2.1).
In particular for \( u = u_0 \in \Lambda^+ \) we have:

\[
t^+ = 1 < \left[ \frac{\| \nabla u_0 \|^2}{(p-1) \| u \|_p^p} \right]^{1/(p-2)}.
\]  
(2.14)

Let \( \varepsilon > 0 \) sufficiently small to have:

\[
1 < \frac{\| \nabla (u_0 - w) \|^2}{(p-1) \| u_0 - w \|_p^p}
\]

for \( \| w \| < \varepsilon \).

From Lemma 2.4, let \( t(w) > 0 \) satisfy \( t(w) (u_0 - w) \in \Lambda \) for every \( \| w \| < \varepsilon \).

Since \( t(w) \to 1 \) as \( \| w \| \to 0 \), we can always assume that

\[
t(w) < \left[ \frac{\| \nabla (u_0 - w) \|^2}{(p-1) \| u_0 - w \|_p^p} \right]^{1/(p-2)}
\]

for every \( w : \| w \| < \varepsilon \).

Namely, \( t(w)(u_0 - w) \in \Lambda^+ \) and for \( 0 < s < \left[ \frac{\| \nabla (u_0 - w) \|^2}{(p-1) \| u_0 - w \|_p^p} \right]^{1/(p-2)} \) we have,

\[
I(s(u_0 - w)) \geq I(t(w)(u_0 - w)) \geq I(u_0).
\]

From (2.14) we can take \( s = 1 \) and conclude:

\[
I(u_0 - w) \geq I(w), \quad \forall w \in H, \quad \| w \| < \varepsilon.
\]

Furthermore if \( f \geq 0 \), take \( t_0 = t^- (\| u_0 \|) > 0 \) with \( t_0 \| u_0 \| \in \Lambda^+ \).

Necessarily \( t_0 \geq 1 \), and

\[
I(t_0 \| u_0 \|) \leq I(\| u_0 \|) \leq I(u_0).
\]

So we can always take \( u_0 \geq 0 \).

To obtain the proof when \( f \) satisfies \((*)_0\) we shall apply an approximation argument. To this purpose, notice that if \( f \) satisfies \((*)_0\) then \( f_\varepsilon = (1 - \varepsilon) f \) satisfies \((*) \) \( \forall \varepsilon \in (0, 1) \).

Set

\[
I_\varepsilon(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p} \int_{\Omega} |u|^p + (1 - \varepsilon) \int_{\Omega} fu, \quad u \in H.
\]

Let \( u_\varepsilon \in \Lambda^+_\varepsilon \subset \{ u \in H : \langle I'_\varepsilon(u), u \rangle = 0, \| \nabla u \|_2^2 - (p-1) \| u \|_p^p > 0 \} \) satisfy:

\[
I_\varepsilon(u_\varepsilon) = \inf_{\Lambda^+_\varepsilon} I_\varepsilon = c_\varepsilon
\]

and

\[
\langle I'_\varepsilon(u_\varepsilon), w \rangle = 0, \quad \forall w \in H.
\]

(2.15)

Clearly \( \| \nabla u_\varepsilon \|_2 \leq C_2 \), for \( 0 < \varepsilon < 1 \) and \( C_2 > 0 \) a suitable constant.
Let \( u \in \Lambda^+ \), necessarily \( \int_\Omega fu > 0 \) and consequently
\[
(1 - \varepsilon) \int_\Omega fu > 0, \quad 0 < \varepsilon < 1.
\]
From Lemma 2.1 applied we find:
\[
0 < t^-_\varepsilon < \left[ \frac{||\nabla u||^2}{(p-1)||u||^p_p} \right]^{1/(p-2)}
\]
with \( t^-_\varepsilon \in \Lambda^+_\varepsilon \).

Since \( 1 < \frac{||\nabla u||^2}{(p-1)||u||^p_p} \), from (2.13) it follows that
\[
I_\varepsilon(t^-_\varepsilon) \leq I_\varepsilon(u)
\]
and consequently:
\[
c_3 \leq I_\varepsilon(t^-_\varepsilon) \leq I_\varepsilon(u) \leq I(u) + \varepsilon ||f'||_{H^{-1}} \| \nabla u \|_2 \leq I(u) + \varepsilon C_3
\]
(with \( C_3 > 0 \) a suitable constant).

