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Relaxation for a class of nonconvex functionals defined on measures

by

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ABSTRACT. – We characterize in a suitable integral form like

$$\bar{F}(\lambda)=\int_{\Omega}\bar{f}\left(x,\frac{d\lambda}{d\mu}\right)d\mu+\int_{\Omega\setminus A_\lambda}\bar{\varphi}(x,\lambda^s)+\int_{A_\lambda}\bar{g}(x,\lambda(x))\,d\#$$

the lower semicontinuous envelope $\bar{F}$ of functionals F defined on the space $\mathcal{M}(\Omega;\mathbf{R}^n)$ of all $\mathbf{R}^n$-valued measures with finite variation on Ω.

RÉSUMÉ. – On établit une représentation intégrale de la forme:

$$\bar{F}(\lambda)=\int_{\Omega}\bar{f}\left(x,\frac{d\lambda}{d\mu}\right)d\mu+\int_{\Omega\setminus A_\lambda}\bar{\varphi}(x,\lambda^s)+\int_{A_\lambda}\bar{g}(x,\lambda(x))\,d\#$$

pour la régularisée semicontinue inférieure $\bar{F}$ d'une fonctionnelle F définie sur l'espace $\mathcal{M}(\Omega,\mathbf{R}^n)$ des mesures à variation bornée sur Ω à valeurs dans $\mathbf{R}^n$.

Classification A.M.S. : 49 J 45 (Primary), 46 G 10, 46 E 27.

1. INTRODUCTION

In a previous paper [3] we introduced a new class of nonconvex functionals defined on the space $\mathcal{M}(\Omega; \mathbf{R}^n)$ of all $\mathbf{R}^n$-valued measures with finite variation on Ω of the form

$$F(\lambda)=\int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu+\int_{\Omega \setminus A_\lambda} \varphi(x, \lambda^s)+\int_{A_\lambda} g(x, \lambda(x))\, d\# \quad (1.1)$$

where $(d\lambda/d\mu)\mu+\lambda^s$ is the Lebesgue-Nikodym decomposition of λ, A_λ is the set of atoms of λ, $\lambda(x)$ denotes the value $\lambda(\{x\})$, and $\#$ is the counting measure (we refer to Section 2 for further details). For this kind of functionals we proved in [3] (*see* Theorem 2.4 below), under suitable hypotheses on f, φ, g, a lower semicontinuity result with respect to the weak* $\mathcal{M}(\Omega; \mathbf{R}^n)$ convergence.

In a subsequent paper [4] we characterized all weakly* lower semicontinuous functionals on $\mathcal{M}(\Omega; \mathbf{R}^n)$ satisfying the additivity condition

$$F(\lambda+\nu)=F(\lambda)+F(\nu) \quad \text{for every } \lambda, \quad \nu \in \mathcal{M}(\Omega; \mathbf{R}^n) \quad \text{with } \lambda \perp \nu \quad (1.2)$$

and we proved that they are all of the form (1.1) for suitable integrands f, φ, g.

In the present paper we deal with funtionals F of the form

$$F(\lambda)=\begin{cases} \displaystyle\int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu+\int_{A_\lambda} g(x, \lambda(x))\, d\# & \text{if } \lambda^s=0 \text{ on } \Omega\setminus A_\lambda \\ +\infty & \text{otherwise} \end{cases}$$

and we consider their (sequential) lower semicontinuous envelope $\bar{F}$ defined by

$$\bar{F}=\sup\{G : G \leqq F,\ G \text{ sequentially weakly* l.s.c. on } \mathcal{M}(\Omega; \mathbf{R}^n)\}.$$

We prove in Theorem 3.1 that $\bar{F}$ satisfies the additivity condition (1.2) so that, by the results of [4], it can be written in the integral form

$$\bar{F}(\lambda)=\int_\Omega \bar{f}\left(x, \frac{d\lambda}{d\bar{\mu}}\right) d\bar{\mu}+\int_{\Omega \setminus A_\lambda} \bar{\varphi}(x, \lambda^s)+\int_{A_\lambda} \bar{g}(x, \lambda(x))\, d\#$$

for suitable $\bar{\mu}$, $\bar{f}$, $\bar{\varphi}$, $\bar{g}$. An explicit way to construct $\bar{\mu}$, $\bar{f}$, $\bar{\varphi}$, $\bar{g}$ in terms of μ, f, g is given (*see* Theorem 3.2), and this is applied in Example 3.4 to the case $f(x, s)=|s|^p$ and $g(x, s)=|s|^q$ with $p \in [1+\infty]$ and $q \in [0, 1]$.

2. NOTATION AND PRELIMINARY RESULTS

In this section we fix the notation we shall use in the following; we recall them only briefly because they are the same used in Bouchitté &

Buttazzo [3] and [4], to which we refer for further details. In all the paper $(\Omega, \mathscr{B}, \mu)$ will denote a measure space, where Ω is a separable locally compact metric space with distance d, $\mathscr{B}$ is the σ-algebra of all Borel subsets of Ω, and $\mu : \mathscr{B} \to [0, +\infty[$ is a positive, finite, non-atomic measure. We shall use the following symbols:

– $C_0(\Omega; \mathbf{R}^n)$ is the space of all continuous functions $u: \Omega \to \mathbf{R}^n$ "vanishing on the boundary", that is such that for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ with $|u(x)| < \varepsilon$ for all $x \in \Omega \setminus K_\varepsilon$.

– $\mathscr{M}(\Omega; \mathbf{R}^n)$ is the space of all vector-valued measures $\lambda : \mathscr{B} \to \mathbf{R}^n$ with finite variation on Ω.

– $|\lambda|$ is the variation of $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$ defined for every $B \in \mathscr{B}$ by

$$|\lambda|(B) = \sup \left\{ \sum_{h=1}^{\infty} |\lambda(B_h)| : \bigcup_{h=1}^{\infty} B_h \subset B,\ B_h \text{ pairwise disjoint} \right\}.$$

– $\lambda_h \to \lambda$ indicates the convergence of λ_h to λ in the weak* topology of $\mathscr{M}(\Omega; \mathbf{R}^n)$ deriving from the duality between $\mathscr{M}(\Omega; \mathbf{R}^n)$ and $C_0(\Omega; \mathbf{R}^n)$.

– $\lambda \ll \mu$ indicates that λ is absolutely continuous with respect to μ, that is $|\lambda|(B) = 0$ whenever $B \in \mathscr{B}$ and $\mu(B) = 0$.

