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## Liouville theorems for semilinear equations on the Heisenberg group

by

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ABSTRACT. – In this paper we consider problems of the type

$$\begin{cases} \Delta_H u + h(x)u^p \leq 0, & \text{in } D \subset \mathbb{R}^{2n+1}, \\ u \geq 0 & \text{in } D, \end{cases} \quad (1)$$

where  $\Delta_H$  is the Heisenberg Laplacian,  $D$  is an unbounded domain and  $h$  is a non negative function.

We prove that, under suitable conditions on  $h$ ,  $p$  and  $D$ , the only solution of (1) is  $u \equiv 0$ .

*Key words:* Liouville property, Heisenberg group.

RÉSUMÉ. – Dans ce travail nous considérons des problèmes du type

$$\begin{cases} \Delta_H u + h(x)u^p \leq 0, & \text{dans } D \subset \mathbb{R}^{2n+1}, \\ u \geq 0 & \text{dans } D, \end{cases} \quad (1)$$

où  $\Delta_H$  est le Laplacien de Heisenberg,  $D$  est un domaine non borné et  $h$  est une fonction positive.

Nous démontrons que sous certaines hypothèses sur  $h$ ,  $p$  et  $D$ , la seule solution de (1) est  $u \equiv 0$ .

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### 1. INTRODUCTION

In this paper we establish some Liouville type theorems for positive functions  $u$  satisfying, for example,

$$\begin{cases} \Delta_H u + h(\xi)u^p \leq 0 & \text{in } D, \\ u \geq 0 & \text{in } D, \end{cases} \tag{1.1}$$

where  $D$  is an unbounded domain of the Heisenberg group  $H^n$ . We recall that  $H^n$  is the Lie group  $(\mathbb{R}^{2n+1}, \circ)$  equipped with the group action

$$\xi_0 \circ \xi = \left( x + x_0, y + y_0, t + t_0 + 2 \sum_{i=1}^n (x_i y_{0_i} - y_i x_{0_i}) \right), \tag{1.2}$$

for  $\xi := (x_1, \dots, x_n, y_1, \dots, y_n, t) := (x, y, t) \in \mathbb{R}^{2n+1}$  and  $\Delta_H$  is the subelliptic Laplacian on  $H^n$  defined by

$$\Delta_H = \sum_{i=1}^n X_i^2 + Y_i^2$$

with

$$\begin{cases} X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \\ Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}. \end{cases}$$

It is easy to check that  $\Delta_H$  is a degenerate elliptic operator satisfying the Hormander condition of order one (see Section 2).

As an example of our results for the case where  $D = H^n$  we prove that, under some conditions on the non negative coefficient  $h$  and suitable restriction on the power  $p$ , any non negative smooth solution  $u$  of (1.1) is identically zero. More precisely, denoting by  $Q = 2n + 2$  the homogeneous dimension of  $H^n$  and by  $|\xi|_H$  the intrinsic distance of the point  $\xi$  to the origin (see [6], [7]), namely

$$|\xi|_H = \left( \sum_{i=1}^n (x_i^2 + y_i^2) + t^2 \right)^{\frac{1}{4}}, \tag{1.3}$$

we have:

**THEOREM 1.1.** – *Let  $u$  be a non negative solution of*

$$\Delta_H u(\xi) + a|\xi|_H^\gamma u^p(\xi) \leq 0 \text{ in } H^n, \tag{1.4}$$

where  $a$  is a positive constant and  $\gamma > -2$ .

Then, if  $1 < p \leq \frac{Q+\gamma}{Q-2}$ ,  $u \equiv 0$ .

A generalized version of this theorem is proved in section 3 below, where also several variants covering the cases when the equation holds in a half space or some “cone” in  $H^n$  are considered (see Theorem 3.2, 3.3, 3.4).

Let us point out that a common feature of our results is that we do not impose any condition on the behaviour of  $u$  for large  $|\xi|_H$ , thus allowing  $u$  to be, *a priori*, singular at infinity.

Therefore our results can be viewed as the analogues, in the present degenerate elliptic setting, of previous ones due to Gidas-Spruck [10] for the uniformly elliptic case. However, our method of proof is rather inspired by [1], where Liouville type results are established for non negative solutions of

$$\Delta u + a|x|^\gamma u^p \leq 0$$

in a cone of  $\mathbb{R}^n$ .