Estimate (2.6) with \( f = f^-_\varepsilon \) and the above inequality imply:
\[
- \frac{1}{16N} [(N+2) ||f'||_{H^{-1}}]^2 \leq - \frac{1}{16N} [(N+2) ||f^-_\varepsilon||_{H^{-1}}]^2 \leq c_3 \leq c_0 + \varepsilon C_3.
\]

Let \( \varepsilon_n \to 0, n \to + \infty \) and \( u_0 \in H \) satisfy:
(a) \( c_{\varepsilon_n} \to \tilde{c} \leq c_0, n \to + \infty \)
(b) \( u_{\varepsilon_n} \to u_0, n \to + \infty \) weakly in \( H \).

From (2.15) it follows \( \langle I'(u_0), w \rangle = 0, \forall w \in H \) (i.e. \( u_0 \) is a critical point for \( I \)) and \( I(u_0) \leq c_0 \).

In particular \( u_0 \in \Lambda \) and necessarily \( I(u_0) = c_0 \), (i.e. \( u_{\varepsilon_n} \to u_0 \) strongly in \( H \)).

This completes the proof. \( \square \)

3. THE PROOF OF THEOREMS 2 AND 4

The functional \( I \) involves the limiting Sobolev exponent \( p = \frac{2N}{N-2} \). This compromises its compactness properties, and a possible failure of the P.S. condition is to be expected.

Our first task is to locate the levels free from this noncompactness effect.

We refer to [B] and [S] for a survey on related problems where such an approach has been successfully used.

In this direction we have:

**PROPOSITION 3.1.** Every sequence \( \{u_n\} \subset H \) satisfying:

(a) \( I(u_n) \to c \) with \( c < c_0 + \frac{1}{N} S^{N/2} \)

\( [c_0 \text{ as defined in (1.3)].} \)

(b) \( \| I'(u_n) \| \to 0 \)

as a convergent subsequence.

Namely the (P.S) condition holds for all level \( c < c_0 + \frac{1}{N} S^{N/2} \).

**Proof.** It is not difficult to see that (a) and (b) imply that \( \| \nabla u_n \|_2 \) is uniformly bounded.

Hence for a subsequence of \( u_n \) (which we still call \( u_n \)), we can find a \( w_0 \in H \) such that

\[ u_n \rightharpoonup w_0 \quad \text{weakly in } H. \]

Consequently from (b) we obtain:

\[ \langle I'(w_0), w \rangle = 0, \quad \forall w \in H. \tag{3.1} \]

That is \( w_0 \) is a solution in \( H_0^1(\Omega) \) for (1.2). In particular \( w_0 \neq 0 \), \( w_0 \in \Lambda \) and \( I(w_0) \geq c_0 \).

Write \( u_n = w_0 + v_n \) with \( v_n \to 0 \) weakly in \( H \).

By a Lemma of Brezis-Lieb [B.L.] we have:

\[ \| u_n \|_p^p = \| w_0 + v_n \|_p^p = \| w_0 \|_p^p + \| v_n \|_p^p + o(1). \]

Hence, for \( n \) large, we conclude:

\[ c_0 + \frac{1}{N} S^{N/2} > I(w_0 + v_n) = I(w_0) + \frac{1}{2} \| \nabla v_n \|_2^2 - \frac{1}{p} \| v_n \|_p^p + o(1) \]

\[ \geq c_0 + \frac{1}{2} \| \nabla v_n \|_2^2 - \frac{1}{p} \| v_n \|_p^p + o(1). \]

which gives:

\[ \frac{1}{2} \| \nabla v_n \|_2^2 - \frac{1}{p} \| v_n \|_p^p < \frac{1}{N} S^{N/2} + o(1). \tag{3.2} \]

Also from (b) follows:

\[ o(1) = \langle I'(u_n), u_n \rangle = \| \nabla w_0 \|_2^2 - \| w_0 \|_p^p - \int_{\Omega} f w_0 + \| \nabla v_n \|_2^2 - \| v_n \|_p^p + o(1) \]

\[ = \langle I'(w_0), w_0 \rangle + \| \nabla v_n \|_2^2 - \| v_n \|_p^p + o(1); \]

and taking into account (3.1) we obtain:

\[ \| \nabla v_n \|_2^2 - \| v_n \|_p^p = o(1). \tag{3.3} \]
We claim that conditions (3.2) and (3.3) can hold simultaneously only if \( \{ v_n \} \) admits a subsequence, say, which converges strongly to zero, i.e. \( \| v_{n_k} \| \rightarrow 0, k \rightarrow +\infty \).

