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NEIL S. TRUDINGER

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Isoperimetric inequalities for quermassintegrals

by

Neil S. TRUDINGER

Centre for Mathematics and its Applications,
The Australian National University,
Canberra, A.C.T. 0200, Australia

ABSTRACT. – The isoperimetric inequalities of Aleksandrov and Fenchel for *quermassintegrals* (cross sectional measures) of convex domains in Euclidean space are established for non-convex domains, subject to natural curvature conditions. The techniques are new and draw upon the theory of Monge-Ampère type equations related to previous work of the author on equations of prescribed curvature.

RÉSUMÉ. – Les inégalités isopérimétriques d'Aleksandrov et Frenchel pour les *quermassintegrals* dans le cas de domaines convexes d'espaces euclidiens sont établies pour un domaine non convexe vérifiant des conditions naturelles de courbure. Les techniques sont neuves et reposent sur des équations de type Monge-Ampère reliées à un travail antérieur de l'auteur sur des équations à courbure prescrite.

1. INTRODUCTION

In this paper we derive isoperimetric inequalities for quermassintegrals (cross sectional measures) of domains in Euclidean space, which extend both the classical isoperimetric inequality and certain cases of the Aleksandrov-

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Fenchel inequalities for convex domains. Our techniques are novel in that they are drawn from the theory of fully nonlinear elliptic equations and exploit connections with prescribed curvature equations, already displayed in our works ([10], [11]).

For a bounded, *convex* domain Ω in Euclidean n-space, $\mathbb{R}^n$, the m^{th} quermassintegral $V_m(\Omega)$ may be defined as the mean integral value of the m dimensional volumes of the projections of Ω on m dimensional subspaces. Explicitly we write

$$V_m(\Omega) = \int_{G_{n,m}} \mathcal{H}^m(P_\nu(\Omega))\, d\omega(\nu), \tag{1.1}$$

where $G_{n,m}$ is the Grassmann manifold of m dimensional subspaces in $\mathbb{R}^n$, $P_\nu(\Omega)$ is the orthogonal projection of Ω on the subspace $\nu \in G_{n,m}$, $\mathcal{H}^m$ denotes m dimensional Hausdorff measure in $\mathbb{R}^n$ and ω denotes the normalized Haar measure on $G_{n,m}$. When $\Omega = B$, the unit ball in $\mathbb{R}^n$, we have $V_m(B) = \omega_m$, the volume of the unit ball in $\mathbb{R}^m$, while for arbitrary Ω we clearly have $V_n(\Omega) = \mathcal{H}^n(\Omega) = |\Omega|$, the Lebesgue n-dimensional measure of Ω. Equivalent definitions may be given in terms of Minkowski mixed volumes or the mean integral value of the m dimensional volumes of the intersections of Ω with m dimensional planes in $\mathbb{R}^n$ (*see* [1], [2], [8] for further details).

For our purposes here, we need to relate $V_m(\Omega)$ to curvatures of the boundary $\partial\Omega$. Assuming $\partial\Omega \in C^2$, we define the m^{th} mean curvature function of $\partial\Omega$ by

$$H_m[\partial\Omega] = \sum \kappa_{i_1} \cdots \kappa_{i_m}, \tag{1.2}$$

where $\kappa_1, \cdots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$ and the sum in (1.2) is taken over increasing m-tuples $(i_1, \cdots, i_m) \subset (1, \cdots, n-1)$. The curvature $\kappa_1, \cdots, \kappa_{n-1}$ are normalized so that they are positive on spheres, and for $m = 0$, we define $H_0[\partial\Omega] \equiv 1$. We then have the following formulae (*see* [1], [8])

$$V_{n-m}(\Omega) = \frac{(n-m)!\,(m-1)!}{n!} \frac{\omega_{n-m}}{\omega_n} \int_{\partial\Omega} H_{m-1}[\partial\Omega]\, d\mathcal{H}^{n-1}, \tag{1.3}$$

for $m = 1, \cdots, n-1$. With $V_0(\Omega) \equiv 1$, we see that (1.3) extends to embrace the case $m = n$. When $m = 1$, we obtain

$$V_{n-1}(\Omega) = \frac{\omega_{n-1}}{n\,\omega_n} \mathcal{H}^{n-1}(\partial\Omega),$$

where $\mathcal{H}^{n-1}(\partial\Omega)$ is the perimeter of Ω. The Aleksandrov-Fenchel inequalities may be written in the form,

$$\left(\frac{V_k(\Omega)}{V_k(B)}\right)^{1/k} \leqq \left(\frac{V_m(\Omega)}{V_m(B)}\right)^{1/m} \tag{1.4}$$

for $m < k$, with equality holding only if Ω is a ball. In the special case $k = n$, we obtain thus, for any $m \geqq 1$,

$$|\Omega|^{1-(m/n)} \leqq \frac{(n-m)!\,(m-1)!}{n!\,\omega_n^{m/n}} \int_{\partial\Omega} H_{m-1}[\partial\Omega].$$

