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Multiple solutions of a semilinear elliptic equation in $\mathbb{R}^N$


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**Multiple solutions of a semilinear elliptic equation in \( \mathbb{R}^N \)**

by

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**ABSTRACT.** — In this paper, we are concerned with the existence of multiple solutions of

\[
-\Delta u + u = \lambda b(x) |u|^{p-1} u + c(x) |u|^{q-1} u
\]

where \( 1 < p, q < \frac{N+2}{N-2} \) if \( N \geq 3 \), \( 1 < p, q < +\infty \) if \( N = 2 \), \( \lambda > 0 \).

We obtain the existence of multiple solutions by using concentrations-compactness method and dual variational principle to establish the corresponding existence of critical points.

**Key words:** Semilinear elliptic equations, variation, critical point, concentration-compactness.

**Résultat.** — Nous obtenons dans cet article un résultat d’existence et de multiplicité de solutions de

\[
-\Delta u + u = \lambda b(x) |u|^{p-1} u + c(x) |u|^{q-1} u
\]

où \( 1 < p, q < \frac{N+2}{N-2} \), \( N \geq 3 \), \( 1 < p, q < +\infty \) si \( N = 2 \), \( \lambda > 0 \).

**A.M.S. Classification:** 35 B 05, 35 J 60.
Ces résultats sont prouvés à l'aide de la méthode de concentration-compacité et de principes variationnels duaux pour obtenir l'existence des points critiques correspondants.

1. INTRODUCTION

We consider the existence of multiple solutions of the following semilinear elliptic equation

\[\begin{cases}
-\Delta u + u = \lambda b(x) |u|^{p-1} u + c(x) |u|^{q-1} u & \text{in } \mathbb{R}^N \\
u \in H^1(\mathbb{R}^N)
\end{cases}\]

where \(1 < p, q < \frac{N+2}{N-2}\) if \(N \geq 3\), \(1 < p, q + \infty\) if \(N = 2\), \(\lambda > 0\) is a real number, \(b(x)\) and \(c(x)\) satisfy

\[\begin{cases}
b(x) \in C(\mathbb{R}^N), & b(x) \geq 0 \text{ in } \mathbb{R}^N, \\
b(x) \to b_\infty > 0, & |x| \to \infty
\end{cases}\]

\[\begin{cases}
c(x) \in C(\mathbb{R}^N), & c(x) \geq 0 \text{ in } \mathbb{R}^N, \\
c(x) \to 0, & |x| \to \infty
\end{cases}\]

Existence of nontrivial solutions (positive solutions, for example) concerning (1.1) has been extensively studied even for more general nonlinearity—see, for instance, W. Strauss [12], H. Berestycki and P. L. Lions [4], W. Y. Ding and W. M. Ni [5], P. L. Lions [9], [10], A. Bahri and P. L. Lions [2] and the references therein. For the multiplicity of solutions we refer to H. Berestycki and P. L. Lions [4], X. P. Zhu [13] and Y. Y. Li [8].

It is known to some extent that the equation

\[\begin{cases}
-\Delta u + u = c(x) |u|^{q-1} u & \text{in } \mathbb{R}^N
\end{cases}\]

may have infinitely many solutions because (1.3) ensures that the corresponding variational functional

\[I^*(u) = \frac{1}{2} \int |\nabla u|^2 + u^2 - \frac{1}{q+1} \int c(x) |u|^{q+1}\]
satisfies the (PS) (Palais-Smale) condition and the dual variational principle of A. Ambrosetti and P. Rabinowitz [1] may be applied. When $\lambda$ is small, (1.1) can be taken as a small perturbation of (1.4) and thus it seems reasonable to hope that (1.1) has more and more solutions as $\lambda$ tends to 0.

As mentioned in P. L. Lions ([9], [10]) that the variational functional corresponding to (1.1) defined by

$$I_\lambda(u) = \frac{1}{2} \int \nabla u^2 + u^2 - \frac{\lambda}{p+1} \int b(x)|u|^{p+1} - \frac{1}{q+1} \int c(x)|u|^{q+1}$$

fails to satisfy the (PS) condition because of the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$.