We wish to mention that non existence results for non negative solutions of semilinear equations on the Heisenberg group have been obtained previously by Garofalo-Lanconelli in [8]. Note, however, that the theorems in [8], based on Rellich-Pohozaev identities, differ considerably from those in the present paper since they require global integrability conditions on  $u$  and on the gradient of  $u$ . (see also [5] for similar results in the uniformly elliptic case).

Finally, we point out that the Liouville theorems presented here are the basic tools for obtaining an *a priori* bound in the sup norm for solutions of the Dirichlet problem

$$\begin{cases} \Delta_H u + f(\xi, u) = 0 & \text{in } \Omega \subset \mathbb{R}^{2n+1}, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

under some growth conditions on  $f$ . This can be done using a blow up technique on the lines of [10], [1], [2] and will be the object of a separate paper [3].

## 2. PRELIMINARY FACTS

In this section we collect for the convenience of the reader some known facts about the Heisenberg group  $H^n$  and the operator  $\Delta_H$  which will be useful later on. For their proof and more informations we refer for example to [6], [7], [8], [12], [13].

As mentioned in the introduction the Heisenberg group  $H^n$  is the Lie group whose underlying manifold is  $\mathbb{R}^{2n+1}$  ( $n \geq 1$ ), endowed with the group action,

$$\xi_0 \circ \xi = \left( x + x_0, y + y_0, t + t_0 + 2 \sum_{i=1}^n (x_i y_{0i} - y_i x_{0i}) \right),$$

for  $\xi = (x_1, \dots, x_n, y_1, \dots, y_n, t) := (x, y, t)$ .

The corresponding Lie Algebra of left-invariant vector fields is generated by  $X_i, Y_i$  for  $i = 1, \dots, n$ , and  $T = \frac{\partial}{\partial t}$ .

It is easy to check that  $X_i$  and  $Y_i$  satisfy  $[X_i, Y_j] = -4T\delta_{i,j}$ ,  $[X_i, X_j] = [Y_i, Y_j] = 0$  for any  $i, j \in \{1, \dots, n\}$ . Therefore, the vector fields  $X_i, Y_i$  ( $i = 1, \dots, n$ ) and their first order commutators span the whole Lie Algebra. Hence, the Hormander condition of order one holds true for  $\Delta_H$  (see [13]); this implies its hypoellipticity (i.e. if  $\Delta_H u \in C^\infty$  then  $u \in C^\infty$  (see [13])) and the validity of the maximum principle (see [4]).

An intrinsic metric can be defined on  $H^n$  by setting

$$d_H(\xi, \eta) = |\eta^{-1} \circ \xi|_H$$

where  $|\cdot|_H$  has been defined in (1.3), see [6]. Clearly in this metric the open ball of radius  $R$  centered at  $\xi_o$  is the set:

$$B_H(\xi_o, r) = \{ \eta \in H^n : d_H(\eta, \xi_o) < r \}.$$

It is also important to observe that  $\xi \rightarrow |\xi|_H$  is homogeneous of degree one with respect to the natural group of dilations (see [6], [7]):

$$\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t). \tag{2.1}$$

Since the base  $\{X_i, Y_i, T\}$  is obtained by the standard one  $\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial t} \}$ , using the transformation

$$B = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \\ 0 & 0 & 1 \end{pmatrix}$$

whose determinant is identically 1, it follows that the Lebesgue measure is the Haar measure on  $H^n$ .

This fact, together with the homogeneity property of  $|\xi|_H$  described above, implies that

$$|B_H(\xi_o, R)| = |B_H(0, 1)|R^Q, \tag{2.2}$$

where  $Q = 2n + 2$  is the homogeneous dimension of  $H^n$  (see [12]) and  $|\cdot|$  denotes the Lebesgue measure.

