FRANK MERLE

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by

Frank MERLE

Université de Cergy-Pontoise, Centre de Mathématiques
Avenue du Parc 8, Le Campus, 95033 Cergy-Pontoise, France

ABSTRACT. In this note, we describe the behavior of a sequence $v_n : \mathbb{R}^N \to \mathbb{C}$ minimal in $L^2$ such that
\[ \frac{1}{2} \int |\nabla v_n|^2 - \frac{1}{N+2} \int |v_n|^\frac{4}{N+2} \leq E_0 \]
and $|v_n|_{H^1} \to +\infty$.

In the present note, we are interested in the behavior of a sequence $v_n : \mathbb{R}^N \to \mathbb{C}$ of $H^1$ functions such that

(1) \[ \int |v_n|^2 = \int Q^2, \]

(2) \[ E(v_n) = \frac{1}{2} \int |\nabla v_n|^2 - \frac{1}{N+2} \int |v_n|^\frac{4}{N+2} \leq E_0, \]

(3) \[ \int |\nabla v_n|^2 \to +\infty, \]

where $Q$ is the radial positive symmetric solution of the equation

(4) \[ \Delta v + |v|^\frac{4}{N} v = 0. \]
This problem is related to the asymptotics of minimal blow-up solutions in \( H^1 \) of the equation

\[
    iu_t = -\Delta u - k(x)|u|^\frac{4}{N}u \quad \text{and} \quad u(0) = \varphi,
\]

where

\[
    \max_{x \in \mathbb{R}^N} k(x) = 1.
\]

Indeed, for all \( \varphi \in H^1 \), there is a unique solution in \( H^1 \) on \([0,T]\) ([2], [4]) and

\[
    T = +\infty \quad \text{or} \quad \lim_{t \to T} \int |\nabla u(t,x)|^2 = +\infty.
\]

In addition, \( \forall \ t \)

\[
    \int |u(t,x)|^2 dx = \int |\varphi(x)|^2 dx
\]

(7)

\[
    E_k(u(t)) = E_k(\varphi)
\]

where

\[
    E_k(v) = \frac{1}{2} \int |\nabla v|^2 - \frac{1}{\frac{4}{N} + 2} \int k(x)|v|^\frac{4}{N} + 2.
\]

From [9]

(9) \( \forall \ v \in H^1 \), \( \frac{1}{\frac{4}{N} + 2} \int |v|^\frac{4}{N} + 2 \leq \frac{1}{2} \left( \frac{\int |v|^2}{\int |Q|^2} \right)^\frac{4}{N} \int |\nabla v|^2 \)

and it follows from (6)-(9) ([6]) that

if \( |\varphi|_{L^2} < |Q|_{L^2} \), then \( T = +\infty \).

Moreover under some conditions on \( k(x) \), for any \( \varepsilon > 0 \) there are blow-up solutions \( u_\varepsilon(t) \) such that

\[
    |u_\varepsilon(0)|^2_{L^2} = |\varphi|^2_{L^2} + \varepsilon \quad ([6]).
\]

Thus the questions are about existence of minimal blow-up solution (that is such that \( u(t) \) blows up in finite time and \( \int |\varphi|^2 = \int Q^2 \) and on the
form of these solutions. In the case where \( k(x) \equiv 1 \), the question has been completely solved (see Merle [5]). The general case is still open. We remark that from (6)-(9), if \( u(t) \) is a blow up solution, the sequences \( v_n = u(t_n) \) as \( t_n \to T \) satisfies (1)-(3) and we ask about the constrains it imply on \( v_n \).

The first result in this direction was obtained by Weinstein in [9]. Using the concentration compactness method, he showed that there is a \( \theta_n \in \mathbb{R} \), \( x_n \in \mathbb{R}^N \) such that

\[
(10) \quad v_n = \lambda_n^{\frac{N}{2}} e^{i\theta_n} Q\left( \frac{\lambda_n^{\frac{N}{2}} (x - x_n)}{\lambda_n} \right) + \varepsilon_n,
\]

where

\[
(11) \quad \lambda_n = \frac{|\nabla v_n|_{L^2}}{|\nabla Q|_{L^2}},
\]

\[
(12) \quad |\varepsilon_n|_{L^2} \xrightarrow{n \to +\infty} 0 \quad \text{and} \quad \frac{|\nabla \varepsilon_n|_{L^2}}{\lambda_n} \xrightarrow{n \to +\infty} 0.
\]

