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Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$

by

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ABSTRACT. – We construct global strong solutions of the Navier-Stokes equations with sufficiently oscillating initial data. We will show that the condition is for the norm in some Besov space to be small enough.

RÉSUMÉ. – Nous construisons des solutions fortes globales des équations de Navier-Stokes, pour des données initiales suffisamment oscillantes. Cette condition se traduit en terme de norme petite dans un certain espace de Besov.

INTRODUCTION

We are interested in the following system, for $x \in \mathbb{R}^3$ and $t > 0$,

$$
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u &= \nu \Delta u - \nabla p \\
\nabla \cdot u &= 0,
\end{align*}
$$

with initial data $u(x, 0) = u_0(x)$. For the sake of simplicity, we suppose that $\nu = 1$; a simple rescaling allows us to obtain any other value. Local
existence and uniqueness in the Sobolev space $H^s(\mathbb{R}^3)$ and the Lebesgue space $L^p(\mathbb{R}^3)$ are known, if $s > 1/2$ and $p > 3$ (see [4]). We have global solutions for small initial data in $L^3(\mathbb{R}^3)$ (see [9] or [4]) and $H^{\frac{3}{2}}(\mathbb{R}^3)$ (see [4] and [5]), or in $L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, with $p > 3$ (see [1]). We shall extend the results of [4], for $s > 1/2$ and $p > 3$. By adapting the auxiliary spaces used in [4], we shall prove the existence and uniqueness of global solutions in $H^s(\mathbb{R}^3)$ provided the initial data are small in a sense which will be made precise later, and in $L^p(\mathbb{R}^3)$ up to additional conditions on $u_0$. Let us define the homogeneous Besov spaces $\dot{B}^s_{p,q}$:

**Definition 1.** Let us choose $\phi \in \mathcal{S}(\mathbb{R}^n)$ a radial function so that $\text{Supp } \hat{\phi} \subset \{|\xi| < 1 + \epsilon\}$, and $\hat{\phi}(\xi) = 1$ for $|\xi| < 1$. Define $\phi_j(x) = 2^{nj}\phi(2^j x)$, $S_j$ the convolution operator with $\phi_j$, and $\Delta_j = S_{j+1} - S_j$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $1 < p, q \leq +\infty$, $f \in \dot{B}^\alpha_{p,q}$ if and only if

$$\left[ \sum_{j=-\infty}^{+\infty} (2^{j\alpha} \|\Delta_j f\|_{L^p})^q \right]^{\frac{1}{q}} < +\infty.$$ 

The reader should consult [12], [2], or [16] where the properties of Besov spaces are exposed in detail. Let us see how homogeneous Besov spaces arise. If we want to construct a global solution, it is useful to control a norm remaining invariant by the rescaling $f(x) \rightarrow \lambda f(\lambda x)$. If this can be achieved in a Besov space with $\alpha < 0$ and therefore bigger than the usual space where we want to obtain a solution, we will have weaker assumptions on $u_0$.

Let us give the results in the case of Sobolev spaces. $BC$ denotes the class of bounded continuous functions.

**Theorem 1.** There exists an universal constant $\beta > 0$ such that, if $s > \frac{1}{2}$, $u_0 \in H^s(\mathbb{R}^3)$, $\nabla \cdot u_0 = 0$ and

$$\| u \|_{\dot{B}^{-1/4}_{4,\infty}} < \beta,$$

then there exists a unique solution $u$ of (1) such that

$$u \in BC([0, \infty), H^s(\mathbb{R}^3)).$$

Moreover, the following properties hold for $u$:

- $\| u(\cdot, t) \|_{L^2}$ is decreasing, and for every $t > 1$,

$$\| u(\cdot, t) - e^{t\Delta} u_0 \|_{L^2} \leq C(\beta, u_0) \frac{t^{1/4}}{t^{1/4}}.$$
For every $t > 1$,
\begin{equation}
\| (-\Delta)^{s/2} u(\cdot, t) \|_{L^2} \leq \frac{C(\beta, u_0, s)}{t^{s/2}}.
\end{equation}

For every $t > 0$,
\begin{equation}
\| u(\cdot, t) \|_{\infty} \leq \frac{C(\beta, u_0)}{\sqrt{t}}.
\end{equation}

If $s \in (1, 3/2]$, for every $t < 1$,
\begin{equation}
\| u(\cdot, t) - e^{t\Delta} u_0 \|_{\infty} \leq C(\beta, u_0).
\end{equation}

Note that the space $\dot{B}^{-1/4}_{4, \infty}$ is invariant under the scaling $f(x) \rightarrow \lambda f(\lambda x)$, and $\dot{H}^{1/2} \subset \dot{B}^{-1/4}_{4, \infty}$. It is very interesting that we do not need a small $H^{1/2}$-norm to obtain a global solution (see [4]). On the other hand, if we want to include the case $1/2$, $u$ is unique in the space
\begin{equation}
\begin{cases}
u \in BC([0, +\infty), H^{1/2}) \\
t^{1/8} u(\cdot, t) \in BC([0, +\infty), L^4) \\
\lim_{t \to 0} t^{1/8} \| u \|_{L^4} = 0.
\end{cases}
\end{equation}

which was used in [4], the starting point of the present work. The weak condition (2) is the only remaining obstacle to the problem of existence of global smooth solutions to the Navier-Stokes equations, and we remark that $\beta$ does not depend on $s$. The decay estimates (4) can be found in [8], in a slightly different context. We recall it here as a natural consequence of the construction of $u$.