– $\lambda \perp \mu$ indicates that λ is singular with respect to μ, that is $|\lambda|(\Omega \setminus B) = 0$ for a suitable $B \in \mathscr{B}$ with $\mu(B) = 0$.

– $u\mu$ with $u \in L^1(\Omega; \mathbf{R}^n; \mu)$, is the measure of $\mathscr{M}(\Omega; \mathbf{R}^n)$ (often indicated simply by u) defined by

$$(u\mu)(B) = \int_B u \, d\mu \quad \text{for every } B \in \mathscr{B}.$$

It is well-known that every measure $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$ which is absolutely continuous with respect to μ is representable in the form $\lambda = u\mu$ for a suitable $u \in L^1(\Omega; \mathbf{R}^n; \mu)$; moreover, by the Lebesgue-Nikodym decomposition theorem, for every $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$ there exists a unique function $u \in L^1_\mu(\Omega; \mathbf{R}^n)$ (often indicated by $d\lambda/d\mu$) and a unique measure $\lambda^s \in \mathscr{M}(\Omega; \mathbf{R}^n)$ such that

$$\begin{cases} \text{(i) } \lambda = u\mu + \lambda^s \\ \text{(ii) } \lambda^s \text{ is singular with respect to } \mu. \end{cases}$$

– $u\lambda$ with $u: \Omega \to \mathbf{R}$ a bounded Borel function and $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$, is the measure of $\mathscr{M}(\Omega; \mathbf{R}^n)$ defined by

$$(u\lambda)(B) = \int_B u \, d\lambda \quad \text{for every } B \in \mathscr{B}.$$

– 1_B with $B \subset \Omega$, is the function

$$1_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \in \Omega \setminus B. \end{cases}$$

– δ_x with $x \in \Omega$, is the measure of $\mathcal{M}(\Omega; \mathbf{R}^n)$

$$\delta_x(B) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \in \Omega \setminus B. \end{cases}$$

– $\mathcal{M}^0(\Omega; \mathbf{R}^n)$ is the space of all non-atomic measures of $\mathcal{M}(\Omega; \mathbf{R}^n)$.

– $\mathcal{M}^{\#}(\Omega; \mathbf{R}^n)$ is the space of all "purely atomic" measures of $\mathcal{M}(\Omega; \mathbf{R}^n)$, that is the measures of the form

$$\lambda = \sum_{i=1}^{\infty} a_i \delta_{x_i} \qquad (x_i \in \Omega,\ a_i \in \mathbf{R}^n).$$

– $\lambda(x)$ with $x \in \Omega$ and $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$, denotes the quantity $\lambda(\{x\})$.

– A_λ is the set of all atoms of λ, that is

$$A_\lambda = \{x \in \Omega : \lambda(x) \neq 0\}.$$

– $\int_B \varphi(x, \lambda)$ with $B \in \mathcal{B}$, $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$, and $\varphi : \Omega \times \mathbf{R}^n \to [0, +\infty]$ a Borel function such that $\varphi(x, .)$ positively 1-homogeneous for every $x \in \Omega$, denotes the quantity

$$\int_B \varphi\left(x, \frac{d\lambda}{d\upsilon}\right) d\upsilon$$

which (*see* for instance Goffman and Serrin [12]) does not depend on υ, when υ varies over all positive measures such that $|\lambda| \ll \upsilon$.

– f^* with $f: \mathbf{R}^n \to]-\infty, +\infty]$ proper function, is the usual conjugate function of f

$$f^*(s) = \sup\{sw - f(w) : w \in \mathbf{R}^n\} \qquad (s \in \mathbf{R}^n).$$

– f^∞ with $f: \mathbf{R}^n \to]-\infty, +\infty]$ proper function, is the usual recession function of f

$$f^\infty(s) = \sup\{f(s+t) - f(t) : t \in \mathbf{R}^n, f(t) < +\infty\} \qquad (s \in \mathbf{R}^n).$$

It is well-known that when f is convex l.s.c. and proper, f^* is convex l.s.c. and proper too, and we have $f^{**} = f$; moreover, in this case, for the recession function f^∞ the following formula holds (*see* for instance Rockafellar [16]):

$$f^\infty(s) = \lim_{t \to +\infty} \frac{f(s_0 + ts)}{t}$$

where s_0 is any point such that $f(s_0) < +\infty$. It can be shown that the definition above does not depend on s_0, and that the function f^∞ turns out to be convex, l.s.c., and positively 1-homogeneous on $\mathbf{R}^n$.

– $\varphi_{f,\mu}$ with $f:\Omega\times\mathbf{R}^n\to[0,+\infty]$ a Borel function such that $f(x,.)$ is convex l.s.c. and proper for μ-a.e. $x\in\Omega$, denotes the function

$$\varphi_{f,\mu}(x,s)=\sup\left\{u(x)s:u\in C_0(\Omega;\mathbf{R}^n),\ \int_\Omega f^*(x,u)\,d\mu<+\infty\right\}$$

defined for every $(x,s)\in\Omega\times\mathbf{R}^n$. The function $\varphi_{f,\mu}(x,s)$ is l.s.c. in $(\mathbf{x},s)$, convex and positively 1-homogeneous in s, and we have (*see* for instance Bouchitté and Valadier [5], Proposition 7)

$$\begin{cases}\varphi_{f,\mu}(x,.)\leqq f^\infty(x,.) & \text{for } \mu\text{-a.e. } x\in\Omega;\\ \varphi_{f,\mu}\geqq f^\infty & \text{if the multimapping } x\to\operatorname{epi} f^*(x,.) \text{ is l.s.c. on } \Omega.\end{cases}$$

– g^0 with $g:\mathbf{R}^n\to[0,+\infty]$ a function such that $g(0)=0$, is the function defined by

$$g^0(s)=\limsup_{t\to0^+}\frac{g(ts)}{t}\qquad(s\in\mathbf{R}^n).$$

– g *subadditive* with $g:\mathbf{R}^n\to[0,+\infty]$ a function such that $g(0)=0$, will mean that

$$g(s_1+s_2)\leqq g(s_1)+g(s_2)\quad\text{for every } s_1, s_2\in\mathbf{R}^n.$$

We remark that g is subadditive if and only if $g^\infty\leqq g$, hence $g^\infty=g$ for every subadditive function g with $g(0)=0$.

– $\alpha\nabla\beta$ with $\alpha,\beta:\mathbf{R}^n\to[0,+\infty]$ denotes the inf-convolution

$$(\alpha\nabla\beta)(s)=\inf\{\alpha(t)+\beta(s-t):t\in\mathbf{R}^n\}.$$

It is easy to see that

$$\begin{cases}f\nabla f^\infty=f & \text{for every } f:\mathbf{R}^n\to[0,+\infty] \text{ convex, l.s.c., proper;}\\ g\nabla g=g & \text{for every } g:\mathbf{R}^n\to[0,+\infty] \text{ subadditive, with } g(0)=0.\end{cases}$$

We also recall some preliminary results which will be used in the following.

