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## **Geodesics on product Lorentzian manifolds (\*)**

by

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**ABSTRACT.** – In this paper, using global variational methods, we prove existence and multiplicity results for geodesics joining two given events of a product Lorentzian manifold  $\mathcal{M}_0 \times \mathbb{R}$ , where  $\mathcal{M}_0$  is a complete Riemannian manifold.

*Key words:* Lorentzian metrics, geodesics, critical point theory.

**RÉSUMÉ.** – Dans cet article, avec des méthodes variationnelles globales, on démontre des résultats d'existence et de multiplicité de géodésiques joignant deux points dans une variété de Lorentz  $\mathcal{M}_0 \times \mathbb{R}$ , où  $\mathcal{M}_0$  est une variété de Riemann complète.

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### **1. INTRODUCTION AND STATEMENT OF THE RESULTS**

Let  $\mathcal{M}$  be a Lorentzian manifold, *i.e.* a smooth manifold equipped with a non-degenerate symmetric (0,2)-tensor field  $g(z)[\cdot, \cdot](z \in \mathcal{M})$  having index 1. (This means that every matricial representation of  $g$  has exactly

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one negative eigenvalue). The points of  $\mathcal{M}$  are called events. A geodesic on  $\mathcal{M}$  is a smooth curve  $\gamma : [a, b] \rightarrow \mathcal{M}$  solving

$$D_s \dot{\gamma}(s) = 0 \quad \text{for all } s \in [a, b]$$

where  $a < b \in \mathbb{R}$ ,  $\dot{\gamma}(s)$  is the derivative of  $\gamma(s)$  and  $D_s \dot{\gamma}(s)$  is the covariant derivative of  $\dot{\gamma}(s)$  with respect to the metric tensor  $g$ .

It is well known that a geodesic is a critical point of the “energy” functional

$$f(\gamma) = \int_a^b g(\gamma(s)) [\dot{\gamma}(s), \dot{\gamma}(s)] ds,$$

and, if  $\gamma$  is a geodesic on  $\mathcal{M}$ , there exists a constant  $E_\gamma \in \mathbb{R}$  such that

$$E_\gamma = g(\gamma(s)) [\dot{\gamma}(s), \dot{\gamma}(s)] \quad \text{for all } s \in [a, b].$$

A geodesic  $\gamma$  is called timelike, null or spacelike if  $E_\gamma$  is less, equal or greater than zero respectively. In General Relativity a timelike geodesic represents the trajectory of a free falling particle. Null geodesics represent the light rays, while spacelike geodesics have a more subtle interpretation: for a suitable local observer they represent “Riemannian” geodesics consisting of simultaneous events.

During the last few years, the existence of geodesics joining two given events in static and stationary Lorentzian manifolds, has been widely studied by global variational methods (*cf.* [2] and references therein).

The aim of this paper is to prove, always by global variational methods, existence and multiplicity results for timelike and spacelike geodesics joining two given events of a product Lorentzian manifold  $\mathcal{M}_0 \times \mathbb{R}$ , in general situations in which the Lorentzian metric of  $\mathcal{M}$  depends on the time variable and has mixed terms.

The results of this paper are different from those of [3] (where the case without mixed terms is considered) and are obtained using different methods. In particular we get a priori estimates for the critical points of the energy functional, using suitable test functions in the weak equation of the critical points and, using the relative category (*see* [6]), we get also a multiplicity result for timelike geodesics.

One of the main difficulties of a global variational approach to the problem is the study of the Palais-Smale compactness condition (*cf.* Definition 2.1) for the energy integral  $f$ . We bypass this difficulty having a set of assumptions which allow to get a priori bounds for the critical points and the Palais-Smale condition, for a suitable penalization functional of  $f$ .

The proofs of the existence results use a Saddle Point Theorem which is a slight device of the classical Saddle Point Theorem of Rabinowitz

(cf. [15]), while the multiplicity results use the topological concept of Relative Category (cf. [4], [6], [7], [17]).

Relative category gives a topological approach which seems simpler than the relative cohomology used in [1] to get infinitely many spacelike geodesics joining two given events, and it allows also to obtain a multiplicity result for timelike geodesics.

Let  $T_z \mathcal{M} = T_x \mathcal{M}_0 \times \mathbb{R}$  be the tangent space to  $\mathcal{M}$  at  $z = (x, t)$ . Assume that the metric tensor  $g$  on  $\mathcal{M}$ , for any  $\zeta = (\xi, \tau) \in T_z \mathcal{M}$ , has the form

$$g(z)[\zeta, \zeta] = \langle \alpha(z)\xi, \xi \rangle_x + 2\langle \delta(z), \xi \rangle_x \tau - \beta(z)\tau^2 \quad (1.1)$$

where  $\langle \cdot, \cdot \rangle_x$  is a Riemannian metric on  $\mathcal{M}_0$ ,  $\alpha(z)[\cdot] = \alpha(x, t)[\cdot]$  is a smooth, symmetric, positive linear operator on  $T_x \mathcal{M}_0$ ,  $\delta$  is a smooth vector field and  $\beta$  is a smooth, positive scalar field. Denote by  $\alpha_t$ ,  $\delta_t$  and  $\beta_t$  the derivative with respect to the variable  $t$  of  $\alpha$ ,  $\delta$ ,  $\beta$  respectively.

In order to get a priori bounds on the critical points of the energy functional we shall make assumptions on  $g$  (on the compact subsets of  $\mathbb{R}$ , globally and at infinity) as follows.

Define

$$A^+(t) = \sup \{ \langle \alpha_t(x, t)\xi, \xi \rangle_x / \langle \alpha(x, t)\xi, \xi \rangle_x : x \in \mathcal{M}_0, \xi \in T\mathcal{M}_0 \},$$

$$A^-(t) = \sup \{ -\langle \alpha_t(x, t)\xi, \xi \rangle_x / \langle \alpha(x, t)\xi, \xi \rangle_x : x \in \mathcal{M}_0, \xi \in T\mathcal{M}_0 \},$$

$$D(t) = \sup \{ \|\delta_t(x, t)\| : x \in \mathcal{M}_0 \},$$

$$B^+(t) = \sup \{ \beta_t(x, t) / \beta(x, t) : x \in \mathcal{M}_0 \},$$

$$\beta^-(t) = \sup \{ -\beta_t(x, t) / \beta(x, t) : x \in \mathcal{M}_0 \},$$

$$B(t) = \sup \{ |\beta_t(x, t)| : x \in \mathcal{M}_0 \}.$$

Moreover set

$$A_{+\infty}^+ = \limsup_{t \rightarrow +\infty} A^+(t),$$

$$A_{-\infty}^- = \limsup_{t \rightarrow -\infty} A^-(t),$$

$$D^+ = \limsup_{t \rightarrow +\infty} D(t),$$

$$D^- = \limsup_{t \rightarrow -\infty} D(t),$$

$$B_{+\infty}^- = \limsup_{t \rightarrow +\infty} B^-(t),$$

$$B_{-\infty}^+ = \limsup_{t \rightarrow -\infty} B^+(t).$$

The global assumptions on  $g$  are the following:

$$(\mathcal{M}_0, \langle \cdot, \cdot \rangle_x) \text{ is a complete Riemannian manifold,} \quad (1.2)$$

$$\inf \{ \langle \alpha(z) \xi, \xi \rangle_x / \langle \xi, \xi \rangle_x : z \in \mathcal{M}, \xi \in T_x \mathcal{M}_0 \} = \lambda > 0, \quad (1.3)$$

$$\inf \{ \beta(z) : z \in \mathcal{M} \} = b > 0. \quad (1.4)$$

About the behavior of  $g(x, t)$  on the compact subsets of  $\mathbb{R}$  we make the following assumptions:

$$A^+, A^-, D \text{ and } B \text{ are bounded on the compact subsets of } \mathbb{R}, \quad (1.5)$$

$$\left. \begin{array}{l} \sup \{ \| \delta(x, 0) \| : x \in \mathcal{M}_0 \} \equiv \delta_0 < +\infty; \\ \sup \{ \beta(x, 0) : x \in \mathcal{M}_0 \} \equiv \beta_0 < +\infty. \end{array} \right\} \quad (1.6)$$

A first reasonable assumption about the behavior of  $g$  at infinity is the following:

$$A_{+\infty}^+, A_{-\infty}^-, D^+, D^-, B_{+\infty}^-, B_{-\infty}^+ \text{ are finite.} \quad (1.7)$$

As observed in Remark 3.5, assumptions (1.2)-(1.6) are sufficient to prove a priori estimates about timelike geodesics. Unfortunately they are not sufficient to prove a priori estimates for spacelike geodesics having energy bounded from above, that we need for the study of the geodesic connectedness (also (1.2)-(1.7) are not sufficient). Indeed the a priori estimates (for spacelike geodesics) do not hold for the Anti-de Sitter space-time, *i.e.*  $\mathbb{R} \times ]-\pi/2, \pi/2[$  with the Lorentz metric

$$ds^2 = \frac{dx^2 - dt^2}{\cos^2 t}.$$

But, by a suitable change of variable, the Anti-de Sitter space-time becomes  $\mathbb{R}^2$  with metric

$$ds^2 = \cosh^2 \vartheta dx^2 - d\vartheta^2$$

which satisfies assumptions (1.2)-(1.7).

Therefore, in order to study the geodesic connectedness of  $\mathcal{M}$  and the multiplicity of spacelike geodesics we reinforce (1.7) in the following way:

$$\left. \begin{array}{l} A_{+\infty}^+, A_{-\infty}^-, B_{+\infty}^-, B_{-\infty}^+ \leq 0 \\ D^+ = D^- = 0 \end{array} \right\} \quad (1.8)$$

In order to prove existence and multiplicity results we also need some assumptions assuring the topological nontriviality of the sublevels of the energy functional. To this aim we assume that *there exist*

$b_0, b_1 \in C^0(\mathcal{M}_0, \mathbb{R}^+)$ ,  $\gamma_0 \in [0, 2[$ ,  $\gamma_1 \in [0, 1[$  such that, for any  $(x, t) \in \mathcal{M}$ ,

$$\frac{1}{\beta(x, t)} \cdot \frac{\langle \alpha(x, t) \xi, \xi \rangle_x}{\langle \xi, \xi \rangle_x} \leq (b_0(x) + b_1(x) |t|^{\gamma_0}) \quad \text{for any } \xi \in T_x \mathcal{M}_0,$$

$$\frac{1}{\beta(x, t)} \cdot \|\delta(x, t)\| \leq (b_0(x) + b_1(x) |t|^{\gamma_1}). \tag{1.9}$$

The first result concerns the existence of a timelike geodesic joining two given events. Let  $z_0 = (x_0, t_0)$ ,  $z_1 = (x_1, t_1) \in \mathcal{M}$  and

$$C^1(x_0, x_1) = \{x \in C^1([0, 1], \mathcal{M}_0) : x(0) = x_0 \text{ and } x(1) = x_1\},$$

$$C^1(t_0, t_1) = \{t \in C^1([0, 1], \mathbb{R}) : t(0) = t_0 \text{ and } t(1) = t_1\}.$$

We shall prove the following:

**THEOREM 1.1.** – *Let  $(\mathcal{M}, g)$  satisfying (1.2)-(1.7) and (1.9). Assume that*

$$\sup_{t \in C^1(t_0, t_1)} \inf_{x \in C^1(x_0, x_1)} \int_0^1 [\langle \alpha(z) \dot{x}, \dot{x} \rangle_x + 2 \langle \delta(z), \dot{x} \rangle_x \dot{t} - \beta(z) \dot{t}^2] ds < 0, \tag{1.10}$$

where  $z = (x, t)$ .