Arguing by contradiction assume that \( \| v_n \| \) is bounded away from zero. That is for some constant \( c_4 > 0 \) we have \( \| v_n \| \geq c_4, \forall n \in \mathbb{N} \).

From (3.3) then it follows:
\[
\| v_n \|_p^{\beta-2} \geq S + o(1),
\]
and consequently
\[
\| v_n \|_p \geq S^{N/2} + o(1).
\]

This yields a contradiction since from (3.2) and (3.3) we have:
\[
\frac{1}{N} S^{N/2} \leq \frac{1}{N} \| v_n \|_p + o(1) = \frac{1}{2} \| \nabla v_n \|_2^2 - \frac{1}{p} \| v_n \|_p^p + o(1) < \frac{1}{N} S^{N/2}
\]
for \( n \) large.

In conclusion, \( u_{nk} \rightarrow w_0 \) strongly. \( \square \)

At this point it would not be difficult to derive Theorem 2, if we had the inequality:
\[
\inf_{\Lambda} I = c_1 < c_0 + \frac{1}{N} S^{N/2} \tag{3.4}
\]

However it appears difficult to derive (3.4) directly.

We shall obtain it by comparison with a mountain-pass value.

To this end, recall that \( u_0 \neq 0 \). Following [B.N.1] we set \( \Sigma \subset \Omega \) to be a set of positive measure such that \( u_0 > 0 \) on \( \Sigma \) (replace \( u_0 \) with \( -u_0 \) and \( f \) with \( -f \) if necessary).

Set \( U_{\varepsilon, a} (x) = \xi_a (x) u_{\varepsilon, a} (x), \quad x \in \mathbb{R}^N; \)
\([u_{\varepsilon, a} \text{ and } \xi_a \text{ defined in (1.6) and (1.7)}].\)

**Lemma 3.1.** - *For every \( R > 0 \) and a.e. \( a \in \Sigma \), there exists \( \varepsilon_0 = \varepsilon_0 (R, a) > 0 \) such that:
\[
I (u_0 + RU_{\varepsilon, a}) < c_0 + \frac{1}{N} S^{N/2}
\]
for every \( 0 < \varepsilon < \varepsilon_0 \).

**Proof.** - We have:
\[
I (u_0 + RU_{\varepsilon, a}) = \int_{\Omega} \frac{|\nabla u_0|^2}{2} + R \int_{\Omega} \nabla u_0 \nabla U_{\varepsilon, a} + \frac{R^2}{2} \int_{\Omega} |\nabla U_{\varepsilon, a}|^2
\]
\[
- \frac{1}{p} \int_{\Omega} |u_0 + RU_{\varepsilon, a}|^p - \int_{\Omega} fu_0 - R \int_{\Omega} fU_{\varepsilon, a} \tag{3.5}
\]
A careful estimate obtained by Brezis-Nirenberg (see formulae (17) and (22) in [B.N.I]) shows that:

\[
\| u_0 + RU_{\varepsilon,a} \|_p^p = \| u_0 \|_p^p + R^p \| U_{\varepsilon,a} \|_p^p + pR \int_{\Omega} |u_0|^{p-2} u_0 U_{\varepsilon,a} \\
+ p R^{p-1} \int_{\Omega} U_{\varepsilon,a}^{p-1} u_0 + o(\varepsilon^{(N-2)/2}) \text{ for a.e. } a \in \Sigma.
\]

Also from [B.N.2] we have:

\[
\| \nabla U_{\varepsilon,a} \|_2^2 = B + O(\varepsilon^{N-2}) \quad \text{ and } \quad \| U_{\varepsilon,a} \|_p^p = A + O(\varepsilon^N)
\]

where

\[
B = \int_{\mathbb{R}^N} |\nabla u_1(x)|^2 \, dx, \quad A = \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^N}
\]

and

\[
S = \frac{B}{A^{2/p}}.
\]

Substituting in (3.5) and using the fact that \( u_0 \) satisfies (1.2) we obtain:

\[
I(u_0 + RU_{\varepsilon,a}) = \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 + R \int_{\Omega} \nabla u_0 \cdot \nabla U_{\varepsilon,a} + \frac{R^2}{2} B - \frac{1}{p} \int_{\Omega} |u_0|^{p-} - \frac{R^p}{p} A \\
- R \int_{\Omega} |u_0| u_0^{p-2} U_{\varepsilon,a} - R^{p-1} \int_{\Omega} fU_{\varepsilon,a} - R \int_{\Omega} fU_{\varepsilon,a} + o(\varepsilon^{(N-2)/2})
\]

\[
= I(u_0) + \frac{R^2}{2} B - \frac{R^p}{p} A - R^{p-1} \int_{\Omega} U_{\varepsilon,a}^{p-1} u_0 + o(\varepsilon^{(N-2)/2})
\]

for a.e. \( a \in \Sigma \).

