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The p-harmonic system with measure-valued right hand side

by

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ABSTRACT. – For $2 - \frac{1}{n} < p < n$ we prove existence of a distributional solution u of the p-harmonic system

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= \mu \quad &\text{in} \quad &\Omega, \\ u &= 0 \quad &\text{on} \quad &\partial\Omega, \end{aligned}$$

where Ω is an open subset of $\mathbb{R}^n$ (bounded or unbounded), $u : \Omega \to \mathbb{R}^m$, and μ is an $\mathbb{R}^m$-valued Radon measure of finite mass. For the solution u we establish the Lorentz space estimate

$$\|Du\|_{L^{q,\infty}} + \|u\|_{L^{q^*,\infty}} \leq C\|\mu\|_{\mathcal{M}}^{\frac{1}{p-1}}$$

with $q = \frac{n}{n-1}(p-1)$ and $q^* = \frac{n}{n-p}(p-1)$. The main step in the proof is to show that for suitable approximations the gradients Du_k converge a.e. This is achieved by a choice of regularized test functions and a localization argument to compensate for the fact that in general $u \notin W^{1,p}$.

Key words: Degenerate elliptic systems, compactness.

RÉSUMÉ. – Soit $2 - \frac{1}{n} < p < n$, Ω un ensemble ouvert de $\mathbb{R}^n$ (borné ou non borné) et μ une mesure de Radon avec masse finite. On démontre

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que le système p-harmonique

$$\begin{aligned} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) &= \mu \qquad \text{dans} \quad \Omega, \\ u &= 0 \qquad \text{sur} \quad \partial\Omega, \end{aligned}$$

possède une solution distributionnelle. Pour cette solution on établit des estimations dans les espaces de Lorentz

$$\|Du\|_{L^{q,\infty}} + \|u\|_{L^{q^*,\infty}} \le C\|\mu\|_{\mathcal{M}}^{\frac{1}{p-1}}.$$

Ici, $q = \frac{n}{n-1}(p-1)$ et $q^* = \frac{n}{n-p}(p-1)$. Le pas le plus important est de démontrer que, pour des approximations propres, les gradients Du_k convergent presque partout. Celà est fait en choisissant des fonctions de test régularisées et en appliquant un argument de localisation pour compenser que, en général, $u \notin W^{1,p}$.

1. INTRODUCTION

For $2 - \frac{1}{n} < p \le n$ Fuchs and Reuling [12] recently proved existence of a distributional solution $u \in \cap_{s<\frac{n}{n-1}(p-1)} W^{1,s}$ of the p-harmonic system

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mu \qquad \text{in} \quad \Omega, \tag{1}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega \tag{2}$$

for an open (bounded) subset Ω of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}^m$, and an $\mathbb{R}^m$-valued Radon measure μ of finite mass which is supported in a compact Lebesgue zero-set. They announce an extension of this result to general Radon measures by an involved argument based on a blow-up lemma of [11]. Here we present an approach to existence for general Radon measures based on ideas of [15] and [5] and establish optimal regularity and decay estimates (*see* the remarks below).

THEOREM 1. – *Suppose that $2 - \frac{1}{n} < p < n$ and let Ω be an open, bounded or unbounded subset of $\mathbb{R}^n$ and let μ be an $\mathbb{R}^m$-valued Radon measure on Ω of finite mass. Then the system* (1) *with boundary values* (2) *has a solution in the sense of distributions such that u and Du lie in the weak spaces $L^{q^*,\infty}$ and $L^{q,\infty}$, respectively, and*

$$\|Du\|_{L^{q,\infty}} + \|u\|_{L^{q^*,\infty}} \le C|\mu|^{\frac{1}{p-1}} \tag{3}$$

with $q = \frac{n}{n-1}(p-1)$ and $q^* = \frac{nq}{n-q} = \frac{n}{n-p}(p-1)$.