*Then there exists a timelike geodesic in  $\mathcal{M}$  joining  $z_0$  and  $z_1$ .*

About the geodesic connectedness of  $\mathcal{M}$  we have

**THEOREM 1.2.** – *Let  $(\mathcal{M}, g)$  satisfying (1.2)-(1.7), and (1.8)-(1.9). Then  $\mathcal{M}$  is geodesically connected, i.e. for every  $z_0, z_1 \in \mathcal{M}$  there exists a geodesic in  $\mathcal{M}$  joining  $z_0$  and  $z_1$ .*

Theorem 1.2 does not generalize Theorem 1.1 of [3], only because of the assumptions of  $\beta_t$ . Indeed here we assume  $B_{+\infty}^-, B_{-\infty}^+ \leq 0$ , while in [3] it is assumed  $\beta_t$  bounded.

Since  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_x)$  is complete, the uniform estimates in  $x$  of assumptions (1.3), (1.5), and (1.6) imply that  $\mathcal{M}$  is globally hyperbolic (for the definition cf. e.g. [12]). Then by a theorem of Geroch (cf. [8]), there exist coordinates such that  $\mathcal{M}$  is a product manifold with metric tensor  $g$  in (1.1) having  $\delta \equiv 0$ . However in our proofs we can not use the Geroch result to reduce us to the case  $\delta \equiv 0$ , because it is not known how the coefficients  $\alpha$  and  $\beta$  become after the change of coordinates.

Eventhough in our case  $\mathcal{M}$  is globally hyperbolic, our results are motivated by the fact that, as far as we know, on globally hyperbolic Lorentz manifolds only existence results for time-like geodesics have been proved (cf. [2]), while here we deal also with multiplicity results about timelike geodesics and existence and multiplicity results about spacelike geodesics. Moreover it is physically relevant to consider product Lorentzian manifolds not satisfying (1.2) (cf. e.g. [9] for Lorentzian stationary product manifolds), and therefore not necessarily globally hyperbolic, and we hope that the techniques used here (in particular the a priori estimates for the critical points of  $f$ , based on the choice of suitable test functions) can be useful in many cases.

Whenever  $\mathcal{M}_0$  has a rich topology multiplicity results for spacelike geodesics and timelike geodesics joining two given events hold.

**THEOREM 1.3.** – *Assume that  $(\mathcal{M}, g)$  satisfies (1.2)-(1.7), and (1.8)-(1.9) and  $\mathcal{M}_0$  is not contractible.*

*Then there exists a sequence  $\{\gamma_k\}_{k \in \mathbb{N}}$  of spacelike geodesics joining  $z_0$  and  $z_1$  in  $\mathcal{M}$  such that*

$$\lim_{k \rightarrow +\infty} E_{\gamma_k} = +\infty.$$

**THEOREM 1.4.** – *Assume that  $(\mathcal{M}, g)$  satisfies (1.2)-(1.7) and (1.9), and denote by  $N(x_0, x_1, t_0, t_1)$  the number of timelike geodesics joining  $(x_0, t_0)$  with  $(x_1, t_1)$ . Moreover assume that  $\mathcal{M}_0$  is not contractible. Then, for any  $x_0, x_1 \in \mathcal{M}_0$ ,*

$$\lim_{|t_1 - t_0| \rightarrow +\infty} N(x_0, x_1, t_0, t_1) = +\infty.$$

The paper is organized as follows. In section 2 we describe the functional tools used for the proofs of the theorems above. In section 3 we prove the a priori estimates for the critical points of the energy functional and the Palais-Smale compactness condition for a suitable penalization of such a functional. In section 4 we prove Theorems 1.1 and 1.2, while in section 5 we prove Theorems 1.3 and 1.4.

## 2. TECHNICAL PRELIMINARIES

Let  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$  and  $g$  as in (1.1) with  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_x)$  complete, connected Riemannian manifold. By the well known embedding Theorem of Nash (see [11]) we can assume that  $\mathcal{M}_0$  is a submanifold of  $\mathbb{R}^N$  (with  $N$

sufficiently large) and  $\langle \cdot, \cdot \rangle_x$  is the Riemannian structure on  $\mathcal{M}_0$  inherited by the Euclidean metric of  $\mathbb{R}^N$ .

We shall prove Theorems 1.1-1.4 by looking for critical points of the energy functional

$$f(z) = \int_0^1 [\langle \alpha(z) \dot{x}, \dot{x} \rangle + 2 \langle \delta(x), \dot{x} \rangle \dot{t} - \beta(z) \dot{t}^2] ds \quad (2.1)$$

(where  $\langle \cdot, \cdot \rangle$  denotes the Euclidean structure of  $\mathbb{R}^N$ ) on the set of the sufficiently smooth curves joining  $z_0 = (x_0, t_0)$  and  $z_1 = (x_1, t_1)$ . More precisely set

$$\Omega^1 \equiv \Omega^1(\mathcal{M}_0, x_0, x_1) = \left\{ x : [0, 1] \rightarrow \mathcal{M}_0 \text{ absolutely continuous :} \right. \\ \left. \int_2^1 \langle \dot{x}, \dot{x} \rangle ds < +\infty, x(0) = x_0, x(1) = x_1 \right\}, \quad (2.2)$$

$$W^1 \equiv W^1(\mathbb{R}, t_0, t_1) = \left\{ t : [0, 1] \rightarrow \mathbb{R} \text{ absolutely continuous :} \right. \\ \left. \int_2^1 \dot{t}^2 ds < +\infty, t(0) = t_0, t(1) = t_1 \right\}. \quad (2.3)$$

It is well known that  $\Omega^1$  is a smooth, complete Hilbert manifold (cfr e.g. [16]) with Riemannian structure given by

$$\langle \xi, \xi \rangle_1 = \int_0^1 \langle \nabla_s \xi, \nabla_s \xi \rangle ds + \int_0^1 \langle \xi, \xi \rangle ds \quad (2.4)$$

(where  $\nabla_s$  denotes the covariant derivative with respect to the Riemannian structure  $\langle \cdot, \cdot \rangle$ ), while the tangant space to  $\Omega^1$  at  $x$  is given by

$$T_x \Omega^1 = \left\{ \xi : [0, 1] \rightarrow T\mathcal{M}_0 \text{ absolutely continuous :} \right. \\ \left. \xi(s) \in T_{x(s)} \mathcal{M}_0, \int_0^1 \langle \nabla_s \xi, \nabla_s \xi \rangle ds < +\infty, \right. \\ \left. \xi(0) = \xi(1) = 0 \right\}, \quad (2.5)$$

where  $T\mathcal{M}_0$  is the tangent bundle of  $\mathcal{M}_0$ .

Moreover  $W^1(\mathbb{R}, t_0, t_1)$  is an affine manifold, that we equip by the Riemannian structure

$$\langle t, t \rangle_1 = \int_0^1 \dot{t}^2 ds + \int_0^1 t^2 ds, \quad (2.6)$$

and whose tangent space is

$$W_0^{1,2} = \left\{ \tau : [0, 1] \rightarrow \mathbb{R} \text{ absolutely continuous :} \right. \\ \left. \int_0^1 \langle \dot{\tau}, \dot{\tau} \rangle ds < +\infty, \tau(0) = 0, \tau(1) = 0 \right\}. \quad (2.7)$$

We shall look for critical points of the functional (2.1) on the Hilbert manifold

$$\mathcal{Z} = \Omega^1 \times W^1. \quad (2.8)$$

Using the Young inequality in  $\mathbb{R}$  (to control the effects on the weak equation satisfied by the critical points of  $f$  due to the mixed term  $\langle \delta(x), \dot{x} \dot{t} \rangle$ ), classical regularization arguments show that any critical point of  $f$  on  $\mathcal{Z}$  is a (smooth) geodesic in  $\mathcal{M}$  joining  $z_0 = (x_0, t_0)$  with  $z_1 = (x_1, t_1)$ .

Consider now a functional  $I \in C^1(X, \mathbb{R})$  where  $X$  is a Hilbert manifold. Let  $c \in \mathbb{R}$ . We recall the following

DEFINITION 2.1. –  $I$  satisfies  $(P.S.)_c$  on  $X$  (the Palais-Smale condition at the level  $c$ ) if any sequence  $\{x_k\}_{k \in \mathbb{N}} \subset X$  such that

$$I(x_k) \xrightarrow[k]{} c \quad \text{and} \quad I'(x_k) \xrightarrow[k]{} 0$$

has a converging subsequence (in  $X$ ).

The following classical Deformation Lemma (cfr. e.g. [13], [15]) is needed to prove the next Lemma 2.3.

LEMMA 2.2. – Let  $X$  be an Hilbert manifold,  $I \in C^2(X, \mathbb{R})$  and  $c \in \mathbb{R}$ . Assume that  $I$  satisfies  $(P.S.)_c$  on  $X$ .

Then, if  $c$  is not a critical level of  $I$  (i.e.  $I'(x) \neq 0$  for any  $x \in I^{-1}(c)$ ), there exist numbers  $0 < \delta_1 < \delta_2 < \bar{\delta}$  and a homeomorphism  $\eta : X \rightarrow X$  such that

$$\eta(I^{-1}([-\infty, c + \delta_1])) \subset I^{-1}([-\infty, c - \delta_1]),$$

and

$$\eta(x) = x \quad \text{for all } x \notin I^{-1}([c - \delta_2, c + \delta_2]).$$

The following Lemma on the existence of critical points (which will be used to prove Theorem 1.1) is a slight variant of a well known Saddle Point Theorem of Rabinowitz (see [14]). The proof is based on Lemma 2.2 and it is the same of Rabinowitz Theorem.

LEMMA 2.3. – Let  $X = \Omega \times H$  where  $\Omega$  is a complete Hilbert manifold and  $H$  is an affine space with  $\dim H < +\infty$ . Let  $I \in C^2(X, \mathbb{R})$ . Assume that

(i) there exist  $\bar{x} \in \Omega$ ,  $e \in H$  and an open neighborhood  $U$  of  $e$  in  $H$  such that

$$b_1 \equiv \sup_{(\bar{x}, \partial U)} I < b_2 \equiv \inf_{(\Omega, e)} I.$$

(ii)  $I$  satisfies (P.S.) $_c$  on  $X$  for all  $c \in [b_1, b_2]$ .

Then  $I$  has a critical value  $c \in [b_2, \sup_{(\bar{x}, \bar{U})} I]$  and  $c$  can be characterized

as follows:

Let  $U_0 = \{\bar{x}\} \times U$ ,  $\partial U_0 = \{\bar{x}\} \times \partial U$ ,  $\bar{U}_0 = \{\bar{x}\} \times \bar{U}$  and

$$S = \{h \in C^0(\bar{U}_0, H) : h(u) = u \text{ for all } u \in \partial U_0\}.$$

Then

$$c = \inf_{h \in S} \sup_{z \in \bar{U}_0} I(h(z)).$$

Actually we do not know if  $f$  satisfies the Palais-Smale condition. For this reason, we consider, for any  $\varepsilon \in ]0, 1]$  a penalized functional defined as follows.