Set \( u_0 = 0 \) outside \( \Omega \), it follows:

\[
\int_{\Omega} U_{\varepsilon,a}^{p-1} u_0 = \int_{\mathbb{R}^N} u_0(x) \frac{\varepsilon^{(N+2)/2}}{(\varepsilon^2 + |x-a|^2)^{(N+2)/2}} dx \\
= \varepsilon^{(N-2)/2} \int_{\mathbb{R}^N} u_0(x) \frac{1}{\varepsilon^N} \Psi_1 \left( \frac{x}{\varepsilon} \right) dx,
\]

where \( \Psi_1(x) = \frac{1}{(1 + |x|^2)^{(N+2)/2}} \in L^1(\mathbb{R}^N) \).

Therefore, setting \( D = \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^{(N+2)/2}} \) we derive:

\[
\int_{\mathbb{R}^N} u_0(x) \frac{1}{\varepsilon^N} \Psi_1 \left( \frac{x}{\varepsilon} \right) dx \to u_0(a) D
\]

for a.e. \( a \in \Sigma \) (see [F]).
In other words,
\[ \int_{\Omega} U_{\epsilon}^{p-1}(x) u_0(x) \, dx = \varepsilon^{(N-2)/2} u_0(a) D + o(\varepsilon^{(N-2)/2}). \]

Consequently:
\[ I(u_0 + RU_{\epsilon,a}) = c_0 + \frac{R^2}{2} B - \frac{R^p}{p} A - R^{p-1} u_0(a) D \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}). \]

Define:
\[ q(s) = \frac{s^2}{2} B - \frac{s^p}{p} A - s^{p-1} u_0(a) D \varepsilon^{(N-2)/2}, \quad s > 0 \]
and assume that \( q(s) \) achieves its maximum at \( s_\varepsilon > 0 \).

Set
\[ S_\varepsilon = \left( \frac{B}{A} \right)^{1/(p-2)}. \]

Since \( s_\varepsilon \) satisfies:
\[ s_\varepsilon B - s_\varepsilon^{p-1} A = (p-1) u_0(a) D \varepsilon^{(N-2)/2} S_\varepsilon^{p-2} \tag{3.7} \]

necessarily \( 0 < s_\varepsilon < S_\varepsilon \) and \( s_\varepsilon \to S_\varepsilon \) as \( \varepsilon \to 0 \).

Write \( s_\varepsilon = S_\varepsilon (1 - \delta_\varepsilon) \). We study the rate at which \( \delta_\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

From (3.7) we obtain:
\[ \left( \frac{B^{p-1}}{A} \right)^{1/(p-2)} (1 - \delta_\varepsilon - (1 - \delta_\varepsilon)^{p-1}) = (p-1) \frac{B}{A} (1 - \delta_\varepsilon)^{p-2} \varepsilon^{(N-2)/2} u_0(a) D; \]
and expanding for \( \delta_\varepsilon \) we derive:
\[ (p-2) \left( \frac{B^{p-1}}{A} \right)^{1/(p-2)} \delta_\varepsilon = (p-1) \frac{B}{A} u_0(a) D \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}). \]

This implies:
\[
I(u_0 + RU_{\epsilon,a}) \leq c_0 + \frac{S_\varepsilon^2}{2} B - \frac{S_\varepsilon^p}{p} A - S_\varepsilon^{p-1} u_0(a) D \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2})
\]
\[ = c_0 + \frac{1}{N} S_{\varepsilon^{p-1}} u_0(a) D \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}). \]

Therefore for \( \varepsilon_0 = \varepsilon_0 (R, a) > 0 \) sufficiently small we conclude

\[
I(u_0 + RU_{\varepsilon, a}) < c_0 + \frac{1}{N} S^{N/2}
\]  

(3.8)

\[ \forall \ 0 < \varepsilon < \varepsilon_0. \qed \]

Our aim is to state a mountain pass principle that produces a value which is below the threshold \( c_0 + \frac{1}{N} S^{N/2} \) but also compares with the value \( c_1 = \inf I_{-} \).