In the Lebesgue spaces, the analogue is

\textbf{Theorem 2.} - Let $p > 3/2$, there exists $\delta(p) > 0$ such that, if $u_0 \in L^p \cap \dot{B}^{-\left(1-\frac{3}{2p}\right)}_{2p, \infty}$, $\nabla \cdot u_0 = 0$ and
\begin{equation}
\| u_0 \|_{\dot{B}^{-\left(1-\frac{3}{2p}\right)}_{2p, \infty}} < \delta(p),
\end{equation}
then there exists a unique solution $u$ such that
\begin{equation}
\begin{cases}
u \in BC([0, +\infty), L^p) \\
t^{\frac{1}{2}-\frac{3}{4p}} u(\cdot, t) \in BC([0, +\infty), L^{2p}) \\
\lim_{t \to 0} t^{\frac{1}{2}-\frac{3}{4p}} \| u \|_{L^{2p}} = 0.
\end{cases}
\end{equation}
The restriction $p > 3/2$ is due to technical considerations, and we could probably obtain 1 instead of $3/2$, by slightly modifying the Besov space involved.

**PROPOSITION 1.** - The constant $\delta(p)$ satisfies:

$$\lim_{p \to +\infty} \delta(p) = 0,$$

$$\lim_{p \to 3/2} \delta(p) = 0.$$

**PROPOSITION 2.** - In Theorem 2, we can replace $u_0 \in L^p \cap \dot{B}^{-1/2}_{2p,\infty}$ by $u_0 \in L^p \cap L^3$, and if $p > 3$ by $L^2 \cap L^p$.

If $u_0 \in H^s, s \geq 1/2$, then as $\hat{H}^{1/2} \subset \dot{B}^{-1/4}_{4,\infty}$, we have a natural candidate for the useful Besov space. On the contrary, if we take $L^p$, we may use two different Besov spaces: the first one is $\dot{B}^{-1/3}_{2p,\infty}$, as $L^p \subset \dot{B}^{-1/3}_{2p,\infty}$. But this space is not invariant by the rescaling. The “right” space is $\dot{B}^{-1/3}_{2p,\infty}$, but unfortunately $L^p \notin \dot{B}^{-1/3}_{2p,\infty}$. This explains the additional condition imposed on $u_0$ in Theorem 2. Both spaces coincide only when $1 - \frac{3}{2p} = \frac{3}{2p}$, which means $p = 3$. The reader should refer to [9] and [4] for details.

**Proofs.** - We first reformulate the problem in order to obtain an integral equation for $u$. This is standard practice, and was first employed by Kato and Fujita (see [10] [11]), and very often used since (see [7] [6] [15]). All these authors use semi-group theory, but in the present case, we do not need this formalism, for the exact expression of the heat kernel in $\mathbb{R}^3$ allows us to obtain directly the estimates we need (see [9]). Let $P$ be the projection operator from $(L^2(\mathbb{R}^3))^3$ onto the subspace of divergence-free vectors, denoted by $PL^2$, and $R_j$ the Riesz transform with symbol $\frac{\xi_j}{|\xi|}$. We easily see that

$$P \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - \begin{pmatrix} R_1 \sigma \\ R_2 \sigma \\ R_3 \sigma \end{pmatrix}$$

where $\sigma = \sum_j R_j u_j$. It is well-known that $P$ can be extended to a bounded operator from $(L^p)^3$ onto $PL^p$, $1 < p < +\infty$, and from $(H^s)^3$ onto $PH^s$, $s \geq 0$. Note that $P$ commutes with $S(t) = e^{t\Delta}$, whereas on an open set $\Omega$, we need to introduce the Stokes operator $-P\Delta$ and the associated semi-group. Note that

$$\text{Ker}P = \{u \mid \exists \phi \text{ such that } u = \nabla \phi\}.$$
Using $P$, (1) becomes an evolution equation

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - P\nabla \cdot (u \otimes u), \\
\nabla \cdot u &= 0, \\
u(x, 0) &= u_0(x).
\end{align*}
$$

(10)

We replace $(u \cdot \nabla)u$ by $\nabla \cdot (u \otimes u)$ to avoid problems of definition, and this is possible only because $\nabla \cdot u = 0$. It is then standard to study (10) via the corresponding integral equation

$$
(11) \quad u(x, t) = S(t)u_0(x) - \int_0^t P S(t - s) \nabla \cdot (u \otimes u)(x, s) ds
$$

in a space of divergence free vectors. The integral should be seen as a Bochner integral. In the general case of evolution equations, a solution of (11) might not be a solution of (10). However, in the case of the Navier-Stokes equations without external forces, it is true without any extra assumptions. Actually, the solutions of (11) are $C^\infty((0, +\infty) \times \mathbb{R}^3)$ and verify the equations (1) in the classical sense, as we recover easily the pressure up to a constant by

$$
(12) \quad -\Delta p = \sum_{i,j} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}.
$$

The reader should refer to [7] [10] or [13] for proofs. We remark that since a solution of (1) is necessarily a solution of (11), uniqueness for (11) guarantees uniqueness for (1). We aim to solve (11) by successive approximations, with the following lemma:

