Annales de l'I. H. P., section C

Paolo Caldiroli
Margherita Nolasco

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Annales de l'I. H. P., section C, tome 15, nº 1 (1998), p. 113-125
<http://www.numdam.org/item?id=AIHPC_1998__15_1_113_0>

Ann. Inst. Henri Poincaré,

Vol. 15, n° 1, 1998, p. 113-125 *Analyse non linéaire*

Multiple homoclinic solutions for a class of autonomous singular systems in $\mathbf{R}^2$

by

Paolo CALDIROLI

SISSA, International School for Advanced Studies
Via Beirut, 2-4, 34013 Trieste, Italy.
E-mail: paolocal@sissa.it

and

Margherita NOLASCO

Università degli Studi dell' Aquila
Dipartimento di Matematica Pura ed Applicata
Via Vetoio, Loc. Coppito, 67010 L'Aquila, Italy.
E-mail: nolasco@mat.uniroma1.it

ABSTRACT. – We look for homoclinic solutions for a class of second order autonomous Hamiltonian systems in $\mathbf{R}^2$ with a potential V having a strict global maximum at the origin and a finite set $S \subset \mathbf{R}^2$ of singularities, namely $V(x) \to -\infty$ as $\text{dist}(x, S) \to 0$. We prove that if V satisfies a suitable geometrical property then for any $k \in \mathbf{N}$ the system admits a homoclinic orbit turning k times around a singularity $\xi \in S$. © Elsevier, Paris

Key words: Hamiltonian systems, singular potentials, homoclinic orbits, minimization argument, Palais Smale sequences.

RÉSUMÉ. – Nous cherchons des solutions homoclines pour une classe de systèmes hamiltoniens autonomes du second ordre dans $\mathbf{R}^2$ définis par un potentiel V ayant un maximum global strict à l'origine et un ensemble fini $S \subset \mathbf{R}^2$ de singularités: $V(x) \to -\infty$ quand $\text{dist}(x, S) \to 0$. Nous montrons que si V vérifie une certaine propriété géométrique, alors le système possède une orbite homocline qui tourne k fois autour d'une singularité $\xi \in S$. © Elsevier, Paris

A.M.S. Classification: 58 E 05, 58 F 05.

1. INTRODUCTION

In this work we deal with a class of autonomous second order Hamiltonian systems defined by a potential V having a strict maximum at the origin. We are interested in finding homoclinic orbits to the unstable equilibrium point $x = 0$, namely non zero solutions to

$$\left.\begin{array}{ll} \ddot{x} = -V'(x) & \\ x(t) \to 0 & \text{as } t \to \pm\infty \\ \dot{x}(t) \to 0 & \text{as } t \to \pm\infty. \end{array}\right\} \tag{1.1}$$

This problem has been widely investigated using variational methods in several papers (see [AB], [BG], [B], [C], [J], [RT], [S], [T] for existence results and [ACZ], [Be], [R], [T2] for multiplicity results).

Here we consider the case of a potential with a unique strict global maximum at the origin. Note that if V is a smooth potential on $\mathbf{R}^N$ in general we cannot expect the existence of homoclinic solutions, as for example in the case of a radially symmetric potential where the only solution to (1.1) is $x(t) \equiv 0$.

In fact we assume V to be singular on a finite set S, *i.e.*, $V(x) \to -\infty$ as $\text{dist}\,(x, S) \to 0$. As it will be clear in the following, this further assumption reflects in the variational formulation of the problem giving a non trivial topology to the sublevel sets of the Lagrangian functional associated to (1.1).

Under these conditions the existence of a homoclinic solution has been proved in [T] and [R]. In particular in the case of planar systems a solution is obtained by a minimizing argument in the class of functions winding around a singularity $\xi \in S$. Moreover, in [R], supposing an additional condition about the ratio between the cost to wind the singularity passing or not through the origin, the existence of a second homoclinic with a winding number sufficiently large is proved.

Aim of this work is to find homoclinics with an arbitrary winding number for planar singular systems. We point out that looking for solutions winding the singularity more than once, a lack of compactness may occur. More precisely, according to the concentration-compactness principle [L], the Palais Smale sequences may exhibit a dichotomy behavior. We show that a suitable geometry of the stable and unstable manifolds near the equilibrium point together with the fact that $V(x) \sim -\frac{1}{2}|x|^2$ as $x \to 0$ permits to recover some compactness. As a consequence, the system admits infinitely many homoclinic orbits characterized by different winding numbers.