Let  $\chi \in C^2(\mathbb{R}, [0, 1])$  such that  $\chi(\sigma) = 0$  for any  $\sigma \leq 0$ ,  $\chi(\sigma) = 1$  for any  $\sigma \geq 1$  and  $\chi'(\sigma) > 0$  for any  $\sigma \in ]0, 1[$ . For any  $\varepsilon > 0$  set

$$\chi_\varepsilon(\sigma) = \begin{cases} \chi(\sigma - 1/\varepsilon) & \text{if } \sigma \geq 1/\varepsilon, \\ 0 & \text{if } -1/\varepsilon \leq \sigma \leq 1/\varepsilon, \\ \chi(-\sigma - 1/\varepsilon) & \text{if } \sigma \leq -1/\varepsilon, \end{cases}$$

$$b_\varepsilon = \sup \{\beta(x, t) : x \in \mathcal{M}_0, t \in [-(1/\varepsilon) - 1, (1/\varepsilon)] \cup [(1/\varepsilon), (1/\varepsilon) + 1]\}$$

(which is finite by (1.5) and (1.6)), and

$$\beta_\varepsilon(x, t) = (1 - \chi_\varepsilon(t))\beta(t, x) + \chi_\varepsilon(t)b_\varepsilon. \tag{2.9}$$

Moreover consider

$$\psi(\sigma) = e^\sigma - (1 + \sigma + \sigma^2/2)$$

and for any  $\varepsilon > 0$  set

$$\psi_\varepsilon(\sigma) = \begin{cases} \psi(s - 1/\varepsilon) & \text{if } \sigma \geq 1/\varepsilon \\ 0 & \text{if } \sigma \leq 1/\varepsilon. \end{cases} \tag{2.10}$$

Finally, for any  $\varepsilon > 0$  define the penalized functional

$$f_\varepsilon(z) = \int_0^1 [\langle \alpha(z) \dot{x}, \dot{x} \rangle + 2 \langle \delta(x), \dot{x} \rangle t - \beta_\varepsilon(z) t^2] ds - \psi_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right). \tag{2.11}$$

In section 3, the Palais-Smale condition for the functional (2.11) and a priori estimates (independently of  $\varepsilon$ ) on its critical points will be proved. The following properties of the function  $\psi(\sigma)$  will be used:

$$\psi(\sigma) \text{ is positive;} \quad (2.12)$$

$$\psi(\sigma) \text{ is convex;} \quad (2.13)$$

$$\psi(\sigma) \leq \psi'(\sigma), \quad \text{for every } \sigma \in \mathbb{R}; \quad (2.14)$$

for every  $p > 0$ , there exists two positive constants  $a_p$  and  $b_p$ , such that for every  $\sigma \geq 0$ :

$$\psi(\sigma) \geq a_p s^p - b_p; \quad (2.15)$$

for every  $s \in \mathbb{R}_+$ :

$$s \psi'(s) \geq \psi(s). \quad (2.16)$$

### 3. PALAIS-SMALE CONDITION AND A PRIORI ESTIMATES FOR THE FUNCTIONAL (2.11)

We begin this section studying the Palais-Smale sequences for the functional  $f_\varepsilon$ .

**PROPOSITION 3.1.** – *Let  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$  and  $g$  as in (1.1). Assume that (1.3)-(1.7), are satisfied. Let  $\{z_n\}_{n \in \mathbb{N}} \subset \mathcal{Z}$  such that*

$$f_\varepsilon(z_n) \leq C \quad (3.1)$$

and

$$-\varepsilon_n \|\zeta\| \leq f'_\varepsilon(z_n)(\zeta) \quad \text{for any } \zeta = (\xi, \tau) \in T_{z_n} \mathcal{Z}, \quad (3.2)$$

where  $\|\zeta\|^2 = \langle \xi, \xi \rangle_1 + \langle t, t \rangle_1$ ,  $\varepsilon_n \rightarrow 0^+$  and  $C$  is a constant independent of  $n$ .

Then,  $\int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle ds$  and  $\int_0^1 t_n^2 ds$  are bounded independently of  $n$ .

*Proof.* – By (2.11), for any  $z = (x, t) \in \mathcal{Z}$  and  $\tau \in W_0^{1,2}$ ,

$$\begin{aligned}
 f'_\varepsilon(z)(0, \tau) &= \int_0^1 [(\alpha_t(z) \dot{x}, \dot{x}) \tau + 2 \langle \delta_t(x, t), \dot{x} \rangle t \tau + 2 \langle \delta(x, t), \dot{x} \rangle \dot{t} \\
 &\quad - (\beta_\varepsilon)_t(z) \dot{t}^2 \tau - 2 \beta_\varepsilon(z) \dot{t} \dot{\tau}] ds \\
 &\quad - 2 \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \int_0^1 \dot{t} \dot{\tau} ds.
 \end{aligned} \tag{3.3}$$

Let

$$t_*(s) = (1 - s) t_0 + s t_1, \tag{3.4}$$

and

$$\tau_n = t_n - t_*. \tag{3.5}$$

By (3.2)-(3.5), since  $\tau_n \in W_0^{1,2}$ ,

$$\begin{aligned}
 &- \varepsilon_n \|\dot{\tau}_n\|_{L^2} \\
 &\leq \int_0^1 [(\alpha_t(z_n) \dot{x}_n, \dot{x}_n) (t_n - t_*) \\
 &\quad 2 \langle \delta_t(x_n, t_n), \dot{x}_n \rangle \dot{t}_n (t_n - t_*) + 2 \langle \delta(x_n, t_n), \dot{x}_n \rangle (\dot{t}_n - \dot{t}_*) \\
 &\quad - (\beta_\varepsilon)_t(z_n) \dot{t}_n^2 (t_n - t_*)] ds \\
 &\quad - \int_0^1 2 \beta_\varepsilon(z_n) \dot{t}_n^2 ds + 2(t_1 - t_0) \int_0^1 \beta_\varepsilon(z_n) \dot{t}_n ds \\
 &\quad - \left( \int_0^1 \dot{t}_n^2 ds \right) 2 \psi'_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right) \\
 &\quad + (t_1 - t_0)^2 2 \psi'_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right).
 \end{aligned}$$

Now by (1.5) and (1.7) there exists  $A > 0$  (independent of  $n$ ) such that

$$\langle \alpha_t(z_n) \dot{x}_n, \dot{x}_n \rangle (t_n - t_*) \leq A \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle |t_n - t_*|.$$

Moreover, by (1.5)-(1.7), there exists  $D > 0$  such that

$$\|\delta_t(x, t)\| \leq D \quad \text{and} \quad \|\delta(x, t)\| \leq \delta_0 + D |t|, \quad \text{for any } (x, t) \in \mathcal{M},$$

where  $\delta_0$  is defined at (1.6), while by (1.5) and (1.6), there exist  $m_\varepsilon$  and  $M_\varepsilon$  such that

$$|(\beta_\varepsilon)_t(z)| \leq M_\varepsilon \quad \text{and} \quad \beta_\varepsilon(z) \leq m_\varepsilon \quad \text{for any } z \in \mathcal{M}.$$

Therefore

$$\begin{aligned}
& -\varepsilon_n \|\dot{t}_n - \dot{t}_*\|_{L^2} \\
& \leq \int_0^1 [A \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle |t_n - t_*| + 2D \|\dot{x}_n\| |\dot{t}_n| |t_n - t_*| \\
& \quad + 2(\delta_0 + D|t_n|) \|\dot{x}_n\| |\dot{t}_n - \dot{t}_*| + M_\varepsilon |\dot{t}_n^2| |t_n - t_*|] ds \\
& \quad - \int_0^1 2\beta_\varepsilon(z_n) \dot{t}_n^2 ds + 2m_\varepsilon |t_1 - t_0| \int_0^1 |\dot{t}_n| ds \\
& \quad - \left( \int_0^1 \dot{t}_n^2 ds \right) 2\psi'_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right) + (t_1 - t_0)^2 2\psi'_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right) \\
& \leq \|t_n - t_*\|_{L^\infty} \int_0^1 [A \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle \\
& \quad + 2D \|\dot{x}_n\| |\dot{t}_n| + M_\varepsilon |\dot{t}_n^2|] ds \\
& \quad + \int_0^1 2(\delta_0 + D|t_n|) \|\dot{x}_n\| [|\dot{t}_n| + |t_1 - t_0|] ds \\
& \quad - \int_0^1 2\beta_\varepsilon(z_n) \dot{t}_n^2 ds + 2m_\varepsilon |t_1 - t_0| \int_0^1 |\dot{t}_n| ds \\
& \quad - \left( \int_0^1 \dot{t}_n^2 ds \right) 2\psi'_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right) \\
& \quad + (t_1 - t_0)^2 2\psi'_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right).
\end{aligned}$$

Now, by (1.3) and the Young inequality,

$$\begin{aligned}
& \|t_n - t_*\|_{L^\infty} \int_0^1 2D \|\dot{x}_n\| |\dot{t}_n| ds \\
& \leq \|t_n - t_*\|_{L^\infty} \left[ D \int_0^1 \|\dot{x}_n\|^2 ds + D \int_0^1 |\dot{t}_n|^2 ds \right] \\
& \leq \|t_n - t_*\|_{L^\infty} \left[ D \int_0^1 \frac{1}{\lambda} \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle ds \right. \\
& \quad \left. + D \int_0^1 |\dot{t}_n|^2 ds \right], \tag{3.7}
\end{aligned}$$

and

$$\begin{aligned}
 & \int_0^1 2(\delta_0 + D|t_n|) \|\dot{x}_n\| [|\dot{t}_n| + |t_1 - t_0|] \\
 & \leq \int_0^1 \|\dot{x}_n\|^2 ds \\
 & \quad + \int_0^1 (\delta_0 + D|t_n|^2) [|\dot{t}_n| + |t_1 - t_0|]^2 ds \\
 & \leq \int_0^1 \frac{1}{\lambda} \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle ds \\
 & \quad + \int_0^1 (\delta_0 + D|t_n|)^2 [|\dot{t}_n| + |t_1 - t_0|]^2 ds;
 \end{aligned}$$

hence there exists a positive constant  $N_1$  (independent of  $n$ ) such that

$$\begin{aligned}
 & \int_0^1 2(\delta_0 + D|t_n|) \|\dot{x}_n\| [|\dot{t}_n| + |t_1 - t_0|] \\
 & \leq \int_0^1 \frac{1}{\lambda} \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle ds + N_1 \left[ 1 + \left( \int_0^1 |\dot{t}_n|^2 ds \right)^2 \right]. \quad (3.8)
 \end{aligned}$$