**Lemma 1.** Let $E$ and $F$ be two Banach functional spaces, endowed with the norms $\| \cdot \| = \| \cdot \|_E$ and $\| \cdot \| = \| \cdot \|_F$. A continuous bilinear operator from $F \times F \rightarrow E$ and $F \times F \rightarrow F$:

\[ \| B(u, v) \| \leq \eta \| u \| \| v \|, \]  
\[ | B(u, v) | \leq \gamma \| u \| \| v \|, \]

and define the sequence $X_0 = 0, X_{n+1} = Y + B(X_n, X_n)$, where $Y$ belongs to $E$ and to $F$. If

$$
(13) \quad 4\gamma \| Y \| < 1,
$$

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then the sequence converges in both spaces $E$ and $F$, and the limit $X$ satisfies

$$X = Y + B(X, X)$$

and

$$|X| < 2|Y|.$$  

The proof is left to the reader. Note that the value of $\eta$ has no influence on the convergence. Now we have to study the following bilinear operator

$$B(u, v) = \int_0^t PS(t - s) \nabla \cdot (u \otimes u) ds.$$  

In order to simplify the notations, we limit ourselves to the following scalar operator

$$B(f, g) = \int_0^t \frac{1}{(t - s)^2} \theta \left( \frac{\cdot}{(t - s)^2} \right) * f g(s) ds.$$  

As $PS(t - s) \nabla \cdot$ is a matrix of convolution operators, the components are all operators like (17), with

$$e^{i|\xi|^2} \hat{\theta}(\xi) = \frac{\xi_j \xi_k \xi_l}{|\xi|^2} (\text{with } a = -\xi_j \text{ on the diagonal}).$$  

**Lemma 2.**  $\theta(x) \in C^\infty(\mathbb{R}^3)$ and $\theta \in L^1 \cap L^\infty$.  

This can be easily seen on the Fourier transform of $\theta$.  

In what follows, $C$ denotes a constant which may vary from one line to another.

**Proof of Theorem 1**

**Proposition 3.**  Let $1/2 < s \leq 3/4$, then there exists a solution $u$ of (11) such that

$$u \in BC([0, +\infty), \dot{H}^s) = E,$$

$$\omega(t)u(x, t) \in BC([0, +\infty), L^4) = F,$$

where $\omega(t) = t^{3/8 - s/2}$ if $0 < t < 1$ and $\omega(t) = t^{3/8}$ if $t \geq 1$.  

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We want to apply Lemma (1) where E and F are defined by the norms
\[ \| u \| = \sup_t \| u \|_{H^s}, \]
\[ |u| = \sup_t \omega(t) \| u \|_{L^4}. \]

If we use Hölder and Young inequalities for \( B(f, g) \), \( A \) being the operator with symbol \( |\xi|^s \),

\begin{align*}
(20) \quad \| A^s B(f, g)(t) \|_{L^2} &\leq \| \theta \|_{L^1} \| f \|_\infty \| g \|_\infty \int_0^t \frac{1}{(t-\tau)^{1/2+s/2} \omega^2(\tau)} \, d\tau, \\
(21) \quad \| B(f, g)(t) \|_{L^4} &\leq \| \theta \|_{L^{4/3}} \| f \|_\infty \| g \|_\infty \int_0^t \frac{1}{(t-\tau)^{7/8} \omega^2(\tau)} \, d\tau.
\end{align*}

We shall then verify that, for all \( t > 0 \),
\[ I_1 = \int_0^t \frac{1}{(t-\tau)^{1/2+s/2} \omega^2(\tau)} \, d\tau < +\infty, \]
\[ I_2 = \int_0^t \frac{1}{(t-\tau)^{7/8} \omega^2(\tau)} \, d\tau < +\infty. \]

Easy calculations actually show that for \( t < 1 \),
\[ I_i < Ct^{s/2-1/4} \]
and for \( t > 1 \)
\[ I_i < Ct^{1/4-s/2}. \]

The continuity at \( t = 0 \) comes from the estimate when \( t < 1 \). In order to include the case \( s = 1/2 \), we have to impose \( \lim_{t \to 0} t^{1/8} \| u \|_{L^4} = 0 \) (see [4]). Note that the constant \( \gamma \) of Lemma 1 is
\[ (23) \quad \gamma = \int_0^1 \frac{\| \theta \|_{L^{4/3}}}{(1-\tau)^{7/8}} \tau^{1/4} \, d\tau. \]

Therefore, if \( S(t)u_0 \) satisfies condition (13), we obtain \( u \in BC([0, +\infty), \tilde{H}^s) \).

**Proposition 4.** - We have
\[ (24) \quad u \in BC([0, +\infty), L^2). \]
Let $G = BC([0, +\infty), L^2)$; $B$ is bicontinuous from $G \times F$ to $G$:

\begin{equation}
(25) \quad \| B(f, g)(t) \|_{L^2} \leq g \left( \sup_{[0,t]} \| f \|_{L^2} \right) \| \theta \|_{L^{4/3}} e^t \int_0^t \frac{1}{(t-\tau)^{7/8} \omega(\tau)} d\tau.
\end{equation}

Let

\[ I_3 = \int_0^1 \frac{\| \theta \|_{L^{4/3}}}{(1-\tau)^{7/8} \tau^{7/8}} d\tau, \]

for $t < 1$, $I_3 < C t^{s/2-1/4}$, and for all $t$,

\[ I_3 \leq \int_0^1 \frac{\| \theta \|_{L^{4/3}}}{(1-\tau)^{7/8} \tau^{7/8}} d\tau = \rho. \]

$G$ being a Banach space, we can use a contraction argument to show that the sequence defined previously converges in $G$. It is sufficient that $2 | u | \rho < 1$, which is true as $\rho \leq \gamma$ and $u$ verifies (15). Therefore, we proved (24) and hence Proposition 3, and shown that $\| u(\cdot, t) \|_{L^2}$ is uniformly bounded.