To conclude this section we recall some simple properties of  $\Delta_H$ . Observe first that

$$\Delta_H = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial t^2}.$$

It is easy to check that the operator  $\Delta_H$  is homogeneous of degree 2 with respect to the dilation  $\delta_\lambda$  defined in (2.1), namely

$$\Delta_H(\delta_\lambda) = \lambda^2 \delta_\lambda(\Delta_H);$$

also, for any fixed  $\xi^o$ , by the left invariance of the vector fields  $X_i, Y_i$  with respect to the group action we have:

$$\Delta_H(u(\xi^o \circ \xi)) = (\Delta_H u)(\xi^o \circ \xi) \quad \forall \xi \in H^n.$$

The next remark concerns the action of  $\Delta_H$  on functions  $u$  depending only on  $\rho := |\xi|_H$ . It is easy to show that

$$\Delta_H u(\rho) = \psi \left[ \frac{\partial^2 u}{\partial \rho^2} + \frac{Q-1}{\rho} \frac{\partial u}{\partial \rho} \right], \tag{2.3}$$

where the function  $\psi$  is defined by

$$\psi(\xi) = \frac{\sum_{i=1}^n (x_i^2 + y_i^2)}{\rho^2} = |\nabla_H \rho|^2 \quad \text{for } \xi \neq 0, \tag{2.4}$$

where with  $\nabla_H u$  we denote the vector field  $(X_i u, Y_i u)$ , for  $i = 1, \dots, n$ .

It is useful to observe that

$$\Delta_H = \operatorname{div}(\sigma^T \sigma \nabla) \quad \text{with } \sigma = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \end{pmatrix}.$$

### 3. LIOUVILLE TYPE THEOREMS

In this section we will generalize to the Heisenberg group some Liouville type results which hold for positive solutions of superlinear equations associated to the laplacian, see [1], [2], [10].

**THEOREM 3.1.** – *Let  $u$  be a non negative solution of*

$$\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \quad \text{in } H^n, \tag{3.1}$$

where  $f$  is a non negative function satisfying

$$f(\xi, u) \geq h(\xi)u^p \tag{3.2}$$

for some function  $h(\xi) \geq 0$  such that, for  $|\xi|_H$  large,

$$h(\xi) \geq K\psi|\xi|_H^\gamma$$

for some  $K > 0$  and  $\gamma > -2$ .

If  $1 < p \leq \frac{Q+\gamma}{Q-2}$ , then  $u \equiv 0$ .

Before the proof let us introduce a cut-off function  $\phi_R$  which will be used throughout this section. Consider  $\phi_R(\rho) := \phi(\frac{\rho}{R})$ , where  $\rho := |\xi|_H$ ,  $R > 0$ , and  $\phi$  satisfies:

$$\left\{ \begin{array}{l} \phi \in C^\infty[0, +\infty), \quad 0 \leq \phi \leq 1, \\ \phi \equiv 1 \quad \text{on} \quad \left[0, \frac{1}{2}\right], \\ \phi \equiv 0 \quad \text{on} \quad [1, +\infty), \\ -\frac{C}{R} \leq \frac{\partial \phi_R}{\partial \rho} \leq 0, \\ \text{and} \quad \left| \frac{\partial^2 \phi_R}{\partial \rho^2} \right| \leq \frac{C}{R^2} \quad \text{for some constant } C > 0. \end{array} \right. \tag{3.3}$$

*Proof.* – Set, for  $R > 0$ ,

$$I_R := \int_{H^n} h(\xi)u^p \phi_R^q d\xi \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.4}$$

Observe that  $I_R \geq 0$ . Moreover, by equation (3.1) and (3.2)

$$I_R \leq \int_{B_H(0,R)} f(\xi, u)\phi_R^q d\xi \leq - \int_{B_H(0,R)} \Delta_H u \phi_R^q d\xi; \tag{3.5}$$

hence an integration by parts yields,

$$\begin{aligned} I_R &\leq - \int_{B_H(0,R)} u \Delta_H (\phi_R^q) d\xi + \int_{\partial B_H(0,R)} u \nabla_H (\phi_R^q) \cdot \nu_H dH_{2n} \\ &\quad - \int_{\partial B_H(0,R)} \phi_R^q \nabla_H u \cdot \nu_H dH_{2n} \leq - \int_{B_H(0,R)} u \Delta_H (\phi_R^q) d\xi \\ &\quad + \int_{\partial B_H(0,R)} u q \phi_R^{q-1} \phi'_R \nabla_H \rho \cdot \nu_H dH_{2n} \leq - \int_{B_H(0,R)} u \Delta_H (\phi_R^q) d\xi, \end{aligned}$$