For convenience, we recall the definition of the Marcinkiewicz space $L^{q,\infty}(\Omega)$: It consists of all measurable functions f on Ω having the property that $\sup_{t>0} t^{1/q} f^*(t) =: \|f\|^*_{L^{q,\infty}} < \infty$ where $f^*(t) := \inf\{y > 0 : \lambda_f(y) \le t\}$ is the nonincreasing rearrangement of f and $\lambda_f(y) := \mathcal{L}^n\{|f| > y\}$ (the measure of the superlevel set to level y) is the distribution function of f. Here, $\|f\|^*_{L^{q,\infty}}$ is a pseudo norm and induces a topology on $L^{q,\infty}(\Omega)$. This topology is metrizable, in fact $\|f\|_{L^{q,\infty}} := \sup_{t>0} t^{1/q} f^{**}(t)$ is a norm for $f^{**}(t) = \frac{1}{t}\int_0^t f^*(y)\,dy$ with $\|f\|^*_{L^{q,\infty}} \le \|f\|_{L^{q,\infty}} \le \frac{q}{q-1}\|f\|^*_{L^{q,\infty}}$ provided $q > 1$.

Remarks. – (1) If Ω is bounded then (3) implies in particular that

$$u \in \bigcap_{s<q} W^{1,s}(\Omega)\,.$$

If Ω is unbounded and $\partial\Omega$ is not Lipschitz then (2) holds in the sense that there exists a sequence $u_k \in C_0^\infty(\Omega)$ such that

$$u_k \to u \qquad \text{in } \; W^{1,s}(\Omega \cap B(0,R)) \;\; \text{for all } R < \infty.$$

The exponents in (3) are optimal as can be seen from the nonlinear Green's function $G(x) = c|x|^{-n/q^*}$ (*see* [9] and [10]) and $L^{q,\infty}$ cannot be replaced by L^q.

(2) Decay estimates: By standard regularity results for solutions of the homogeneous p-harmonic system in the unit ball and scaling one easily deduces

$$|u(x)| \le C(\mathrm{dist}(\mathrm{x}, \partial\Omega \cup \mathrm{spt}\mu))^{-n/q^*}.$$

For this scaling argument it is crucial to bound u in $L^{q^*,\infty}$ and not just in L^s for all $s < q^*$.

To finish this introduction let us briefly comment on some related work. Estimates for ∇u in L^q and BMO are discussed by DiBenedetto and Manfredi in [6] if μ lies in the dual space of $W^{1,p}$. The case of a single equation with a Radon measure on the right hand side is known for some time: *see* Boccardo and Gallouët [2], [3], Rakotoson [18] (and references therein), Boccardo and Murat [4], Bénilan *et al.* [1]. In particular for equations the range $1 < p \le 2 - \frac{1}{n}$ is also treated in the mentioned articles by considering renormalized or entropy solutions. It is not clear how to extend these results to systems since truncation behaves quite differently for scalar and vector functions.

2. APPROXIMATION OF THE SOLUTION

We approximate a solution of (1)-(2) by the solution u_k of

$$-\operatorname{div}\sigma(\nabla u_k) = f_k \qquad \text{in } \Omega, \tag{4}$$

$$u_k = 0 \qquad \text{on } \partial\Omega, \tag{5}$$

where we here and subsequently write

$$\sigma(F) := |F|^{p-2}F.$$

For f_k we choose the standard mollification

$$f_k(x) := \int_{\mathbb{R}^n} \phi_k(x-y)\, d\mu(y)$$

where, for $k \in \mathbb{N}$, $\phi_k(x) = k^n\phi_0(kx) \geq 0$ with a function $\phi_0 \in C_0^\infty(B(0,1))$, $\|\phi_0\|_{L^1} = 1$. Thus we have $f_k \in C^\infty \cap L^1 \cap L^\infty$ for each k and

$$f_k \overset{*}{\rightharpoonup} \mu \qquad \text{in } \mathcal{M}.$$

Since σ is monotone the solution of (4), (5) corresponds to a minimizer of

$$E(u) = \int_\Omega \left(\frac{1}{p}|Du|^p - f_k u\right) dx\,.$$

Hence for fixed k we find a solution $u_k \in X_0^{1,p}(\Omega, \mathbb{R}^m)$ of (4)-(5) by the direct method of the calculus of variation, where $X_0^{1,p}(\Omega, \mathbb{R}^m)$ denotes the closure of $C_0^\infty(\Omega, \mathbb{R}^m)$ in the space $X^{1,p}(\Omega, \mathbb{R}^m) = \{u \in L^{p^*}(\Omega, \mathbb{R}^m) : |Du| \in L^p(\Omega)\}$ equipped with norm $\|u\| := \|u\|_{p^*} + \|Du\|_p$. Note that for bonded sets Ω we have $X_0^{1,p}(\Omega) = W_0^{1,p}(\Omega)$.