We now show (6): the following estimation is verified by the heat kernel,

\begin{equation}
(26) \quad \sup_{[0,t]} \sqrt{t} \| S(t)u_0 \|_\infty \leq C.
\end{equation}

We have

\[ \| B(f, g)(t) \|_\infty \leq \| \theta \|_{L^{4/3}} \int_0^t \frac{\| f(s) \|_{L^4} \| g(s) \|_\infty}{(t-s)^{7/8}} ds. \]

Let us denote $W(f, t) = \sup_{[0,t]} \sqrt{t} \| f(\cdot, s) \|_\infty$, then

\begin{equation}
(27) \quad W(B(f, g), t) \leq \| f \|_\infty W(g, t) \| \theta \|_{L^{4/3}} \int_0^t \frac{\sqrt{t}}{(t-s)^{7/8}s^{5/8}} ds.
\end{equation}

Let

\[ I_4 = \int_0^1 \frac{\| \theta \|_{L^{4/3}}}{(1-\mu)^{7/8} \mu^{5/8}} d\mu, \]

then, as $I_4 \leq 2 \gamma$, we have $2W(S(t)u_0, t)I_4 < 1$. Therefore,

\begin{equation}
(28) \quad \sup_{[0,t]} \sqrt{t} \| u(\cdot, t) \|_\infty \leq \frac{C}{1 - 2I_4} \| S(t)u_0 \|.
\end{equation}

Now we can prove (4) as follows:

\[ \| B(f, g)(t) \|_{L^2} \leq \int_0^t \frac{C}{(t-s)^{\frac{1}{2} - \frac{1}{2q}}} \| g \|_{L^2} \| f \|_{L^\alpha} ds, \]
where
\[ \frac{1}{2} = \frac{1}{q} + \frac{1}{2} + \frac{1}{\beta} - 1. \]

If we take \( q \) such that \( \frac{3}{2q} = 1 + \varepsilon, \varepsilon > 0 \), using interpolation and (28) we get, for \( t > 1 \),

\begin{equation}
\| f \|_{L^p} \leq \| f \|_{L^2}^{\frac{2}{p}} \| f \|_{\infty}^{1-\frac{2}{p}}.
\end{equation}

and

\begin{equation}
\| B(f, g)(t) \|_{L^2} \leq \int_0^t \frac{C}{(t-s)^{1-\varepsilon} s^{1+\frac{1}{2}\varepsilon}} ds.
\end{equation}

On the other hand, we know by (26) that \( \forall q \geq 2 \),

\begin{equation}
\sup_{[0,t]} t^{\frac{3}{2}(\frac{1}{2} - \frac{1}{q})} \| S(t)u_0 \|_q \leq C.
\end{equation}

Therefore, as \( u \) satisfies (14), we will improve (30) in the following way: let

\begin{equation}
B_1(f, g) = \int_0^1 \frac{1}{(t-s)^2} \theta \left( \frac{\cdot}{\sqrt{t-s}} \right) * fg(s) ds.
\end{equation}

\begin{equation}
B_2(f, g) = \int_1^t \frac{1}{(t-s)^2} \theta \left( \frac{\cdot}{\sqrt{t-s}} \right) * fg(s) ds.
\end{equation}

The term \( B_1 \) can be handled very easily, so that \( \forall \eta > 0 \),

\[ \| B_1(f, g)(t) \|_{L^2} \leq \frac{C}{t^{1-\eta}}. \]

Now, we split \( B_2(u, u) \) in three parts. By (31) we have

\[ \| B_2(S(t)u_0, S(t)u_0) \|_{L^2} \leq \int_1^t \frac{C}{(t-s)^{2-\frac{3}{2q}} s^{\frac{3}{2}(1-\alpha) - \frac{1}{2}}} ds \]

\[ \leq \frac{C}{t^{1-\frac{3}{2q} + \frac{3}{2}(1-\alpha)}} \]

\[ \leq \frac{C}{t^{1/4}}. \]
We remark that the exponent $1/4$ cannot be improved, as it does not depend on $\gamma, \alpha$ and $\beta$.

**Lemma 3.** Suppose that for $0 < \mu$

$$\| B(u, u)(t) \|_{L^2} \leq \frac{C}{t^{\mu}},$$

then

$$\| B_2(S(t)u_0, B(u, u)) \|_{L^2} \leq \frac{C}{t^{1/4 + \mu}},$$

and there exists $\nu > 0$ such that

$$\| B_2(B(u, u), B(u, u)) \|_{L^2} \leq \frac{C}{t^{\mu + \nu}}.$$

By (31)

$$\| B_2(S(t)u_0, B(u, u)) \|_{L^2} \leq \int_1^t \frac{C}{(t - s)^{2 - \frac{1}{2\gamma}} s^{\frac{3}{2} - (\frac{1}{2} - \frac{1}{\beta})}} ds$$

$$\leq \frac{C}{t^{1/4 + \mu}},$$

and, by (28) and (29)

$$\| B_2(B(u, u), B(u, u)) \|_{L^2} \leq \int_1^t \frac{C}{(t - s)^{2 - \frac{3}{4\gamma}} s^{\frac{1}{2} - (\frac{1}{2} - \frac{1}{\gamma}) + \frac{3}{4} \mu}} ds$$