In [5], Merle then showed that for all \( R > 0 \), there is a \( c > 0 \) such that

\[
(13) \quad \int_{|x - x_n| > R} |\nabla v_n|^2 \leq c.
\]

We now claim the following result

**THEOREM.** Let \((v_n)\) be a sequence of \( H^1 \) functions satisfying (1)-(3) and \( \theta_n(x) \) be such that \( v_n = |v_n|e^{i\theta_n} \).

i) **Phase estimates.** There is a \( c > 0 \) such that

\[ \forall \ n, \quad \int |v_n|^2 |\nabla \theta_n|^2 \leq c. \]

ii) **Asymptotics on the modulus.**
There is a \( \varepsilon_n(x), \ x_n \in \mathbb{R}^N \), and \( c > 0 \) such that

\[ \forall \ x, \ |v_n(x)| = \lambda_n^{\frac{N}{2}} Q(\lambda_n(x - x_n)) + \varepsilon_n(x) \]

where

\[ |\nabla \varepsilon_n|_{L^2} \leq c, \quad |\varepsilon_n|_{L^2} \leq \frac{c}{\lambda_n} \quad \text{and} \quad \lambda_n \left( \frac{|\nabla Q|_{L^2}}{|\nabla v_n|_{L^2}} \right) \xrightarrow{n \to +\infty} 1. \]

**Remark.** This Theorem simplifies some proofs in [5], [6]. The case where \( v_n \) is real valued is also related to similar problems for the generalized Korteweg-de Vries equation with critical nonlinearity.
Remark. – The Theorem implies in particular for blow-up solution of equation (5) \( u(t, x) = |u(t, x)| e^{i\theta(t, x)} \) and \( \int |u(t)|^2 = \int Q^2 \) the phase gradient is uniformly bounded at the blow-up: there is a \( c > 0 \) such that

\[
\forall \ 0 < t < T, \quad \int |u(t, x)|^2 |\nabla \theta(t, x)|^2 dx \leq c.
\]

(Of course, we still have \( \int |\nabla u|^2(t, x) dx \to +\infty \) as \( t \to T \).

Remark. – It is easy to check that the result is optimal. We remark that the residual term in the theorem \( \epsilon_n = O(1) \) (compared to \( o(|\nabla v_n|_{L^2}) \) in [9]).

Proof of the Theorem. – Let \( (v_n) \) a sequence of \( H^1 \) function satisfying (1)-(3) and \( \theta_n(x) \) such that \( v_n = |v_n| e^{i\theta_n} \). We have that

\[
\frac{1}{2} \int |\nabla v_n|^2 = \frac{1}{2} \left( |\nabla v_n|^2 + \int |v_n|^2 |\nabla \theta_n|^2 \right)
\]

and

\[
E(v_n) = \frac{1}{2} \int |v_n|^2 |\nabla \theta_n|^2 + E(|v_n|).
\]

The idea is to apply the variational identity (9) not with \( v_n \) but with \( |v_n| \). Indeed, since \( v_n \in H^1 \) we have that \( |v_n| \in H^1 \). From (9) (applied with \( |v_n| \))

\[
\frac{1}{n+2} \int |v_n|^n + 2 \leq \frac{1}{2} \left( \int \frac{|v_n|^2}{Q^2} \right)^\frac{n}{4} \int |\nabla v_n|^2 \leq \frac{1}{2} \int |\nabla v_n|^2,
\]

or equivalently

\[
E(|v_n|) \geq 0.
\]

Thus (2), (15), (17) imply that

\[
\frac{1}{2} \int |v_n|^2 |\nabla \theta_n|^2 \leq E_0
\]

(19)

\[
E(|v_n|) \leq E_0.
\]

Part i). – It is implied by (18).
Part ii). – We claim that it is as a consequence of (18)-(19). We prove it in three steps:

- from Weinstein’s results, we first obtain rough estimates on $|v_n|$, 
- we then choose appropriate approximations parameters, 
- we conclude the proof using a convexity property in certain directions of $E$ near $Q$ (and use in a crucial way that $|v_n|$ is a real-valued function).