where  $\nu_H(\xi) = \sigma(\xi)\nu(\xi)$  and  $\nu$  is the normal to  $\partial\Omega$ ;  $dH_{2n}$  denotes the  $2n$ -dimensional Hausdorff measure. On the other hand, as observed in Section 2 (see (2.3)),

$$\Delta_H(\phi_R^q) = \psi \left[ \frac{\partial^2}{\partial \rho^2}(\phi_R^q) + \frac{Q-1}{\rho} \frac{\partial}{\partial \rho}(\phi_R^q) \right]. \tag{3.6}$$

Thus we get, using the hypotheses on  $\phi_R$  and denoting by  $\Sigma_R := B_H(0, R) \setminus B_H(0, \frac{R}{2})$ ,

$$\begin{aligned} I_R &\leq - \int_{\Sigma_R} u\psi \left[ q\phi_R^{q-1}\phi_R'' + \frac{Q-1}{\rho}q\phi_R^{q-1}\phi_R' \right] d\xi \\ &\leq \frac{C}{R^2} \int_{\Sigma_R} u\psi\phi_R^{q-1} d\xi. \end{aligned}$$

Hence, the Hölder inequality yields:

$$I_R \leq \frac{C}{R^2} \left[ \int_{\Sigma_R} u^p \rho^\gamma \psi \phi_R^{(q-1)p} d\xi \right]^{\frac{1}{p}} \left[ \int_{B_H(0,R)} \psi \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}}. \tag{3.7}$$

Choosing  $R > 0$  sufficiently large, in  $\Sigma_R$ ,  $h$  satisfies  $h \geq \psi K \rho^\gamma$ . Therefore,

$$I_R \leq C \left[ \int_{\Sigma_R} u^p h \phi_R^q d\xi \right]^{\frac{1}{p}} R^{\left(\frac{-\gamma}{p} + \frac{q}{q} - 2\right)}, \tag{3.8}$$

as  $0 \leq \psi \leq 1$ . Then,

$$I_R^{1-\frac{1}{p}} \leq CR^{\left(\frac{-\gamma}{p} + \frac{q}{q} - 2\right)}.$$

Hence, if  $1 < p < \frac{Q+\gamma}{Q-2}$ , letting  $R \rightarrow +\infty$ , we obtain

$$I := \int_{H^n} hu^p d\xi = 0.$$

This implies  $u \equiv 0$  for  $\rho$  large, since  $h$  is strictly positive outside of a set of measure zero and  $u$  is *a priori* non negative.

The claim follows now by the maximum principle (see [4]). In fact, choose  $\bar{R} > 0$  in such a way that, for  $\rho \geq \bar{R}$ ,  $h > 0$ . Then,  $u \equiv 0$  on the complementary of  $B_H(0, \bar{R})$ , as we proved. Hence,  $u$  satisfies:

$$\begin{cases} u \geq 0 & \text{in } B_H(0, \bar{R} + \delta), \\ \Delta_H u \leq 0 & \text{in } B_H(0, \bar{R} + \delta), \\ u \equiv 0 & \text{for } \bar{R} \leq \rho \leq \bar{R} + \delta, \end{cases}$$



for some  $\delta > 0$ . Therefore, by the maximum principle, since  $u$  is not strictly positive,  $u$  has to be identically zero.

If  $p = \frac{Q+\gamma}{Q-2}$ , we obtain, by (3.7), that  $I$  is finite and that the right hand side of (3.7) tends to zero when  $R$  goes to infinity. This yields  $I = 0$  and we can conclude as above.

*Remark 3.1.* – If  $h = K > 0$ , we get by the previous theorem that, for  $1 < p \leq \frac{Q}{Q-2}$ , the unique solution of

$$\Delta_H u + K u^p \leq 0 \quad \text{in } H^n \tag{3.9}$$

is  $u \equiv 0$ .