PROPOSITION 2.1: (*see* Bouchitté and Buttazzo [3], Proposition 2.2). – *Let $g:\mathbf{R}^n\to[0,+\infty]$ be a subadditive l.s.c. function, with $g(0)=0$. Then we have*:

(i) *the function $g^0:\mathbf{R}^n\to[0,+\infty]$ is convex, l.s.c., and positively 1-homogeneous;*

(ii) $g^0(s)=\sup_{t>0}\frac{g(ts)}{t}=\lim_{t\to0^+}\frac{g(ts)}{t}$ *for every $s\in\mathbf{R}^n$.*

PROPOSITION 2.2: (*see* Bouchitté and Buttazzo [3], Proposition 2.4). – *Let $\alpha,\beta:\mathbf{R}^n\to[0,+\infty]$ be two convex l.s.c. and proper functions, with α*

such that

$$\lim_{|s| \to +\infty} \alpha(s) = +\infty.$$

Then we have:

(i) $\alpha \nabla \beta$ *is l.s.c. and* $\alpha \nabla \beta = (\alpha^* + \beta^*)^*$;

(ii) $\alpha \nabla \beta_h \uparrow \alpha \nabla \beta$ *for every sequence* $\beta_h : \mathbf{R}^n \to [0, +\infty]$ *of l.s.c. functions with* $\beta_h \uparrow \beta$.

PROPOSITION 2.3. — *Let* $f, g : \mathbf{R}^n \to [0, +\infty]$ *be two subadditive l.s.c. functions with* $f(0) = g(0) = 0$. *Assume that for a suitable* $\alpha > 0$ *it is*

$$f(s) \geqq \alpha |s| \quad \text{for every } s \in \mathbf{R}^n. \tag{2.1}$$

Then we have

$$(f \nabla g)^0 = f^0 \nabla g^0.$$

Proof. — The inequalities $(f \nabla g)^0 \leqq f^0$ and $(f \nabla g)^0 \leqq f^0$ imply that

$$(f \nabla g)^0 \leqq f^0 \nabla g^0.$$

Let us prove the opposite inequality. Let us fix $s \in \mathbf{R}^n$ with $(f \nabla g)^0 (s) = C < +\infty$ and for every $t > 0$ let $s_t \in \mathbf{R}^n$ be such that

$$(f \nabla g)(ts) = f(ts_t) + g(ts - ts_t). \tag{2.2}$$

By (2.1) and (2.2) we have for every $t > 0$

$$\alpha |s_t| \leqq \frac{f(ts_t)}{t} \leqq \frac{(f \nabla g)(ts)}{t} \leqq (f \nabla g)^0 (s) = C$$

so that we may assume $s_t \to z$ as $t \to 0$. For every $\varepsilon > 0$ and $w \in \mathbf{R}^n$ set

$$f_\varepsilon(w) = \sup \{ ww^* : tw^* \leqq f(t) \quad \text{for every } |t| \leqq \varepsilon \}$$
$$g_\varepsilon(w) = \sup \{ ww^* : tw^* \leqq g(t) \quad \text{for every } |t| \leqq \varepsilon \}.$$

Fix $\varepsilon > 0$; by Proposition 2.3 of Bouchitté and Buttazzo [3] we have for every t small enough

$$\frac{f(ts_t) + g(ts - ts_t)}{t} \geqq f_\varepsilon(s_t) + g_\varepsilon(s - s_t),$$

so that, passing to the lim inf as $t \to 0$, and taking into account (2.2)

$$(f \nabla g)^0 (s) \geqq f_\varepsilon(z) + g_\varepsilon(s - z).$$

Finally, passing to the limit as $\varepsilon \to 0$, by Proposition 2.3 of [3] again, we get

$$(f \nabla g)^0 (s) \geqq f^0(z) + g^0(s - z) \geqq (f^0 \nabla g^0)(s). \quad \blacksquare$$

We shall deal with functionals defined on $\mathcal{M}(\Omega; \mathbf{R}^n)$ of the form

$$F(\lambda)=\int_{\Omega} f\left(x, \frac{d\lambda}{d\mu}\right) d\mu+\int_{\Omega \setminus A_\lambda} \varphi(x, \lambda^s)+\int_{A_\lambda} g(x, \lambda(x))\, d\#. \quad (2.3)$$

For this kind of functionals we proved in [3] a result of lower semicontinuity with respect to the weak* convergence in $\mathcal{M}(\Omega; \mathbf{R}^n)$. More precisely, the following theorem holds.

THEOREM 2.4. – *Let* $\mu \in \mathcal{M}(\Omega)$ *be a non-atomic positive measure and let* $f, \varphi, g: \Omega \times \mathbf{R}^n \rightarrow [0, +\infty]$ *be three Borel functions such that*

(H_1) $f(x, .)$ *is convex and l.s.c. on* $\mathbf{R}^n$, *and* $f(x, 0)=0$ *for* μ-a.e. $x \in \Omega$,

(H_2) $f^{\infty}(x, .)=\varphi(x, .)=\varphi_{f, \mu}(x, .)$ *for* μ-a.e. $x \in \Omega$,

(H_3) g *is l.s.c. on* $\Omega \times \mathbf{R}^n$, *and* $g(x, 0)=0$ *for every* $x \in \Omega$,

(H_4) $g(x, .)$ *is subadditive for all* $x \in \Omega$, *and* $g \leqq \varphi_{f, \mu}$ *on* $\Omega \times \mathbf{R}^n$,

(H_5) $g^0=\varphi$ *on* $(\Omega \setminus N) \times \mathbf{R}^n$, *where* N *is a suitable countable subset of* Ω,

Then the functional F *defined in* (2.3) *is sequentially weakly* l.s.c. on* $\mathcal{M}(\Omega; \mathbf{R}^n)$.

Remark 2.5. – The assumption $\varphi=\varphi_{f, \mu}$ on $(\Omega \setminus N) \times \mathbf{R}^n$ with N countable, of Theorem 3.3 of Bouchitté & Buttazzo [3], has been replaced here by the weaker one $\varphi=\varphi_{f, \mu}$ on $(\Omega \setminus M) \times \mathbf{R}^n$ with $\mu(M)=0$. A careful inspection of our proof shows indeed that this weaker condition is still sufficient to provide the lower semicontinuity of F.