Combining (3.6), (3.7) and (3.8) gives

$$\begin{aligned}
 & -\varepsilon_n \|\dot{t}_n - \dot{t}_*\|_{L^2} \\
 & \leq [(A + (D/\lambda)) \|t_n - t_*\|_{L^\infty} + (1/\lambda)] \int_0^1 \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle ds \\
 & \quad + [(D + M_\varepsilon) \|t_n - t_*\|_{L^\infty}] \int_0^1 |\dot{t}_n|^2 ds + N_1 \left[ 1 + \left( \int_0^1 |\dot{t}_n|^2 ds \right)^2 \right] \\
 & \quad - \int_0^1 2\beta_\varepsilon(z_n) \dot{t}_n^2 ds + 2m_\varepsilon |t_1 - t_0| \int_0^1 |\dot{t}_n| ds \\
 & \quad - \left( \int_0^1 \dot{t}_n^2 ds \right) 2\psi'_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right) \\
 & \quad + (t_1 - t_0)^2 2\psi'_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right). \quad (3.9)
 \end{aligned}$$

Now, by (2.11),

$$\begin{aligned}
& \int_0^1 \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle ds \\
&= f_\varepsilon(z_n) - 2 \int_0^1 \langle \delta(x_n, t_n), \dot{x}_n \rangle \dot{t}_n ds \\
&\quad + \int_0^1 \beta_\varepsilon(z_n) \dot{t}_n^2 ds + \psi_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right) \\
&\leq f_\varepsilon(z_n) + \int_0^1 2(\delta_0 + D|t_n|) \|\dot{x}_n\| |\dot{t}_n| ds \\
&\quad + \int_0^1 \beta_\varepsilon(z_n) \dot{t}_n^2 ds + \psi_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right) \\
&\leq f_\varepsilon(z_n) + \eta \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle ds + \eta^{-1} \int_0^1 (\delta_0 + D|t_n|)^2 \dot{t}_n^2 ds \\
&\quad + \int_0^1 \beta_\varepsilon(z_n) \dot{t}_n^2 ds + \psi_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right) \\
&\leq f_\varepsilon(z_n) + \eta \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle ds + \eta^{-1} N_2 \left[ 1 + \left( \int_0^1 |\dot{t}_n|^2 ds \right)^2 \right] \\
&\quad + \int_0^1 \beta_\varepsilon(z_n) \dot{t}_n^2 ds + \psi_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right), \tag{3.10}
\end{aligned}$$

for any positive real number  $\eta$  and for a suitable positive constant  $N_2$  independent of  $n$  and  $\eta$ .

Therefore, by (3.10), (1.3) and (3.1), choosing  $\eta$  such that

$$1 - \eta/\lambda = 1/2$$

gives,

$$\begin{aligned}
& \int_0^1 \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle ds \leq 2C + 2N_2 \eta^{-1} \left[ 1 + \left( \int_0^1 |\dot{t}_n|^2 ds \right)^2 \right] \\
&\quad + 2 \int_0^1 \beta_\varepsilon(z_n) \dot{t}_n^2 ds + 2\psi_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right). \tag{3.11}
\end{aligned}$$

Combining (3.9) and (3.11) gives

$$\begin{aligned}
 & -\varepsilon_n \|\dot{t}_n - \dot{t}_*\|_{L^2} \\
 & \leq [(A + (D/\lambda)) \|t_n - t_*\|_{L^\infty} + (1/\lambda)] \\
 & \quad \times \left[ 2C + 2N_2 \eta^{-1} \left[ 1 + \left( \int_0^1 |\dot{t}_n|^2 ds \right)^2 \right] \right. \\
 & \quad \left. + 2 \int_0^1 \beta_\varepsilon(z_n) \dot{t}_n^2 ds + 2\psi_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right) \right] \\
 & + [(D + M_\varepsilon) \|t_n - t_*\|_{L^\infty}] \int_0^1 |\dot{t}_n|^2 ds \\
 & + N_1 \left[ 1 + \left( \int_0^1 |\dot{t}_n|^2 ds \right)^2 \right] \\
 & - \int_0^1 2\beta_\varepsilon(z_n) \dot{t}_n^2 ds + 2m_\varepsilon |t_1 - t_0| \int_0^1 |\dot{t}_n| ds \\
 & - \left( \int_0^1 \dot{t}_n^2 ds \right) 2\psi'_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right) \\
 & + (t_1 - t_0)^2 2\psi'_\varepsilon \left( \int_0^1 \dot{t}_n^2 ds \right). \tag{3.12}
 \end{aligned}$$

Since

$$\|t_n - t_*\|_{L^\infty} \leq \left( \int_0^1 |\dot{t}_n - \dot{t}_*|^2 ds \right)^{1/2},$$

and  $\beta_\varepsilon$  is bounded, (3.12), (2.14) and (2.15) imply that  $\int_0^1 \dot{t}_n^2 ds$  is bounded independently of  $n$ . Moreover by (1.5), (3.1), (2.11), and (1.3) also  $\int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle ds$  is bounded independently of  $n$ .  $\square$

Proposition 3.1, and standard arguments (cf. e.g. [1], [9]) allows to get the following

**PROPOSITION 3.2.** – *Under the assumptions of Proposition 3.1, if (1.2) holds, then  $f_\varepsilon$  satisfies (P.S.) $_c$  for any  $c \in \mathbb{R}$  and for any  $\varepsilon > 0$ .*

Now we shall prove the *a priori* estimates on the critical points of  $f_\varepsilon$  starting with an *a priori* estimate in  $L^\infty$  for the time variable.

**PROPOSITION 3.3.** – *Let  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$  and  $g$  as in (1.1). Assume that (1.2)-(1.6) and (1.8) are satisfied. Let  $z = (x, t) \equiv z_\varepsilon = (x_\varepsilon, t_\varepsilon)$  be a*

critical point of  $f_\varepsilon$  such that

$$f_\varepsilon(z) \leq C \quad (3.13)$$

where  $C$  is a constant independent of  $\varepsilon$ .

Then  $t(s) \equiv t_\varepsilon(s)$  is uniformly bounded on  $\varepsilon > 0$  and  $s \in [0, 1]$ .

In order to prove Proposition 3.3 the following Lemma is needed.

LEMMA 3.4. – Let  $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$  and  $L > 0$  such that

$$0 \leq \varphi'(\sigma) \leq L\varphi(\sigma) \quad \text{for any } \sigma \in \mathbb{R}^+ \quad (3.14)$$

and

$$\lim_{\sigma \rightarrow +\infty} \varphi(\sigma) = +\infty. \quad (3.15)$$

Let  $\bar{t} \in \mathbb{R}^+$  and consider

$$\Gamma(\vartheta, \varphi, \bar{t}) = \left\{ t \in W^{1,2}([0, 1], \mathbb{R}) : \begin{array}{l} \text{for any } [a, b] \subset [0, 1] \text{ satisfying} \\ t(a) = \bar{t}, t(b) = \bar{t}, \text{ and } t(s) > \bar{t} \text{ for any } s \in ]a, b[, \\ \text{the following inequality holds :} \\ \int_a^b \dot{t}^2(s) \varphi(t(s)) ds \leq \vartheta \int_a^b \varphi(t(s)) ds \end{array} \right\}, \quad (3.16)$$

where  $\vartheta$  is a positive number such that

$$L\sqrt{\vartheta} < 2. \quad (3.17)$$

Then there exists a constant  $S = S(\varphi, L, \vartheta, \bar{t})$  such that

$$\sup \{t(s) : s \in [0, 1], t \in \Gamma(\vartheta, \varphi, \bar{t})\} \leq S. \quad (3.18)$$

*Proof.* – Set, for any  $t \in \Gamma(\vartheta, \varphi, \bar{t})$ ,

$$S(t) = \sup \{t(s) : s \in [0, 1]\}.$$

Assume, by contradiction, that there exists  $\{t_n\}_{n \in \mathbb{N}} \subset \Gamma(\vartheta, \varphi, \bar{t})$  such that

$$S(t_n) \xrightarrow[n]{} +\infty, \quad (3.19)$$

and let  $[a_n, b_n]$  be an interval satisfying the properties in (3.16). By (3.14), Hölder inequality and (3.16),

$$\begin{aligned}
 \sqrt{\varphi(S(t_n))} - \sqrt{\varphi(\bar{t})} &= \sqrt{\varphi(S(t_n))} - \sqrt{\varphi((t_n(a_n)))} \\
 &\leq \frac{1}{2} \int_{a_n}^{b_n} \varphi^{-1/2}(t_n(s)) (\varphi'_n(t_n(s))) |\dot{t}_n(s)| ds \\
 &\leq \frac{1}{2} L \int_{a_n}^{b_n} \varphi^{1/2}(t_n(s)) |\dot{t}_n(s)| ds \\
 &\leq \frac{1}{2} L (b_n - a_n)^{1/2} \left( \int_{a_n}^{b_n} \varphi(t_n(s)) |\dot{t}_n(s)|^2 ds \right)^{1/2} \\
 &\leq \frac{1}{2} L \left( \int_{a_n}^{b_n} \varphi(t_n(s)) |\dot{t}_n(s)|^2 ds \right)^{1/2} \\
 &\leq \frac{1}{2} L \sqrt{\vartheta} \left( \int_{a_n}^{b_n} \varphi(t_n(s)) ds \right)^{1/2} \leq \frac{1}{2} L \sqrt{\vartheta} \cdot \sqrt{\varphi(S(t_n))}.
 \end{aligned}$$

Therefore, by (3.19) and (3.17) we get a contradiction, proving (3.18).  $\square$

*Proof of Proposition 3.3.* – If  $z = (x, t) \equiv z_\varepsilon = (x_\varepsilon, t_\varepsilon)$  is a critical point of  $f_\varepsilon$ , it satisfies the differential equation

$$D_s \dot{z} - \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) (0, \ddot{t}) = 0, \tag{3.20}$$

where  $0$  is the null vector in  $T_x \mathcal{M}_0$  and  $\ddot{t}$  is the second derivative of  $t$ . Multiplying by  $\dot{z}$  and integrating give the existence of  $E_z \in \mathbb{R}$  such that

$$\langle \dot{z}, \dot{z} \rangle - \dot{t}^2 \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) = E_z, \tag{3.21}$$

where

$$\langle \dot{z}, \dot{z} \rangle = \langle \alpha(x, t) \dot{x}, \dot{x} \rangle + 2 \langle \delta(x, t), \dot{x} \rangle \dot{t} - \beta_\varepsilon(x, t) \dot{t}^2.$$

Moreover, integrating (3.21) in the interval  $[0, 1]$ , give

$$E_z = f_\varepsilon(z) - \left( \int_0^1 \dot{t}^2 ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) + \psi_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right). \tag{3.22}$$

By (1.8), for any  $\mu > 0$ , there exists  $t_+ = t_+(\mu) > \max(t_0, t_1)$  such that

$$\begin{aligned}
 \langle \alpha_t(x, t) \xi, \xi \rangle_x &\leq \mu \langle \alpha(x, t) \xi, \xi \rangle_x \quad \text{for any } x \in \mathcal{M}_0, \xi \in T_x \mathcal{M}_0, t \geq t_+, \\
 \|\delta_t(x, t)\| &\leq \mu \quad \text{for any } x \in \mathcal{M}_0, t \geq t_+, \\
 -\beta_t(x, t) &\leq \mu \beta(x, t) \quad \text{for any } x \in \mathcal{M}_0, t \geq t_+.
 \end{aligned} \tag{3.23}$$