**Step 1:** First asymptotics. – Since

\[
\int |\nabla v_n|^2 = \int |\nabla|v_n|_2|^2 + \int \frac{|v_n||\nabla \theta_n|^2}{n \to +\infty} + \infty,
\]

and

\[
\forall n, \int |v_n|^2|\nabla \theta_n|^2 \leq c,
\]

we have

\[
\int |\nabla v_n|^2 \longrightarrow +\infty.
\]

Moreover,

\[
\int |v_n|^2 = \int Q^2 \text{ and } E(|v_n|) \leq E_0.
\]

We conclude from Weinstein’s result on the existence of $\hat{x}_n, \hat{\varepsilon}_n$ such that

\[
|v_n|(x) = \hat{\lambda}_n^{N/2}Q \left( \hat{\lambda}_n x - \hat{x}_n \right) + \hat{\varepsilon}_n(x)
\]

where

\[
\hat{\lambda}_n = \frac{|\nabla v_n|_{L^2}}{|\nabla Q|_{L^2}},
\]

\[
|\nabla \hat{\varepsilon}_n|_{L^2} = o\left( \frac{1}{\hat{\lambda}_n} \right), \quad |\hat{\varepsilon}_n|_{L^2} = o(1).
\]

In order to obtain better estimates on the rest (that is $|\nabla \varepsilon_n|_{L^2} \leq c$), we have to choose appropriate parameters $\lambda_n, x_n$ and use the structure of the functional $E(\cdot)$ near $Q$.

**Step 2:** Choice of the parameters of approximation. – Let us first renormalize the problem. We consider

\[
w_n, \lambda_1, x_1 (x) = \left( \frac{\lambda_1}{\lambda_n} \right)^{N/2} |v_n| \left( (\lambda_1 x + \hat{x}_n + x_1) \frac{1}{\lambda_n} \right)
\]
We have from (23),

\[ w_{n,\lambda_1, x_1}(x) = \lambda_1^{N/2} Q(\lambda_1 x + x_1) + \tilde{\varepsilon}_{n,\lambda_1, x_1}(x) \]

where

\[ \frac{\| \nabla \tilde{\varepsilon}_n \|_{L^2}}{\lambda_1} + \| \tilde{\varepsilon}_n \|_{L^2} \xrightarrow{n \to +\infty} 0. \]

We write (28) as follows

\[ w_{n,\lambda_1, x_1}(x) = Q(x) + \varepsilon_{n,\lambda_1, x_1}(x) \]

where

\[ \varepsilon_{n,\lambda_1, x_1}(x) = \left[ \lambda_1^{N/2} Q(\lambda_1 x + x_1) - Q(x) \right] + \tilde{\varepsilon}_{n,\lambda_1, x_1}(x). \]

From the implicit function Theorem, we derive easily for \(|\varepsilon_n|_{H^1}\) small enough the existence of \(\lambda_{1,n}, x_{1,n}\) such that

\[ \forall i = 1, \ldots, N, \quad \int \varepsilon_{n,\lambda_{1,n}, x_{1,n}, x_i} Q = 0 \]

\[ \int \varepsilon_{n,\lambda_{1,n}, x_{1,n}} |x|^2 Q = 0. \]

Moreover, from (29)

\[ (\lambda_{1,n}, x_{1,n}) \xrightarrow{n \to +\infty} (1, 0). \]

Indeed, let us note

for \(i = 1, \ldots, N\),

\[ \rho_i(\lambda_1, x_1) = \int \varepsilon_{n,\lambda_1, x_1} x_i Q, \]

\[ \rho_{N+1}(\lambda_1, x_1) = \int \varepsilon_{n,\lambda_1, x_1} |x|^2 Q. \]

From (30), we have

\[ \frac{\partial \varepsilon_{n,1,0}}{\partial x_{1,i}} = \partial_i Q + \partial_i \tilde{\varepsilon}_{n,1,0} \]

\[ \frac{\partial \varepsilon_{n,1,0}}{\partial \lambda_1} = \frac{N}{2} Q + x \nabla Q + \left( \frac{N}{2} \tilde{\varepsilon}_{n,1,0} + x \nabla \tilde{\varepsilon}_{n,1,0} \right), \]

where \(x_1 = (x_{1,1}, \ldots, x_{1,N})\).
Therefore, from (29) and integration by parts, for $i = 1, \ldots, N$, and $j = 1, \ldots, N$,