To this end observe that under assumption \((*)\), the manifold \( \Lambda^- \) disconnects \( H \) in exactly two connected components \( U_1 \) and \( U_2 \).

To see this, notice that for every \( u \in H, \| u \| = \| \nabla u \|_2 = 1 \) by Lemma 2.1 we can find a unique \( t^+(u) > 0 \) such that

\[
t^+(u) \in \Lambda^- \quad \text{and} \quad I(t^+(u)) = \max_{t \in [0, t_{\max}]} I(tu).
\]

The uniqueness of \( t^+(u) \) and its extremal property give that \( t^+(u) \) is a continuous function of \( u \).

Set

\[
U_1 = \left\{ u = 0 \text{ or } u : \| u \| < t^+ \left( \frac{u}{\| u \|} \right) \right\}
\]

and

\[
U_2 = \left\{ u : \| u \| > t^+ \left( \frac{u}{\| u \|} \right) \right\}.
\]

Clearly \( H - \Lambda^- = U_1 \cup U_2 \) and \( \Lambda^+ \subset U_1 \).

In particular \( u_0 \in U_1 \).

**The Proof of Theorem 4**

Easy computations show that, for suitable constant \( C_5 > 0 \) we have:

\[
0 < t^+(u) < C_5, \quad \forall u : \| u \| = 1.
\]

Set \( R_0 = \left( \frac{1}{B} |C_5^2 - \| u_0 \|^2| \right)^{1/2} + 1 \) and fix \( a \in \Sigma \) such that Lemma 3.2 applies, and the estimate (3.8) holds for all \( 0 < \varepsilon < \varepsilon_0 \).

We claim that

\[
w_\varepsilon := u_0 + R_0 \varepsilon a u_{\varepsilon, a} \in U_2 \quad (3.9)
\]

for \( \varepsilon > 0 \) small.
Indeed
\[ \|\nabla w_\varepsilon\|^2 = \|\nabla (u_0 + R_0 \zeta_\varepsilon U_{\varepsilon, a})\|^2 \]
\[ = \|u_0\|^2 + R_0^2 B + o(1) \geq C_\varepsilon^2 \left( t^\varepsilon \left( \frac{|w_\varepsilon|}{\|w_\varepsilon\|} \right) \right)^2, \]
for \( \varepsilon > 0 \) small enough.

For such a choice of \( R_0 \) and \( a \in \Sigma \), fix \( \varepsilon > 0 \) such that both (3.8) and (3.9) hold.

Set
\[ \mathcal{F} = \left\{ h : [0, 1] \rightarrow H \text{ continuous, } h(0) = u_0 \right\} \]
\[ h(1) = R_0 \zeta_\varepsilon u_{\varepsilon, a} \]
Clearly \( h : [0, 1] \rightarrow H \) given by \( h(t) = u_0 + t R_0 \zeta_\varepsilon u_{\varepsilon, a} \) belongs to \( \mathcal{F} \). So by Lemma 2.3 we conclude:
\[ c = \inf_{h \in \mathcal{F}} \max_{t \in [0, 1]} I(h(t)) < c_0 + \frac{1}{N} S^{N/2} \quad (3.10) \]
Also, since the range of any \( h \in \mathcal{F} \) intersect \( \Lambda^- \), we have
\[ c \geq c_1 = \inf_{\Lambda^-} I. \quad (3.11) \]
At this point the conclusion of Theorem 4 follows by Lemma 3.1 and a straightforward application of the mountain-pass lemma (cf. [A.R.]). \( \Box \)

The Proof of Theorem 2

Analogously to the proof of Theorem 1, one can show that the Ekeland’s variational principle gives a sequence \( \{u_n\} \subset \Lambda^- \) satisfying:
\[ I'(u_n) \rightarrow c_1 \]
\[ \|I'(u_n)\| \rightarrow 0 \]
But from (3.10) and (3.11), we have:
\[ c_1 < c_0 + \frac{1}{N} S^{N/2}. \]
Thus, by Lemma 3.1, we obtain a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) and \( u_1 \in H \) such that:
\[ u_{n_k} \rightarrow u_1 \text{ strongly in } H. \]
Consequently \( u_1 \) is a critical point for \( I, u_1 \in \Lambda^- \) (since \( \Lambda^- \) is closed) and \( I(u_1) = c_1 \).
Finally to see that \( f \geq 0 \) yields \( u_1 \geq 0 \), let \( t^* > 0 \) satisfy
\[ t^* |u_1| \in \Lambda^-. \]
From Lemma 2.1 we conclude:
\[ I(u_1) = \max_{t \geq t_{\max}} I(t^+ u_1) \geq I(t^+ |u_1|). \]
So we can always take \( u_1 \geq 0 \).