3. *A PRIORI* ESTIMATES

LEMMA 1. – *Suppose $\Omega \subset \mathbb{R}^n$ is an open set, bounded or unbounded, and $2 - \frac{1}{n} < p < n$. Suppose for*

$$u \in X_0^{1,p}(\Omega, \mathbb{R}^m),$$
$$f \in L^1(\Omega, \mathbb{R}^m)$$

there holds

$$-\mathrm{div}\sigma(Du) = f \tag{6}$$

in distributional sense. Then

$$u \in L^{q^*,\infty}(\Omega, \mathbb{R}^m),$$
$$Du \in L^{q,\infty}(\Omega, \mathbb{R}^{m\times n}),$$

where

$$q = \frac{n}{n-1}(p-1), \quad q^* = \frac{nq}{n-q} = \frac{n}{n-p}(p-1),$$

and

$$\|Du\|_{L^{q,\infty}} + \|u\|_{L^{q^*,\infty}} \le C\|f\|_{L^1}^{\frac{1}{p-1}}. \tag{7}$$

Proof. – By scaling it suffices to consider $\|f\|_{L^1} = 1$. Now, we use similar arguments as Grüter and Widman in [13] for a linear scalar equation with L^∞ coefficients. Let $T_\alpha(y) = \min(1, \frac{\alpha}{|y|})y$ be a truncation function and note that $|T_\alpha(y)| \le \alpha$ and

$$DT_\alpha(y) = \begin{cases} \mathrm{Id} & \text{for } |y| < \alpha, \\ \dfrac{\alpha}{|y|}\left(\mathrm{Id} - \dfrac{y}{|y|} \otimes \dfrac{y}{|y|}\right) & \text{for } |y| > \alpha. \end{cases}$$

In particular we have

$$F : DT_\alpha(y)F \ge 0 \qquad \text{for all } y \in \mathbb{R}^m,$$

where $A : B = \mathrm{tr}\,(A^T B)$ denotes the inner product on $m \times n$ matrices. If we test (6) with $T_\alpha(u)$ we thus obtain

$$\int_{|u|<\alpha} |Du|^p dx \le \alpha. \tag{8}$$

Since $|D|u|| \le |Du|$ it follows from the Proposition below that

$$u \in L^{q^*,\infty}(\Omega, \mathbb{R}^m)$$

and

$$\|u\|_{L^{q^*,\infty}} \le C_1. \tag{9}$$

From (8) and (9) one deduces that the distribution function $\lambda_{|Du|}$ satisfies for all $\alpha > 0$

$$\begin{aligned}\lambda_{|Du|}(\beta) := \mathcal{L}^n\{|Du| > \beta\} &\le \frac{\alpha}{\beta^p} + \lambda_{|u|}(\alpha)\\ &\le \frac{\alpha}{\beta^p} + C_1^{q^*}\alpha^{-q^*}.\end{aligned}$$

The choice $\alpha = \beta^{\frac{p}{q^*+1}} = \beta^{\frac{n-p}{n-1}}$ yields

$$\lambda_{|Du|}(\beta) \le C_2\beta^{-q}$$

and the Lemma is proved. □

The following estimate has been used by Talenti [19] in connection with quasilinear elliptic equations and later also by Bénilan *et al.* [1, Lemma 4.1] for solutions of p-Laplace equations.

PROPOSITION 1. – *Let* $f \in X_0^{1,p}(\Omega)$, $1 \le p < n$, *and suppose that* $f \ge 0$ *and that for all* $\alpha > 0$

$$\int_{f<\alpha} |Df|^p dx \le \alpha.$$

Then

$$\|f\|_{L^{q^*,\infty}} \le C_1,$$

where $q^* = \frac{n}{n-p}(p-1)$.

Proof. – Let $f_\alpha(x) = \min(f(x), \alpha)$. Then

$$\int_\Omega |Df_\alpha|^p dx = \int_{f<\alpha} |Df|^p dx \le \alpha\,.$$

Since $f_\alpha \in X_0^{1,p}(\Omega)$ the Sobolev embedding theorem yields

$$\int_\Omega |f_\alpha|^{p^*}\, dx \le C\alpha^{p^*/p}.$$

Therefore we conclude

$$\begin{aligned}\lambda_f(\alpha) &\le \alpha^{-p^*}\int_\Omega |f_\alpha|^{p^*} dx\\ &\le C\,\alpha^{\frac{p^*}{p}-p^*}\\ &= C\,\alpha^{-q^*}\end{aligned}$$