The main result of our work is to establish these inequalities for *non-convex* domains Ω, subject to natural curvature conditions. At the same time we supply a completely new proof of (1.4), independent of previous proofs involving Minkowski mixed volumes, which are strongly tied to the convex case (*see* [1], [2]). A new proof of the fundamental mixed volume inequalities (for convex sets) is also given in the recent doctoral dissertation of Andrews (Australian National University 1993). Note that the identity (1.3) provides an extension of the quermassintegral definition to smooth non-convex domains (*see* [8] for the connection between this and integral means aristing from intersections with planes).

To formulate the appropriate conditions on Ω we say that the boundary $\partial\Omega \in \bar{\Gamma}_k$, $0 \leqq k \leqq n-1$, if $H_j[\partial\Omega] \geqq 0$ for all $j = 0, 1, \cdots, k$. If $\partial\Omega$ is connected, $\partial\Omega \in \bar{\Gamma}_k$, if only $H_k[\partial\Omega] \geqq 0$. We will sometimes refer to domains with boundary $\partial\Omega \in \bar{\Gamma}_k$ as being k convex. An arbitrary domain is clearly 0–convex while a C^2 domain is convex if and only if it is $(n-1)$-convex.

THEOREM 1.1. – *The isoperimetric inequalities (1.4) are valid for any bounded C^2 domain Ω which is $n-m-1$ convex.*

From Theorem 1.1 we deduce more precise versions of the Sobolev inequalities used in [10], [11]. In an ensuing paper [14], we will treat versions of inequalities (1.4), (1.5) in arbitrary domains, which have interesting applications to extremal problems in analysis and geometry.

The plan of this paper is as follows. In Section 2, we derive an integration formula for elementary symmetric functions of Jacobians of vector fields which extends corresponding formulae in [7] and [10]. The case of (1.5) for convex domains then follows immediately from an existence theorem of the Monge-Ampère type equations in Section 3. The integration formulae are further developed in Section 4, so that in Section 5 we can derive Theorem 1.1 in the general case by consideration of further Monge-Ampère type equations, treated in the author's papers ([9], [13]). Finally in Section 6, we provide some remarks concerning related work.

The special case of Theorem 1.1, $k = n$, was announced in our previous papers, but our proof followed the lines of Section 3. The present proof resulted from earlier considerations on solution bounds for curvature quotient equations [2] and recent observations pertaining to the monotonicity of the integrands in Section 4 with regard to parameter t.

2. INTEGRATION FORMULAE

We derive, in this section, integration formulae for vector fields, which extend the classical divergence theorem in general and corresponding integral identities of Reilly [7] for particular cases. Adopting the notation of [10], for a real $n \times n$ matrix $\mathcal{A} = [a_{ij}]$, we let $S_m = [\mathcal{A}]_m$ be the sum of its $m \times m$ principal minors. If Ω is a bounded domain in $\mathbb{R}^n$, with boundary $\partial\Omega \in C^2$, we let $\gamma \in C^1(\bar{\Omega})$ be an extension to $\bar{\Omega}$ of the outer unit normal vector field on $\partial\Omega$ and introduce the extended tangential gradient operator ∂, given by

$$\partial = D - (\gamma \cdot D)\,\gamma, \tag{2.1}$$

where $D = (D_1, \cdots, D_n)$ is the gradient in $\mathbb{R}^n$. The eigenvalues of the matrix $[\partial\gamma]$, restricted to $\partial\Omega$, are then $\kappa_1, \cdots, \kappa_{n-1}, 0$ where $\kappa_1, \cdots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$. Accordingly we have the formulae

$$\begin{aligned} H_m[\partial\Omega] &= [\partial\gamma]_m \\ &= [D\gamma]_m \quad (\text{if } \gamma \cdot D\gamma = 0 \text{ on } \partial\Omega). \end{aligned} \tag{2.2}$$