\[ \square \]

4. APPENDIX

The Proof of Lemma 2.2

Let \( \{u_n\} \) be a minimizing sequence for (2.1) such that for \( u_0 \in H \) we have \( u_n \rightharpoonup u_0 \) weakly in \( H \) and \( u_n \to u_0 \) pointwise a.e. in \( \Omega \).

In general \( \|u_0\|_p \leq 1 \). We are done once we show \( \|u_0\|_p = 1 \).

To obtain this, we shall argue by contradiction and assume
\[ \|u_0\|_p < 1. \]

Hence write \( u_n = u_0 + w_n \) where \( w_n \to 0 \) weakly in \( H \).

We have
\[ \mu_0 + o(1) = c_n \|\nabla u_n\|^{(N+2)/2} - \int_{\Omega} f u_n = c_N \left( \|\nabla u_0\|^2 + \|\nabla w_n\|^2 \right)^{(N+2)/4} \]
\[ - \int_{\Omega} f u_0 + o(1) \quad (4.1) \]

On the other hand,
\[ 1 = \|u_0 + w_n\|_p^p = \|u_0\|_p^p + \|w_n\|_p^p + o(1) \]
(see [B.L.]), which gives:
\[ \|w_n\|_p^p = (1 - \|u_0\|_p^p)^{2/p} + o(1). \]

So from (4.1) we conclude:
\[ \mu_0 + o(1) = c_N \left( \|\nabla u_0\|^2 + \|\nabla w_n\|^2 \right)^{(N+2)/4} - \int_{\Omega} f u_0 \]
\[ \geq c_N \left( \|\nabla u_0\|^2 + S \left( 1 - \|u_0\|_p^p \right)^{2/p} + o(1) \right)^{(N+2)/4} - \int_{\Omega} f u_0. \]

That is,
\[ c_N \left[ \|\nabla u_0\|^2 + S \left( 1 - \|u_0\|_p^p \right)^{2/p} \right]^{(N+2)/4} - \int_{\Omega} f u_0 \leq \mu_0. \quad (4.2) \]

Following [B.N.1] for every \( u \in H \), \( \|u\|_p < 1 \) and \( a \in \Omega \) let \( c_e = c_e(a) > 0 \) satisfy the following:
\[ \|u + c_e U_{e,a}\|_p = 1 \]

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[recall \( U_{e,a}(x) = \tilde{\xi}_a(x) u_{e,a}(x) \) with \( \tilde{\xi}_a \) and \( u_{e,a} \) given in (1.6) and (1.7)].

We have:

\[
\|\nabla (u + c_\varepsilon U_{e,a})\|_2^2 = \|\nabla u\|_2^2 + c_\varepsilon^2 \|\nabla U_{e,a}\|_2^2 + o(1)
= \|\nabla u\|_2^2 + c_\varepsilon^2 B + o(1) \quad (4.3)
\]

and

\[
1 = \|u + c_\varepsilon U_{e,a}\|^p_p = \|u\|^p_p + c_\varepsilon^p \|U_{e,a}\|^p_p + o(1) = \|u\|^p_p + c_\varepsilon^p A + o(1)
\]

[A, B as given in (3.6)].

Thus

\[
c_\varepsilon^2 = \frac{1}{A^{2/p}} (1 - \|u\|^p_p)^{2/p} + o(1). \quad (4.4)
\]

Substituting in (4.3) we obtain:

\[
\|\nabla (u + c_\varepsilon U_{e,a})\|_2^2 = \|\nabla u\|_2^2 + \frac{B}{A^{2/p}} (1 - \|u\|^p_p)^{2/p} + o(1)
= \|\nabla u\|_2^2 + S (1 - \|u\|^p_p)^{2/p} + o(1).
\]