We remark that under our assumptions, the condition given in [R] to obtain the second solution is always verified.

The geometrical property assumed in the present paper is satisfied for example by potentials with some discrete rotational symmetry (see theorem 2.4) and by potentials given by the sum of a smooth radial term and a localized singular perturbation (see corollary 2.8).

Let us remark that the interest of this result lies in the fact that very little is known about the multiplicity of homoclinic solutions for conservative systems; we mention here a recent work [BS] where infinitely many homoclinic orbits of multibump type are obtained for a different class of autonomous Hamiltonian systems.

Finally we point out that the above considerations on compactness apply for multiplicity results also in different settings, as for instance, in the problem of heteroclinic solutions between strict global maxima (see [R2]).

2. STATEMENT OF THE RESULTS AND FUNCTIONAL SETTING

Let S be a finite subset of $\mathbf{R}^2 \setminus \{0\}$. Let us consider a potential $V : \mathbf{R}^2 \setminus S \to \mathbf{R}$ satisfying:

(V1) $V \in C^{1,1}(\mathbf{R}^2 \setminus S, \mathbf{R})$;
(V2) $V(x) < 0$ for every $x \in \mathbf{R}^2 \setminus (S \cup \{0\})$;
(V3) $V(0) = 0$ and $V'(x) = -x + o(|x|)$ as $x \to 0$;
(V4) $V(x) \to -\infty$ as $\text{dist}\,(x, S) \to 0$ and there is a neighborhood N_S of S and a function $U \in C^1(N_S \setminus S, \mathbf{R})$ such that $|U(x)| \to \infty$ as $\text{dist}\,(x, S) \to 0$ and $V(x) \leq -|U'(x)|^2$ for every $x \in N_S \setminus S$;
(V5) there are $\bar{R} > 0$ and a function $U_\infty \in C^1(\mathbf{R}^2 \setminus B_{\bar{R}}, \mathbf{R})$ such that $|U_\infty(x)| \to \infty$ as $|x| \to \infty$ and $V(x) \leq -|U'_\infty(x)|^2$ for every $x \in \mathbf{R}^2 \setminus B_{\bar{R}}$, being $B_{\bar{R}} = \{x \in \mathbf{R}^2 : |x| < \bar{R}\}$.

Remark 2.1. – The assumptions (V2) and (V3) imply that the origin is a strict global maximum point for the potential V and $V(x) = -\frac{1}{2}|x|^2 + o(|x|^2)$ as $x \to 0$.

Remark 2.2. – The assumption (V4) corresponds to the strong force condition, introduced by Gordon in [G]. This condition governs the rate at which $V(x) \to -\infty$ as $\text{dist}\,(x, S) \to 0$. In particular (V4) is satisfied in the case $V(x) = -(\text{dist}\,(x, S))^{-\alpha}$ in a neighborhood of S and $\alpha \geq 2$.

The assumption (V5) is formally very similar to (V4) and concerns the behavior of the potential V at infinity. Precisely (V5) says that $V(x)$ can go to 0 as $|x| \to \infty$ but not too fast. For example (V5) holds if $V(x) \leq -a\,|x|^{-2}$ for $|x|$ large, being $a > 0$.

Let us introduce the open subset of $H^1(\mathbf{R},\mathbf{R}^2)$

$$\Lambda = \{ u \in H^1(\mathbf{R},\mathbf{R}^2) \,:\, \mathrm{dist}\,(u(t),S) > 0 \;\forall t \in \mathbf{R} \}$$

and the action functional

$$\varphi(u) = \int_{\mathbf{R}} \left(\frac{1}{2}|\dot{u}|^2 - V(u) \right) dt$$

defined for every $u \in \Lambda$. It is known that $\varphi \in C^1(\Lambda,\mathbf{R})$ and the critical points of φ in Λ, *i.e.*, the functions $u \in \Lambda$ such that $\varphi'(u) = 0$, are classical homoclinic solutions to (1.1). We set $K = \{ u \in \Lambda \,:\, \varphi'(u) = 0\,,\, u \not\equiv 0\}$ and, for any $c \in \mathbf{R}$, $K(c) = \{ u \in K \,:\, \varphi(u) = c \}$.