Remark 2.6. – A slightly more general form of the lower semicontinuity Theorem 2.4 can be given (*see* Bouchitté and Buttazzo [4]) by requiring, instead of (H_4), that

(i) the set D_g has no accumulation points,

(H'_4) (ii) the function g^{∞} is l.s.c. on $\Omega \times \mathbf{R}^n$,

(iii) $g^{\infty} \leqq \varphi_{f, \mu}$ nd $g^{\infty} \leqq \hat{g}$ on $\Omega \times \mathbf{R}^n$,

where D_g and $\hat{g}$ are defined by

$$D_g=\{x \in \Omega: g(x, .) \text{ is not subadditive}\}$$
$$\hat{g}(x, s)=\liminf_{\substack{(y, t) \rightarrow (x, s) \\ y \neq x}} g(y, t).$$

The fact that all additive sequentially weakly* l.s.c. functionals on $\mathcal{M}(\Omega; \mathbf{R}^n)$ are of the form (2.3) has been shown in [4], where the following result is proved.

THEOREM 2.7: (*see* Bouchitté and Buttazzo [4], Theorem 2.3). – *Let* $F: \mathcal{M}(\Omega, \mathbf{R}^n) \rightarrow [0, +\infty]$ *be a functional such that*

(i) F *is additive* (i.e. $F(\lambda+\nu)=F(\lambda)+F(\nu)$ *whenever* $\lambda \perp \nu$);

(ii) F *is sequentially weakly* l.s.c. on* $\mathcal{M}(\Omega; \mathbf{R}^n)$.

Then there exist a non-atomic positive measure $\mu \in \mathcal{M}(\Omega)$ *and three Borel funtions* f, φ, g: $\Omega \times \mathbf{R}^n \to [0, +\infty]$ *which satisfy*

(H_1) $f(x, .)$ *is convex and l.s.c. on* $\mathbf{R}^n$, *and* $f(x,0)=0$ *for* μ-a.e. $x \in \Omega$,

(H_2) $f^\infty(x, .) = \varphi_{f,\mu}(x, .)$ *for* μ-a.e. $x \in \Omega$,

(H_3) g *and* g^∞ *are l.s.c. on* $\Omega \times \mathbf{R}^n$, *and* $g(x,0)=0$ *for every* $x \in \Omega$,

(H_4) $g^\infty \leqq \varphi_{f,\mu}$ *and* $g^\infty \leqq \hat{g}$ *on* $\Omega \times \mathbf{R}^n$,

(H_5) $g^0 = \varphi = \varphi_{f,\mu}$ *on* $(\Omega \setminus \mathrm{N}) \times \mathbf{R}^n$, *where* N *is a suitable countable subset of* Ω, *and such that for every* $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$ *the integral representation* (2.3) *holds.*

3. RELAXATION

The main application of Theorem 2.7 consists in representing into an integral form the relaxed functionals associated to additive functionals defined on $\mathcal{M}(\Omega; \mathbf{R}^n)$. More precisely, given a functional $\mathrm{F}: \mathcal{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$, we consider its relaxed functional $\bar{\mathrm{F}}$ defined by

$$\bar{\mathrm{F}} = \sup \{ \mathrm{G} : \mathrm{G} \leqq \mathrm{F},\ \mathrm{G} \text{ sequentially weakly* l.s.c. on } \mathcal{M}(\Omega; \mathbf{R}^n) \}.$$

The functional $\bar{\mathrm{F}}$ above is sequentially weakly* l.s.c. and less than or equal to F on $\mathcal{M}(\Omega; \mathbf{R}^n)$. We shall apply Theorem 2.7 to $\bar{\mathrm{F}}$ thanks to the following result.

THEOREM 3.1. – *Let* $\mathrm{F}: \mathcal{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$ *be additive; then* $\bar{\mathrm{F}}$ *is additive too.*

Our goal is to characterize the functional $\bar{\mathrm{F}}$ when F is of the form

$$\mathrm{F}(\lambda) = \begin{cases} \displaystyle\int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\mathrm{A}_\lambda} g(x, \lambda(x))\, d\# & \text{if } \lambda^s = 0 \text{ on } \Omega \setminus \mathrm{A}_\lambda \\ +\infty & \text{otherwise} \end{cases}$$

where $\mu \in \mathcal{M}(\Omega)$ is a non-atomic positive measure and f, $g: \Omega \times \mathbf{R}^n \to [0, +\infty]$ are two Borel functions satisfying the following assumptions:

$f(x, .)$ is convex and l.s.c. on $\mathbf{R}^n$, and $f(x, 0) = 0$ for μ-a.e. $x \in \Omega$ (3.1)

There exist $\alpha > 0$ and $\beta \in \mathrm{L}^1_\mu$ such that:

$$f(x, s) \geqq \alpha |s| - \beta(x), \qquad \forall (x, s) \in \Omega \times \mathbf{R}^n \tag{3.2}$$

g is l.s.c. on $\Omega \times \mathbf{R}^n$, and $g(x, 0) = 0$ for every $x \in \Omega$ (3.3)

$g(x, .)$ is subadditive for every $x \in \Omega$ (3.4)

$g^0(x, s) \geqq \alpha |s|$ for every $(x, s) \in \Omega \times \mathbf{R}^n$. (3.5)

By Theorem 3.1 we may apply the integral representation Theorem 2.7 to $\bar{F}$ and we obtain

$$\bar{F}(\lambda)=\int_{\Omega}\bar{f}\left(x, \frac{d\lambda}{d\bar{\mu}}\right)d\bar{\mu}+\int_{\Omega\setminus A_\lambda}\bar{\varphi}(x, \lambda^s)+\int_{A_\lambda}\bar{g}(x, \lambda(x))\,d\#.$$

for a suitable non-atomic positive measure $\bar{\mu}\in\mathcal{M}(\Omega)$ and suitable Borel functions $\bar{f}$, $\bar{\varphi}$, $\bar{g}:\Omega\times\mathbf{R}^n\to[0, +\infty]$ satisfying conditions (H_1)-(H_5) of Theorem 2.7. In order to characterize these integrands we introduce the functional

$$F_1(\lambda)=\int_{\Omega}f_1\left(x, \frac{d\lambda}{d\mu}\right)d\mu+\int_{\Omega\setminus A_\lambda}\varphi_1(x, \lambda^s)+\int_{A_\lambda}g_1(x, \lambda(x))\,d\#$$

where

$$f_1=f\,\nabla\,\varphi_{f,\mu}\,\nabla\,g^0,\qquad \varphi_1=\varphi_{f,\mu}\,\nabla\,g^0,\qquad g_1=\varphi_{f,\mu}\,\nabla\,g.$$

The main result of this paper is the following relaxation theorem.