*Remark 3.2.* – The upper bound of the exponent  $p$  is optimal. Indeed, we claim that the function  $v(\rho) = C_\varepsilon(1 + \rho^2)^{-\frac{\alpha}{2}}$  with  $\alpha = Q - 2 - \varepsilon$  and a suitable choice of  $C_\varepsilon$  is a positive solution of

$$\Delta_H u(\xi) + \psi(\xi)\rho^\gamma u^p(\xi) \leq 0 \quad \text{in } H^n, \tag{3.10}$$

for  $p \geq \frac{Q+\gamma-\varepsilon}{Q-2-\varepsilon}$ .

Indeed, let  $u(\rho) = (1 + \rho^2)^{-\frac{\alpha}{2}}$ . Then  $u$  satisfies:

$$\begin{aligned} -\Delta_H u &= -\psi \left[ \frac{\partial^2 u}{\partial \rho^2} + \frac{Q-1}{\rho} \frac{\partial u}{\partial \rho} \right] \\ &= \psi \alpha(1 + \rho^2)^{-\left(\frac{\alpha}{2}+2\right)} [Q(1 + \rho^2) - (\alpha + 2)\rho^2] \\ &= \psi \alpha(1 + \rho^2)^{-\left(\frac{\alpha}{2}+2\right)} [\rho^2(Q - \alpha - 2) + Q] \\ &\geq \psi \alpha(Q - \alpha - 2)(1 + \rho^2)^{-\left(\frac{\alpha}{2}+1\right)}. \end{aligned} \tag{3.11}$$

Hence, if we impose that

$$Q - 2 > \alpha, \quad p \frac{\alpha}{2} - \frac{\gamma}{2} \geq \left(\frac{\alpha}{2} + 1\right), \tag{3.12}$$

we can choose  $c = (\alpha(Q - \alpha - 2))^{\frac{1}{p-1}}$  and  $v = cu$  satisfies:

$$-\Delta_H v \geq \psi (\alpha(Q - \alpha - 2))^{\frac{p}{p-1}} (1 + \rho^2)^{-p\frac{\alpha}{2} + \frac{\gamma}{2}} \geq \psi \rho^\gamma v^p.$$

Now just choose  $\alpha = Q - 2 - \varepsilon$  then (3.12) holds if  $p \geq \frac{Q+\gamma-\varepsilon}{Q-2-\varepsilon}$  for any  $\varepsilon$  positive.

The idea of the function  $v$  was taken from Ramon Soranzo (personal communication to I.B.) who gave a similar counterexample for the Laplacian.

The next result concern the case where the unbounded domain  $D$  is an half-space.

THEOREM 3.2. – Let  $D \subset H^n$  be the set

$$D = \left\{ \xi \in H^n : \sum_{i=1}^n a_i x_i + b_i y_i + d > 0, \right. \\ \left. \text{with } (a, b) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}, d \in \mathbb{R} \right\}.$$

Let  $u$  be a non negative solution of

$$\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \text{ in } D, \tag{3.13}$$

where  $f$  is as in Theorem 3.1 with  $\gamma > -1$ .

If  $1 < p \leq \frac{Q+\gamma}{Q-1}$ , then  $u \equiv 0$  in  $D$ .

A similar result is valid for half-spaces which do not contain the  $t$ -direction or for particular cones. However, the upper bound of the exponent  $p$  is lower than in the previous case.

The following results hold:

THEOREM 3.3. – Let  $D \subset H^n$  be the set

$$D = \left\{ \xi \in H^n : \sum_{i=1}^n a_i x_i + b_i y_i + ct + d > 0 \right\}, \\ \text{for } a, b \in \mathbb{R}^n, c \in \mathbb{R} \setminus \{0\}, d \in \mathbb{R},$$

and let  $u$  be a non negative solution of

$$\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \text{ in } D, \tag{3.14}$$

with  $f$  as in theorem 3.1 and  $\gamma > 0$ .

Then, if  $1 < p \leq \frac{Q+\gamma}{Q}$ ,  $u \equiv 0$  in  $D$ .