If, by contradiction,  $t_\varepsilon$  is not uniformly bounded from above, there exist a critical point  $(x, t) \equiv (x_\varepsilon, t_\varepsilon)$  of  $f_\varepsilon$  and an interval  $I \subset [0, 1]$ , such that

$$t(s) \geq t_+ \quad \text{for any } s \in I, \quad \text{and} \quad t|_{\partial I} = t_+. \quad (3.24)$$

Now, by (2.9),

$$\begin{aligned} (\beta_\varepsilon)_t(x, t) &= \chi'_\varepsilon(t)(b_\varepsilon - \beta(x, t)) + (1 - \chi_\varepsilon(t))\beta_t(x, t) \\ &\geq (1 - \chi_\varepsilon(t))\beta_t(x, t). \end{aligned}$$

Then putting, in (3.3),

$$\tau(s) = \begin{cases} \sinh(\omega(t(s) - t_+)) & \text{if } s \in I \\ 0 & \text{if } s \notin I, \end{cases} \quad (3.25)$$

(where  $\omega \in \mathbb{R}^+ \setminus \{0\}$ ), (3.23) give

$$\begin{aligned} 0 &\leq \int_I [\mu \langle \alpha(z) \dot{x}, \dot{x} \rangle |\tau| + 2\mu \|\dot{x}\| |\dot{t}| |\tau| + \mu \beta_\varepsilon \dot{t}^2 |\tau|] ds \\ &\quad + \omega \int_I 2 \langle \delta(z), \dot{x} \rangle \dot{t} \cosh(\omega(t - t_+)) ds \\ &\quad - 2\omega \int_I \beta_\varepsilon(z) \dot{t}^2 \cosh(\omega(t - t_+)) ds \\ &\quad - 2\omega \left( \int_I \dot{t}^2 \cosh(\omega(t - t_+)) ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right). \end{aligned} \quad (3.26)$$

Now, by (3.21) and (3.22),

$$\begin{aligned} 2 \langle \delta(z), \dot{x} \rangle \dot{t} &= f_\varepsilon(z) - \left( \int_0^1 \dot{t}^2 ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \\ &\quad + \psi_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) + \dot{t}^2 \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \\ &\quad - \langle \alpha(z) \dot{x}, \dot{x} \rangle + \beta_\varepsilon(z) \dot{t}^2. \end{aligned} \quad (3.27)$$

Combining (3.26) and (3.27) gives

$$\begin{aligned}
 0 &\leq \int_I [\mu \langle \alpha(z) \dot{x}, \dot{x} \rangle |\tau| + 2\mu \|\dot{x}\| |\dot{t}| |\tau| + \mu \beta_\varepsilon \dot{t}^2 |\tau|] ds \\
 &\quad + \omega f_\varepsilon(z) \int_I \cosh(\omega(t-t_+)) ds \\
 &\quad + \omega \left[ - \left( \int_0^1 \dot{t}^2 ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \right. \\
 &\quad \quad \left. + \psi_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \right] \int_I \cosh(\omega(t-t_+)) ds \\
 &\quad + \omega \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \left( \int_I \dot{t}^2 \cosh(\omega(t-t_+)) ds \right) \\
 &\quad - \omega \int_I \langle \alpha(z) \dot{x}, \dot{x} \rangle \cosh(\omega(t-t_+)) \\
 &\quad + \omega \int_I \beta_\varepsilon(z) \dot{t}^2 \cosh(\omega(t-t_+)) \\
 &\quad - 2\omega \int_0^1 \beta_\varepsilon(z) \dot{t}^2 \cosh(\omega(t-t_+)) ds \\
 &\quad - 2\omega \left( \int_I \dot{t}^2 \cosh(\omega(t-t_+)) ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \\
 &= \int_I [\mu \langle \alpha(z) \dot{x}, \dot{x} \rangle |\tau| + 2\mu \|\dot{x}\| |\dot{t}| |\tau| + \mu \beta_\varepsilon \dot{t}^2 |\tau|] ds \\
 &\quad + \omega f_\varepsilon(z) \int_I \cosh(\omega(t-t_+)) ds \\
 &\quad + \omega \left[ - \left( \int_0^1 \dot{t}^2 ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \right. \\
 &\quad \quad \left. + \psi_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \right] \int_I \cosh(\omega(t-t_+)) ds \\
 &\quad + \omega \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \left( \int_I \dot{t}^2 \cosh(\omega(t-t_+)) ds \right), \\
 &\quad - \omega \int_I \langle \alpha(z) \dot{x}, \dot{x} \rangle \cosh(\omega(t-t_+)) \\
 &\quad - \omega \int_I \beta_\varepsilon(z) \dot{t}^2 \cosh(\omega(t-t_+)) ds \\
 &\quad - 2\omega \left( \int_I \dot{t}^2 \cosh(\omega(t-t_+)) ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right).
 \end{aligned}$$

Therefore, by the choice of  $\psi_\varepsilon$  [cf. (2.14)-(2.16)],

$$\begin{aligned}
0 &\leq \int_I [\mu \langle \alpha(z) \dot{x}, \dot{x} \rangle |\tau| + 2\mu \|\dot{x}\| |\dot{t}| |\tau| + \mu \beta_\varepsilon \dot{t}^2 |\tau|] ds \\
&\quad + \omega f_\varepsilon(z) \int_I \cosh(\omega(t - t_+)) ds \\
&\quad - \omega \int_I \langle \alpha(z) \dot{x}, \dot{x} \rangle \cosh(\omega(t - t_+)) \\
&\quad - \omega \int_I \beta_\varepsilon(z) \dot{t}^2 \cosh(\omega(t - t_+)) ds. \tag{3.28}
\end{aligned}$$

Moreover, by (1.3)

$$\begin{aligned}
&\int_I 2\mu \|\dot{x}_n\| |\dot{t}_n| |\tau| ds \\
&\leq \mu \int_I \frac{1}{\lambda} \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle |\tau| ds + \mu \int_I |\dot{t}_n|^2 |\tau| ds. \tag{3.29}
\end{aligned}$$

Then, since

$$|\tau(s)| \leq \cosh(\omega(t(s) - t_+(s))) \quad \text{for any } s \in [0, 1],$$

choosing  $\omega$  such that

$$\omega \geq (\mu + \mu/\lambda),$$

(3.28), and (3.29) give

$$\begin{aligned}
0 &\leq \int_I \mu \dot{t}^2 \cosh(\omega(t - t_+)) ds \\
&\quad + \omega f_\varepsilon(z) \int_I \cosh(\omega(t - t_+)) ds \\
&\quad - \int_I [\omega - \mu] \beta_\varepsilon(z) \dot{t}^2 \cosh(\omega(t - t_+)) ds,
\end{aligned}$$

hence by (3.13), (1.4), and the choice of  $\beta_\varepsilon$  [cf. (2.9)], if

$$\omega/2 \geq \mu \quad \text{and} \quad b \omega/4 \geq \mu,$$

we have

$$\int_I \dot{t}^2 \cosh(\omega(t - t_+)) \leq (4C/b) \int_I \cosh(\omega(t - t_+)) ds. \tag{3.30}$$

Finally by (1.8),  $\mu$  and  $\omega$  can be chosen so small that

$$\omega(4C/b)^{1/2} < 2,$$

hence, by Lemma 3.4,  $t \equiv t_\epsilon$  is uniformly bounded from above (independently of  $\epsilon$ ). Analogously the uniform boundedness from below can be obtained using the analogous of Lemma 3.4 in  $\mathbb{R}^-$ , concluding the proof of Proposition 3.3.  $\square$

*Remark 3.5.* – By the proof of Proposition 3.3 it turns out that the *a priori* estimate in  $L^\infty$  for the timelike geodesics requires only assumptions (1.7) instead of (1.8), because, in this case,  $C < 0$ .

Moreover, about the geodesic connectedness, it is clear that (1.8) can be weakened, asking that the constant in (1.8) are small with respect to the smallest critical level  $C$  of the energy functional.

**PROPOSITION 3.6.** – *Let  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$  and  $g$  as in (1.1). Assume that (1.2)-(1.6) are satisfied. Let  $z = (x, t) \equiv z_\epsilon = (x_\epsilon, t_\epsilon)$  be a critical point of  $f_\epsilon$  such that*

$$f_\epsilon(z_\epsilon) \leq C \tag{3.31}$$

where  $C$  is a constant independent of  $\epsilon$ , and

$$t_\epsilon \text{ is uniformly bounded (independently of } \epsilon). \tag{3.32}$$

Then  $\int_0^1 \langle \dot{x}_\epsilon, \dot{x}_\epsilon \rangle ds$ , and  $\int_0^1 \dot{t}_\epsilon^2 ds$  are bounded independently of  $\epsilon$ .

*Proof.* – Consider

$$\tau = \sinh(\omega(t - t_*)), \tag{3.33}$$

where  $t_*$  is defined by (3.4) and  $\omega \in \mathbb{R}^+ \setminus \{0\}$  will be chosen later. By (3.32), (1.5), (1.6), and the definition of  $\beta_\epsilon$  [cf. (2.9)], choosing in formula (3.3)  $\tau$  as defined in (3.33), give the existence of  $A, D_1, B_1 \in \mathbb{R}^+$  such that

$$\begin{aligned} 0 &\leq \int_0^1 [A \langle \alpha(z) \dot{x}, \dot{x} \rangle |\tau| + 2D_1 \|\dot{x}\| |\dot{t}| |\tau| + B_1 \dot{t}^2 \tau] ds \\ &+ \omega \int_0^1 2 \langle \delta(z), \dot{x} \rangle \dot{t} \cosh(\omega(t - t_*)) ds \\ &- 2\omega(t_1 - t_0) \int_0^1 \langle \delta(z), \dot{x} \rangle \cosh(\omega(t - t_*)) ds \end{aligned}$$

$$\begin{aligned}
& - 2\omega \int_0^1 \beta(z) \dot{t}^2 \cosh(\omega(t-t_*)) ds \\
& + 2\omega(t_1-t_0) \int_0^1 \beta(z) \dot{t} \cosh(\omega(t-t_*)) ds \\
& - 2\omega \left( \int_0^1 \dot{t}^2 \cosh(\omega(t-t_*)) ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \\
& + 2\omega(t_1-t_0) \left( \int_0^1 \dot{t} \cosh(\omega(t-t_*)) ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right). \quad (3.34)
\end{aligned}$$