THEOREM 3.2. – *For every* $\lambda\in\mathcal{M}(\Omega;\mathbf{R}^n)$ *we have*

$$\bar{F}(\lambda)=F_1(\lambda).$$

Remark 3.3. – We may consider on g the following weaker assumptions instead of (3.4):
There exists a subset D *of* Ω, *which has no accumulation points, such that* $g(x, .)$ *is subadditive for every* $x\in\Omega\setminus D$, *and the function* g^∞ *is l.s.c. in* (x, s).

The conclusion will be the same.

Example 3.4. – Let $p\in[1, +\infty]$, $q\in[0, 1]$, and let

$$f(s)=|s|^p,\qquad g(s)=|s|^q.$$

In the case $p=+\infty$ we set $f=\chi_{\{|s|\leq 1\}}$ (*i.e.* the function which is 0 if $|s|\leq 1$ and $+\infty$ otherwise), and in the case $q=0$ we set $g=1_{\mathbf{R}\setminus\{0\}}$ (*i.e.* the function which is 1 if $s\neq 0$ and 0 if $s=0$). Then we have

$$\begin{aligned} p>1,\quad q<1 &\Rightarrow \bar{f}=f,\quad \bar{g}=g\\ p=1,\quad q=1 &\Rightarrow \bar{f}=f,\quad \bar{g}=g \end{aligned}$$

that is the associated functional F is sequentially weakly* lower semicontinuous. In the remaining cases, F is not sequentially weakly* lower semicontinuous and, after some calculations, one finds

$$\begin{aligned} p>1,\quad q=1 &\Rightarrow \bar{g}=g,\quad \bar{f}(s)=(f\,\nabla\,|.|)(s),\\ p=1,\quad q<1 &\Rightarrow \bar{f}=f,\quad \bar{g}(s)=(g\,\nabla\,|.|)(s). \end{aligned}$$

It is

$$(f \nabla |.|)(s)=\begin{cases} |s|^p & \text{if } |s| \leqq p^{1/(1-p)} \\ |s|+p^{p/(1-p)}-p^{1/(1-p)} & \text{if } |s|>p^{1/(1-p)} \end{cases}$$

$$(g \nabla |.|)(s)=|s| \wedge |s|^q.$$

Of course, in the case $p=+\infty$ and $q=1$ it is

$$\bar{f}(s)=\begin{cases} 0 & \text{if } |s| \leqq 1 \\ |s|-1 & \text{if } |s|>1, \end{cases}$$

while, in the case $p=1$ and $q=0$ it is

$$\bar{g}(s)=|s| \wedge 1.$$

4. PROOF OF THE RESULTS

In this section we shall prove Theorem 3.1 and Theorem 3.2; some preliminary lemmas will be necessary.

LEMMA 4.1. – *Let* $\lambda_h \to \lambda$, *let* C *be a compact subset of* Ω, *and for every* $t>0$ *let*

$$C(t)=\{x \in \Omega : \text{dist}(x, C)<t\}.$$

Then there exists a sequence $t_h \to 0$ *such that*

$$1_{C(t_h)} \lambda_h \to 1_C \lambda.$$

Proof. – Since $C(r)$ is relatively compact, we have

$$1_{C(r)} \lambda_h \to 1_{C(r)} \lambda$$

as soon as $\partial C(r)$ is $|\lambda|$-negligible, hence for all $r \in \mathbf{R}^+ \setminus N$ with N at most countable. Choose $r_k \in \mathbf{R}^+ \setminus N$ with $r_k \to 0$; then

$$\begin{cases} 1_{C(r_k)} \lambda_h \to 1_{C(r_k)} \lambda & (\text{as } h \to \infty) \quad \text{for every } k \in \mathbf{N}, \\ 1_{C(r_k)} \lambda \to 1_C \lambda & (\text{as } k \to \infty). \end{cases}$$

Therefore, the conclusion follows by a standard diagonalization procedure. ■

Remark 4.2. – For every functional $G : \mathcal{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$ we define

$$G'(\lambda)=\inf\{\liminf_{h \to \infty} G(\lambda_h) : \lambda_h \to \lambda\} \quad \text{for every } \lambda \in \mathcal{M}(\Omega; \mathbf{R}^n).$$

It is possible to prove (*see* for instance Buttazzo [7], Proposition 1.3.2) that if Ξ is the set of all countable ordinals and for every $\xi \in \Xi$ we define by transfinite induction

$$F_0 = F$$
$$F_{\xi+1} = (F_\xi)'$$
$$F_\xi = \inf\{F_\eta : \eta < \xi\} \qquad \text{if} \quad \xi \text{ is a limit ordinal,}$$

we have

$$\bar{F} = \inf\{F_\xi : \xi \in \Xi\}.$$

LEMMA 4.3. – *For every* $\varepsilon > 0$ *and* $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$ *let us define*

$$F_\varepsilon(\lambda) = F(\lambda) + \varepsilon \|\lambda\|. \tag{4.1}$$

Then we have

$$F' = \inf\{F'_\varepsilon : \varepsilon > 0\}.$$

Proof. – The inequality $\leqq$ is obvious. In order to prove the opposite inequality, fix $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$ and $r > 0$; there exists $\lambda_h \to \lambda$ such that, setting $M = \sup\{\|\lambda_h\| : h \in \mathbf{N}\}$, it is

$$F'(\lambda) \geqq \liminf_{h \to \infty} F(\lambda_h) = \liminf_{h \to \infty} [F_\varepsilon(\lambda_h) - \varepsilon \|\lambda_h\|] \geqq F'_\varepsilon(\lambda) - \varepsilon M.$$

The conclusion follows by letting $\varepsilon \to 0$. ■

Proof of Theorem 3.1. – By Remark 4.2 it is enough to show that

$$F \text{ additive} \quad \Rightarrow \quad F' \text{ additive}.$$

Moreover, setting F_ε as in (4.1) and applying Lemma 4.3, it is enough to prove that F'_ε is additive for every $\varepsilon > 0$. By Proposition 1.3.5 and Remark 1.3.6 of Buttazzo [7] it is

$$F'_\varepsilon = \bar{F}_\varepsilon \quad \text{for every } \varepsilon > 0;$$

in particular, F'_ε is weakly* l.s.c. on $\mathscr{M}(\Omega; \mathbf{R}^n)$. We prove first that for every $r > 0$, $\lambda \in \mathscr{M}(\Omega; \mathbf{R}^n)$, and $B_1, B_2 \in \mathscr{B}$ with $B_1 \cap B_2 = \varnothing$ it is

$$r + F'_\varepsilon(1_{B_1 \cup B_2} \lambda) \geqq F'_\varepsilon(1_{B_1} \lambda) + F'_\varepsilon(1_{B_2} \lambda). \tag{4.2}$$