Now let $g = (g_1, \cdots, g_n)$ be a smooth vector field on Ω with $g_i \in C^1(\bar{\Omega}) \cap (C^2(\Omega)$, $i = 1, \cdots, n$. Following [9], we write

$$S_m[Dg] = [Dg]_m = \frac{1}{m!} \sum \delta \begin{pmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{pmatrix} \prod_{k=1}^{m} D_{i_k} g_{j_k}, \tag{2.3}$$

where $i_1, \cdots, i_m$ are distinct indices and $\delta = \pm 1$ according as $j_1, \cdots, j_m$ is an ever or odd permutation of $i_1, \cdots, j_m$ and zero otherwise. Writing also

$$\begin{aligned} S^{ij}_{m-1}[Dg] &= \frac{\partial A_m}{\partial a_{ij}}[Dg] \\ &= \frac{1}{(m-1)!} \sum \delta \begin{pmatrix} i & i_2 & \cdots & i_m \\ j & j_2 & \cdots & j_m \end{pmatrix} \prod_{k=2}^{m} D_{i_k} g_{j_k}, \end{aligned} \tag{2.4}$$

we then have

$$m\,[Dg]_m = D_i\,(S^{ij}_{m-1}\,g_j) \tag{2.5}$$

so that by the divergence theorem

$$\begin{aligned} m\int_\Omega [Dg]_m &= \int_{\partial\Omega} S^{ij}_{m-1}\,\gamma_i\,g_i \\ &= \frac{1}{(m-1)\,!}\int_{\partial\Omega}\sum \delta\begin{pmatrix} i & i_2 & \cdots & i_m \\ j & j_2 & \cdots & j_m\end{pmatrix} \\ &\times\ \gamma_i\,g_j\prod_{k=2}^m D_{i_k}\,g_{j_k} \\ &= \frac{1}{(m-1)\,!}\int_{\partial\Omega}\sum \delta\begin{pmatrix} i & i_2 & \cdots & i_m \\ j & j_2 & \cdots & j_m\end{pmatrix} \\ &\times\ \gamma_i\,g_j\prod_{k=2}^m D_{i_k}\,g_{j_k} \\ &= \frac{1}{(m-1)\,!}\int_{\partial\Omega}\sum \delta\begin{pmatrix} i & i_2 & \cdots & i_m \\ j & j_2 & \cdots & j_m\end{pmatrix} \\ &\times\ \gamma_i\,g_j\prod_{k=2}^m D_{i_k}\,g_{j_k} \\ &= \frac{1}{(m-1)\,!}\int_{\partial\Omega}\sum \delta\begin{pmatrix} i & i_2 & \cdots & i_m \\ j & j_2 & \cdots & j_m\end{pmatrix} \\ &\times\ \gamma_i\,g_j\prod_{k=2}^m \{(g\cdot\gamma)\,\partial_{i_k}\,\gamma_{j_k} \\ &\qquad\qquad + \gamma_{i_k}\,\partial_{j_k}\,(g\cdot\gamma) + \partial_{i_k}\,g'_{j_k}\}, \end{aligned} \tag{2.6}$$

where g' is the tangential projection of g on $\partial\Omega$ given by

$$g' = g - (g\cdot\gamma)\,\gamma. \tag{2.7}$$

By splitting the sum in (2.6) into parts where $i = j$ and $i \neq j$, we then obtain

$$\begin{aligned} m\int_\Omega [Dg]_m = \frac{1}{(m-1)\,!}&\int_{\partial\Omega}\sum \delta\begin{pmatrix} i_2 & \cdots & i_m \\ j_2 & \cdots & j_m\end{pmatrix} \\ &\times\ (g\cdot\gamma)\prod_{k=2}^m \{(g\cdot\gamma)\,\partial_{i_k}\,\gamma_{j_k} + \partial_{i_k}\,g'_{j_k}\} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{(m-2)\,!}\int_{\partial\Omega}\sum\delta\begin{pmatrix} i & i_3 & \cdots & i_m \\ j & j_3 & \cdots & j_m\end{pmatrix} \\
&\qquad\times\ \partial_i\,(g\cdot\gamma)\,g'_j\prod_{k=3}^{m}\{(g\cdot\gamma)\,\partial_{i_k}\gamma_{j_k}+\partial_{i_k} g'_{j_k}\} \\
&=\int_{\partial\Omega}(g\cdot\gamma)\,S_{m-1}\,[(g\cdot\gamma)\,\partial\gamma+\partial g'] \\
&-\int_{\partial\Omega}S^{ij}_{m-2}\,[(g\cdot\gamma)\,\partial\gamma+\partial g']\,\partial_i\,(g\cdot\gamma)\,g'_j. \qquad (2.8)
\end{aligned}
$$

Formula (2.8) agrees with that of Reilly [7] for the case when $g = Df$ for some function $f \in C^2(\bar{\Omega})$. Furthermore, when g is normal on $\partial\Omega$, so that g' vanishes there, then we obtain from (2.8),

$$
m\int_{\Omega}[Dg]_m=\int_{\partial\Omega}(g\cdot\gamma)^m\,H_{m-1}\,[\partial\Omega], \qquad (2.9)
$$

as asserted in [9], Lemma 3. For our later estimation, we need to eliminate the derivatives of the normal component $g \cdot \gamma$ [from (2.8)]. Certain special cases of the inequalities (1.5) are readily inferred from the simpler formula (2.9). We treat these in the next section and take up the further development of (2.8) in the following section. Note that inequalities (2.8), (2.9) will continue to hold if we only assume $g_i \in C^{0,1}(\bar{\Omega})$.