\[
\frac{\partial \rho_i}{\partial x_{1,j}}(1,0) = \int \partial_j Q x_i Q + o(1) = -2 \delta_{i,j} \int Q^2 + o(1),
\]
\[
\frac{\partial \rho_i}{\partial \lambda_1}(1,0) = \int \left( \frac{N}{2} Q + x.\nabla Q \right) x_i Q + o(1) = o(1),
\]
\[
\frac{\partial \rho_{N+1}}{\partial x_{1,j}}(1,0) = \int \partial_j Q |x|^2 Q + o(1) = o(1),
\]
\[
\frac{\partial \rho_{N+1}}{\partial \lambda_1}(1,0) = \int \left( \frac{N}{2} Q + x.\nabla Q \right) |x|^2 Q + o(1)
\]
\[
= \frac{N}{2} \int |x|^2 Q^2 - \frac{N}{2} \int |x|^2 Q^2 - \frac{1}{2} \int x.x Q^2 + o(1)
\]
\[
= -\frac{1}{2} \int |x|^2 Q^2 + o(1).
\]

Therefore, the implicit function theorem implies the existence of $(\lambda_{1,n}, x_{1,n})$ such that (31)-(33) hold.

In conclusion, we have proved the following. There exist $(\lambda_{1,n}, x_{1,n}) \xrightarrow{n \to +\infty} (1, 0)$ such that

\begin{equation}
(34) \quad w_{n,\lambda_{1,n},x_{1,n}}(x) = Q(x) + \varepsilon_{n,\lambda_{1,n},x_{1,n}}(x)
\end{equation}

where

\begin{equation}
(35) \quad \forall \ i = 1, \ldots, n, \quad \int \varepsilon_{n,\lambda_{1,n},x_{1,n}} x_i Q = 0,
\end{equation}

\begin{equation}
(36) \quad \int \varepsilon_{n,\lambda_{1,n},x_{1,n}} |x|^2 Q = 0,
\end{equation}

\begin{equation}
(37) \quad |\varepsilon_{n,\lambda_{1,n},x_{1,n}}|_{H^1} \xrightarrow{n \to +\infty} 0.
\end{equation}

We now note

\[
\begin{align*}
    w_n &= w_{n,\lambda_{1,n},x_{1,n}}, \\
    \varepsilon_n &= \varepsilon_{n,\lambda_{1,n},x_{1,n}}.
\end{align*}
\]

**Step 3**: Conclusion of the proof. – Geometry of energy functions at $Q$. 

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We now use convexity properties of a functional (related to $E$) and the fact $\int w_n^2 = \int Q^2$ to conclude the proof. Let

$$H(v) = \frac{1}{2} \int |\nabla v|^2 - \frac{1}{N+2} \int |v|^{\frac{4}{N+2}} + \frac{1}{2} \int v^2 = E(v) + \frac{1}{2} \int v^2,$$

and

$$H_2(v) = \frac{1}{2} \int |\nabla v|^2 - \frac{4}{N} \int Q^4 v^2 + \frac{1}{2} \int v^2.$$

We know that $Q$ is a critical point of $H$, and $H_2$ is the quadratic part of $H$ near $Q$ (where $v$ is real-valued). Moreover, it is classical that for $|\varepsilon|_{H^1} \leq 1$,

$$H(Q + \varepsilon) - H(Q) = H_2(\varepsilon) + \tilde{H}_2(\varepsilon)$$

where $|\tilde{H}_2(\varepsilon)| = o(|\varepsilon|^2_{H^1}).$

From a result of Weinstein [8] (see also Kwong [4]), we have the following convexity property of $H_2$ at $Q$.

**Proposition 1.** See [8]. – (Directions of convexity of $H$ at $Q$ in the set of real-valued functions.) There is a constant $c_1 > 0$ such that $\forall \varepsilon \in H^1$

If

(i) \[ \forall \, i = 1, \ldots, N, \quad \int \varepsilon \, x_i Q = 0 \]

(ii) \[ \int \varepsilon |x|^2 Q = 0, \]

(iii) \[ \int \varepsilon Q = 0, \]

then

$$H_2(\varepsilon) \geq c_1 \left( \int |\nabla \varepsilon|^2 + \varepsilon^2 \right) = c_1 |\varepsilon|^2_{H^1}.$$  

Remark. – We have here a strict convexity property (up to the invariance of the equation) except in the direction $Q$ which is not true for the quadratic part of $H$ for complex valued functions (see [8]).

Remark. – This proposition is optimal. Other functions can be chosen also.