Let $\lambda_h \to 1_{B_1 \cup B_2} \lambda$ be such that

$$r + F'_\varepsilon(1_{B_1 \cup B_2} \lambda) \geqq \liminf_{h \to \infty} F_\varepsilon(\lambda_h), \tag{4.3}$$

and let $K_i \subset B_i$ be compact sets ($i = 1, 2$). By Lemma 4.1 we have

$$1_{K_i(t_h)} \lambda_h \to 1_{K_i} \lambda \qquad (i = 1, 2)$$

for a suitable sequence $t_h \to 0$, so that

$$\liminf_{h \to \infty} F_\varepsilon(\lambda_h) \geqq \liminf_{h \to \infty} F_\varepsilon(1_{K_1(t_h)} \lambda_h) + \liminf_{h \to \infty} F_\varepsilon(1_{K_2(t_h)} \lambda_h) \tag{4.4}$$
$$\geqq F'_\varepsilon(1_{K_1} \lambda) + F'_\varepsilon(1_{K_2} \lambda).$$

Now, (4.2) (hence the superadditivity of F'_ε) follows from (4.3) and (4.4) by taking the supremum as $K_1 \uparrow B_1$ and $K_2 \uparrow B_2$. Finally, we prove that for every $r>0$, $\lambda \in \mathcal{M}(\Omega; \mathbf{R}^n)$, and $B_1, B_2 \in \mathcal{B}$ with $B_1 \cap B_2 = \varnothing$, it is

$$F'_\varepsilon(1_{B_1 \cup B_2}\lambda) \leqq F'_\varepsilon(1_{B_1}\lambda) + F'_\varepsilon(1_{B_2}\lambda) + r. \tag{4.5}$$

Let $\lambda_{1,h} \to 1_{B_1}\lambda$ and $\lambda_{2,h} \to 1_{B_2}\lambda$ be such that

$$\liminf_{h \to \infty} F_\varepsilon(\lambda_{i,h}) \leqq F'_\varepsilon(1_{B_i}\lambda) + \frac{r}{2} \qquad (i=1, 2), \tag{4.6}$$

and let $K_i \subset B_i$ be compact sets ($i=1, 2$). By Lemma 4.1 we have

$$1_{K_i(t_h)}\lambda_{i,h} \to 1_{K_i}\lambda \qquad (i=1, 2)$$

for a suitable sequence $t_h \to 0$, so that

$$\liminf_{h \to \infty}[F_\varepsilon(\lambda_{1,h}) + F_\varepsilon(\lambda_{2,h})] \geqq \liminf_{h \to \infty}[F_\varepsilon(1_{K_1(t_h)}\lambda_{1,h}) + F_\varepsilon(1_{K_2(t_h)}\lambda_{2,h})]$$
$$= \liminf_{h \to \infty} F_\varepsilon(1_{K_1(t_h)}\lambda_{1,h} + 1_{K_2(t_h)}\lambda_{2,h}) \geqq F'_\varepsilon(1_{K_1 \cup K_2}\lambda).$$

Now, (4.5) (hence the subadditivity of F'_ε) follows from (4.6) and (4.7) by taking the supremum as $K_1 \uparrow B_1$ and $K_2 \uparrow B_2$. ■

LEMMA 4.4. – *There exists a countable subset* N *of* Ω *such that*

(i) $\bar{g} \leqq g$ *on* $\Omega \times \mathbf{R}^n$,

(ii) $\bar{g} \leqq \varphi_{f,\mu}$ *on* $\Omega \times \mathbf{R}^n$,

(iii) $\bar{\varphi} \leqq g^0$ on $(\Omega \setminus N) \times \mathbf{R}^n$,

(iv) $\bar{\varphi} \leqq \varphi_{f,\mu}$ on $(\Omega \setminus N) \times \mathbf{R}^n$.

Proof. – Property (i) follows immediately from the fact that $\bar{F} \leqq F$ on $\mathcal{M}(\Omega; \mathbf{R}^n)$.

Let us prove property (ii). Denoting by F_0 the functional

$$F_0(\lambda) = \begin{cases} F(\lambda) & \text{if } \lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.8}$$

by using Theorem 4 of Bouchitté and Valadier [5] and Proposition 2.2 we have

$$\bar{F}_0(\lambda) = \int_\Omega (f \nabla \varphi_{f,\mu})\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_\Omega \varphi_{f,\mu}(x, \lambda^s), \qquad \forall \lambda \in \mathcal{M}(\Omega; \mathbf{R}^n) \tag{4.9}$$

so that, if $\lambda = s\,\delta_x$,

$$\bar{g}(x, s) = \bar{F}(s\,\delta_x) \leqq \bar{F}_0(s\,\delta_x) = \int_\Omega \varphi_{f,\mu}(x, s\,\delta_x) = \varphi_{f,\mu}(x, s).$$

Let us prove property (iii). By the integral representation Theorem 2.7 we have for a suitable countable subset N of Ω

$$\bar{\varphi}=(\bar{g})^0 \qquad \text{on } (\Omega\setminus N)\times \mathbf{R}^n,$$

so that (iii) follows from (i).

Finally, let us prove property (iv). If F_0 is the functional defined in (4.8), we have

$$\frac{1}{t}\bar{F}(t\lambda)\leqq \frac{1}{t}\bar{F}_0(t\lambda), \qquad \forall t>0, \quad \forall \lambda\in \mathcal{M}^0(\Omega; \mathbf{R}^n).$$

Letting $t\to+\infty$ and taking (4.9) into account, we get for every $\lambda\in\mathcal{M}^0(\Omega; \mathbf{R}^n)$

$$\int_\Omega (\bar{f})^\infty\left(x, \frac{d\lambda}{d\bar{\mu}}\right)d\bar{\mu}+\int_\Omega \bar{\varphi}(x, \lambda^s)=(\bar{F})^\infty(\lambda)\leqq(\bar{F}_0)^\infty(\lambda)$$

$$=\int_\Omega (f\nabla\varphi_{f,\mu})^\infty\left(x, \frac{d\lambda}{d\mu}\right)d\mu+\int_\Omega \varphi_{f,\mu}(x, \lambda^s)=\int_\Omega \varphi_{f,\mu}(x, \lambda)$$

since $\varphi_{f,\mu}(x, .)\leqq f^\infty(x, .)$ for μ-a.e. $x\in\Omega$. By Theorem 2.7 it is $(\bar{f})^\infty(x, .)=\bar{\varphi}(x, .)$ for $\bar{\mu}$-a.e. $x\in\Omega$, and we obtain

$$\int_\Omega \bar{\varphi}(x, \lambda)\leqq\int_\Omega \varphi_{f,\mu}(x, \lambda), \qquad \forall\lambda\in\mathcal{M}^0(\Omega; \mathbf{R}^n),$$

so that (iv) follows from Proposition 3.2 of Bouchitté and Buttazzo [3]. ■

LEMMA 4.5. – *The functional* F_1 *is sequentially weakly* l.s.c. on* $\mathcal{M}(\Omega; \mathbf{R}^n)$ *and verifies the inequality* $F_1\leqq F$.