Since we deal with planar systems and since any $u \in \Lambda$ is a continuous function such that $u(t) \to 0$ as $t \to \pm\infty$, any $u \in \Lambda$ describes a closed curve in $\mathbf{R}^2$. Hence, fixed $\xi \in S$ we can associate an integer $\mathrm{ind}_\xi(u)$ giving the winding number of u about ξ. We recall that if $u, v \in \Lambda$ and $|u(t) - v(t)| < |u(t) - \xi|$ for every $t \in \mathbf{R}$ then $\mathrm{ind}_\xi(u) = \mathrm{ind}_\xi(v)$. For every $k \in \mathbf{Z}$ we set

$$\Lambda_k(\xi) = \{ u \in \Lambda \,:\, \mathrm{ind}_\xi(u) = k \}$$
$$c_k(\xi) = \inf \{ \varphi(u) \,:\, u \in \Lambda_k(\xi) \}.$$

We remark that $c_k(\xi) > 0$ for any $k \in \mathbf{Z} \setminus \{0\}$. Indeed given $u \in \Lambda$ with $\mathrm{ind}_\xi(u) \neq 0$ there exist $s_u < t_u$ such that $|u(s_u)| = \frac{|\xi|}{3}$, $|u(t_u)| = \frac{2|\xi|}{3}$ and $\frac{|\xi|}{3} \leq |u(t)| \leq \frac{2|\xi|}{3}$ for every $t \in [s_u, t_u] = I_u$. In particular $\int_{I_u} |\dot{u}|\, dt \geq \frac{|\xi|}{3}$. Then, since $\min_{\frac{|\xi|}{3} \leq |x| \leq \frac{2|\xi|}{3}} -V(x) = m > 0$, we get $\varphi(u) \geq \frac{1}{2} \int_{I_u} |\dot{u}|^2\, dt + m|I_u| \geq \frac{|\xi|^2}{18} \frac{1}{|I_u|} + m|I_u| \geq \bar{m} > 0$ with $\bar{m}$ a positive constant independent of u. We also point out that $\Lambda_{-k}(\xi) = \{u_- \,:\, u \in \Lambda_k(\xi) \}$ where $u_-(t) = u(-t)$. For the potential V is time independent, $\varphi(u_-) = \varphi(u)$ for all $u \in \Lambda$. Consequently $c_k(\xi) = c_{-k}(\xi)$. Moreover we notice that $c_k(\xi) \leq c_{k+1}(\xi)$ for any $k \in \mathbf{N}$. Indeed if $u \in \Lambda_{k+1}(\xi)$ then there is $I = [t_1, t_2] \subset \mathbf{R}$ such that $u(t_1) = u(t_2)$ and $\mathrm{ind}_\xi(u|_I) = 1$. Then defining $v(t) = u(t)$ for $t \leq t_1$ and $v(t) = u(t - t_1 + t_2)$ for $t \geq t_1$, we get $v \in \Lambda_k(\xi)$ and $\varphi(v) \leq \varphi(u)$.

We state a preliminary result, already discussed in [R] and [T], about the existence of a homoclinic orbit describing a simple curve around a singularity $\xi \in S$.

THEOREM 2.3. – *Let* $V \,:\, \mathbf{R}^2 \setminus S \to \mathbf{R}$ *satisfy* (V1)-(V5). *Then for* $\xi \in S$ *(1.1) admits a homoclinic solution* $v_1 \in \Lambda_1(\xi)$ *and* $\varphi(v_1) = c_1(\xi)$.

We remark that the existence of a homoclinic solution is given in [R] for a potential merely C^1 with a periodic time dependence and in [T] for systems in $\mathbf{R}^N$.

Here we are interested in finding a homoclinic orbit $v_k \in \Lambda_k(\xi)$ for any $k \in \mathbf{Z} \setminus \{0\}$, being $\xi \in S$ fixed.

We prove the existence of infinitely many solutions when the potential presents a discrete radial symmetry.