$$\leq \frac{C}{t^{-1/4 + \frac{3}{4} + (\frac{3}{4} - \frac{1}{4})\mu}}.$$

We can start with $\mu = 1/6 - \varepsilon$, and obtain any exponent $\eta > 1/4$. Thus,

$$\| B_2(u, u) \|_{L^2} \leq \frac{C}{t^{1/4}}.$$

We constructed $u$ for $s \leq 3/4$. Now we will see that if $s > 3/4$, $u$ as above is actually in $H^s$. We limit ourselves to the bilinear form (17), as the term $S(t)u_0$ satisfies at least the same estimates.
LEMMA 4. – Let $f, g \in H^s(\mathbb{R}^3), 3/4 < s < 3/2,$

\begin{equation}
\| \Lambda^{2s-3/2}(fg) \|_{L^2} \leq C \| \Lambda^s f \|_{L^2} \| \Lambda^s g \|_{L^2}.
\end{equation}

For a proof see the Appendix. Suppose now that $s > 3/4$, and $u$ is the solution of Proposition 3 for $s = 3/4$. Then, if $\eta \leq 1/4$, we obtain for $t < 1$

\begin{equation}
\| \Lambda^{3/4+\eta} B(f, g)(t) \|_{L^2} \leq \int_0^t \frac{C}{(t-\tau)^{3/4+s/2}} d\tau
\end{equation}

using Lemma 4 and the boundedness of $f$ in $H^{3/4}$, so that

\begin{equation}
\| \Lambda^{3/4+\eta} B(f, g)(t) \|_{L^2} \leq C t^{3/4-\eta},
\end{equation}

which gives the continuity at zero. For $t > 1$, we have by (29)

\begin{equation}
\| B(f, g) \|_{L^4} \leq \frac{C}{t^{1/4}}
\end{equation}

which allows us to improve (22), for $s \leq 3/4$

\begin{equation}
\| \Lambda^s B(f, g)(t) \|_{L^2} \leq \int_0^1 \frac{\| \theta \|_{L^1}}{(t-\tau)^{1/2+s/2} \omega^2(\tau)} d\tau \| f \|_{L^1} | g |
\end{equation}

\begin{equation}
+ \int_1^t \frac{C \| \theta \|_{L^1}}{(t-\tau)^{1/2+s/2} \tau^{1/2}} d\tau
\end{equation}

\begin{equation}
\leq \frac{C}{t^{s/2}}.
\end{equation}

Then

\begin{equation}
\| \Lambda^{3/4} f(t) \|_{L^2} \leq \frac{C}{t^{3/8}}
\end{equation}

and,

\begin{equation}
\| \Lambda^{3/4+\eta} B_1(f, g) \|_{L^2} \leq \frac{C}{(t-1)^{3/8+\eta/4}}
\end{equation}

and

\begin{equation}
\| \Lambda^{3/4+\eta} B_2(f, g) \|_{L^2} \leq \int_1^t \frac{C}{(t-\tau)^{3/8+\eta/4} \tau^{3/8}} d\tau
\end{equation}

then, for all $t$

\begin{equation}
\| \Lambda^{3/4+\eta} B(f, g)(t) \|_{L^2} \leq \frac{C}{1 + t^{3/8+\eta/4}}.
\end{equation}
We have thus obtained \( u \in H^{3/2}_0 \). By applying the same argument we can reach the value \( s > 3/2 \), as

\[
\Lambda^{s+\eta} B(f,g)(t) = \int_0^t \frac{1}{(t-\tau)^2} \Lambda^{\eta+3/2-s}\theta * \Lambda^{2s-3/2}(f)(\tau) d\tau,
\]

with \( \eta + 3/2 - s < 1 \). Before dealing with the case \( s > 3/2 \), let us briefly show (7). By Sobolev’s injection theorem (see [14]), if \( s < 3/2 \) then

\[
\| f \|_{L^p} \leq C \| \Lambda^s f \|_{L^2},
\]

with \( 1/p = 1/2 - s/3 \). If \( s = 1 + \alpha, \alpha < 1/2 \), we obtain, for \( t < 1 \)

\[
\| B(f,g)(t) \|_{L^\infty} \leq \int_0^t \frac{C}{(t-\tau)^{1-\alpha}} d\tau.
\]

For small \( t \), \( B(f,g) \) is bounded and tends to zero as \( t \) goes to zero. Now we treat the case where \( s > 3/2 \), using the following estimate

**Lemma 5.** – Let \( f, g \in H^s(\mathbb{R}^3), s > 3/2 \),

\[
\| \Lambda^s(fg) \|_{L^2} \leq C(s) \left( \| \Lambda^s f \|_{L^2} \| g \|_{L^\infty} + \| \Lambda^s g \|_{L^2} \| f \|_{L^\infty} \right).
\]

For a proof, see the appendix. We will then show

**Lemma 6.** – Let \( s > 3/2 \), for all \( t > 0 \)

\[
\| \Lambda^s u(\cdot, t) \|_{L^2} \leq \frac{C(s)}{1 + t^{3/2}}.
\]

This can be achieved by successive iterations, starting from the previous estimate for \( s = 2 \), and applying Lemma 5. Let us see how it works at each step. Let \( \eta < 1 \), we first treat the case \( t < 1 \).