Such a failure creates difficulties for the application of standard variational techniques. In section 2, arguing as P. L. Lions [10], we show by using the concentration-compactness principle that $I_\lambda(u)$ satisfies $(PS)_c$ condition if $c$ belongs to an interval depending on $\lambda$ which becomes large as $\lambda$ tends to 0. In section 3, using a variant of the dual variational principle (dealing with unbounded even functionals) of A. Ambrosetti and P. Rabinowitz [1] we obtain the existence of multiple solutions by establishing the corresponding existence of critical points of $I_\lambda(u)$ with critical values in the interval in which $I_\lambda(u)$ satisfies $(PS)_c$ condition.

We conclude this introduction by remarking that some more general nonlinearities can be considered and similar existence results can be obtained by the arguments in this paper.

### 2. EXISTENCE OF A POSITIVE SOLUTION

In this section, we are concerned with the existence of a positive solution of (1.1). As preparations and for the discussion of next section, we first give some notations, definitions and auxiliary results.

Define

$$M_\lambda = \{ u \in H^1(\mathbb{R}^N) \mid u \neq 0, I'_\lambda(u)u = 0 \}$$

$$M^\infty_\lambda = \{ u \in H^1(\mathbb{R}^N) \mid u \neq 0, I'_{\lambda^\infty}(u)u = 0 \}$$

where $I_\lambda(u)$ is defined by (1.6), $I_{\lambda^\infty}(u)$ is defined by

$$I_{\lambda^\infty}(u) = \frac{1}{2} \int \nabla u^2 + u^2 - \frac{\lambda}{p+1} \int b_\infty|u|^{p+1}$$

Let

$$I_\lambda = \inf \{ I_\lambda(u) \mid u \in M_\lambda \}$$

$$I_{\lambda^\infty} = \inf \{ I_{\lambda^\infty}(u) \mid u \in M_{\lambda^\infty} \}$$

We have

**Proposition 2.1.** - For each \( \lambda > 0 \), \( I_\lambda \leq I^* \).

**Proof.** - If \( c(x) \equiv 0 \), then \( I^* = +\infty \), thus \( I_\lambda \leq I^* \). In what follows, we assume \( c(x) \neq 0 \).

Suppose \( u \in H^1(\mathbb{R}^N) \), \( u \neq 0 \) such that

\[
(2.8) \quad \int |\nabla u|^2 + u^2 = \int c(x)|u|^{q+1}.
\]

Let \( v = \sigma u \) such that \( v \in M_\lambda \), i.e.,

\[
(2.9) \quad \int |\nabla u|^2 + u^2 = \sigma^{-p-1} \int \lambda b(x)|u|^{p+1} + \sigma^{-q-1} \int c(x)|u|^{q+1}.
\]

Comparing (2.8) and (2.9) we deduce that such \( \sigma \) exists and \( \sigma \in (0, 1) \).

Letting

\[
h(\sigma) = \frac{\sigma^2}{2} \int |\nabla u|^2 + u^2 - \sigma^{q+1} \int c(x)|u|^{q+1},
\]

we have

\[
h'(\sigma) = \sigma \left( \int |\nabla u|^2 + u^2 - \sigma^{-q-1} \int c(x)|u|^{q+1} \right) > 0 \quad \text{for} \quad \sigma \in (0, 1).
\]

Thus \( I_\lambda \leq I^* \) and we have proved Proposition 2.1.

**Proposition 2.2.** - We have

\[
I^*_\lambda = \frac{p-1}{2(p+1)} S^{(p+1)/(p-1)} (\lambda b_\infty)^{-(2/(p-1))}.
\]
Proof. – We can easily find that

\begin{equation}
S = \inf \left\{ \int |\nabla u|^2 + u^2 \mid u \in H^1(\mathbb{R}^N), \int |u|^{p+1} = 1 \right\}
\end{equation}

which has a positive minimum \( \bar{u} \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \) satisfying

\begin{equation}
-\Delta u + u = S |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N
\end{equation}

(see W. Strauss [12], P. L. Lions [9], [10] for examples). By Gidas, Ni and Nirenberg [7] we may assume \( \bar{u} \) is radial.