THEOREM 2.4. – *Let* $V : \mathbf{R}^2 \setminus S \to \mathbf{R}$ *satisfy* (V1)-(V5) *and*

(V6) $V(Rx) = V(x)$ *for every* $x \in \mathbf{R}^2 \setminus S$, *where* R *is the rotation around the origin of an angle* $2\pi/m$ *with* $m \geq 5$.

Then, given $\xi \in S$, *the system (1.1) admits a sequence* $(v_k)_{k\in\mathbf{Z}\setminus\{0\}}$ *of homoclinic orbits with* $v_k \in \Lambda_k(\xi)$ *and* $\varphi(v_k) = c_k(\xi)$.

As we will see, the fact that $c_k(\xi)$ is a critical level is related to the existence of homoclinic solutions at the levels $c_1(\xi), \ldots, c_{k-1}(\xi)$ which stay asymptotically inside a cone of width strictly less than $\pi/2$.

In fact theorem 2.4 follows from this more general result.

THEOREM 2.5. – *Let* $V : \mathbf{R}^2 \setminus S \to \mathbf{R}$ *satisfy* (V1)-(V5). *If*

(h_{k-1}) *there are* $v_j^+, v_j^- \in \Lambda_j(\xi) \cap K(c_j(\xi))$ $(j = 1, \ldots, k-1)$ *(possibly* $v_j^+ = v_j^-$*) such that for every pair* $(i, j) \in \{1, \ldots, k-1\}^2$ *with* $i + j \leq k$

$$\liminf_{\substack{s\to-\infty\\ t\to+\infty}} \frac{v_i^+(s) \cdot v_j^-(t)}{|v_i^+(s)|\,|v_j^-(t)|} > 0 \tag{2.6}$$

then (1.1) admits a homoclinic solution $v_k \in \Lambda_k(\xi)$ *and* $\varphi(v_k) = c_k(\xi)$.

Remark 2.7. – It is possible to prove that if $u \in K$ then there exists $\lim_{t\to\pm\infty} \frac{u(t)}{|u(t)|} = x_\pm(u)$. (This is essentially due to the fact that, thanks to (V3), $|u(t)| \sim C\exp(-|t|)$ and $\frac{u(t)}{|u(t)|} \wedge \frac{\dot{u}(t)}{|\dot{u}(t)|} \to 0$ as $t \to \pm\infty$). In this way the condition (2.6) reduces to $x_-(v_i^+) \cdot x_+(v_j^-) > 0$.

Finally we give other examples of systems for which (h_{k-1}) holds for any $k \in \mathbf{N}$.

COROLLARY 2.8. – *Let* $V : \mathbf{R}^2 \setminus S \to \mathbf{R}$ *satisfy* (V1)-(V5) *and one of the following conditions: either*

(V6)′ $V(x) = V_0(|x|) + V_s(x)$ *where*

$V_0 \in C^{1,1}(\mathbf{R}_+, \mathbf{R})$;

$V_s \in C^{1,1}(\mathbf{R}^2 \setminus S, \mathbf{R})$ *and* $V_s(x) \to -\infty$ *as* $\mathrm{dist}\,(x, S) \to 0$

$\mathrm{supp}\, V_s \subseteq \mathrm{span}_+\{x_1, x_2\}$ *for some* $x_1, x_2 \in \mathbf{R}^2$ *with* $x_1 \cdot x_2 > 0$;

or

(V6)″ *there are* $x_1, x_2 \in \mathbf{R}^2$ *with* $x_1 \cdot x_2 > 0$ *such that* $\xi \in \operatorname{span}_+\{x_1, x_2\}$ *for some* $\xi \in S$ *and* $V(p(x)) \geq V(x)$ *for every* $x \in \mathbf{R}^2 \setminus S$.

Then the system (1.1) admits a sequence $(v_k)_{k \in \mathbf{Z}\setminus\{0\}}$ *of homoclinic solutions with* $\operatorname{ind}_\xi(v_k) = k$.

(We denote $\operatorname{span}_+\{x_1, x_2\} = \{\lambda_1 x_1 + \lambda_2 x_2 \ : \ \lambda_1, \lambda_2 > 0\}$ and p the projection on its closure.)

3. PROOFS

We fix $\xi \in S$ and we put $\Lambda_k = \Lambda_k(\xi)$ and $c_k = c_k(\xi)$.

First we prove theorem 2.3. To begin we give some properties of the sequences $(u_n) \subset \Lambda$ with $\varphi(u_n)$ bounded.