which proves the Proposition. □

4. L^1-CONVERGENCE OF Du_k

From Section 2 we infer that

$$u_k \in X_0^{1,p}(\Omega, \mathbb{R}^m),$$

$$\|Du_k\|_{L^s} \leq C \quad \text{for some } s > p-1,$$

(if Ω is unbounded then the latter assertion holds in $\Omega \cap B(0,R)$ for all $R < \infty$)

$$\|f_k\|_{L^1} \leq C,$$

and

$$-\mathrm{div}\sigma(Du_k) = f_k,$$

where $\sigma(F) := |F|^{p-2}F$. We choose a subsequence such that

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{in } W^{1,s}_{\mathrm{loc}} \text{ and a.e.,} \\ \sigma(Du_k) &\rightharpoonup \overline{\sigma} \quad \text{in } L^{\frac{s}{p-1}}_{\mathrm{loc}}, \\ |Du_k - Du| &\rightharpoonup h \quad \text{in } L^s_{\mathrm{loc}}, \\ f_k &\overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}. \end{aligned}$$

Then we claim that $h = 0$ a.e. and hence that $Du_k \to Du$ in L^1_{loc}.

Proof. – We combine ideas of Evans in [7], of Chen, Hong and Hungerbühler in [5] and of Müller in [15]. One would like to use a truncation of $u_k - u$ as a testfunction. This runs into problems since in general $Du \notin L^p$. Therefore one fixes a function v (which later will be chosen as a linear Taylor polynomial of u) and uses a truncation of $u_k - v$. Let

$$\begin{aligned} &\eta \in C_0^\infty(\mathbb{R}^m), \\ &\phi \in C_0^\infty(\Omega), \\ &\psi(z) = \alpha(|z|)\frac{z}{|z|}, \end{aligned}$$

such that $1 \geq \eta \geq 0$, $0 \leq \phi \leq 1$, $\alpha \in C^1(0,\infty)$, α and α' bounded and non-negative, and $\psi|_{\mathrm{spt}\eta} = \mathrm{Id}$. Then

$$\begin{aligned} (10) \quad &\int_\Omega |Du_k - Du|\,\eta(u_k - v)\,\phi\,dx \\ &\leq \int_\Omega |Du_k - Dv|\,\eta(u_k - v)\,\phi\,dx + \int_\Omega |Dv - Du|\,\eta(u_k - v)\,\phi\,dx \end{aligned}$$

$$\leq \left(\int_\Omega |Du_k - Dv|^p \, \eta(u_k - v) \, \phi \, dx \right)^{\frac{1}{p}} \left(\int_\Omega \eta(u_k - v) \, \phi \, dx \right)^{\frac{p-1}{p}}$$
$$+ \int_\Omega |Dv - Du| \, \eta(u_k - v) \, \phi \, dx.$$

Now, suppose for the moment that $p \geq 2$. Then, since $\psi|_{\mathrm{spt}\eta} = \mathrm{Id}$,

$$\begin{aligned}
T_k := c \int_\Omega |Du_k - Dv|^p \, \eta(u_k - v) \, \phi \, dx \\
&\leq \int_\Omega (\sigma(Du_k) - \sigma(Dv)) \, (Du_k - Dv) \, \eta(u_k - v) \, \phi \, dx \\
&= \int_\Omega (\sigma(Du_k) - \sigma(Dv)) \, D(\psi(u_k - v)) \, \eta(u_k - v) \, \phi \, dx \\
&= \int_\Omega \sigma(Du_k) \, D(\psi(u_k - v)) \, \phi \, dx \\
&\qquad - \int_\Omega \sigma(Du_k) \, D(\psi(u_k - v)) \, (1 - \eta(u_k - v)) \, \phi \, dx \\
&\qquad - \int_\Omega \sigma(Dv) \, (Du_k - Dv) \, \eta(u_k - v) \, \phi \, dx \\
&=: I + II + III.
\end{aligned}$$

We discuss the terms I and II separately:

$$I = \int_\Omega f_k \, \psi(u_k - v) \, \phi \, dx - \int_\Omega \sigma(Du_k) \, \psi(u_k - v) \, D\phi \, dx.$$

To estimate II note that

$$D\psi(z) = \alpha'(|z|) \frac{z}{|z|} \otimes \frac{z}{|z|} + \frac{\alpha(|z|)}{|z|} \left(\mathrm{Id} - \frac{z}{|z|} \otimes \frac{z}{|z|} \right)$$

so that

$$F : D\psi(z) F \geq 0 \qquad \text{for all } \; z \in \mathbb{R}^m \qquad \text{and} \qquad F \in \mathbb{R}^{m \times n}.$$