On the other hand, there exists a positive radial function \( \bar{v} \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \) achieving \( I_{\lambda}^\infty \) such that \( \bar{u} \) satisfying

\begin{equation}
-\Delta u + \lambda b_{\infty} |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N
\end{equation}

(see also W. Strauss [12], P. L. Lions [9], [10] for examples).

Let \( \bar{u} = \left( \frac{S}{\lambda b_{\infty}} \right)^{1/(p-1)} \), then \( \bar{v} > 0 \) in \( \mathbb{R}^N \) and solves (2.13). By the uniqueness of radial positive solution due to M. K. Kwong [11] we deduce \( \bar{v} \equiv \bar{u} \) and thus

\begin{equation}
I_{\lambda}^\infty = I_{\lambda}^\infty(\bar{u}) = \frac{p-1}{2(p+1)} \int | \nabla \bar{u} |^2 + \bar{u}^2 = \frac{p-1}{2(p+1)} \frac{S^{(p+1)/(p-1)}(\lambda b_{\infty})^{-(2/(p-1))}}
\end{equation}

proving Proposition 2.2.

Lemma 2.3. – \( I_{\lambda}(u) \) satisfies \((PS)_c\) condition if

\begin{equation}
c \in (-\infty, I_{\lambda}^\infty).
\end{equation}

Proof. – Suppose \( \{u_n\} \subset H^1(\mathbb{R}^N) \) such that

\begin{equation}
I_{\lambda}(u_n) \to c \in (-\infty, I_{\lambda}^\infty)
\end{equation}

\begin{equation}
I_{\lambda}'(u_n) \to 0 \quad \text{in} \quad H^1(\mathbb{R}^N)
\end{equation}

It is easy to deduce from (2.16) and (2.17) that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \). By choosing subsequence if necessary we assume

\begin{equation}
u_0 \to u_0 \quad \text{weakly in} \quad H^1(\mathbb{R}^N).
\end{equation}

By the method of concentration-compactness, as in A. Bahri and P. L. Lions [2], P. L. Lions [10], V. Benci and G. Cerami [3] we deduce that there exist a nonnegative integer \( k \), \( \{x_i^k\} \) \( (1 \leq i \leq k) \) in \( \mathbb{R}^N \), solutions \( \bar{u}_i \in H^1(\mathbb{R}^N) \) \( (1 \leq i \leq k) \) of (2.14) such that (extracting subsequence if necessary)

\begin{equation}
\| u_n - u_0 - \sum_{i=1}^k \bar{u}_i(x-x_i^k) \|_n \to 0
\end{equation}

\[ c = I_\lambda(u_0) + \sum_{i=1}^{n} I_\lambda^\infty(\overline{u}_i). \]

Since \( I_\lambda^\infty(\overline{u}_i) = \frac{p-1}{2(p+1)} \int |\nabla \overline{u}_i|^2 + \overline{u}_i^2 \geq 0 \) for \( i = 1, \ldots, k \) if for some \( i, \overline{u}_i \neq 0 \), then \( I_\lambda^\infty(\overline{u}_i) \geq I_\lambda^\infty \) which implies \( c \geq I_\lambda^\infty \) because \( I_\lambda(u_0) \geq 0 \). Thus \( \overline{u}_i \equiv 0 \) for \( 1 \leq i \leq k \). Hence \( u_n \) converges to \( u_0 \) strongly and therefore Lemma 2.3 has been proved.

We are now going to use the preceding result to obtain the existence of a positive solution.