3. CONVEX DOMAINS

For purposes of illustration, we supply here a specialization of our approach to convex domains where the technicalities are relatively simple. The starting point is an existence theorem for the Dirichet problem for the equation of prescribed Gauss curvature, which is proved in [3] and [15]. For the graph $\mathcal{S}$ of a function $u \in C^2(\Omega)$, the principal curvatures (with respect to the downwards directed normal) are the eigenvalues of the Jacobian matrix $D\nu$ where ν is the vector field given by

$$
\nu=\frac{Du}{\sqrt{1+|Du|^2}}. \qquad (3.1)
$$

The m^{th} mean curvature function of $\mathcal{S}$ is thus given by

$$
H_m\,[\mathcal{S}]=[D\nu]_m, \qquad (3.2)
$$

so that in particular the Gauss curvature $H_n[\mathcal{S}]$ is given by

$$H_n[\mathcal{S}] = (1 + |Du|^2)^{-(n+2)/2} \det [D^2 u]. \tag{3.3}$$

THEOREM 3.1. – ([3], [15]). *Let Ω be a uniformly convex domain $\mathbb{R}^d$ with boundary $\partial\Omega \in C^3$ and ψ positive function in Ω such that $\psi^{1/n} \in C^2(\bar{\Omega})$, $\psi = 0$ on $\partial\Omega$ and*

$$\int_\Omega \psi < \omega_n. \tag{3.4}$$

Then there exists a unique convex solution $u \in C^3(\Omega) \cap C^{1,1}(\bar{\Omega})$ of the Dirichlet problem,

$$\begin{aligned} F[u] = H_n[\mathcal{S}] = \psi \quad &in\ \Omega, \\ u = 0 \quad &on\ \partial\Omega. \end{aligned} \tag{3.5}$$

To deduce (1.5) from Theorem 3.1, we use the inequality

$$\psi^{m/n} = H_n^{m/n} \leqq \frac{H_m}{\binom{n}{m}} = \frac{[D\nu]_m}{\binom{n}{m}}. \tag{3.6}$$

By the integration formula (2.9), we have

$$m \int_\Omega \psi^{m/n} \leqq \frac{1}{\binom{n}{m}} \int_{\partial\Omega} H_{m-1}[\partial\Omega]$$

and letting ψ approach the constant $\omega_n/|\Omega|$, we conclude (1.5). By using classical existence theorems of Ivochkina [4] for the prescribed m^{th} order mean curvature equation, we can in fact extend the inequality (1.5) from convex domains to domains Ω which are m-convex, *i.e.* $\partial\Omega \in \bar{\Gamma}_m$. Also if $E \subset \Omega$ and we let

$$\psi \to \frac{\omega_n}{|E|} \chi_E,$$

we obtain by integration over E the classical isoperimetric inequality (1.5) with $m = n - 1$ and no geometric restrictions.

4. INTEGRATION FORMULAE REVISITED

In this section we develop an extension of (2.9) to general vector fields. First we need to re-examine (2.8) with a view towards controlling the terms involving $g \cdot \gamma$.

If $\mathcal{A} = [a_{ij}]$, $\mathcal{B} = [b_{ij}]$ are real $n \times n$ matrices, we can expand

$$S_m [t \mathcal{A} + \mathcal{B}] = \sum_{k=0}^{m} t^k S_{m,k} [\mathcal{A}, \mathcal{B}], \tag{4.1}$$

where the mixed invariant $S_{m,k}$ is given by

$$S_{m,k} [\mathcal{A}, \mathcal{B}] = \frac{1}{m!} \sum \delta \begin{pmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{pmatrix} \times (a_{i_1 j_1}, \cdots, a_{i_k j_k}) (b_{i_{k+1} j_{k+1}}, \cdots, b_{i_m j_m}). \tag{4.2}$$

Similarly we have the expansion

$$S_m^{ij} [t \mathcal{A} + \mathcal{B}] = \sum_{k=0}^{m} t^k S_{m,k}^{ij} [\mathcal{A}, \mathcal{B}], \tag{4.3}$$

where

$$S_{m,k}^{ij} [\mathcal{A}, \mathcal{B}] = \frac{1}{m!} \sum \delta \begin{pmatrix} i_1 & \cdots & i_m & i \\ j_1 & \cdots & j_m & j \end{pmatrix} \times (a_{i_1 j_1}, \cdots, a_{i_k j_k}) (b_{i_{k+1} j_{k+1}}, \cdots, b_{i_m j_m}). \tag{4.4}$$