Combining (3.27) and (3.34) gives

$$\begin{aligned}
0 & \leq \int_0^1 [A \langle \alpha(z) \dot{x}, \dot{x} \rangle |\tau| + 2D_1 \|\dot{x}\| |\dot{t}| |\tau| + B_1 \dot{t}^2 \tau] ds \\
& + \omega f_\varepsilon(z) \int_0^1 \cosh(\omega(t-t_*)) ds \\
& + \omega \left[ - \left( \int_0^1 \dot{t}^2 ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \right. \\
& \quad \left. + \psi_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \right] \int_0^1 \cosh(\omega(t-t_*)) ds \\
& + \omega \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \left( \int_0^1 \dot{t}^2 \cosh(\omega(t-t_*)) ds \right) \\
& - \omega \int_0^1 \langle \alpha(z) \dot{x}, \dot{x} \rangle \cosh(\omega(t-t_*)) \\
& - 2\omega(t_1-t_0) \int_0^1 \langle \delta(z), \dot{x} \rangle \cosh(\omega(t-t_*)) ds \\
& - \omega \int_0^1 \beta(z) \dot{t}^2 \cosh(\omega(t-t_*)) ds \\
& + 2\omega(t_1-t_0) \int_0^1 \beta(z) \dot{t} \cosh(\omega(t-t_*)) ds \\
& - 2\omega \left( \int_0^1 \dot{t}^2 \cosh(\omega(t-t_*)) ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right) \\
& + 2\omega(t_1-t_0) \left( \int_0^1 \dot{t} \cosh(\omega(t-t_*)) ds \right) \psi'_\varepsilon \left( \int_0^1 \dot{t}^2 ds \right). \quad (3.35)
\end{aligned}$$

Since  $\dot{t} - \dot{t}_* = \dot{t} - (t_1 - t_0)$ ,

$$\begin{aligned} \int_0^1 \dot{t} \cosh(\omega(t - t_*)) ds &= \int_0^1 (\dot{t} - \dot{t}_*) \cosh(\omega(t - t_*)) ds \\ &\quad + (t_1 - t_0) \int_0^1 \cosh(\omega(t - t_*)) ds \\ &= (t_1 - t_0) \int_0^1 \cosh(\omega(t - t_*)) ds. \end{aligned} \tag{3.36}$$

Obviously we can assume that  $\int_0^1 \dot{t}^2 ds$  is large as we want. Then, (3.14), (3.35) and (3.36) give

$$\begin{aligned} 0 &\leq \int_0^1 [A \langle \alpha(z) \dot{x}, \dot{x} \rangle |\tau| + 2 D_1 \|\dot{x}\| |\dot{t}| |\tau| + B_1 \dot{t}^2 \tau] ds \\ &\quad + \omega f_\varepsilon(z) \int_0^1 \cosh(\omega(t - t_*)) ds \\ &\quad - \omega \int_0^1 \langle \alpha(z) \dot{x}, \dot{x} \rangle \cosh(\omega(t - t_*)) ds \\ &\quad - 2\omega(t_1 - t_0) \int_0^1 \langle \delta(z), \dot{x} \rangle \cosh(\omega(t - t_*)) ds \\ &\quad - \omega \int_0^1 \beta(z) \dot{t}^2 \cosh(\omega(t - t_*)) ds \\ &\quad + 2\omega(t_1 - t_0) \int_0^1 \beta(z) \dot{t}^2 \cosh(\omega(t - t_*)) ds. \end{aligned} \tag{3.37}$$

Moreover by (3.32), (1.5), (1.6) and (1.3), there exists a positive constant  $D_2$  such that

$$\begin{aligned} &\|\delta(x, t)\| \leq D_2, \\ &\left| 2\omega(t_1 - t_0) \int_0^1 \langle \delta(z), \dot{x} \rangle \cosh(\omega(t - t_*)) ds \right| \\ &\leq 2\omega |t_1 - t_0| D_2 \int_0^1 \langle \dot{x}, \dot{x} \rangle^{1/2} \cosh(\omega(t - t_*)) ds \\ &\leq \omega |t_1 - t_0| D_2 (\eta_1/\lambda) \int_0^1 \langle \alpha(z) \dot{x}, \dot{x} \rangle \cosh(\omega(t - t_*)) ds \\ &\quad + \omega |t_1 - t_0| D_2 \eta_1^{-1} \int_0^1 \cosh(\omega(t - t_*)) ds, \end{aligned} \tag{3.38}$$

for any  $\eta_1 > 0$ , while by (1.3)

$$\begin{aligned} & \int_0^1 2D_1 \|\dot{x}_n\| \|\dot{t}_n\| |\tau| ds \\ & \leq D_1 \int_0^1 \frac{1}{\lambda} \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle |\tau| ds + D_1 \int_0^1 |\dot{t}_n|^2 |\tau| ds. \end{aligned} \quad (3.39)$$

Then, since

$$|\tau(s)| \leq \cosh(\omega(t(s) - t_*(s))) \quad \text{for any } s \in [0, 1],$$

choosing  $\eta_1$  such that

$$|t_1 - t_0| D_2 (\eta_1/\lambda) = 1/4$$

and  $\omega$  such that

$$\omega \geq 4(D_1/\lambda + A),$$

(3.37), (3.38) and (3.39) give

$$\begin{aligned} 0 & \leq \int_0^1 (D_1 + B_1) \dot{t}^2 \cosh(\omega(t - t_*)) ds \\ & \quad + \omega f_\varepsilon(z) \int_0^1 \cosh(\omega(t - t_*)) ds \\ & \quad + \omega |t_1 - t_0| D_2 \eta_1^{-1} \int_0^1 \cosh(\omega(t - t_*)) ds \\ & \quad - \omega \int_0^1 \beta(z) \dot{t}^2 \cosh(\omega(t - t_*)) ds \\ & \quad + 2\omega(t_1 - t_0) \int_0^1 \beta(z) \dot{t}^2 \cosh(\omega(t - t_*)) ds. \end{aligned} \quad (3.40)$$

Now, by (3.32), (1.5) and (1.6) there exists a positive constant  $B_2$  such that

$$\beta(x, t) \leq B_2 \quad \text{for any } (x, t),$$

therefore,

$$\begin{aligned} & \left| 2\omega(t_1 - t_0) \int_0^1 \beta(z) \dot{t} \cosh(\omega(t - t_*)) ds \right| \\ & \leq \omega |t_1 - t_0| B_2 \int_0^1 (\eta_2 \dot{t}^2 + \eta_2^{-1}) \cosh(\omega(t - t_*)) ds, \end{aligned} \quad (3.41)$$

for any  $\eta_2 > 0$ , and, choosing  $\eta_2$  such that

$$|t_1 - t_0| B_2 \eta_2 = b/4,$$

[cf. (1.4)], and  $\omega$  such that

$$\omega \geq 4(D_1 + B_1)/b$$

combining (1.4), (3.40) and (3.41), gives

$$\begin{aligned} 0 \leq \omega f_\varepsilon(z) & \int_0^1 \cosh(\omega(t - t_*)) ds \\ & + \omega |t_1 - t_0| D_2 \eta_1^{-1} \int_0^1 \cosh(\omega(t - t_*)) ds \\ & - (\omega b)/2 \int_0^1 t^2 \cosh(\omega(t - t_*)) ds \\ & + \omega |t_1 - t_0| B_2 \eta_2^{-1} \int_0^1 \cosh(\omega(t - t_*)) ds. \end{aligned}$$

Therefore by (3.31) there exists a real constant

$$K = K(C, |t_1 - t_0|, A, D_1, D_2, B_1, B_2, b, \lambda) > 0$$

such that

$$\int_0^1 t^2 \cosh(\omega(t - t_*)) ds \leq K \int_0^1 \cosh(\omega(t - t_*)) ds. \tag{3.42}$$

Finally, by (3.32) and (3.42), since  $\cosh(\sigma) \geq 1$  for any  $\sigma$ , there exist a constant  $H$  independently of  $\varepsilon$  such that

$$\int_0^1 t^2 ds \leq \int_0^1 t^2 \cosh(\omega(t - t_*)) ds \leq H. \tag{3.43}$$

Moreover, (2.11), (3.32), (1.3), (1.5), (1.6), (3.33) and (3.43) imply also that  $\int_0^1 \langle \dot{x}, \dot{x} \rangle ds$  is bounded independently of  $\varepsilon$ .  $\square$

*Remark 3.7.* – By remark 3.5 and Proposition 3.6, the a priori estimates in  $H^{1,2}$  for timelike geodesics can be obtained assuming only (1.7) instead of (1.8).

*Remark 3.8.* – By the choice of  $\psi_\varepsilon$  [cf. (2.10)], under the assumptions of Proposition 3.6, if  $z_\varepsilon$  is a critical point of  $f_\varepsilon$  and  $\varepsilon$  is sufficiently small, then  $z_\varepsilon$  is a critical point of  $f$ .

#### 4. PROOF OF THEOREMS 1.1 AND 1.2

Since the functional  $f_\varepsilon$  is unbounded both from below and from above, to overcome these difficulties we shall use a finite dimensional approximation on the space of the time variable.

For  $k \in \mathbb{N}$  we set

$$\mathcal{Z}_k = \Omega^1 \times W_k, \quad W_k = (t_* + W_{k,0}), \quad (4.1)$$

where  $t_*(s) = t_0 + (t_1 - t_0)s$  and  $W_{k,0} = \text{span} \{\sin(\pi qs), q = 1, \dots, k\}$ .

Following the proof of Proposition 3.1 and the ideas of the proof of Lemma (3.4) of [1] give the following result which allows us to look for critical points of  $f_\varepsilon$  on a manifold which is finite dimensional in the variable  $t$ .

LEMMA 4.1. – For any  $k \in \mathbb{N}$  let  $z_k \in \mathcal{Z}_k$  be a critical point of  $f_{\varepsilon|_{\mathcal{Z}_k}}$ . Assume that there exists  $c_1, c_2 \in \mathbb{R}$ , independent of  $k$  such that

$$c_1 \leq f_\varepsilon(z_k) \leq c_2.$$

Then  $\{z_k\}_{k \in \mathbb{N}}$  contains a subsequence which converges in  $\mathcal{Z}$  to a critical point  $z$  of  $f_\varepsilon$  and

$$c_1 \leq f_\varepsilon(z) \leq c_2.$$

Remark 4.2. – By the same proof of Propositions 3.1 and 3.2 we see that for all  $c \in \mathbb{R}$  and for all  $k \in \mathbb{N}$ ,  $f_{\varepsilon|_{\mathcal{Z}_k}}$  satisfies (P.S.) $_c$ .

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* – Fix  $k \in \mathbb{N}$  and put  $f_\varepsilon^k = f_{\varepsilon|_{\mathcal{Z}_k}}$  where  $\mathcal{Z}_k$  is defined by (4.1). By assumption (1.10) there exists  $\bar{x} \in \Omega^1$ ,  $\lambda_1, \lambda_2 > 0$  (independent of  $k \in \mathbb{N}$  and  $\varepsilon \in ]0, \varepsilon_0]$ ,  $\varepsilon_0 > 0$ ) such that

$$\sup_{t \in W_k} f_\varepsilon^k(\bar{x}, t) \leq -\lambda_1 < 0, \quad (4.2)$$

while

$$\inf_{x \in \Omega^1} f_\varepsilon^k(x, t_*) \geq -\lambda_2. \quad (4.3)$$

Moreover by (2.11) and assumption (1.9), there exist an open subset  $\Lambda_k$  of  $W_k$  such that  $t_* \in \Lambda_k$  and

$$\sup_{t \in \partial\Lambda_k} f_\varepsilon(\bar{x}, t) < -\lambda_2 \quad (4.4)$$

where  $\lambda_2$  is defined at (4.3).