LEMMA 3.1. – *Given a sequence* $(u_n) \subset \Lambda$ *such that* $\sup \varphi(u_n) < \infty$ *it holds that*

(i) *there is* $R > 0$ *such that* $\|u_n\|_{L^\infty} \leq R$;

(ii) *there is* $\rho > 0$ *such that* $|u_n(t) - \xi| \geq \rho$ *for all* $t \in \mathbf{R}$ *and* $n \in \mathbf{N}$;

(iii) (u_n) *is bounded in* $H^1(\mathbf{R}, \mathbf{R}^2)$.

Proof. – (i) By the contrary, assume that for some subsequence, denoted again (u_n), $\|u_n\|_{L^\infty} \to \infty$ holds. Using the invariance under translation, without loss of generality we can assume that $\|u_n\|_{L^\infty} = |u_n(0)|$. Moreover since $u_n(t) \to 0$ as $t \to +\infty$, there exists $N \in \mathbf{N}$ such that for any $n \geq N$ there is $t_n > 0$ such that $|u_n(t_n)| = \bar{R}$ and $|u_n(t)| \geq \bar{R}$ for $t \in (0, t_n)$, being $\bar{R}$ given by (V5). Then (V5) yields

$$
\begin{aligned}
|U_\infty(u_n(0))| &\leq |U_\infty(u_n(t_n))| + \int_0^{t_n} |U'_\infty(u_n) \cdot \dot{u}_n| \, dt \\
&\leq \max_{|x|=\bar{R}} |U_\infty(x)| + \left(\int_0^{t_n} -V(u_n) \, dt \right)^{1/2} \left(\int_0^{t_n} |\dot{u}_n|^2 \, dt \right)^{1/2} \\
&\leq \max_{|x|=\bar{R}} |U_\infty(x)| + 2\,\varphi(u_n).
\end{aligned}
$$

But $|U_\infty(u_n(0))| \to \infty$, while $\varphi(u_n)$ is bounded. Thus we get a contradiction.

A similar argument can be followed to prove (ii).

(iii) Fixed $\delta > 0$ and setting $T_n(\delta) = \{\, t \in \mathbf{R} \ : \ |u_n(t)| > \delta \,\}$ we claim that there is $\bar{T}(\delta) > 0$ such that $\operatorname{meas} T_n(\delta) \leq \bar{T}(\delta)$ for all $n \in \mathbf{N}$; indeed let $\beta(\delta) = \inf\{\, |V(x)| \,:\, |x-\xi| \geq \rho\,,\ \delta < |x| \leq R\}$ where $R > 0$ and $\rho > 0$ are given by (i) and (ii) respectively. Then

$$\beta(\delta)\,\mathrm{meas}\,T_n(\delta) \leq \int_{T_n(\delta)} |V(u_n)|\,dt \leq \varphi(u_n).$$

Since $\varphi(u_n)$ is bounded and $0 < \beta(\delta) < \infty$, also $\mathrm{meas}\,T_n(\delta)$ is bounded uniformly with respect to $n \in \mathbf{N}$. Now we observe that, for (V2),

$$\frac{1}{2}\int_{\mathbf{R}} |\dot{u}_n|^2\,dt \leq \varphi(u_n)$$

and so $(\dot{u}_n)$ is bounded in L^2. Now, let us take $\delta \in (0, |\xi|\,)$ such that $\frac{1}{4}|x|^2 \leq |V(x)|$ for $|x| \leq \delta$. Then

$$\begin{aligned}\int_{\mathbf{R}} |u_n|^2\,dt &= \int_{\mathbf{R}\setminus T_n(\delta)} |u_n|^2\,dt + \int_{T_n(\delta)} |u_n|^2\,dt \\ &\leq 4\int_{\mathbf{R}\setminus T_n(\delta)} |V(u_n)|\,dt + R^2\,\mathrm{meas}\,T_n(\delta) \\ &\leq 4\,\varphi(u_n) + R^2\,\bar{T}(\delta).\end{aligned}$$

where $R > 0$, $T_n(\delta)$ and $\bar{T}(\delta)$ are defined as above. Hence, using again the boundedness of $\varphi(u_n)$ we conclude that (u_n) is bounded in L^2 and thus in H^1. □

Lemma 3.1 says in particular that any Palais Smale sequence for φ is bounded in $H^1(\mathbf{R}, \mathbf{R}^2)$. Then, since we are dealing with autonomous systems it is possible to characterize in a sharp way the PS sequences, as already done in [CZES] and [CZR] in the periodic case, with a concentration-compactness argument [L]. In particular it holds that any PS sequence (u_n) admits a subsequence which is generated by a finite set of critical points $w_1, \ldots, w_l \in \Lambda$ and definitively the winding number of its elements can be related to that one of $w_1, \ldots, w_l$.