THEOREM 3.4. – Let  $\Sigma$  be the cone

$$\Sigma = \left\{ \xi \in H^n : \sum_{i=1}^n (a_i x_i - b_i y_i)(b_i x_i + a_i y_i) > 0 \right\},$$

and let  $u$  be a non negative solution of

$$\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \text{ in } \Sigma, \tag{3.15}$$

with  $f$  as in theorem 3.1 and  $\gamma > 0$ .

If  $1 < p \leq \frac{Q+\gamma}{Q}$ ,  $u \equiv 0$  in  $\Sigma$ .

The proofs of theorems 3.2, 3.3, 3.4 follow from the next lemma.

LEMMA 3.1. – Let  $D \subset H^n$  be an unbounded domain. Assume that  $\eta$  satisfies:

$$\begin{cases} \eta > 0 & \text{in } D, \\ \Delta_H \eta \geq 0 & \text{in } D, \\ \eta = 0 & \text{on } \partial D, \end{cases}$$

and let  $u$  be a non negative solution of

$$\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \quad \text{in } D, \quad (3.16)$$

with  $f$  as in Theorem 3.1. Then, for

$$I_R := \int_{D_R} h(\xi) u^p \phi_R^q \eta^q d\xi,$$

the following estimate holds

$$I_R \leq I_R^{\frac{1}{p}} \left( \frac{C}{R^2} \left[ \int_{\Omega_R} \eta^q \psi \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}} + \frac{C}{R} \left[ \int_{\Omega_R} \psi |\nabla_H \eta \cdot \nabla_H \rho|^q \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}} \right) \quad (3.17)$$

for  $R > 0$  large enough, where  $D_R := B_H(0, R) \cap D$ ,  $\Omega_R := (B_H(0, R) \setminus B_H(0, \frac{R}{2})) \cap D$ , and  $q$  is the conjugate exponent of  $p$ .

*Proof.* – From equation (3.16), assumption (3.2) and the divergence's theorem we get:

$$\begin{aligned} I_R &\leq - \int_{D_R} u \Delta_H (\eta^q \phi_R^q) d\xi + \int_{\partial D_R} u \nabla_H (\eta^q \phi_R^q) \cdot \nu_H dH_{2n} \\ &\quad - \int_{\partial D_R} \eta^q \phi_R^q \nabla_H u \cdot \nu_H dH_{2n}. \end{aligned}$$

Moreover, since  $\phi_R = 0$  on  $\partial B_H(0, R)$ ,  $\eta = 0$  on  $\partial D$ , and  $q > 1$ , the integrals on the boundary of  $D_R$  vanish and therefore,

$$I_R \leq - \int_{D_R} u \Delta_H ((\eta \phi_R)^q) d\xi.$$

Thus, using the properties of  $\phi_R$  and observing that, by the hypotheses made on  $\eta$ ,

$$\Delta_H (\eta^q) = q(q-1)\eta^{q-2} |\nabla_H \eta|^2 + q\eta^{q-1} \Delta_H \eta > 0 \quad (3.18)$$

it results:

$$I_R \leq - \int_{\Omega_R} u \eta^q \Delta_H(\phi_R^q) d\xi - 2 \int_{\Omega_R} u \nabla_H(\eta^q) \cdot \nabla_H(\phi_R^q) d\xi.$$

Using the properties of  $\phi_R$ , as in the proof of Theorem 3.1 we obtain

$$I_R \leq \frac{C}{R^2} \int_{\Omega_R} u \eta^q \psi \phi_R^{q-1} d\xi + \frac{C}{R} \int_{\Omega_R} u \eta^{q-1} \psi \phi_R^{q-1} \nabla_H \eta \cdot \nabla_H \rho d\xi. \quad (3.19)$$

Thus, the Hölder inequality yields:

$$\begin{aligned} I_R &\leq \frac{C}{R^2} \left[ \int_{\Omega_R} \psi \rho^\gamma u^p (\eta \phi_R)^{(q-1)p} d\xi \right]^{\frac{1}{p}} \left[ \int_{\Omega_R} \eta^q \psi \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}} \\ &\quad + \frac{C}{R} \left[ \int_{\Omega_R} \psi \rho^\gamma u^p (\eta \phi_R)^{(q-1)p} d\xi \right]^{\frac{1}{p}} \left[ \int_{\Omega_R} |\nabla_H \eta \cdot \nabla_H \rho|^q \psi \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}} \\ &\leq I_R^{\frac{1}{p}} \left( \frac{C}{R^2} \left[ \int_{\Omega_R} \eta^q \psi \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \frac{C}{R} \left[ \int_{\Omega_R} \psi |\nabla_H \eta \cdot \nabla_H \rho|^q \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}} \right), \end{aligned} \quad (3.20)$$

for  $R > 0$  large enough. The statement is proved.