\[
\| \Lambda^{s+\eta} B(f,g)(t) \|_{L^2} \leq \int_0^t \frac{1}{(t-\tau)^2} \| \Lambda^\eta \theta \left( \frac{\cdot}{\sqrt{t-\tau}} \right) * \Lambda^s(fg) \|_{L^2} d\tau,
\]

and as \( f \) and \( g \) are bounded in \( L^\infty \) and in \( H^s \),

\[
\| \Lambda^{s+\eta} B(f,g)(t) \|_{L^2} \leq C(s) t^{3/2-\eta/2}.
\]

For \( t > 1 \),

\[
\| \Lambda^{s+\eta} B_1(f,g) \|_{L^2} \leq \int_0^1 \frac{1}{(t-\tau)^{3/2+\eta-1/2}} \| \Lambda^{s+\eta-1/2} \theta \|_{L^1} \| L \| \Lambda f \|_{L^2} \| \Lambda g \|_{L^2} d\tau,
\]

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by using Lemma 4. Then
\[ \| \Lambda^{s+\eta} B_1(f, g) \|_{L^2} \leq \frac{C(s)}{(t - 1)^{\frac{s+\eta+1/2}{2}}}. \]

For \( t > 1 \),
\[ \| \Lambda^{s+\eta} B_2(f, g) \|_{L^2} \leq \int_1^t \frac{1}{(t - \tau)^2} \| \Lambda^\eta \theta \left( \frac{\cdot}{\sqrt{t - \tau}} \right) * \Lambda^s(f g) \|_{L^2} d\tau, \]

and using Lemma 5 we deduce the estimate for \( s+\eta \) from the estimate for \( s \):

\begin{equation}
\| \Lambda^{s+\eta} B_2(f, g) \|_{L^2} \leq \frac{C(s)}{t^{\frac{s+\eta}{2}}}. \tag{41}
\end{equation}

This achieves the proof of the existence of \( u \in BC([0, +\infty), H^n) \). Now, we observe that, as we have local existence and uniqueness for \( s > 1/2 \) (see [4]), our solution is unique by applying this theorem on intervals covering \([0, \infty)\). In the case \( s = 1/2 \), it is necessary to establish uniqueness directly, (see [4] or [11]). The reader should refer to [11] or [7], in order to see why a solution of (11) is actually a solution in the classical sense.

We can nevertheless make a few remarks. By the same process we use to gain the regularity \( s - 3/4 \), we can establish, independently of \( s \), estimates in \( H^r, r > s \): for all \( t > 0 \), there exists \( \pi(r) > 0 \)
\[ \| \Lambda^r u(\cdot, t) \|_{L^2} \leq \frac{C(r)}{t^{\pi(r)}}, \]

and \( \Lambda^r u \) is holderian on every interval \([t_0, t_1]\), provided \( t_1 > t_0 > 0 \). This provides the regularity in the space variables. As for regularity in time, it suffices to use the relation, which can be established without knowing (10)

\begin{equation}
u(t) = S(t - \varepsilon) u(\varepsilon) + \int_\varepsilon^t P S(t - \tau) \nabla \cdot (u \otimes u) d\tau, \tag{42}
\end{equation}


**Lemma 7.** - Let \( u(t) = \int_0^t e^{-(t-s)} \Delta f(s) ds, t \in [0, T], f \in C^n([0, T], B), \eta < 1, B \) a Banach space. Then \( u \in C^{1+\nu}((0, T], B), Au \in C^\nu((0, T], B), \)

\[ \partial_t u = -\Delta u + f, \]

for all \( \nu < \eta \).

We then obtain the \( C^\infty \) regularity of \( u \), for \( t > 0 \), with a bootstrap argument. Let us see how condition (13) can be expressed on \( u_0 \) in terms

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of Besov spaces. We set \(|S(t)u_0| < \beta\), where \(\beta\) has been chosen so that our scheme converges in \(F\). Remember that

\[ |S(t)u_0| = \sup_t \omega(t) \| S(t)u_0 \|_{L^4}. \]

Therefore, as \(3/8 - \inf(s, 3/4)/2 < 1/8\),

\[ \sup_t t^{3/8 - \inf(s, 3/4)/2} \| S(t)u_0 \|_{L^4} < \beta \]

and

\[ \sup_t t^{1/8} \| S(t)u_0 \|_{L^4} < \beta. \]

**Lemma 8.** Let \(u_0 \in S(\mathbb{R}^3)\), \(\alpha > 0\), and \(\gamma > 1\); \(\sup_t t^{\alpha/2} \| S(t)u_0 \|_{L^\gamma}\) is a norm on \(\dot{B}^{-\alpha}_{\beta, \infty}\) which is equivalent to the classical dyadic norm.

We refer to [4] or [12] for a proof. In our case, except for \(s = 3/4\), the condition on \(u_0\) is equivalent to

\[
\left\{ \begin{aligned}
\| u_0 \|_{\dot{B}^{-1/4}_{4, \infty}} &\leq \beta, \\
\| u_0 \|_{\dot{B}^{-(-1/4 - \inf(s, 3/4))}_{4, \infty}} &\leq \beta.
\end{aligned} \right.
\]

Thus, as \(\dot{H}^{1/2} \subset \dot{B}^{-1/4}_{4, \infty}\) and \(\dot{H}^{\inf(s, 3/4)} \subset \dot{B}^{-(-1/4 - \inf(s, 3/4))}_{4, \infty}\), \(u_0\) belongs to both Besov spaces. If \(u\) is a solution with initial condition \(u_0\), \(\lambda u(\lambda x, \lambda^2 t)\) is a solution with \(\lambda u_0(\lambda x)\) as initial data. The condition (44) is independent of \(\lambda\) for the norm is invariant by scaling. And (43) can be forced by a suitable choice of \(\lambda\). For \(s = 3/4\), we know that \(H^{3/4} \subset L^4\), and we conclude in the same way. This ends the proof.