Proof. – The inequality $F_1\leqq F$ is an obvious consequence of the definition of f_1, φ_1, g_1. We shall apply the lower semicontinuity Theorem 2.4 by showing that the functions f_1, φ_1, g_1 satisfy conditions (H_1)-(H_5). Conditions (H_1) and (H_3) follow immediately from Proposition 2.2 (i), and condition (H_5) follows from Proposition 2.3.

Let us prove condition (H_4). The subadditivity of $g_1(x, .)$ is an easy consequence of the subadditivity of $g(x, .)$ and $\varphi_{f,\mu}(x, .)$; it remains to prove that $g_1\leqq\varphi_{f_1,\mu}$ on $\Omega\times\mathbf{R}^n$, or equivalently $(g_1)^0\leqq\varphi_{f_1,\mu}$ on $\Omega\times\mathbf{R}^n$. Setting

$$\Gamma_f(x)=\operatorname{dom}(\varphi_{f,\mu})^*(x, .)$$

$$\Gamma_{f_1}(x)=\operatorname{dom}(\varphi_{f_1,\mu})^*(x, .)$$

$$\Gamma_0(x)=\operatorname{dom}(g^0)^*(x, .)$$

and using Proposition 2.2 (i), it remains to show that

$$\Gamma_0(x)\cap\Gamma_f(x)\subset\Gamma_{f_1}(x), \qquad \forall x\in\Omega.$$

Since g^0 is coercive and l.s.c., the multimapping $x \mapsto \Gamma_0(x)$ is l.s.c. and its values are with nonempty interior. The same holds true for $\Gamma_f(x)$ and $\Gamma_{f_1}(x)$. Moreover, by Proposition 6 of Bouchitté and Valadier [5] we have

$$\Gamma_f(x) = \mathrm{cl}\{s \in \mathbf{R}^n : f^*(.,s) \text{ is locally } \mu\text{-integrable around } x\} \quad (4.10)$$

$$\Gamma_{f_1}(x) = \mathrm{cl}\{s \in \mathbf{R}^n : (f_1)^*(.,s) \text{ is locally } \mu\text{-integrable around } x\}. \quad (4.11)$$

Let us now fix $x \in \Omega$ and $s \in \mathrm{int}(\Gamma_0(x) \cap \Gamma_f(x))$. The lower semicontinuity of the multimapping Γ_0 implies (*see* for instance Lemma 15 of [6]) that for a suitable neighbourhood V of x

$$s \in \Gamma_0(y), \qquad \forall y \in \mathrm{V}.$$

By (4.10) we can choose V such that

$$\int_{\mathrm{V}} f^*(.,s)\,d\mu < +\infty.$$

Therefore

$$\int_{\mathrm{V}} f_1^*(.,s)\,d\mu = \int_{\mathrm{V}} [f^*(.,s) + (g^0)^*(.,s) + \varphi_{f,\mu}^*(.,s)]\,d\mu$$

$$= \int_{\mathrm{V}} f^*(.,s)\,d\mu < +\infty$$

that is, by (4.11), $s \in \Gamma_{f_1}(x)$. Hence

$$\mathrm{int}(\Gamma_0(x) \cap \Gamma_f(x)) \subset \Gamma_{f_1}(x).$$

The conclusion now follows by recalling that $\Gamma_{f_1}(x)$ is closed, and that $\mathrm{cl}(\mathrm{int}\,\mathrm{K}) = \mathrm{cl}\,\mathrm{K}$ for every convex set $\mathrm{K} \subset \mathbf{R}^n$ with nonempty interior.

Finally, let us prove condition (H_2). Since $f_1 \leqq \varphi_1$ on $\Omega \times \mathbf{R}^n$, we have $f_1^\infty \leqq \varphi_1^\infty = \varphi_1$ on $\Omega \times \mathbf{R}^n$. By conditions (H_4) and (H_5) already proved, we have for a countable set $\mathrm{N} \subset \Omega$

$$\varphi_1 = g_1^0 \leqq (\varphi_{f_1,\mu})^0 = \varphi_{f_1,\mu} \quad \text{on } (\Omega \setminus \mathrm{N}) \times \mathbf{R}^n.$$

Finally, the inequality

$$\varphi_{f_1,\mu}(x,.) \leqq f_1^\infty(x,.) \qquad \text{for} \quad \mu\text{-a.e. } x \in \Omega,$$

is a general property of the functions of the form $\varphi_{f,\mu}$ (*see* Section 2). ■

LEMMA 4.6. – *Setting*

$$\mathrm{E} = \{x \in \Omega : \bar{f}(x,.) \neq \bar{\varphi}(x,.)\}$$

we have that there exists $\alpha \in \mathrm{L}_\mu^1(\Omega)$ *such that* $\alpha\mu = 1_{\mathrm{E}}\bar{\mu}$.