Applying the expansion (4.3) to the second integrand in (2.8), we obtain

$$\begin{aligned} S_{m-2}^{ij} &[(g \cdot \gamma) \partial \gamma + \partial g'] \partial_i (g \cdot \gamma) g'_j \\ &= \sum_{k=0}^{m-2} (g \cdot \gamma)^k S_{m-2,k}^{ij} [\partial \gamma, \partial g'] \partial_i (g \cdot \gamma) g'_j \\ &= \sum_{k=0}^{m-2} S_{m-2,k}^{ij} [\partial \gamma, \partial g'] \frac{1}{k+1} \partial_i (g \cdot \gamma)^{k+1} g'_j. \end{aligned}$$

Next it is readily checked that

$$\partial_i \{ S_{m-2,k}^{ij} [\partial \gamma, \partial g'] \}$$

has vanishing tangential component on $\partial \Omega$, so that in particular

$$\partial_i \{ S_{m-2,k}^{ij} [\partial \gamma, \partial g'] g'_j \} = S_{m-2,k}^{ij} [\partial \gamma, \partial g'] \partial_i g'_j \tag{4.5}$$

and hence by integration by parts we obtain from the second integral in (2.8),

$$\int_{\partial \Omega} S_{m-2}^{ij} [(g \cdot \gamma) \partial \gamma + \partial g'] \partial_i (g \cdot \gamma) g'_j$$

$$= -\int_{\partial\Omega} \sum_{k=0}^{m-2} \frac{1}{k+1} S_{m-2,\,k}^{ij}$$
$$\times\ [\partial\gamma,\ \partial g']\,(g\cdot\gamma)^{k+1}\,\partial_i\, g'_j$$
$$= -\int_{\partial\Omega} \sum_{k=0}^{m-2} \frac{m-k-1}{k+1} S_{m-1,\,k}^{ij}$$
$$\times\ [\partial\gamma,\ \partial g']\,(g\cdot\gamma)^{k+1}. \tag{4.6}$$

Combining (4.6) with the expansion (4.1) for the first integrand in (2.8), we thus obtain the formula

$$\int_{\Omega} [Dg]_m = \int_{\partial\Omega} \sum_{k=0}^{m-1} \frac{1}{k+1} S_{m-1,\,k}\,[\partial\gamma,\ \partial g']\,(g\cdot\gamma)^{k+1}. \tag{4.7}$$

Setting

$$P_m(t) = \sum_{k=0}^{m-1} \frac{1}{k+1} S_{m-1,\,k}\,[\partial\gamma,\ \partial g']\,t^{k+1}, \tag{4.8}$$

we then have

$$P'_m(t) = \sum_{k=0}^{m-1} S_{m-1,\,k}\,[\partial\gamma,\ \partial g']\,t^{k}$$
$$= S_{m-1}\,[t\,\partial\gamma + \partial g'], \tag{4.9}$$

so that the formula (4.7) may be written in the form,

$$\int_{\Omega} S_m\,[Dg] = \int_{\partial\Omega} \int_0^{\gamma\cdot g} S_{m-1}\,[t\,\partial\gamma + \partial g']\,dt\,d\mathcal{H}^{n-1}. \tag{4.10}$$

Consequently if

$$S_{m-1}\,[t\,\partial\gamma + \partial g'] \geqq 0 \tag{4.11}$$

for all $t \geqq g\cdot\gamma$, we estimate, for $g_0 = \max_{\partial\Omega}(g\cdot\gamma)$,

$$\int_{\Omega} [Dg]_m \leqq \int_{\partial\Omega} \sum_{k=0}^{m-1} \frac{g_0^{k+1}}{k+1} S_{m-1,\,k}\,[\partial\gamma,\ \partial g']$$
$$= \frac{g_0^m}{m} \int_{\partial\Omega} H_{m-1}\,[\partial\Omega] \tag{4.12}$$

by virtue of (4.5). In the next section, we will take g as a gradient field so all the above matrices are in fact symmetric.

5. PROOF OF THE ISOPERIMETRIC INEQUALITY

Our approach is to exhibit an appropriate vector field g satisfying (4.11). This will be accomplished by solving certain Hessian equations, rather than the curvature type equation (3.5) and taking g as the gradient of a solution, instead of the vector field (3.1). Specifically we shall use the following existence theorem from [9] or [13].