LEMMA 3.2. – *Let $(u_n) \subset \Lambda$ be a PS sequence for φ at the level $b > 0$. Then there are $l \in \mathbf{N}$, $w_1, \ldots, w_l \in K$, a subsequence of (u_n), denoted again (u_n), and corresponding sequences $(t_n^1), \ldots, (t_n^l) \subseteq \mathbf{R}$ such that, as $n \to \infty$, $|t_n^{j+1} - t_n^j| \to +\infty \quad (j = 1, \ldots, l-1)$ and*

(i) $\quad \|u_n - (w_1(\cdot - t_n^1) + \ldots + w_l(\cdot - t_n^l))\| \to 0$

(ii) $\quad b = \varphi(w_1) + \cdots + \varphi(w_l)$

(iii) $\quad \mathrm{ind}_\xi(u_n) = \mathrm{ind}_\xi(w_1) + \cdots + \mathrm{ind}_\xi(w_l).$

Proof. – We refer to [CZR] to prove (i) and (ii). The property (iii) follows from the fact that, given $v_1, v_2 \in \Lambda$ and a sequence $|t_n| \to \infty$ then definitively $v_1 + v_2(\cdot - t_n) \in \Lambda$ and $\text{ind}_\xi(v_1 + v_2(\cdot - t_n)) = \text{ind}_\xi(v_1) + \text{ind}_\xi(v_2)$. Indeed, setting $\rho_i = \inf_{t\in\mathbf{R}} |v_i(t) - \xi|$ and $\rho = \min\{\rho_1, \rho_2\}$, we can find $u_i \in \Lambda$ with compact support such that $\|v_i - u_i\|_{L^\infty} < \frac{1}{3}\rho$ and $\text{ind}_\xi(u_i) = \text{ind}_\xi(v_i)$ follows. Now since $|t_n| \to \infty$ we can take $\bar{n} \in \mathbf{N}$ such that for $n \geq \bar{n}$, u_1 and $u_2(\cdot - t_n)$ have disjoint supports and moreover $|v_1(t) + v_2(t - t_n) - \xi| \geq \frac{2}{3}\rho$. Hence $|(v_1(t) + v_2(t - t_n)) - (u_1(t) + u_2(t - t_n))| < |v_1(t) + v_2(t - t_n) - \xi|$ for all $t \in \mathbf{R}$. Then $u_1 + u_2(\cdot - t_n) \in \Lambda$ and $\text{ind}_\xi(v_1 + v_2(\cdot - t_n)) = \text{ind}_\xi(u_1 + u_2(\cdot - t_n)) = \text{ind}_\xi(u_1) + \text{ind}_\xi(u_2(\cdot - t_n)) = \text{ind}_\xi(v_1) + \text{ind}_\xi(v_2)$. □

With the above results we can prove the existence of a first solution $v_1 \in \Lambda_1$ at the level c_1.

Proof of theorem 2.3. – By lemma 3.1 (ii), the set $\Lambda_1 \cap \{\varphi \leq c_1 + 1\}$ is closed in $H^1(\mathbf{R}, \mathbf{R}^2)$. Then, for the Ekeland principle, there is a PS sequence $(u_n) \subset \Lambda_1$ at the level c_1. Consequently, by lemma 3.2, we have $u_n = w_1(\cdot - t_n^1) + \ldots + w_l(\cdot - t_n^l) + y_n$ where $l \in \mathbf{N}$, $w_1, \ldots, w_l \in K$ and $\|y_n\| \to 0$. Now we exclude the case $l > 1$. Indeed, if $l > 1$, it must be $\text{ind}_\xi(w_j) = 0$ for all $j = 1, \ldots, l$. Otherwise, if $|\text{ind}_\xi(w_j)| = m \geq 1$ for some j then, by lemma 3.2 (ii), $c_1 > \varphi(w_j) \geq c_m \geq c_1$, a contradiction. On the other hand the fact that $\text{ind}_\xi(w_j) = 0$ for all $j = 1, \ldots, l$ is in contradiction with lemma 3.2 (iii). Consequently $l = 1$. Hence, by (ii) and (iii) of lemma 3.2, w_1 is a critical point of φ such that $c_1 = \varphi(w_1)$ and $\text{ind}_\xi(w_1) = 1$. Thus the theorem is proved. □