Then assumption (i) of Lemma 2.3 is satisfied with  $e = t_*$ ,  $b_1 < -\lambda_2$ ,  $b_2 \geq -\lambda_2$  and  $U = \Lambda_k$ .

Moreover, by Remark 4.2, also assumption (ii) of Lemma 2.3 is satisfied. Then, by Lemma 2.3, for all  $k \in \mathbb{N}$ ,  $f_\varepsilon^k$  has a critical point  $z_k = (x_k, t_k)$  in  $\mathcal{Z}_k$  such that

$$-\lambda_2 \leq f_\varepsilon(x_k, t_k) \leq -\lambda_1 < 0$$

with  $\lambda_1$  and  $\lambda_2$  independent of  $k$  and  $\varepsilon$ .

Therefore, by Lemma 4.1,  $f_\varepsilon$  has as critical point  $z_\varepsilon$  in  $\mathcal{Z}$  such that

$$-\lambda_2 \leq f_\varepsilon(z_\varepsilon) \leq -\lambda_1 < 0.$$

Finally, Remarks 3.7 and 3.8 gives that, for  $\varepsilon$  small enough,  $z_\varepsilon$  is a critical point of  $f$ , i.e. a geodesic joining the given events  $z_0$  and  $z_1$ , such that

$$f(z_\varepsilon) \leq -\lambda_1 < 0.$$

This concludes the proof of Theorem 1.1.  $\square$

*Proof of Theorem 1.2.* – The proof of Theorem 1.2 is the same of Theorem 1.1 taking account of Propositions 3.3, 3.6 and Remark 3.8.  $\square$

## 5. PROOF OF THEOREMS 1.3 AND 1.4

In order to get the multiplicity results we use the concept of Relative Category (cf. [4], [6], [7], [17]), which is an extension of the classical Lusternik and Schnirelmann category (cf. e.g. [16]). We recall the definition for the convenience of the reader.

**DÉFINITION 5.1.** – *Let  $X$  be a topological space,  $A, Y$  subsets of  $X$ ,  $A \neq \emptyset$ . The relative category of  $A$  in  $X$ , with respect to  $Y$  [denoted by  $\text{cat}_{X, Y}(A)$ ] is the smallest integer  $p$  (possibly  $+\infty$ ) such that  $A \subset \bigcup_{i=0}^p A_i$  where the  $A_i$ 's have the following properties:*

$$\text{for any } i = 1, \dots, p, \quad A_i \text{ is closed and contractible in } X; \quad (5.1)$$

$$A_0 \supset Y \cap A \text{ and } A_0 \text{ is closed}; \quad (5.2)$$

there exists  $h \in C^0([0, 1] \times A_0, X)$  such that

- (i)  $h(0, x) = x$  for any  $x \in A_0$ ,
- (ii)  $h(1, x) \in Y$  for any  $x \in A_0$ ,
- (iii)  $h(s, y) = y$  for any  $s \in [0, 1]$ ,  $y \in Y \cap A_0$ . (5.3)

*Remark 5.2.* – If  $Y$  is a strong deformation retract of  $A$ , then  $\text{cat}_{X, Y}(A) = 0$ .

*Remark 5.3.* – If  $Y = \emptyset$ ,  $\text{cat}_{X, Y}(A)$  is the Lusternik and Schnirelmann category of  $A$  in  $X$ .

*Remark 5.4.* – The use of Relative Category, together with a Galerkin approximation argument is used in [6] to get interesting results about multiplicity results of critical points for strongly indefinite functionals. Unfortunately, to prove Theorem 1.3, we can not directly apply Theorem 6.3 of [6] for four reasons. First in our case assumption 6.3*b*) does not hold. Secondly the presence of the penalization term in (2.11) requires estimates independently of  $\varepsilon$  on the critical levels. Moreover we also need a sequence of critical values of the functional (2.1) going to  $+\infty$  and we can not, in general, use the cuplength of  $\Omega^1$  to estimate the relative category. Indeed, whenever  $\mathcal{M}_0$  is not contractible it is not known, in general, if the cuplength of  $\Omega^1$  is infinite.

For any  $k \in \mathbb{N}$ , we use the relative category on the manifold

$$\mathcal{Z}_k = \Omega^1 \times W_k$$

[cf. (4.1)], to get multiple critical levels for the functional

$$f_\varepsilon^k = f_{\varepsilon|_{\mathcal{Z}_k}}. \quad (5.4)$$

Fix  $k \in \mathbb{N}$  and, for any  $R > 0$ , put

$$B_k(R) = \{t \in W_k : \|t - t_*\|_{W^{1,2}} \geq R\},$$

[where  $t_*$  is defined at (4.1)]. Moreover, for any  $\mu \in \mathbb{R}$ , put

$$(f_\varepsilon^k)^\mu = \{z \in \mathcal{Z}_k : f_\varepsilon^k(z) \leq \mu\}.$$

Assumptions (1.3), (1.5) and (1.6) imply that

$$d \equiv \inf \{f_\varepsilon^k(x, t_*) : x \in \Omega^1, \varepsilon \in ]0, 1], k \in \mathbb{N}\} > -\infty. \quad (5.5)$$

The constant  $d$  plays a crucial role in the proofs of Theorems 1.3 and 1.4, and, in particular in the definition of the following class of subsets of  $\mathcal{Z}_k$ .

For any  $m \in \mathbb{N}$ ,  $k \in \mathbb{N}$ ,  $R > 0$  and  $\varepsilon \in ]0, 1]$ , put

$$\Gamma_m^k \equiv \Gamma_m^k(R, \varepsilon) = \{B \subset \mathcal{Z}_k : B \text{ is closed,}$$

$$B \cap (\Omega^1 \times B_k(R)) \neq \emptyset,$$

$$\sup f_\varepsilon^k(B \cap (\Omega^1 \times B_k(R))) \leq d - 1,$$
(5.6)

$$\text{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(B) \geq m\}. \tag{5.7}$$

The following Lemma holds:

LEMMA 5.5. – Assume  $\mathcal{M}_0$  non contractible and 1-connected. Then for any  $m \in \mathbb{N}$ , there exists a compact subset  $K_m$  of  $\mathcal{Z}_k$  and  $R_m > 0$  such that

$$K_m \in \Gamma_m^k(R, \varepsilon) \text{ for any } R \geq R_m, \quad k \in \mathbb{N}, \quad \varepsilon \in ]0, 1]. \tag{5.8}$$

*Proof.* – Fix  $m \in \mathbb{N}$ . Let  $D^k$  and  $S^k$  be the unit disk in  $W_k$  and its boundary respectively. Since  $\mathcal{M}_0$  is not contractible and 1-connected, by a recent of Fadell and Husseini (cf. [5]), there exists  $C_m$ , compact subset of  $\Omega^1 \times D^k$  such that

$$\text{cat}_{\Omega^1 \times D^k, \Omega^1 \times S^k}(C_m) \geq m. \tag{5.9}$$

Now let  $K_m = K_m(R) = \{(x, t_* + R(t - t_*)) : (x, t) \in C_m\}$ . By (5.9)

$$\text{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(K_m) \geq m. \tag{5.10}$$

Moreover, by assumptions (1.4) and (1.9) and the compactness of  $C_m$ , there exists  $R_m$  (independent of  $k$  and  $\varepsilon$ ) such that

$$\sup f_\varepsilon^k(K_m(R) \cap (\Omega^1 \times B_k(R))) \leq d - 1$$

$$\text{for any } R \geq R_m, \quad k \in \mathbb{N}, \quad \varepsilon \in ]0, 1],$$

concluding the proof of lemma 5.5.  $\square$

We shall need the following Lemma about the invariance of the class  $\Gamma_m^k(R, \varepsilon)$  with respect to the flow generated by the curves of maximal slope of the functional  $f_\varepsilon$ .

LEMMA 5.6. – Let  $\Gamma_m^k(R, \varepsilon) \neq \emptyset$  and  $c$  be a regular value for  $f$ ,  $c \geq d$  [cf. (5.5)].

Then there exists  $\sigma_0 = \sigma_0(c) > 0$  such that for any  $\sigma \in [0, \sigma_0]$  there exists a homeomorphism  $\Phi_\sigma : \mathcal{Z}_k \rightarrow \mathcal{Z}_k$ , satisfying

$$\Phi_\sigma(\{f_\varepsilon(z) \leq c + \sigma\}) \subset \{f_\varepsilon(z) \leq c - \sigma\}, \quad (5.11)$$

$$\Phi_\sigma(z) = z \quad \text{if} \quad f_\varepsilon(z) \notin [c - 2\sigma, c + 2\sigma], \quad (5.12)$$

and,

$$\Phi_\sigma(x) = x \quad \text{for any } x \in \{f_\varepsilon(z) \leq d - 1\} \quad (5.13)$$

Moreover, for any  $B \in \Gamma_m^k(R, \varepsilon)$ ,

$$\text{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(\Phi_\sigma(B)) \geq \text{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(B). \quad (5.14)$$

*Proof.* – Since  $c$  is a regular value for  $f_\varepsilon$  and  $(P.S.)_c$  holds, there exists  $\sigma_0 \in ]0, 1/4[$ , such that  $[c - \sigma_0, c + \sigma_0]$  consists of regular values for  $f_\varepsilon$ .

Now let  $\sigma \in ]0, \sigma_0[$  and let  $\Phi_\sigma$  be the homeomorphism given by the solution (at the instant  $2\sigma$ ) of the Cauchy problem

$$\begin{cases} \dot{\eta}_\sigma = -\chi_\sigma(f_\varepsilon(\eta_\sigma)) f'_\varepsilon(\eta_\sigma) / \|f'_\varepsilon(\eta_\sigma)\|^2 \\ \eta_\sigma(0) = x, \end{cases} \quad (5.15)$$

[i.e.  $\Phi_\sigma(z) = \eta_\sigma(2\sigma, z)$ ], where  $\chi_\sigma$  is a Lipschitz continuous real function with values on  $[0, 1]$ , such that  $\chi_\sigma(s) = 1$  if  $s \in [c - \sigma, c + \sigma]$  and  $\chi_\sigma(s) = 0$  if  $s \notin [c - 2\sigma, c + 2\sigma]$  (cf. [15]). Clearly  $\Phi_\sigma$  satisfies (5.11)-(5.13).

It remains to prove (5.14). To this aim let  $A_0, \dots, A_k$  be closed sets covering  $\Phi_\sigma(B)$ , satisfying (5.1)-(5.3), and consider  $B_0 = \Phi_\sigma^{-1}(A_0) \cap B, \dots, B_k = \Phi_\sigma^{-1}(A_k) \cap B$ . Clearly,  $B_0, \dots, B_k$  are closed subsets covering  $B$  and satisfying (5.1)-(5.2). Then it remains to prove that  $B_0$  satisfies (5.3).