**Proof of Theorem 2.** We introduce as before two Banach spaces \(E = BC([0, +\infty), L^p)\) with the natural norm

\[ \| f \| = \sup_t \| f(\cdot, t) \|_{L^p}, \]

and \(F = \{ f \mid t^{3/4 - \frac{1}{2p}} f \in BC([0, +\infty), L^{2p}) \}\) with the norm

\[ \| f \| = \sup_t t^{\frac{3}{4} - \frac{1}{2p}} \| f(\cdot, t) \|_{L^{2p}}. \]

Then, we see that

\[ \| B(f, g)(t) \|_{L^p} \leq \| f \| \| g \| \| \theta \|_{L^q} \int_0^t \frac{1}{(t-s)^{2-\frac{3}{2p}} s^{\frac{1}{2} - \frac{3}{4p}}} ds, \]

\[ \| B(f, g)(t) \|_{L^{2p}} \leq \| f \| \| g \| \| \theta \|_{L^s} \int_0^t \frac{1}{(t-s)^{2-\frac{3}{2p}} s^{1-\frac{3}{2p}}} ds, \]

where \(\frac{1}{p} = \frac{1}{q} + \frac{1}{p} + \frac{1}{2p} - 1\).
which gives the continuity of $B$ from $F \times F \to F$ and $F \times E \to E$, with constants $\gamma(p)$ and $\eta(p)$.

$$\gamma(p) = \| \theta \|_{L^q} t^{\frac{3}{4p} - \frac{3}{2p}} \int_0^t \frac{1}{(t-s)^{\frac{2}{p} - \frac{3}{2q} - \frac{3}{2p}}} \, ds,$$

$$\eta(p) = \| \theta \|_{L^q} \int_0^t \frac{1}{(t-s)^{\frac{2}{p} - \frac{3}{2q} - \frac{3}{2p}}} \, ds,$$  

and a simple rescaling shows both quantities are bounded. Then if we use the same sequence as before, Lemma 1 gives us the convergence in $F$, and we obtain the convergence in $E$ by an contraction argument, as $\eta(p) \leq \gamma(p)$, we obtain $2|u|\eta(p) < 1$. The continuity at $t = 0$ comes from a slight modification of (45), as we can replace $|f|$ by $\sup_{[0,t]} \tau^{\frac{1}{2} - \frac{3}{4p}} \| f(\cdot, \tau) \|_{L^{2p}}$, which tends to zero with $t$. Actually, the value of $t^{\frac{1}{2} - \frac{3}{4p}} \| f(\cdot, t) \|_{L^{2p}}$ could only be zero: the first term $u_1 = S(t)u_0$ tends to zero, for if we consider a sequence of $C_0^\infty$ functions $(v_j)_j$ which approximate $u_0$,

$$t^{\frac{1}{2} - \frac{3}{4p}} \| S(t)u_0 \|_{L^{2p}} = t^{\frac{1}{2} - \frac{3}{4p}} \| S(t) \| \| u_0 - v_j \|_{L^p} + \| S(t) \| t^{\frac{1}{2} - \frac{3}{4p}} \| v_j \|_{L^{2p}}.$$

By Lemma 8 the condition on $u_0$ becomes,

$$\| u_0 \|_{\dot{B}^{-\left(1 - \frac{3}{2p}\right)}_{2p,\infty}} \leq \delta(p),$$

where $\delta(p) \approx 1/\gamma(p)$. This proves Proposition 1. Proposition 2 results from the inclusion of $L^3$ in $\dot{B}^{-\left(1 - \frac{3}{2p}\right)}_{2p,\infty}$. Note that for $p = 2$, we impose the condition

$$t^{\frac{1}{2}} \| S(t)u_0 \|_{L^4} < \delta,$$

which is equivalent to the condition (2). For a general $u_0 \in L^2$, we only know

$$t^{\frac{3}{8}} \| S(t)u_0 \|_{L^4} < +\infty.$$

In other words, we do not know enough about low frequencies, and a sufficient condition is (2), of which $u_0 \in L^3$ or $u_0 \in H^{\frac{1}{2}}$ with small norms are particular cases. We obtained existence and uniqueness in a ball of $F$ with Lemma 1 and uniqueness in the whole space can be obtained directly as in [11] or [4]. As in the Sobolev case, it is possible to obtain estimates on $L^q$ norms of $u(\cdot, t)$, $q > p$, in order to show the $C^\infty$ regularity for $t > 0$.  