THEOREM 5.1. – *Let B be a ball in $\mathbb{R}^n$ and ψ a non-negative function in $C^{1,1}(\bar{B})$. Then there exists a unique convex solution $u \in C^{1,1}(B) \cap C^{0,1}(\bar{B})$ of the Dirichlet problem,*

$$\left.\begin{aligned} \det D^2 u &= S_m(D^2 u)\,\psi^{n-m} && \textit{in } B, \\ u &= 0 && \textit{on } \partial B, \end{aligned}\right\} \tag{5.1}$$

for any $m = 0, 1, \cdots, n-1$, with $u \in C^3\{\psi > 0\}$.

In order to use Theorem 5.1, we suppose that the domain Ω lies in the ball B_ρ of radius ρ centred at the origin and take $B = B_R$ to be the larger concentric ball of radius $R > \rho$. The function ψ is chosen initially to satisfy

$$0 \leqq \psi \leqq \chi_\Omega. \tag{5.2}$$

From the Aleksandrov estimate ([3], Lemma 9.2), we then have

$$\begin{aligned} \omega_n (\sup_\Omega |u|)^n &\leqq (R+\rho)^n \int_\Omega S_m(D^2 u) \\ &\leqq \frac{(R+\rho)^n}{m} \sup_\Omega |Du|^m \int_{\partial\Omega} H_{m-1} \end{aligned}$$

provided $\partial\Omega \in \bar{\Gamma}_{m-1}$, $m \geqq 1$, by (4.12). Again using the convexity of u, we may estimate

$$\begin{aligned} \sup_\Omega |Du|^n &\leqq (R-\rho)^{-n} \sup_\Omega |u|^n \\ &\leqq \left(\frac{R+\rho}{R-\rho}\right)^n \frac{1}{m\,\omega_n} \sup_\Omega |Du|^m \int_{\partial\Omega} H_{m-1}, \end{aligned}$$

whence we obtain the estimate

$$\sup_\Omega |Du|^{n-m} \leqq \frac{1}{m\,\omega_n}\left(\frac{R+\rho}{R-\rho}\right)^n \int_{\partial\Omega} H_{m-1}. \tag{5.3}$$

When $m = 0$, we obtain, instead

$$\sup_{\Omega} |Du|^n \leqq \frac{1}{\omega_n} \left(\frac{R+\rho}{R-\rho} \right)^n |\Omega|. \tag{5.4}$$

To proceed further, we use the Newton-Maclaurin inequalities,

$$\left[\frac{\binom{n}{m} S_n}{S_m} \right]^{1/(n-m)} \leqq \left[\frac{\binom{n}{m} S_k}{\binom{n}{k} S_m} \right]^{1/(k-m)}, \tag{5.5}$$

for $0 \leqq m \leqq k \leqq n$, to obtain from equation (5.1), the differential inequality,

$$S_k (D^2 u) \geqq c \psi^{k-m} S_m (D^2 u) \tag{5.6}$$

with constant c given by

$$c = \binom{n}{k} \binom{n}{m}^{(k-m)/(n-m)}. \tag{5.7}$$

Let us now specify ψ further so that

$$\psi = 1 \quad \text{in } \Omega_\delta = \{ x \in \Omega | \mathrm{dist}(x, \partial\Omega) > \delta \} \tag{5.8}$$

where $\delta > 0$, is chosen sufficiently small to ensure $\partial\Omega_\delta \in \bar{\Gamma}_{k-1}$ if $\partial\Omega_g \in \bar{\Gamma}_{k-1}$. Integrating (5.8) over Ω_δ we thus obtain from (4.10),

$$0 \leqq \int_{\partial\Omega_\delta} \int_0^{\gamma \cdot Du} (S_{k-1} - c\, S_{m-1}) \, [t\, \partial\gamma + \partial^2 u] \, dt \, d\mathcal{H}^{n-1}. \tag{5.9}$$

We now need to show that the integrand in (5.9) is non-negative for $t \geqq \gamma \cdot Du$. Fixing $y \in \partial\Omega$, we choose a principal coordinate system at y, so that in particular the x_n axis is directed along the unit inner normal at y (*see* [3], Ch. 14). Writting

$$\mathcal{A} = t\, \partial\gamma + \partial^2 u,$$

we then have

$$\left. \begin{aligned} \mathcal{A}_{ij} &= (t + D_u\, u)\, \chi'_i\, \delta_{ij} + D_{ij}\, u, \qquad i, j = 1, \cdots, n-1, \\ \mathcal{A}_{in} &= \mathcal{A}_{nj} = 0, \qquad i, j = 1, \cdots, n \end{aligned} \right\} \tag{5.10}$$