*Proof of Theorem 3.2.* – Consider, without loss of generality, the half space  $\{x_1 > 0\}$ .

The claim is proved by using the estimate (3.17) applied to  $D = \{x_1 > 0\}$  and  $\eta = x_1$ .

Indeed, by the maximum principle, to show that  $u \equiv 0$ , it is enough to check that

$$I_R := \int_{\{x_1 > 0\}} h u^p \phi_R^q x_1^q d\xi \rightarrow 0 \quad \text{when } R \rightarrow \infty, \quad (3.21)$$

where  $\phi_R$  is as in (3.3).

If  $D_R := B_H(0, R) \cap \{x_1 > 0\}$ , then (3.17) becomes:

$$I_R \leq I_R^{\frac{1}{p}} \left( \frac{C}{R^2} \left[ \int_{\Omega_R} x_1^q \psi \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}} + \frac{C}{R} \left[ \int_{\Omega_R} \psi |\nabla_H \rho|^q \rho^{\frac{-\gamma q}{p}} d\xi \right]^{\frac{1}{q}} \right).$$

Therefore, as  $0 \leq \psi \leq 1$  and  $x_1 \leq CR$  in  $\Omega_R$ , for  $p \leq \frac{Q+\gamma}{Q-1}$  we get:

$$I_R \leq C I_R^{\frac{1}{p}} R^{(-\frac{\gamma}{p} + \frac{Q}{q} - 1)}, \quad (3.22)$$

and we can conclude using the same arguments as in Theorem 3.1.

*Proof of Theorem 3.3.* – As in the proof of Theorem 3.2, the claim is proved using the estimate (3.17) of Lemma 3.1 with  $\eta = A \cdot x + B \cdot y + ct + d$  and  $D_R := B_H(0, R) \cap D$ .

Let us consider the integral

$$I_R := \int_D hu^p \phi_R^q \eta^q d\xi, \tag{3.23}$$

where  $\phi_R$  is as in (3.3). By (3.17), using the fact that

$$\begin{aligned} \eta &\leq CR^2 \\ |\nabla_H \eta| &= |(A + 2cy, B - 2cx)| \leq CR \end{aligned} \tag{3.24}$$

we obtain:

$$\begin{aligned} I_R &\leq I_R^{\frac{1}{p}} \left( \frac{C}{R^2} \left[ \int_{\Omega_R} \eta^q \psi \rho^{-\frac{\gamma q}{p}} d\xi \right]^{\frac{1}{q}} + \frac{C}{R} \left[ \int_{\Omega_R} \psi |\nabla_H \eta \cdot \nabla_H \rho|^q \rho^{-\frac{\gamma q}{p}} d\xi \right]^{\frac{1}{q}} \right) \\ &\leq CI_R^{\frac{1}{p}} R^{(-\frac{\gamma}{p} + \frac{Q}{q})}. \end{aligned} \tag{3.25}$$

If  $1 < p \leq \frac{Q+\gamma}{Q}$  we can conclude as in the previous cases.

*Proof of Theorem 3.4.* – This result follows from the estimate (3.17) by choosing  $\eta := \sum_{i=1}^n (a_i x_i - b_i y_i)(b_i x_i + a_i y_i)$  and  $D := \Sigma$ . Since the function  $\eta$  has the same behaviour as the function  $\eta$  chosen in the proof of Theorem 3.3, we can conclude in the same way.