Hence

$$II \leq \int_\Omega \sigma(Du_k) \; : \; [D\psi(u_k - v)] Dv \, (1 - \eta(u_k - v)) \phi \, dx \, .$$

Since $u_k \to u$ a.e. and η, ψ and $D\psi$ are continuous and bounded it follows that

$$
\begin{aligned}
(11)\qquad T_\infty := \limsup_{k\to\infty} T_k
&\leq \sup|\psi| \int_\Omega \phi\, d\mu \\
&\quad - \int_\Omega \overline{\sigma}\, \psi(u-v)\, D\phi\, dx \\
&\quad + \int_\Omega \overline{\sigma} : [D\psi(u-v)]\, Dv\, (1-\eta(u-v))\, \phi\, dx \\
&\quad - \int_\Omega \sigma(Dv)\,(Du-Dv)\,\eta(u-v)\,\phi\, dx.
\end{aligned}
$$

Moreover by (10)

$$
\begin{aligned}
(12)\qquad \int_\Omega h\,\eta(u-v)\,\phi\, dx &\leq c^{-1/p}\, T_\infty^{1/p} \Big(\int_\Omega \eta(u-v)\,\phi\, dx \Big)^{\frac{p-1}{p}} \\
&\quad + \int_\Omega |Dv - Du|\, \eta(u-v)\,\phi\, dx.
\end{aligned}
$$

We are still free to choose v and η, ψ and ϕ. Let a be simultaneously a Lebesgue point of u, Du, h and $\overline{\sigma}$, and let

$$
\begin{aligned}
\phi_r(x) &= \frac{1}{r^n}\widetilde{\phi}\Big(\frac{x-a}{r}\Big), &\quad \widetilde{\phi} \in C_0^\infty(B(0,1)), &\quad \int \widetilde{\phi} = 1, \\
\eta_r(y) &= \widetilde{\eta}\Big(\frac{y}{r}\Big), & \widetilde{\eta} \in C_0^\infty(B(0,1)), & \quad \widetilde{\eta}|_{B(0,\frac{1}{2})} \equiv 1, \\
\psi_r(y) &= r\widetilde{\psi}\Big(\frac{y}{r}\Big),
\end{aligned}
$$

with $\widetilde{\psi}$ as in the above discussion. We choose v to be the linear Taylor polynomial of u in a, *i.e.*

$$
v(x) = u(a) + (Du(a))(x-a)\,.
$$

Then, for a.e. a, we have as $r \to 0$

$$\begin{aligned} \fint_{B(a,r)} \left| \frac{v-u}{r} \right| dx &\to 0, \\ \fint_{B(a,r)} |Du - Dv|\, dx &\to 0, \\ \fint_{B(a,r)} |h(x) - h(a)|\, dx &\to 0, \\ \fint_{B(a,r)} |\overline{\sigma}(x) - \overline{\sigma}(a)|\, dx &\to 0 \end{aligned}$$

(*see* e.g. [8]). Furthermore we may assume that

$$\limsup_{r \to 0} \frac{\mu(B(a,r))}{r^n} < \infty .$$

Hence by (11) we conclude

$$\limsup_{r \to 0} T_{\infty,r} = 0$$

(where $T_{\infty,r}$ denotes the expression (11) with η and ϕ replaced by η_r and ϕ_r, respectively) and (12) yields

$$h(a) = 0 \qquad \text{for a.e. } \ a \in \Omega.$$

Thus the claim is proved in case $p \geq 2$. For $2 - \frac{1}{n} < p < 2$ notice that for all $m \times n$ matrices F and G

$$(\sigma(F)F - \sigma(G)G) : (F - G) \geq c|F - G|^2(|F| + |G|)^{p-2}$$

for a constant $c > 0$. Now we replace (10) by

$$\begin{aligned} (10') \quad & \int_\Omega |Du_k - Du|\, \eta(u_k - v)\, \phi\, dx \\ & \leq \int_\Omega |Du_k - Dv|\, \eta(u_k - v)\, \phi\, dx + \int_\Omega |Dv - Du|\, \eta(u_k - v)\, \phi\, dx \\ & \leq \left(\int_\Omega |Du_k - Dv|^2(|Du_k| + |Dv|)^{p-2} \eta(u_k - v)\, \phi\, dx \right)^{\frac{1}{2}} \cdot \\ & \quad \cdot \left(\int_\Omega (|Du_k| + |Dv|)^{2-p} \eta(u_k - v)\, \phi\, dx \right)^{\frac{1}{2}} \\ & \quad + \int_\Omega |Dv - Du|\, \eta(u_k - v)\, \phi\, dx . \end{aligned}$$