Proof. – Let us consider $\lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n)$ with $\lambda \perp \mu$; taking into account that $\mathrm{F}_1 \leqq \bar{\mathrm{F}}$ (by Lemma 4.6) and $\bar{\varphi} \leqq \varphi_1$ (by Lemma 4.5) we have

$$\bar{\mathrm{F}}(\lambda) \geqq \mathrm{F}_1(\lambda) = \int_\Omega \varphi_1(x,\lambda) \geqq \int_\Omega \bar{\varphi}(x,\lambda) = (\bar{\mathrm{F}})^\infty(\lambda).$$

Since $\bar{F} \leqq (\bar{F})^\infty$ on $\mathcal{M}(\Omega; \mathbf{R}^n)$, we obtain

$$\bar{F}(\lambda) = (\bar{F})^\infty(\lambda) \quad \text{for every } \lambda \in \mathcal{M}^0(\Omega; \mathbf{R}^n) \quad \text{with } \lambda \perp \mu. \tag{4.12}$$

Consider now the Lebesgue-Nikodym decomposition of $1_E \bar{\mu}$ with respect to μ

$$1_E \bar{\mu} = \alpha\mu + \nu \qquad \text{with} \quad \alpha \in L^1_\mu(\Omega), \qquad \nu \perp \mu,$$

and let

$$\lambda = u\, 1_E \nu \qquad \text{with} \quad u \in L^1_\nu(\Omega).$$

We have, by (4.12)

$$\int_E \bar{f}(x, u)\, d\nu = \bar{F}(\lambda) = (\bar{F})^\infty(\lambda) = \int_E \bar{\varphi}(x, \lambda) = \int_E \bar{\varphi}(x, u)\, d\nu.$$

Since $u \in L^1_\nu(\Omega)$ is arbitrary, we get

$$\bar{f}(x, .) = \bar{\varphi}(x, .) \qquad \nu\text{-a.e. on E},$$

and, by definition of E, this implies $\nu(E) = 0$, that is $\nu = 0$. ■

Proof of Theorem 3.2. – By Lemma 4.5 it is enough to show that

$$\bar{F} \leqq F_1 \quad \text{on } \mathcal{M}(\Omega; \mathbf{R}^n),$$

that is

$$\bar{g} \leqq g_1 \quad \text{on } \Omega \times \mathbf{R}^n \tag{4.13}$$

$$\bar{\varphi} \leqq \varphi_1 \quad \text{on } (\Omega \setminus N) \times \mathbf{R}^n \tag{4.14}$$

$$1_E \bar{\mu} = \alpha\mu \tag{4.15}$$

$$f_1(x, s) \geqq \begin{cases} \alpha(x) \bar{f}\left(x, \dfrac{s}{\alpha(x)}\right) & \text{if } \alpha(x) \neq 0 \\ \bar{\varphi}(x, s) & \text{if } \alpha(x) = 0 \end{cases} \quad \text{on } (\Omega \setminus M) \times \mathbf{R}^n \tag{4.16}$$

where N is a suitable countable subset of Ω, M is a suitable Borel subset of Ω with $\mu(M) = 0$, and α is a suitable function in $L^1_\mu(\Omega)$.

Conditions (4.13) and (4.14) follow from Lemma 4.4, whereas (4.15) follows from Lemma 4.6. Let us now prove (4.16). Take $u \in L^1_\mu(\Omega; \mathbf{R}^n)$ and $\lambda = u\mu$. We have

$1_{\{\alpha \neq 0\} \cap E} \lambda = \dfrac{u}{\alpha} 1_{\{\alpha \neq 0\} \cap E} \bar{\mu}$ so that

$$\bar{F}(1_{\{\alpha \neq 0\} \cap E} \lambda) = \int_{\{\alpha \neq 0\}} \alpha \bar{f}\left(x, \frac{u}{\alpha}\right) d\mu \tag{4.17}$$

$1_{\{\alpha \neq 0\} \setminus E} \lambda = 0$ because $\alpha = 0$ μ-a.e. on $\Omega \setminus E$, hence

$$\bar{F}(1_{\{\alpha \neq 0\} \setminus E} \lambda) = 0 \tag{4.18}$$

$1_{\{\alpha\neq 0\}\cap E}\lambda \perp \bar{\mu}$ because $\bar{\mu}(\{\alpha=0\}\cap E)=0$, hence

$$\bar{F}(1_{\{\alpha=0\}\cap E}\lambda)=\int_{\{\alpha=0\}\cap E}\bar{\varphi}(x,\lambda) \tag{4.19}$$

$\bar{f}=\bar{\varphi}$ on $(\Omega\setminus E)\times\mathbf{R}^n$ so that

$$\bar{F}(1_{\{\alpha=0\}\setminus E}\lambda)=\int_{\{\alpha=0\}\setminus E}\bar{\varphi}(x,\lambda). \tag{4.20}$$

Collecting (4.17)-(4.20) we get

$$\begin{aligned}\int_\Omega f(x,u)\,d\mu &= F(\lambda)\geqq\bar{F}(\lambda)\\ &=\bar{F}(1_{\{\alpha\neq 0\}\cap E}\lambda)+\bar{F}(1_{\{\alpha\neq 0\}\setminus E}\lambda)+\bar{F}(1_{\{\alpha=0\}\cap E}\lambda)+\bar{F}(1_{\{\alpha=0\}\setminus E}\lambda)\\ &=\int_{\{\alpha\neq 0\}}\alpha\bar{f}\left(x,\frac{u}{\alpha}\right)d\mu+\int_{\{\alpha=0\}}\bar{\varphi}(x,u)\,d\mu.\end{aligned}$$

Since $u\in L^1_\mu(\Omega;\mathbf{R}^n)$ was arbitrary, we obtain for a suitable $B\in\mathscr{B}$ with $\mu(M)=0$

$$f(x,s)\geqq\begin{cases}\alpha(x)\bar{f}\left(x,\dfrac{s}{\alpha(x)}\right) & \text{if } \alpha(x)\neq 0\\ \bar{\varphi}(x,s) & \text{if } \alpha(x)=0\end{cases} \tag{4.21}$$

for every $(x,s)\in(\Omega\setminus M)\times\mathbf{R}^n$. Now, (4.16) comes out easily from (4.21). Indeed, for μ-a.e. $x\in\Omega$ with $\alpha(x)=0$, we have, using (4.14) and (4.21):

$$\bar{\varphi}(x,.)\leqq\inf\{\varphi_1(x,.),f(x,.)\}\leqq\varphi_1(x,.)\,\nabla\, f(x,.)=f_1(x,.).$$

On the other hand, by Theorem 2.7 and (4.14) we get

$$\bar{f}(x,.)\leqq(\bar{f})^\infty(x,.)\leqq\bar{\varphi}(x,.)\leqq\varphi_1(x,.)$$

$\bar{\mu}$-a.e. on Ω, hence μ-a.e. on $\{\alpha\neq 0\}$, so that by (4.21):

$$\alpha(x)\bar{f}\left(x,\frac{s}{\alpha(x)}\right)\leqq\inf\{\varphi_1(x,s),f(x,s)\}\leqq f_1(x,s)$$

on $(\Omega\setminus M)\times\mathbf{R}^n$ with $\mu(M)=0$. Therefore (4.16) is proved, and the proof of Theorem 3.2 is completely achieved. ■

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