**Theorem 2.4.** Suppose \( I_\lambda < I_\lambda^\infty \). Then (1.1) has a positive solution.

**Proof.** By Ekeland's variational principle [6] and the definition of \( I_\lambda \), there exists a minimizing sequence \( \{ u_n \} \) such that \( \{ u_n \} \subset M_\lambda \)

\[ I_\lambda(u_n) \to I_\lambda \]

(2.21)

\[ I'_\lambda|_{M_\lambda}(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N). \]

(2.22)

\[ I'_\lambda(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N). \]

(2.23)

Indeed, from (2.21), \( u_n \in M_\lambda \), using Sobolev inequality we can find \( C_1, C_2 > 0 \) such that

\[ C_1 < \int |\nabla u_n|^2 + u_n^2 < C_2 \quad \text{for all} \quad n = 1, 2, \ldots \]

(2.24)

Letting \( J_\lambda(u) = \int |\nabla u|^2 + u^2 - \int \lambda b(x)|u|^{p+1} - \int c(x)|u|^{q+1} \), we have

\[ M_\lambda = \{ u \in H^1(\mathbb{R}^N) \setminus \{ 0 \} \mid J_\lambda(u) = 0 \}. \]

Thus

\[ I'_\lambda(u_n) = I'_\lambda|_{M_\lambda}(u_n) - \theta_n J'_\lambda(u_n) \]

(2.26)

for some \( \theta_n \in \mathbb{R} \).

Since \( u_n \in M_\lambda \), we have from (2.26)

\[ I'_\lambda|_{M_\lambda}(u_n) - \theta_n J'_\lambda(u_n) = I'_\lambda(u_n) u_n = 0 \]

(2.27)

\[ J'_\lambda(u_n) u_n = 2 \int |\nabla u_n|^2 + u_n^2 - (p+1) \int \lambda b(x)|u_n|^{p+1} 
\]

\[ - (q+1) \int c(x)|u_n|^{q+1} 
\]

\[ = -(p-1) \int \lambda b(x)|u_n|^{p+1} - (q-1) \int c(x)|u_n|^{q+1}. \]

(2.28)
Thus from (2.24), (2.28) and $\nu_n \in M_\lambda$ we have
\begin{equation}
(2.29) \quad -C_3 < J'(u_n) u_n < -C_4
\end{equation}
for some constants $C_3, C_4 > 0$ independent of $n$.

From $I'_{n|M_\lambda}(u_n) \to 0$, we obtain by (2.27) and (2.29) that $\theta_n \to 0$ which combined with (2.26) deduces $I'_n(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$. Thus (2.23) holds.

Following Lemma 2.3, we can assume (by choosing subsequence if necessary)
\[ u_n \to u_0 \quad \text{strongly in } H^1(\mathbb{R}^N). \]

By Sobolev inequality, we have $I_\lambda > 0$. Thus $u_0$ is a nontrivial solution of (1.1). Letting $u_0 = u_0^+ + u_0^-$, where $u_0^+ = \max\{u_0, 0\}$, $u_0^- = u_0 - u_0^+$, we have $I_\lambda(u_0) = I_\lambda(u_0^+) + I_\lambda(u_0^-)$.

Since $I'_\lambda(u_\lambda) u_\lambda^+ = 0$, i.e., $u_\lambda^+ \in M_\lambda$ if $u_\lambda^+ \neq 0$ we have $I'_\lambda(u_\lambda) \geq I_\lambda$ if $u_\lambda^+ \neq 0$. Therefore $u_\lambda^+ \equiv 0$ or $u_\lambda^- \equiv 0$. Without loss of generality, assume $u_\lambda^+ \equiv 0$. Thus $u_0 \geq 0$ in $\mathbb{R}^N$. It follows from standard regularity method and maximum principle that $u_0 \in C^2(\mathbb{R}^N), u_0 > 0$ in $\mathbb{R}^N$. Thus, we conclude the proof of Theorem 2.4.