Using now crucially estimates on the $L^2$ norm, we obtain the following
Proposition 2 (Control of the $Q$ direction by the $L^2$ norm).
Assume

(i) \[ \forall i = 1, \ldots, N, \int \varepsilon x_i Q = 0, \]

(ii) \[ \int \varepsilon |x|^2 Q = 0, \]

(iii) \[ \int (Q + \varepsilon)^2 = \int Q^2, \]

then there are $c_1 > 0$ and $c_2 > 0$ such that

\[ |\nabla \varepsilon|_{L^2} + |\varepsilon|_{L^2} \leq c_2 \text{ implies } H_2(\varepsilon) \geq c_1 (|\nabla \varepsilon|_{L^2}^2 + |\varepsilon|_{L^2}^2). \]

Remark. – We need control on 3 directions to obtain estimates on $|\varepsilon|_{H^1}$ with $H_2(Q + \varepsilon)$. Two directions can be controlled using the invariance of the equation. The last one is controlled by the condition of minimality on the $L^2$ norm (among sequence satisfying (2)).

Proof of Proposition 2. – Let us note

\[ \tilde{H}_2(v_1, v_2) = \frac{1}{2} \int \nabla v_1 \nabla v_2 - \frac{4}{N} \int Q^\frac{N}{2} v_1 v_2 + \frac{1}{2} \int v_1 v_2. \]

We can write

\[ \varepsilon = z + aQ + b|x|^2 Q \]

with

\[ \int zQ = \int z x_i Q = \int z |x|^2 Q = 0 \text{ for } i = 1, \ldots, N. \]

Indeed $a$ and $b$ have to satisfy

\[ \int \varepsilon Q = a \int Q^2 + b \int |x|^2 Q^2, \]

\[ o = a \int |x|^2 Q^2 + b \int |x|^4 Q^2. \]
or equivalently
\[ b = -a \left( \frac{\int |x|^2 Q^2}{\int |x|^4 Q^2} \right) \]
\[ a \left( \frac{\int Q^2 \int |x|^4 Q^2 - \left( \int |x|^2 Q^2 \right)^2}{\int |x|^4 Q^2} \right) = \int \varepsilon Q \]
(which has always a solution since from the Schwarz inequality and the fact \( |x|^2 Q \neq Q \), \( \int |x|^2 Q^2 < \left( \int Q^2 \int |x|^4 Q^4 \right)^{1/2} \)).

On the other hand, we have from \( \int (Q + \varepsilon)^2 = \int Q^2 \)

\[ 2 \int Q\varepsilon = - \int \varepsilon^2 \]
\[ 2 \left( a \int Q^2 + b \int |x|^2 Q^2 \right) = - \int \varepsilon^2 \]
\[ 2a \left( \frac{\int Q^2 \int |x|^4 Q^2 - \left( \int |x|^2 Q^2 \right)^2}{\int |x|^4 Q^2} \right) = - \int z^2 + O(a^2 + b^2) \]
or equivalently,
\[ ac_0 = - \int z^2 + O(a^2) \quad \text{where } c_0 \neq 0 \]
which implies that
\[ a = O \left( \int z^2 \right) \quad \text{and} \quad b = O \left( \int z^2 \right) \]
and for \( |\varepsilon|_{H^1} \) small enough
\[ |\varepsilon|_{H^1}^2 \geq |z|_{H^1}^2 \geq \frac{1}{2} |\varepsilon|_{H^1}^2. \]

On the other hand, by bilinearity and Proposition 1, we have for \( |\varepsilon|_{H^1} \) small enough
\[ H_2(\varepsilon) = H_2(z) + 2a\tilde{H}_2(z, Q) + 2b\tilde{H}_2(z, |x|^2 Q) + 2ab\tilde{H}_2(Q, |x|^2 Q) \]
\[ + a^2 H_2(Q) + b^2 H_2(|x|^2 Q) \]
\[ \geq H_2(z) - c(|z|_{H^1}(|a| + |b|) + a^2 + b^2) \]
\[ \geq H_2(z) - c(|z|_{H^1}^3 + |z|_{H^1}^4) \]
\[ \geq c_1 \left( |z|_{H^1}^2 \right) - c \left( |z|_{H^1} + |z|_{H^1}^4 \right) \]
\[ \geq \frac{c_1}{2} |z|_{H^1}^2 \]
\[ \geq \frac{c_1}{4} |\varepsilon|_{H^1}^2. \]
This concludes the proof of Proposition 2.