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APPENDIX

We recall that if $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a radial function so that $\text{Supp } \hat{\phi} \subset \{ |\xi| < 1 + \varepsilon \}$, and $\hat{\phi}(\xi) = 1$ for $|\xi| < 1$, we define $\phi_j(x) = 2^{nj} \phi(2^j x)$, $S_j$ the convolution operator with $\phi_j$, and $\Delta_j = S_{j+1} - S_j$. Then

$$I = \sum_{-\infty}^{+\infty} \Delta_j$$

and $f(x) \in \dot{H}^s(\mathbb{R}^n)$ if and only if, $\forall j$,

$$\|\Delta_j(f)\|_{L^2} \leq 2^{-js} \|f\|_{\dot{H}^s}$$

where $\sum \varepsilon_j^2 \leq 1$. We will show the two following inequalities, which are homogeneous variants of well-known inequalities:

for $s < \frac{n}{2}$,

$$\|\Lambda^{2s-\frac{n}{2}}(fg)\|_{L^2} \leq C\|\Lambda^s f\|_{L^2}\|\Lambda^s g\|_{L^2},$$

for $s > \frac{n}{2}$,

$$\|\Lambda^s (fg)\|_{L^2} \leq C(s)(\|\Lambda^s f\|_{L^2}\|g\|_{\infty} + \|\Lambda^s g\|_{L^2}\|f\|_{\infty}).$$

Let us start with the first case: we will use a paraproduct decomposition (see [3]): for $f, g \in \mathcal{S}$,

$$f(x)g(x) = \sum_j \Delta_j(f) \sum_l \Delta_l(g)$$

$$= \sum_{|j-l| \leq 1} \Delta_j(f) \Delta_l(g) + \sum_{|j-l| \geq 1} \Delta_j(f) \Delta_l(g).$$

The second sum is, by reordering the terms, a finite sum of terms like $S_2 = \sum_j S_{j-1} \Delta_j(g)$. We will treat only $S_2$, as the other ones are of the same kind. The Fourier transform of $S_2$ is supported in an annulus $[2^{j-1}(1 - 2\varepsilon), 2^{j+1}(1 + 2\varepsilon)]$. Using Bernstein’s lemma,

$$\|\Delta_j(f)\|_{\infty} \leq C 2^{j\frac{n}{2}} \|\Delta_j f\|_{L^2} \leq C 2^{j\frac{n}{2} - s} \|f\|_{\dot{H}^s}.$$ 

Then,

$$\|S_j(f)\|_{\infty} \leq C \sum_{-\infty}^{j} 2^{q(\frac{n}{2} - s)} \varepsilon_q \|f\|_{\dot{H}^s}.$$
If $j < 0$, 

$$
\sum_{-\infty}^{j} 2^{q(\frac{3}{2}-s)}\varepsilon_q = 2^{j(\frac{3}{2}-s)}\varepsilon_j,
$$

and

$$
\varepsilon_j = \sum_{-\infty}^{0} 2^{q(\frac{3}{2}-s)}\varepsilon_{j+q}
$$

is a convolution product between $l^1$ and $l^2$, therefore in $l^2$. For $j \geq 0$, 

$$
\sum_{-\infty}^{j} 2^{q(\frac{3}{2}-s)}\varepsilon_q \leq C(1 + \ldots + 2^{j(\frac{3}{2})}\varepsilon_j).
$$

if

(52)

$$
2^{j(\frac{3}{2}-s)}\varepsilon_j = 1 + \ldots + 2^{j(\frac{3}{2}-s)}\varepsilon_j,
$$

$(\varepsilon_j)$ is in $l^2$ for the same reason as $\varepsilon_j$. This gives

$$
\|S_j(f)\|_{\infty} \leq C2^{j(\frac{3}{2}-s)}\|f\|_{\dot{H}^s}\eta_j.
$$

where $(\eta_j)_j \in l^2$. Then, if $(\mu_j)_j$ is associated to $g$,

$$
\|S_{j-1}(f)\Delta_j(g)\|_{L^2} \leq 2^{j(\frac{3}{2}-s)}\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^s}\eta_j\mu_j,
$$

and as $(\eta_j\mu_j)_j \in l^1 \subset l^2$, $S_1 \in \dot{H}^{2s-\frac{3}{2}}$. The terms of the first sum in (51) are like $S_1 = \sum_j \Delta_j(f)\Delta_j(g)$, and in this case we only know that the support of the Fourier transform of $\Delta_j(f)\Delta_j(g)$ is in $\{\|\xi\| \leq C2^j\}$, and

$$
\|\Delta_j(f)\Delta_j(g)\|_{L^1} \leq \varepsilon_j\mu_j2^{-2js}\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^s}.
$$

**Lemma 9.** - If $u \in L^1$, supp $\hat{u} \subset B(0, R)$, and $\|\hat{u}(\xi)\|_{\infty} \leq R^{-2s}$, then

$$
\|\Lambda^{2s-\frac{n}{2}}u\|_{L^2} \leq \int_{S^2} dS.
$$

This comes from

$$
\int_{|\xi| \leq R} (\|\xi\|^2)^{2s-\frac{n}{2}}|\hat{u}(\xi)|^2 d\xi \leq R^{-4s} \int_{|\xi| \leq R} |\xi|^{4s-n} d\xi
$$

$$
\leq R^{-4s} \int_{S^2} \int_{0}^{R} r^{4s-1} dr dS.
$$

then, applying Lemma 9 to $\Delta_j(f)\Delta_j(g)$,

$$
\|S_{j-1}(f)\Delta_j(g)\|_{H^{2s-\frac{3}{2}}} \leq C\varepsilon_j\mu_j\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^s}.
$$

As $(\eta_j\mu_j)_j \in l^1$, this ends the proof. The second inequality can be proved by the same estimates, except that we have a better estimate for $\|S_j(f)\|_{\infty}$ and $\|\Delta_j(f)\|_{\infty}$, both bounded by $\|f\|_{\infty}$.
REFERENCES


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