Then, we have as before

$$T_k' := c\int_\Omega |Du_k - Dv|^2(|Du_k| + |Dv|)^{p-2}\eta(u_k - v)\,\phi\,dx \le I + II + III$$

and the rest of the proof is analogue to the case $p \ge 2$. Notice that we need $n \ge 2$ to conclude that the second factor on the right of (10′)

$$\int_\Omega (|Du_k| + |Dv|)^{2-p}\eta_r(u_k - v)\,\phi_r\,dx$$

remains finite in the localization process for almost all $a \in \Omega$. □

From

$$Du_k \to Du \qquad \text{strongly in } L^1_{\text{loc}}(\Omega)$$

and

$$Du_k \rightharpoonup Du \qquad \text{weakly in } L^s_{\text{loc}}(\Omega)$$

we conclude that for all $s' < s$

$$Du_k \to Du \qquad \text{strongly in } L^{s'}_{\text{loc}}(\Omega)$$

and hence we may pass to the limit $k \to \infty$ in (4) and Theorem 1 follows.

REFERENCES

[1] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L. Vazquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, Vol. **22**, 1995, pp. 241-273.

[2] L. Boccardo and T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, Vol. **87**, 1989, pp. 149-169.

[3] L. Boccardo and T. Gallouët, Nonlinear elliptic equations with right-hand side measures. *Comm. Part. Diff. Eqn.*, Vol. **17**, 1992, pp. 641-655.

[4] L. Boccardo and F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.*, Vol. **19**, 1992, pp. 581-597.

[5] Y. Chen, M.-C. Hong and N. Hungerbühler, Heat flow of p-harmonic maps with values into spheres. *Math. Z.*, Vol. **215**, 1994, pp. 25-35.

[6] E. DiBenedetto and J. Manfredi, On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems. *Am. J. Math.*, Vol. **115**, 1993, pp. 1107-1134.

[7] L. C. Evans, Weak convergence methods for nonlinear partial differential equations. *CBMS Regional Conf. Ser. in Math.*, No. 74, *Am. Math. Soc.*, Providence, RI, 1990.

[8] L. C. Evans and R. Gariepy, Measure theory and fine properties of functions. *CRC Press, Boca Raton* (etc.), 1992.

[9] M. Fuchs, The Green-matrix for elliptic systems which satisfy the Legendre-Hadamard condition. *Manuscr. Math.*, Vol. **46**, 1984, pp. 97-115.

[10] M. Fuchs, The Green matrix for strongly elliptic systems of second order with continuous coefficients. *Z. Anal. Anwend.*, Vol. **5** (6), 1986, pp. 507-531.

[11] M. Fuchs, The blow-up of p-harmonic maps. *Manuscr. Math.*, Vol. **81**, 1993, pp. 89-94.

[12] M. FUCHS and J. REULING, Non-linear elliptic systems involving measure data. *Rend. Math. Appl.*, Vol. **15**, 1995, pp. 311-319.
[13] M. GRÜTER and K.-O. WIDMAN, The Green function for uniformly elliptic equations. *Manuscr. Math.*, Vol. **37**, 1982, pp. 303-342.
[14] P. L. LIONS and F. MURAT, Solutions renormalisées d'equations elliptiques (to appear).
[15] S. MÜLLER, Det = det. A remark on the distributional determinant. *C.R. Acad. Sci., Paris*, Sér. I, Vol. **311**, 1990, pp. 13-17.
[16] F. MURAT, Équations elliptiques non linéaires avec second membre L^1 ou mesure. Preprint.
[17] F. MURAT, Soluciones renormalizadas de EDP elipticas no lineales. Publications du Laboratoire d'Analyse Numérique **R93023**, 1993.
[18] J. M. RAKOTOSON, Generalized solutions in a new type of sets for problems with measure data. *Differ. Integral Eqn.*, Vol. **6**, 1993, pp. 27-36.
[19] G. TALENTI, Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces. *Ann. Mat. Pura Appl., IV. Ser.*, Vol. **120**, 1979, pp. 159-184.

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