*Remark 3.3.* – Let us observe that, instead of inequality (3.17), one can similarly obtain

$$I_R \leq I_R^{\frac{1}{p}} \left( \frac{1}{R^2} \left[ \int_{\Omega_R} \eta^q \psi h^{-\frac{q}{p}} d\xi \right]^{\frac{1}{q}} + \frac{1}{R} \left[ \int_{\Omega_R} \psi h^{-\frac{q}{p}} |\nabla_H \eta \cdot \nabla_H \rho|^q d\xi \right]^{\frac{1}{q}} \right), \tag{3.26}$$

provided  $f$  satisfies (3.2) for some  $h \geq 0$  such that the right hand side of (3.26) exists.

Consequently, if  $h$  verifies:

$$\lim_{R \rightarrow +\infty} \frac{1}{R^q} \int_0^R h^{-\frac{q}{p}}(\rho\omega) \rho^{Q-1} d\rho = 0$$

where  $\omega = \frac{\xi}{|\xi|_H}$ , then the conclusion of Theorem 3.2 holds true. Similar conditions on  $h$  and  $p$  can be given for Theorems 3.3 and 3.4.

For the sake of completeness, we will also prove a Liouville theorem for bounded solutions of  $\Delta_H u = 0$  in the whole space  $H^n$ .

**THEOREM 3.5.** – *If  $u$  is a bounded function such that  $\Delta_H u = 0$  in the whole space  $H^n$ , then  $u$  is a constant.*

The proof is based on the following representation formula for Heisenberg harmonic functions. This formula can be proved easily by using the divergence’s theorem, see e.g. Gaveau ([9]) for details.

**LEMMA 3.2.** – *Let  $w$  satisfy  $\Delta_H w = 0$  in  $H^n$ . Then, for any  $\xi \in H^n$ ,*

$$w(\xi) = \frac{C_Q}{R^Q} \int_{B_H(\xi, R)} w(\eta)\psi(\eta)d\eta, \tag{3.27}$$

where  $\psi$  is defined in (2.4), and  $C_Q = |B_H(\xi, 1)|^{-1}$ .

*Proof of Theorem 3.5.* – Let us first prove that  $\frac{\partial w}{\partial t} \equiv 0$ . Observe that, in view of the Hormander condition, the vector field  $T = \frac{\partial}{\partial t}$  commutes with  $X_i$  and  $Y_i$ , i.e.  $T(X_i) = X_i(T)$  and  $T(Y_i) = Y_i(T)$ . Hence,

$$\Delta_H(Tw) = T(\Delta_H w) = 0.$$

Therefore, applying the previous lemma, we get:

$$\begin{aligned} \frac{\partial w}{\partial t}(\xi) &= \frac{C_Q}{R^Q} \int_{B_H(\xi, R)} \frac{\partial w}{\partial t}(\eta)\psi(\eta)d\eta \\ &= -\frac{C_Q}{R^Q} \int_{B_H(\xi, R)} \frac{\partial \psi}{\partial t}(\eta)w(\eta)d\eta + \frac{C_Q}{R^Q} \int_{\partial B_H(\xi, R)} w\psi\nu_t dH_{2n}, \end{aligned}$$

where  $\nu_t$  is the  $t$ -component of the exterior unit normal vector to  $B_H(\xi, R)$ . Since

$$\begin{aligned} \left| \frac{\partial \psi}{\partial t} \right| &= \frac{|\psi| |t|}{\rho^4} \leq \frac{1}{\rho^2} \\ |\nu_t| &= \frac{|t|}{2\rho^3} \leq \frac{1}{\rho|\nabla\rho|}, \end{aligned}$$

from (2.2) we obtain that

$$\left| \frac{\partial w}{\partial t}(\xi) \right| \leq \frac{C\|w\|_{L^\infty}}{R^2}$$

for any  $\xi \in H^n$  and for any  $R > 0$ . Thus, letting  $R$  go to infinity, we get  $\frac{\partial w}{\partial t}(\xi) = 0$  for any  $\xi \in H^n$ . Then,  $w$  is a bounded solution of

$$\sum_{i=1}^n \frac{\partial^2 w}{\partial x_i^2} + \frac{\partial^2 w}{\partial y_i^2} + \frac{\partial^2 w}{\partial t^2} = 0 \quad \text{in } \mathbb{R}^{2n+1}.$$

Therefore it has to be constant by the classical Liouville theorem (see e.g. [11]).

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