**Corollary 2.5.** Suppose (1.2) holds, $c(x)$ satisfies
\begin{equation}
(2.30) \quad \begin{cases}
  c(x) \in C(\mathbb{R}^N), & c(x) \geq 0 \quad \text{in } \mathbb{R}^N,
  c(x) \to 0, & c(x) \neq 0 \quad \text{in } \mathbb{R}^N.
\end{cases}
\end{equation}

Then (1.1) has a positive solution provided
\begin{equation}
(2.31) \quad \lambda \in \left(0, \frac{p-1}{2(p+1)I^*} \right)^{(p-1)/2} \left[ s^{(p+1)/2} b^{-1} \right].
\end{equation}

**Proof.** From (2.31) we have
\begin{equation}
(2.32) \quad I^* < \frac{p-1}{2(p+1)} S^{(p+1)/(p-1)} (\lambda b_\infty)^{-(2/(p-1))} = I_\lambda^\infty
\end{equation}
which combined with Proposition 2.1 implies
\begin{equation}
(2.33) \quad I_\lambda < I_\lambda^\infty.
\end{equation}

Thus, by Theorem 2.4 we know (1.1) has a positive solution.

We end this section by a few remarks.

**Remark 2.6.** The fact that if $I_\lambda < I_\lambda^\infty$ then $I_\lambda$ has a minimum has been proved in P. L. Lions ([9], [10]). We reprove this fact for the sake of completeness.

**Remark 2.7.** Consider the following equation
\begin{equation}
(2.35) \quad -\Delta u + u = Q(x) |u|^{p-1} u \quad \text{in } \mathbb{R}^N
\end{equation}
where $Q(x) \in C(\mathbb{R}^N), Q(x) \geq 0$ in $\mathbb{R}^N, Q(x) \to Q > 0$ as $|x| \to \infty$.
(2.35) can be obtained by taking $\lambda = 1$, $Q(x) \equiv b(x)$, $c(x) \equiv 0$ in (1.1). From Theorem 2.4 we can deduce the corresponding results concerning the existence of positive solution of (2.35) in section 3 of W. Y. Ding and W. M. Ni [5] [for the case $Q(x) \to \bar{Q}$ as $|x| \to \infty$]. Corollary 2.5 gives a type of precise condition under which $I_{\lambda} < I_0^\infty$.

Suppose $Q(x) = \lambda b(x) + c(x)$, where $b(x)$ satisfies (1.2) and

$$(2.36) \quad (b_\infty - b(x)) \log(1 + |x|) \to +\infty \quad \text{as} \quad |x| \to \infty$$

c(x) satisfies (2.30) with $\text{supp} \ c(x)$ bounded.

Corollary 2.5 ensures the existence of positive solution if $\lambda$ is properly small. It should be pointed out that in this case $Q(x)$ does not satisfy the condition proposed by A. Bahri and P. L. Lions in [2].

### 3. Existence of Multiple Solutions

First of all, let us state a variant of the dual variational principle of A. Ambrosetti and P. Rabinowitz [1] dealing with unbounded even functionals.

Let $E$ be a Banach space, $B_r$ be the ball in $E$ centered at $0$ with radius $r$, $\partial B_r$ be the boundary of $B_r$. $A \subset E$ is called symmetric if $u \in A$ implies $-u \in A$. Let

$$(3.1) \quad \Sigma = \{A \mid A \subset E \setminus \{0\}, \ A \text{ is closed and symmetric}\}$$

For $A \subset \Sigma$, $v(A)$ denotes the genus of $A$. We set for $f \in C^1(E, \mathbb{R})$

$$(3.2) \quad E_+ = \{u \in E \mid f(u) \geq 0\}$$

$$(3.3) \quad H = \{h \mid h \in C(E, E), \ h \text{ is odd homeomorphism } h(B_1) \subset E_+\}$$

$$(3.4) \quad \Gamma_n = \{A \in \Sigma \mid A \text{ is compact, } v(A \cap h(\partial B_1)) \geq n \text{ for any } h \in H\}$$

Replacing (PS) by condition, we have the following lemma proved exactly as in [1].