This yields:

\[
\mu_0 \leq c_N \|\nabla (u + c_\varepsilon U_{e,a})\|_{(N+2)/2}^{(N+2)/2} \int_\Omega f(u + c_\varepsilon U_{e,a})
= c_N (\|\nabla u\|_2^2 + S (1 - \|u\|^p_p)^{(N+2)/4} - \int_\Omega f u + o(1),
\]

and passing to the limit as \( \varepsilon \to 0 \), we derive:

\[
\mu_0 \leq c_N [\|\nabla u\|_2^2 + S (1 - \|u\|^p_p)^{(N+2)/4} - \int_\Omega f u, \quad \forall u \in H, \quad \|u\|_p < 1.
\]

Therefore from (4.2) we conclude:

\[
c_N [\|\nabla u_0\|_2^2 - S (1 - \|u_0\|^p_p)^{(N+2)/4} - \int_\Omega f u = \mu_0 \quad (4.5)
\]

and that for every \( w \in H \) necessarily:

\[
\frac{d}{dt} \left[ c_N [\|\nabla (u_0 + tw)\|_2^2 + S (1 - \|u_0 + tw\|^p_p)^{(N+2)/4} - \int_\Omega f(u_0 + tw) \right]_{t=0} = 0.
\]

That is:

\[
\frac{N+2}{2} c_N \left[\|\nabla u_0\|_2^2 + S (1 - \|u_0\|^p_p)^{(N+2)/4}\right]^{(N-2)/4}
\times \left[\int_\Omega \nabla u_0 \cdot \nabla w - S (1 - \|u_0\|^p_p)^{(2-p)/p} \int_\Omega \|u_0\|^p_p - \int_\Omega \right] w = w = 0, \quad \forall w \in H.
\]
So setting \( \sigma_0 = \frac{N+2}{2} c_N \left[ \| \nabla u_0 \|_2^2 + S (1 - \| u_0 \|_p^{2/p})^{(N-2)/4} \right] > 0 \) and

\[ \lambda_0 = \frac{S}{1 - \| u_0 \|_p^{(p-2)/p}} \]

we obtain that \( u_0 \) weakly satisfies:

\[-\Delta u_0 = \lambda_0 \| u_0 \|^{p-2} u_0 + \frac{1}{\sigma_0} f. \quad (4.5)\]

Since \( f \neq 0 \), in particular, we have that \( u_0 \neq 0 \).

Hence for a set of positive measure \( \Sigma \subset \Omega \) we have:

\[ u_0 (a) > 0, \quad \forall \ a \in \Sigma, \]

(replace \( u_0 \) with \( -u_0 \) and \( f \) with \( -f \) if necessarily).

Let \( a \in \Sigma \) and \( c_\varepsilon = c_\varepsilon (a) \) satisfy:

\[ \| u_0 + c_\varepsilon U_{\varepsilon, a} \|_p = 1. \]

We will reach a contradiction by showing that

\[ I (u_0 + c_\varepsilon U_{\varepsilon, a}) < \mu_0 \]

for a suitable choice of \( a \in \Sigma \) and \( \varepsilon > 0 \) small enough.

To this end, let \( c_0^p = \frac{1 - \| u_0 \|_p^p}{A} \). From \( (4.4) \) it follows that \( c_\varepsilon > c_0 \) as \( \varepsilon \to 0 \). Set \( c_\varepsilon = c_0 (1 - \delta_\varepsilon) \), \( \delta_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). In [B.N.1], Brezis-Nirenberg have obtained a precise rate at which \( \delta_\varepsilon \to 0 \), by showing that, for a.e. \( a \in \Sigma \), one has:

\[ \delta_\varepsilon A c_0^p = \varepsilon^{(N-2)/2} \left[ c_0 \int \Omega \left| u_0 (x) \right| u_0^{p-2} (x) \xi_a (x) \frac{dx}{|x-a|^{N-2}} + c_0^{p-1} u_0 (a) D \right] + o (\varepsilon^{(N-2)/2}) \quad (4.7) \]

with

\[ D = \int_{\mathbb{R}^N} \frac{dx}{(\varepsilon^2 + |x|^2)^{(N+2)/2}}. \quad (See \ formula \ (2.9) \ in \ [B.N.1].) \]