We remark that theorem 2.3 can be proved in a different way as done in [R] just studying the minimizing sequences. Here we prove this result by using the characterization of PS sequences (lemma 3.2) which is basic in our argument to get multiple homoclinics.

To prove theorems 2.4 and 2.5, firstly we will show that with the additional information given by the further assumption (h_{k-1}), $\varphi|_{\Lambda_k}$ satisfies the PS condition at the level c_k, for $k > 1$. To get this, we have to compare the value c_k with the sums $c_{k_1} + \cdots + c_{k_l}$ where $k_1, \ldots, k_l$ are arbitrary integers such that $k_1 + \cdots + k_l = k$.

The first step is given by the following technical lemma.

LEMMA 3.3. – *For any $\theta \in [0, \frac{\pi}{2})$ there is $\delta_\theta \in (0, |\xi|)$ such that for every $\delta \in (0, \delta_\theta)$ there exists $\bar{T} = \bar{T}(\theta, \delta) \geq 0$ for which for any $x_-, x_+ \in \mathbf{R}^2$*

satisfying $\frac{x_- \cdot x_+}{|x_-|\,|x_+|} \geq \cos\theta$, $|x_-| = |x_+| = \delta$ *and denoting*

$$m_T[x_-, x_+] = \inf\left\{ \int_{-T}^{T} \left(\frac{1}{2}|\dot{u}|^2 - V(u)\right) dt \ :\ u \in H^1([-T,T];\mathbf{R}^2),\right.$$
$$\left. u(\pm T) = x_\pm,\ \|u\|_{L^\infty([-T,T])} \leq \delta_\theta \right\}$$

then $\liminf_{T\to+\infty} m_T[x_-, x_+] > m_{\bar{T}}[x_-, x_+]$. *(For* $T = 0$ *and* $x_- \neq x_+$ *we agree that* $m_T[x_-, x_+] = +\infty$.*)*

Proof. – To begin, for $\gamma > 0$, $x_-, x_+ \in \mathbf{R}^2$ let us introduce

$$m_T[\gamma; x_-, x_+] = \inf\left\{ \int_{-T}^{T} \frac{1}{2}(|\dot{u}|^2 + \gamma^2|u|^2)\, dt \ :\right.$$
$$\left. u \in H^1([-T,T];\mathbf{R}^2),\ u(\pm T) = x_\pm \right\}$$

One can easily calculate the explicit expression given by:

$$m_T[\gamma; x_-, x_+] = \frac{\gamma}{2}\left(\frac{|x_+|^2 + |x_-|^2}{\tanh 2\gamma T} - \frac{2\,x_- \cdot x_+}{\sinh 2\gamma T}\right).$$

In particular we get that $\lim_{T\to+\infty} m_T[\gamma; x_-, x_+] = \frac{\gamma}{2}(|x_+|^2 + |x_-|^2)$ and if $\frac{x_- \cdot x_+}{|x_-|\,|x_+|} \geq \cos\theta$ for some $\theta \in [0, \frac{\pi}{2})$, then there exists $\bar{T} = \bar{T}(\gamma; \theta, |x_-|, |x_+|) < +\infty$ such that

$$\inf_{T\geq 0} m_T[\gamma; x_-, x_+] = m_{\bar{T}}[\gamma; x_-, x_+] = \frac{\gamma}{2}[(|x_+|^2 + |x_-|^2)^2 - (2x_- \cdot x_+)^2]^{\frac{1}{2}}.$$