**Lemma 3.1.** — Suppose $f \in C^1(E, \mathbb{R})$ satisfies

(C1) $f(0) = 0$ and there exist $\rho, \alpha > 0$ such that $f(u) > 0$ for any $u \in B_\rho \setminus \{0\}, f(u) \geq \alpha$ for all $u \in \partial B_\rho$;

(C2) for any finite dimensional subspace $E^n \subset E$, $E^n \cap E_+$ is bounded;

(C3) $f(u) = f(-u)$.

Set

$$(3.5) \quad b_n = \inf_{A \in \Gamma_n} \sup_{u \in A} \{f(u)\}, \quad n = 1, 2, \ldots$$

Then

(i) $\Gamma_n \neq \emptyset$ for $n = 1, 2, \ldots$, $b_n \geq \alpha$;

(ii) $b_n$ is a critical level if $f$ satisfies (PS)$_c$ condition for $c = b_n$. 

Annales de l'Institut Henri Poincaré - Analyse non linéaire
Furthermore, if \( b = b_n = \ldots = b_{n+m} \), then \( \nu(K_{b_n}) \geq m + 1 \), where
\[
K_b = \{ u \in E \mid f(u) = b, f'(u) = 0 \}.
\]

In what follows, we always take \( E = H^1(\mathbb{R}^N) \) and use the same notations \( \Sigma, B_\ast, \partial B_\ast \) and \( \nu(A) \). Let

\[
\begin{align*}
E_\lambda &= \{ u \in H^1(\mathbb{R}^N) \mid I_\lambda(u) \geq 0 \} \\
E_\ast &= \{ u \in H^1(\mathbb{R}^N) \mid I^*(u) \geq 0 \} \\
H_\lambda &= \{ h \in C(H_1(\mathbb{R}^N), H^1(\mathbb{R}^N)), h \text{ is odd homeomorphism,} \} \\
H_\ast &= \{ h \in C(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)), h \text{ is odd homeomorphism,} \}
\end{align*}
\]

Obviously \( E_\lambda \subset E_\ast, H_\lambda \subset H_\ast \).

**Proposition 3.2.** — If \( b(x) \) satisfies (1.2), \( c(x) \) satisfies
\[
\begin{align*}
&c(x) \in C(\mathbb{R}^N), \quad c(x) \geq 0 \quad \text{in} \quad \mathbb{R}^N, \\
&\text{meas}\{ x \in \mathbb{R}^N \mid c(x) = 0 \} = 0, \\
c(x) \to 0 \quad \text{as} \quad |x| \to \infty
\end{align*}
\]
Then \( I_\lambda(u) \) and \( I^*(u) \) satisfy (C1), (C2) and (C3) in the previous lemma.

**Proof.** — The verification of (C1) and (C3) is trivial. We only show that (C2) holds for \( I_\lambda(u) \) [resp. \( I^*(u) \)]. We argue by way of contradiction. Suppose there exists a \( m \) dimensional subspace \( E_m \subset H^1(\mathbb{R}^N) \), a sequence \( \{ u_n \} \subset E_m \cap E_\lambda \) (resp. \( \{ u_n \} \subset E_m \cap E_\ast \)) such that \( \| u_n \| \to + \infty \). Let \( e_1, e_2, \ldots, e_m \) be the basis of \( E_m \). Then
\[
u_n = t_1^n e_1 + \ldots + t_m^n e_m
\]
for some \( t_n = (t_1^n, \ldots, t_m^n) \in \mathbb{R}^m \).
Set \( |t_n| = \max_{1 \leq i \leq m} |t_i^n| \), we have \( |t_n| \to + \infty \).

\[
\begin{align*}
\int |\nabla u_n|^2 + u_n^2 &= 0 (\ |t_n|^2) \\
\int b(x) |u_n|^{p+1} &\geq 0 \\
\int c(x) |u_n|^{q+1} &\geq C_5 |t_n|^{q+1} \quad \text{for} \quad n \text{ large enough}
\end{align*}
\]
where \( C_5 > 0 \) is some constant.