so that for $t = \gamma \cdot Du = -D_n u$, we have

$$\begin{aligned} S_{k-1}(\mathcal{A}) - c\, S_{m-1}(\mathcal{A}) &\geqq S_{k-1}(\mathcal{A}) - \frac{S_k(D^2 u)\, S_{m-1}(\mathcal{A})}{S_m(D^2 u)} \\ &\geqq \frac{1}{S_m(D^2 u)} \{ S_{k-1}(\mathcal{A})\, S_m(D^2 u) \\ &\quad - S_k(D^2 u)\, S_{m-1}(\mathcal{A})\} \\ &\geqq \frac{1}{S_m(D^2 u)} \{ S_{k-1}(\mathcal{A})\, S_m(\mathcal{A}) \\ &\quad - S_k(\mathcal{A})\, S_{m-1}(\mathcal{A}) \} \\ &\geqq 0 \end{aligned} \tag{5.11}$$

by the Newton-Maclaurin inequalities and the convexity of u. Next we note that the function

$$F(\mathcal{A}) = \left(\frac{S_{k-1}}{S_{m-1}} \right)^{1/(k-m)} \tag{5.12}$$

is concave on the cone

$$\bar{\Gamma}_{k-1} = \{ \mathcal{A} \in \$^{n-1} \mid S_j(\mathcal{A}) \geqq 0,\ j = 1, \cdots, k-1 \}$$

and hence if $\partial\Omega_\delta \in \bar{\Gamma}_{k-1}$, $t \geqq \gamma.\ Du$, we also have

$$S_{k-1}(\mathcal{A}) - c\, S_{m-1}(\mathcal{A}) \geqq 0. \tag{5.13}$$

Therefore can estimate from (5.9) and (4.5),

$$\begin{aligned} 0 &\leqq \int_{\partial\Omega_\delta} \int_0^{\max_\Omega |Du|} (S_{k-1} - c\, S_{m-1})\,[\mathcal{A}]\, dt\, d\mathcal{H}^{n-1} \\ &\leqq \frac{1}{k} \max_\Omega |Du|^k \int_{\partial\Omega_\delta} H_{k-1} - \frac{c}{m} \max_\Omega |Du|^m \int_{\partial\Omega_\delta} H_{m-1}, \end{aligned} \tag{5.14}$$

so that

$$\int_{\partial\Omega_\delta} H_{m-1} \leqq \frac{m}{ck} \max_\Omega |Du|^{k-m} \int_{\partial\Omega_\delta} H_{k-1}. \tag{5.15}$$

Combining (5.15) and (5.3) and sending δ to zero, R to infinity, yields, for any $1 \leqq m \leqq k < n$,

$$\left(\frac{1}{m \binom{n}{m} \omega_n} \int_{\partial\Omega} H_{m-1} \right)^{1/(n-m)}$$

$$\leqq \left(\frac{1}{k \binom{n}{k} \omega_n} \int_{\partial\Omega} H_{k-1} \right)^{1/(n-k)} \tag{5.16}$$

which agrees with (1.4), $k < n$. When $m = 0$, we obtain, in place of (5.9),

$$\begin{aligned} c|\Omega_\delta| &\leqq \int_{\partial\Omega_\delta} \int_0^{\gamma \cdot Du'} S_{k-1}(\mathrm{A})\, dt\, d\mathcal{H}^{n-1} \\ &\leqq \frac{1}{k} \max_{\Omega} |Du|^k \int_{\partial\Omega_\delta} H_{k-1}, \end{aligned} \tag{5.17}$$

so that we have, for any $1 \leqq k < n$,

$$\left(\frac{1}{\omega_n}|\Omega|\right)^{1/n} \leqq \left(\frac{1}{k \binom{n}{k} \omega_n} \int_{\partial\Omega} H_{k-1} \right)^{1/(n-k)}, \tag{5.18}$$

which agrees with (1.5). This completes the proof of Theorem 1.1.

Remark 5.2. – Theorem 5.1 follows from the estimates presented in [9]. Indeed, the assertion of [9], Theorem 1 extends directly to the more general operators

$$F(D^2 u) = \frac{S_k}{S_m}(D^2 u), \qquad 0 \leqq m < k \leqq n, \tag{5.19}$$

through approximation by the *uniformly elliptic* operators (*cf.* [12]),

$$F_\epsilon(D^2 u) = F(D^2 u + \epsilon I \Delta u), \qquad \epsilon > 0. \tag{5.20}$$

In [13], we treat the classical solvability of equations of this type in general domains and show, for example, that Theorem 5.1 is valid for any uniformly convex domain Ω with boundary $\partial\Omega \in C^4$ and resultant solution $u \in C^{1,1}(\bar{\Omega})$. In our derivation of the isoperimetric inequalities, we could also have approximated χ_Ω by positive functions ψ and taken corresponding solutions $u \in C^4(B)$. Alternatively we could have dispensed with elliptic regularity theory altogether and solved the Dirichlet problem (5.1) in the viscosity sense of [12], using the Perron process.