As a corollary of Proposition 2 and (38), we have

**Corollary.** - There are $c_1 > 0$ and $c_2 > 0$ such that if

(i) $\forall i = 1, \ldots, N, \int \varepsilon x_i Q = 0$

(ii) $\int |\varepsilon|^2 Q = 0$

(iii) $\int (Q + \varepsilon)^2 = \int Q^2$

(iv) $|\nabla \varepsilon|_{L^2} + |\varepsilon|_{L^2} \leq c_2$

then

$$H(Q + \varepsilon) - H(Q) \geq c_1 \left( \int \nabla \varepsilon^2 + \int \varepsilon^2 \right).$$

We now apply the corollary. If $w_n = Q + \varepsilon_n$, we have

$- |\varepsilon_n|_{H^1} \xrightarrow{n \to +\infty} 0$, and in particular there is $n_0$ such that

$\forall n \geq n_0, |\varepsilon_n|_{H^1} \leq c_2$,

- $\forall i = 1, \ldots, N, \int \varepsilon_n x_i Q = 0$.

In addition,

$$H(Q) = \frac{1}{2} \int |\nabla Q|^2 - \frac{1}{4N + 2} \int |Q|^{\frac{4}{N} + 2} + \frac{1}{2} \int Q^2$$

$$= E(Q) + \frac{1}{2} \int Q^2$$

(since the Pohozaev identity for equation (4) yields $E(Q) = 0$), and

$$H(Q + \varepsilon_n) = H(w_n) = H\left( \left( \frac{\lambda_1}{\lambda_n} \right)^{\frac{N}{2}} |v_n| \left( \frac{x \lambda_1}{\lambda_n} + \hat{x_n} + x_1 \right) \right)$$

$$= E\left( \left( \frac{\lambda_1}{\lambda_n} \right)^{\frac{N}{2}} |v_n| \left( \frac{x \lambda_1}{\lambda_n} \right) \right) + \frac{1}{2} \int |v_n|^2$$

$$= \left( \frac{\lambda_1}{\lambda_n} \right)^2 E(|v_n|) + \frac{1}{2} \int Q^2.$$
Therefore $\forall \ n \geq n_0$

\[(40) \quad \left( \frac{\lambda_1}{\lambda_n} \right)^2 E(|v_n|) > c_3 \left( |\varepsilon_n|_{H^1} \right) \]

or equivalently from (19), (24) and the fact that $\lambda_1 \to 1$,

\[(41) \quad |\varepsilon_n|_{H^1}^2 \leq \frac{c}{2} \frac{1}{\int |\nabla v_n|^2} \leq c \frac{1}{\int |\nabla v_n|^2},\]

where $c$ is independent of $n$. Thus,

\[w_n = Q + \varepsilon_n,\]

with

\[(42) \quad |\varepsilon_n|_{H^1}^2 \leq \frac{c}{\int |\nabla v_n|^2}.

Therefore from (26), there is $x_n$ such that

\[(43) \quad |v_n|(x) = \left( \frac{\lambda_n}{\lambda_1} \right)^{\frac{N}{2}} Q \left( x \left( \frac{\lambda_n}{\lambda_1} + x_n \right) + \left( \frac{\lambda_n}{\lambda_1} \right)^{\frac{N}{2}} \varepsilon_n \left( \frac{\lambda_n}{\lambda_1} x + x_n \right) \right).\]

We remark that from (19), (42), the fact that $\lambda_1 \to 1$,

\[
\frac{\lambda_n}{\lambda_1} \frac{1}{\left( \int |\nabla v_n|^2 \right)^{\frac{1}{2}}} = \frac{1}{\lambda_1} \left( \frac{\int |\nabla v_n|^2}{\int |\nabla Q|^2} \right)^{\frac{1}{2}} \to 1, \quad n \to +\infty
\]

\[
\left| \left( \frac{\lambda_n}{\lambda_1} \right)^{\frac{N}{2}} \varepsilon_n \left( \frac{\lambda_n}{\lambda_1} x + x_n \right) \right|_{L^2}^2 = |\varepsilon_n|_{L^2}^2 \leq c \frac{1}{\int |\nabla v_n|^2},
\]

\[
\left| \nabla \left( \frac{\lambda_n}{\lambda_1} \right)^{\frac{N}{2}} \varepsilon_n \left( \frac{\lambda_n}{\lambda_1} x + x_n \right) \right|_{L^2}^2 \leq c \left( \frac{\lambda_n^2}{\int |\nabla v_n|^2} \right) \leq c
\]

conclude the proof of the Theorem.
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