(3.14), (3.15) and (3.16) deduce \( I_\lambda(u_n) < 0 \) for \( n \) large enough [resp. \( I^*(u_n) < 0 \) for \( n \) large enough], which contradicts \( u_n \in E_\lambda \) (resp. \( u_n \in E_\ast \)).

Define

\begin{align}
\Gamma^n_* &= \{ A \subset \Sigma \mid A \text{ is compact and } v(A \cap h(\partial B_1)) \geq n \text{ for any } h \in H_1 \}, \quad n = 1, 2, \ldots, \\
\Gamma^n &= \{ A \subset \Sigma \mid A \text{ is compact and } v(A \cap h(\partial B_1)) \geq n \text{ for any } h \in H_* \}, \quad n = 1, 2, \ldots, \\
c^n_* &= \inf_{A \in \Gamma^n_*} \max \{ I_* (u) \mid u \in A \}, \quad n = 1, 2, \ldots, \\
c^n &= \inf_{A \in \Gamma^n} \max \{ I_* (u) \mid u \in A \}, \quad n = 1, 2, \ldots,
\end{align}

Suppose (3.10) holds then by Proposition 3.2 and Lemma 3.1, \( \Gamma^n_* \neq \emptyset \) for each \( n = 1, 2, \ldots, \), and consequently \( c^n_* < +\infty \).

Let

\[ \lambda_k = \left[ \frac{p - 1}{2 (p + 1) c^n_*} \right]^{(p - 1)/2} S^{(p + 1)/2} b^{-1}_\infty, \quad k = 1, 2, \ldots \]

We have

**Theorem 3.3.** Suppose (1.2) and (3.10) hold. Then for each \( n = 1, 2, \ldots, \), (1.1) has \( n \) pair of solutions \( \{ -u_i, u_i \}, \quad i = 1, \ldots, n \) if \( \lambda \in (0, \lambda_n) \).

**Proof.** By the definition of \( c^n_*, c^n, n = 1, 2, \ldots \) we have

\[ c^n_* = \inf_{A \in \Gamma^n_*} \max \{ I_* (u) \mid u \in A \} \]

\[ \leq \inf_{A \in \Gamma^n_*} \max \{ I_* (u) \mid u \in A \} \]

\[ \leq \inf_{A \in \Gamma^n_*} \max \{ I_* (u) \mid u \in A \} \]

Thus

\[ c^n_* \leq c^n \quad \text{for } n = 1, 2, \ldots \]

Next we claim that for each \( c^k, k = 1, \ldots, n, I_\lambda (u) \) satisfies (PS)$_c$ condition.

Indeed, \( \lambda < \lambda_n \) implies

\[ \lambda < \left[ \frac{p - 1}{2 (p + 1) c^n_*} \right]^{(p - 1)/2} S^{(p + 1)/2} b^{-1}_\infty. \]
Thus
\[ c^n_k < \frac{p-1}{2(p+1)} \cdot S^{(p+1)/(p-1)}(c_k b_\infty)^{-2/(p-1)} = I_k^\infty \]
which combining with (3.23) deduces
\[ c^n_k < I_k^\infty. \]

On the other hand, obviously we have
\[ c^1_k \leq \ldots \leq c^n_k. \]
Thus, by Lemma 2.3, \( I_k(u) \) satisfies \((PS)_c\) condition for \( c^1_k, k = 1, 2, \ldots, n \). Following Lemma 3.1, \( I_k(u) \) has at least \( n \) different critical points \( u_i \in H^1(\mathbb{R}^N) \) \( (1 \leq i \leq n) \) such that \( I_k(u_i) = c^i_k (1 \leq i \leq n) \). Since \( I_k(u) \) is an even functional \( -u \) is critical point either \( (1 \leq i \leq n) \), \( \{ -u_i, u_i \} \) are the solutions we are looking for. Hence we have proved Theorem 3.3.

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