6. FURTHER REMARKS

6.1. Using Theorem 1.1 [inequality (1.5)], we can improve the Sobolev inequalities presented in [10], [11]. For a function $u \in C^2(\Omega)$, we set

$$\gamma = \frac{Du}{|Du|}, \qquad Du \neq 0$$

$$\mathcal{C}_m(u) = \begin{cases} |Du|^{2-m} S_{m-1}[D_{ij} u - \gamma_i \gamma_k D_{kj} u], & Du \neq 0, \\ 0, & Du = 0. \end{cases} \tag{6.1}$$

Then if $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $\mathcal{C}_k[u] \geqq 0$ in Ω, $k = 1, \cdots, m$, we have the estimate

$$\|u\|_{L^{n/(n-m)}(\Omega)} \leqq c(m, n) \int_\Omega \mathcal{C}_m[u], \tag{6.2}$$

where

$$c(m, n) = \frac{(n-m)!(m-1)!}{n!\,\omega_n^{m/n}} = \left[m \binom{n}{m} \omega_n^{m/n}\right]^{-1} \tag{6..3}$$

is the constant in (1.5). In particular (6.2) holds for functions u that are admissible with respect to the m-mean curvature operator (*see* [11]). Inequalities of this type were used by us in [10], [11] to derive *a priori* solution bounds for mean curvature equations. The constant c in (6.2) is optimal and as in the case $m = 1$, it can be approximated by letting u tend to the characteristic function of a ball. We pursue further and more general inequalities of this type in [14], as well as develop more precise bounds for equations of prescribed curvatures and curvature quotients.

6.2. It can be shown that balls are the only domains for which equality is achieved in Theorem 1.1. This follows from Korevaar [5] since the boundary $\partial\Omega$ of any extremal domain Ω, for which equality is achieved, must satisfy

$$\frac{H_{n-m-1}}{H_{n-k-1}}[\partial\Omega] = \text{constant}. \tag{6.4}$$

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REFERENCES

[1] Yu D. BURAGO and V. A. ZALGALLER, *Geometric Inequalities*, Springer-Verlag, Berlin, 1988.
[2] H. BUSEMANN, *Convex Surfaces*, Wiley Interscience, New York, 1958.
[3] D. GILBARG and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Second edition, Springer-Verlag, Berlin, 1983.
[4] N. M. IVOCHKINA, The Dirichlet problem for the equations of curvature of order, m. *St Petersburg Math. J.*, **2**, 1991, pp. 631–654.
[5] N. J. KOREVAAR, Sphere theorems via Aleksandrov for constant Weingarten curvature, *J. Diff. Geom.*, **27**, 1988, pp. 221-223.
[6] P. L. LIONS, N. S. TRUDINGER and J. J. E. URBAS, The Neumann problem for equations of Monge-Ampère type, *Comm. Pure Appl. Math.*, **39**, 1986, pp. 539-563.
[7] R. C. REILLY, On the Hessian of a function and the curvatures of its graph, *Michigan Math. J.* **20**, 1973, pp. 373-383.
[8] L. SANTALO, *Integral Geometry and Geometric Probability*, Addison-Wesley, Cambridge, Mass., 1977.
[9] N. S. TRUDINGER, On degenerate fully nonlinear equations in balls, *Bull. Aust. Math. Soc.*, **35**, 1987, pp. 299-307.
[10] N. S. TRUDINGER, A priori bounds for graphs with prescribed curvature, In *Analysis, etc.*, Academic Press, 1990, pp. 667-676.
[11] N. S. TRUDINGER, *A priori* bounds and necessary conditions for solvability of prescribed curvature equations, *Manuscripta Math.*, **67**, 1990, pp. 99-112.
[12] N. S. TRUDINGER, The Dirichlet problem for the prescribed curvature equations, *Arch. Rat. Mech. Anal.*, **111**, 1990, pp. 153-170.
[13] N. S. TRUDINGER, *On the Dirichlet problem for Hessian equations*, Aust. Nat. Univ. Centre for Math. and its Appl., Research Report M25, 1994.
[14] N. S. TRUDINGER, *On new isoperimetric inequalities and symmetrization*, in preparation.
[15] N. S. TRUDINGER and J. J. E. URBAS, The Dirichlet problem for the equation of prescribed Gauss curvature, *Bull. Aust. Math. Soc.*, **28**, 1983, pp. 217-231.

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