Now fix \( a \in \Sigma \) for which \( (4.7) \) holds and

\[ \int_{\Omega} \left| \frac{u_0^{p-2} u_0 \xi_a}{\varepsilon^2 + |x-a|^2} \right|^{(N-2)/2} \to \int_{\Omega} \left| \frac{u_0^{p-2} u_0 \xi_a}{|x-a|^2} \right|^{(N-2)/2} \quad \text{as} \quad \varepsilon \to 0. \quad (4.8) \]
Using (4.5), (4.7) and the definition of $c_0$ we obtain:

$$I(u_0 + c_0 U_{e,a}) = c_N \left[ \frac{1}{2} \left( \sum_{\Omega} \nabla \cdot u_0 \right)^2 + 2 c_0 \sum_{\Omega} \nabla \cdot \left( \nabla u_0 \cdot \nabla U_{e,a} + c_0^2 \nabla U_{e,a} \right) \right]^{(N+2)/4}$$

$$- \int_{\Omega} f u_0 - c_0 \int_{\Omega} f U_{e,a}$$

$$= c_N \left[ \frac{1}{2} \left( \sum_{\Omega} \nabla \cdot u_0 \right)^2 + 2 c_0 \sum_{\Omega} \nabla \cdot \left( \nabla u_0 \cdot \nabla U_{e,a} + c_0^2 \nabla \right) \right]^{(N+2)/4}$$

$$- \int_{\Omega} f u_0 - c_0 \int_{\Omega} f U_{e,a} = c_N \left[ \frac{1}{2} \left( \sum_{\Omega} \nabla \cdot u_0 \right)^2 + c_0^2 \sum_{\Omega} \nabla \cdot \left( 1 - 2 \delta \right) B + o \left( \varepsilon^{(N-2)/2} \right) \right]$$

$$+ \frac{N+2}{4} c_N \left[ \frac{1}{2} \left( \sum_{\Omega} \nabla \cdot u_0 \right)^2 + c_0^2 \sum_{\Omega} \nabla \cdot \left( 1 - 2 \delta \right) B \right]^{(N-2)/4}$$

$$- 2 c_0^2 \delta B - c_0 \int_{\Omega} f U_{e,a}$$

$$+ o \left( \varepsilon^{(N-2)/2} \right) = \mu_0 + c_0 \left[ \sigma_0 \int_{\Omega} \nabla \cdot \left( u_0 \cdot \nabla U_{e,a} \right) \right]$$

$$- \int_{\Omega} f U_{e,a} - \sigma_0 c_0^2 B \delta + o \left( \varepsilon^{(N-2)/2} \right).$$

Thus from equation (4.6) we derive:

$$I(u_0 + c_0 U_{e,a}) = \mu_0 + \sigma_0 \lambda_0 c_0 \int_{\Omega} \left| u_0 \right|^{p-2} u_0 U_{e,a} - \delta_0 c_0^2 B \delta + o \left( \varepsilon^{(N-2)/2} \right).$$

On the other hand from (4.8) we have:

$$\int_{\Omega} \left| u_0 \right|^{p-2} u_0 U_{e,a} = \varepsilon^{(N-2)/2} \int_{\Omega} \left| u_0(x) \right|^{p-2} u_0(x) \frac{\xi_a(x)}{|x-a|^{N-2}} dx + o \left( \varepsilon^{(N-2)/2} \right).$$

Therefore:

$$I(u_0 + c_0 U_{e,a}) = \mu_0 + \sigma_0 \left[ \varepsilon^{(N-2)/2} \lambda_0 \int_{\Omega} \left| u_0(x) \right|^{p-2} u_0(x) \frac{\xi_a(x)}{|x-a|^{N-2}} - c_0^2 \delta \right] + o \left( \varepsilon^{(N-2)/2} \right)$$

$$= \mu_0 + \sigma_0 \left[ \frac{\varepsilon^{(N-2)/2}}{1 - \left| u_0 \right|^{p(p-2)/2}} \int_{\Omega} \left| u_0 \right|^{p-2} u_0(x) \frac{\xi_a(x)}{|x-a|^{N-2}} - B c_0^2 \delta \right] + o \left( \varepsilon^{(N-2)/2} \right)$$

$$= \mu_0 + \sigma_0 \left[ \frac{\varepsilon^{(N-2)/2}}{\lambda_0 \left| u_0 \right|^{p-2} c_0^2 \delta} \right] + o \left( \varepsilon^{(N-2)/2} \right)$$

$$= \mu_0 + \sigma_0 \left[ \frac{B}{A c_0^2 \delta} \right] + o \left( \varepsilon^{(N-2)/2} \right).$$

Finally, from (4.7) we conclude:

\[ I\left(u_0 + \varepsilon \, U_{\varepsilon, a}\right) = \mu_0 - \sigma_0 \frac{B}{A} u_0(a) D \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}) < \mu_0 \]

for \( \varepsilon > 0 \) sufficiently small.

REFERENCES


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