Towards this goal, note that by (5.13) and (5.6), (setting  $Y = \Omega^1 \times B_k(R)$ ),

$$\begin{aligned} \Phi_\sigma^{-1}(A_0) \cap B \cap Y &= \Phi_\sigma^{-1}(A_0) \cap \Phi_\sigma^{-1}(B \cap Y) \\ &= \Phi_\sigma^{-1}(A_0 \cap B \cap Y) = A_0 \cap B \cap Y. \end{aligned} \quad (5.16)$$

Then, if  $h : [0, 1] \times A_0 \rightarrow \mathcal{Z}_k$  is the homotopy satisfying (5.3) relatively to  $A_0$ ,  $\tilde{h} : [0, 1] \times B_0 \rightarrow \mathcal{Z}_k$ , defined by

$$\tilde{h}(s, z) := \begin{cases} \eta_\sigma(4\sigma s, z) & \text{if } s \leq 1/2 \\ h(2s - 1, \Phi_\sigma(z)) & \text{if } s \geq 1/2 \end{cases},$$

satisfies (5.3) relatively to  $B_0$ . [cf. (5.11)-(5.13)].  $\square$

Now for any  $\varepsilon \in ]0, 1]$ ,  $1 \leq k \in \mathbb{N}$ ,  $1 \leq m \in \mathbb{N}$ , define

$$c_\varepsilon^k(m) = \inf_{B \in \Gamma_m^k} \sup_{z \in B} f_\varepsilon^k(z), \tag{5.17}$$

where  $\Gamma_m^k = \Gamma_m^k(R_m)$ .

The following Lemma holds.

LEMMA 5.7. – Assume  $\mathcal{M}_0$  non contractible and 1-connected. Then

- (i)  $c_\varepsilon^k(m) \geq d$ , for any  $\varepsilon, k, m$ , where  $d$  is defined at (5.5),
- (ii) there exists two positive real constant  $a_1, a_2$  such that, for any  $c \in \mathbb{R}$ , there exists  $m_c \in \mathbb{N}$ , such that

$$c_\varepsilon^k(m) \geq \lambda c - a_1 \sqrt{c} - a_2 \quad \text{for any } m > m_c, \quad k \geq 1, \quad \varepsilon \in ]0, 1],$$

where  $\lambda$  is defined at (1.3).

- (iii) there exists  $c(m)$ , independent of  $k$  and  $\varepsilon$  such that

$$c_\varepsilon^k(m) \leq c(m), \quad *$$

- (iv)  $c_\varepsilon^k(1) \leq c_\varepsilon^k(2) \leq \dots$ ,
- (v)  $c_\varepsilon^k(m)$  is a critical value of  $f_\varepsilon^k$ .

*Proof.* – If, by contradiction,  $c_\varepsilon^k(m) < d$ , by (5.17) and (5.5) there exists  $B \in \Gamma_m^k(R, \varepsilon)$  such that  $B \cap \{(x, t_*) : x \in \Omega^1\} = \emptyset$ . Then

$$\text{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(B) = 0,$$

proving (i).

Fix  $c \in \mathbb{R}$  and put

$$E^c = \left\{ x \in \Omega^1 : \int_0^1 \langle \dot{x}(s), \dot{x}(s) \rangle ds \leq c \right\},$$

and

$$E_c = \left\{ x \in \Omega^1 : \int_0^1 \langle \dot{x}(s), \dot{x}(s) \rangle ds \geq c \right\}.$$

Let  $B \in \Gamma_m^k(R, \varepsilon)$ . Suppose

$$B \cap (E_c \times \{t_*\}) = \emptyset. \tag{5.18}$$

Then

$$B \subset (E^c \times \{t_*\}) \cup (\mathcal{Z}_k \setminus \{(x, t_*), x \in \Omega^1\}),$$

and, since  $\text{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(\mathcal{Z}_k \setminus \{(x, t_*), x \in \Omega^1\}) = 0$ , if  $B$  satisfies (5.18),

$$\text{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(B) \leq \text{cat}_{\Omega^1}(E^c). \tag{5.19}$$

Moreover by a well known result  $E^c$  is a strong deformation retract of a finite dimensional manifold whose dimension depends on  $c$  (cf. [10]). Therefore by the properties of the Lusternik and Schnirelmann category (cf. e.g. [16]) there exists  $m_c$  such that

$$\text{cat}_{\Omega^1}(E^c) \leq m_c.$$

Then by (5.18) and (5.19), if  $m > m_c$ ,

$$B \cap (E_c \times \{t_*\}) \neq \emptyset,$$

hence, [by assumptions (1.3), (1.5), (1.6)], there exists two positive real constant  $a_1$  and  $a_2$  such that

$$\begin{aligned} \sup f_\varepsilon^k(B) &\geq \inf_{x \in \Omega^1} f_\varepsilon^k(E_c \times \{t_*\}) \\ &\geq \inf_{x \in \Omega^1} \left( \lambda \int_0^1 \langle \dot{x}, \dot{x} \rangle ds - a_1 \left( \int_0^1 \langle \dot{x}, \dot{x} \rangle ds \right)^{1/2} - a_2 \right). \end{aligned}$$

Therefore, for any  $B \in \Gamma_m^k(R, \varepsilon)$ ,

$$\sup f_\varepsilon^k(B) \geq \lambda c - a_1 \sqrt{c} - a_2,$$

proving (ii).

In order to prove (iii) choose  $B = K_m$  and  $C_m$  as in Lemma 5.5. Then by (5.17) and (5.4)

$$\begin{aligned} c_\varepsilon^k(m) &\leq \sup f_\varepsilon^k(K_m) \\ &\leq \sup \left\{ \int_0^1 \left( \frac{1}{\beta_\varepsilon(x, t)} \cdot \langle \alpha(x, t) \dot{x}, \dot{x} \rangle \right. \right. \\ &\quad \left. \left. + \frac{2}{\beta_\varepsilon(x, t)} \cdot \|\delta(x, t)\| \cdot \|\dot{x}\| |t| - t^2 \right) \right. \\ &\quad \left. \times \beta_\varepsilon(x, t) ds - \psi_\varepsilon \left( \int_0^1 t^2 ds \right) : \right. \\ &\quad \left. x \in \pi_1(C_m), \quad t \in W^1, \quad \|t - t_*\|_{W^{1,2}} \leq R \right\}, \end{aligned}$$

where  $\pi_1$  is the projection on  $\Omega^1$ . Therefore, (since  $C_m$  is compact) by (1.4) and (1.9) we get (iii), because  $t - t_* \in W_0^{1,2}$  and, in (1.9),  $\gamma_0 < 2$ ,  $\gamma_1 < 1$ .

Since  $\Gamma_{m+1}^k(R, \varepsilon) \subset \Gamma_m^k(R, \varepsilon)$ , (iv) follows immediately. In order to prove (v) assume by contradiction that  $c_\varepsilon^k(m)$  is not a critical value of  $f_\varepsilon^k$ . Then, since  $c_\varepsilon^k(m) \geq d$ , we can use Lemma 5.6 with  $c = c_\varepsilon^k(m)$ .

Now let  $B \in \Gamma_m^k(R, \varepsilon)$  such that

$$\sup f_\varepsilon^k(B) \leq c_\varepsilon^k(m) + \sigma.$$

If  $\sigma$  is sufficiently small, by Lemma 5.6,  $\Phi_\sigma(B) \in \Gamma_m^k(R, \varepsilon)$ , in contradiction with (5.27), because  $\sup f_\varepsilon(\Phi_\sigma(B)) < c_\varepsilon^k(m)$ .  $\square$

Now we are finally ready to prove Theorems 1.3 and 1.4.

*Proof of Theorem 1.3.* – Assume  $\mathcal{M}_0$  1-connected. Choose  $c$  such that

$$\lambda c - a_1 \sqrt{c} - a_2 > d,$$

and  $m_c$  as in (ii) of Lemma 5.7.

Then, by (v) of Lemma 5.7,  $c_\varepsilon^k(m_c)$  is a critical value of  $f_\varepsilon^k$ . Moreover by (iii) of Lemma 5.7, and Lemma 4.1, there exists a critical point  $z_\varepsilon$  of  $f_\varepsilon$  such that

$$\lambda c - a_1 \sqrt{c} - a_2 \leq f_\varepsilon(z_\varepsilon) \leq c(m_c).$$

Finally by Proposition 3.6, if  $\varepsilon$  is sufficiently small  $z_\varepsilon$  is a critical point of  $f_\varepsilon$  (cf. also Remark 3.8), such that

$$f_\varepsilon(z_\varepsilon) \geq \lambda c - a_1 \sqrt{c} - a_2,$$

giving the proof of Theorem 1.3 when  $\mathcal{M}_0$  is simply connected. If the fundamental group of  $\mathcal{M}_0$  is finite, the proof of Theorem 1.3 can be got using the universal covering, while if it is infinite the proof can be got working on the connected components of  $\Omega^1$  which are infinite.  $\square$

*Proof of Theorem 1.4.* – Recalling the conclusion of the proof of Theorem 1.3 we can reduce ourselves to prove Theorem 1.4 whenever  $\mathcal{M}_0$  is 1-connected. The same proofs of Propositions 3.1 and 3.2 and Lemma 4.1 show that  $f_\varepsilon^k$  satisfies the following condition of Palais and Smale (uniformly with respect to  $k$ ) at every level  $c \in \mathbb{R}$  (cf. [6]):

any  $\{z_k\}_{k \in \mathbb{N}} \subset \mathcal{Z}_k$  such that

$$f_\varepsilon^k(z_k) \xrightarrow{k} c, \quad df_\varepsilon^k(z_k) \xrightarrow{k} 0$$

possesses a subsequence which converges in  $\mathcal{Z}$

$$\text{to a critical point of } f_\varepsilon. \tag{5.20}$$

Fix  $m \in \mathbb{N}$  and let  $K_m$  as in Lemma 5.5. The proof of (iii) of Lemma 5.7 shows that, if  $|t_1 - t_0|$  is sufficiently large (depending only by  $m$ ) the critical values  $c_\varepsilon^k(1) \leq c_\varepsilon^k(2) \leq \dots \leq c_\varepsilon^k(m)$  are  $\leq -1$ .

Using (5.20) it is possible to show (cf. [6]) that, if there exists a sequence  $\{k_p\}_{p \in \mathbb{N}}$  such that

$$c_\varepsilon^{k_p}(i) \xrightarrow{p} c_\varepsilon(i), \quad c_\varepsilon^{k_p}(i) - c_\varepsilon^{k_p}(j) \xrightarrow{p} 0,$$

for some  $i \neq j$ , then  $c_\varepsilon(i)$  is a critical value of  $f_\varepsilon$  reached by infinitely many critical points of  $f_\varepsilon$ .

Therefore there exists  $c_\varepsilon(1), c_\varepsilon(2), \dots, c_\varepsilon(m) \in ]-\infty, -1]$ , critical values of  $f_\varepsilon$  such that, if  $c_\varepsilon(i) = c_\varepsilon(j)$  for some  $i \neq j$ , then  $f_\varepsilon$  has infinitely many critical points at the level  $c_\varepsilon(i)$ . Then, by (2.11) and Remark 3.7, for any fixed  $m$  there exists  $\Delta_m$  such that, if

$$|t_1 - t_0| \geq \Delta_m,$$

and  $\varepsilon$  is sufficiently small,  $f$  has at least  $m$  critical points  $z_\varepsilon^1, \dots, z_\varepsilon^m$ , where  $f$  is negative, i.e.  $m$  timelike geodesics joining  $(x_0, t_0)$  with  $(x_1, t_1)$ .  $\square$

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