Let us fix $\epsilon < \frac{1-\sin\theta}{1+\sin\theta}$. We note that by (V3) there exists $\delta_\theta > 0$ such that $-\frac{1}{2}(1+\epsilon)^2|x|^2 \leq V(x) \leq -\frac{1}{2}(1-\epsilon)^2|x|^2$ for all $|x| \leq \delta_\theta$. Let us define $\gamma_- = 1 - \epsilon$, $\gamma_+ = 1 + \epsilon$. We have that for all $T \geq 0$ $m_T[\gamma_-; x_-, x_+] \leq m_T[x_-, x_+] \leq m_T[\gamma_+; x_-, x_+]$. Then we have that $m_T[x_-, x_+] - m_{\bar{T}_+}[x_-, x_+] \geq m_T[\gamma_-; x_-, x_+] - m_{\bar{T}_+}[\gamma_+; x_-, x_+]$ where $\bar{T}_+ = \bar{T}(\gamma_+; \theta, |x_-|, |x_+|)$ and

$$\liminf_{T\to+\infty} m_T[x_-, x_+] - m_{\bar{T}_+}[x_-, x_+]$$
$$\geq \frac{1}{2}(|x_-|^2 + |x_+|^2)\left[\gamma_- - \gamma_+\left(1 - \left(\frac{2x_- \cdot x_+}{|x_-|^2 + |x_+|^2}\right)^2\right)^{\frac{1}{2}}\right].$$

Since we take $|x_-| = |x_+| = \delta \in (0, \delta_\theta)$ and $\epsilon < \frac{1-\sin\theta}{1+\sin\theta}$ we finally get

$$\liminf_{T\to+\infty} m_T[x_-, x_+] - m_{\bar{T}_+}[x_-, x_+] > 0. \quad \square$$

In the next lemma we prove that the assumption on the geometry near the origin of the solutions with index smaller than k implies the PS condition for φ in Λ_k at level c_k

LEMMA 3.4. – *If* $v_1 \in \Lambda_{k_1}$ *and* $v_2 \in \Lambda_{k_2}$ *satisfy*

$$\liminf_{\substack{s\to-\infty\\ t\to+\infty}} \frac{v_2(s)\cdot v_1(t)}{|v_2(s)|\,|v_1(t)|} > 0$$

then there is $v \in \Lambda_{k_1+k_2}$ *such that* $\varphi(v) < \varphi(v_1) + \varphi(v_2)$.

Proof. – Let $\theta \in [0, \frac{\pi}{2})$ be defined by

$$2\cos\theta = \liminf_{\substack{s\to-\infty\\ t\to+\infty}} \frac{v_2(s)\cdot v_1(t)}{|v_2(s)|\,|v_1(t)|}.$$

Let $\delta_\theta > 0$ be given by lemma 3.3 and let $\bar{s}, \bar{t} \in \mathbf{R}$ be such that

$$\frac{v_2(s)\cdot v_1(t)}{|v_2(s)|\,|v_1(t)|} \geq \cos\theta$$

for every $s \leq \bar{s}$ and $t \geq \bar{t}$. Then, fixing $\delta \in (0, \delta_\theta)$ let $s_0 \in (-\infty, \bar{s}]$ and $t_0 \in [\bar{t}, +\infty)$ be such that $|v_1(t_0)| = \delta$, $|v_1(t)| \leq \delta$ for $t \geq t_0$, $|v_2(s_0)| = \delta$, $|v_2(s)| \leq \delta$ for $s \leq s_0$. Choosing any sequence $T_n \to +\infty$, we set

$$u_n(t) = \begin{cases} v_1(t+T_n) & \text{for } t < -1 \\ v_2(t-T_n) & \text{for } t > 1 \end{cases}$$

and for $|t| \leq 1$ we define $u_n(t)$ as a linear function joining $v_1(T_n - 1)$ at $t = -1$ with $v_2(1 - T_n)$ at $t = 1$. It is easy to check that $\varphi(u_n) \to \varphi(v_1) + \varphi(v_2)$ as $n \to \infty$. Now let us set $s_n = s_0 + T_n$, $t_n = t_0 - T_n$, $x_- = v_1(t_0)$ and $x_+ = v_2(s_0)$. We see that $u_n(t_n) = x_-$, $u_n(s_n) = x_+$ and for $n \in \mathbf{N}$ sufficiently large $s_n - t_n \geq 2\bar{T}$, being $\bar{T} = \bar{T}(\theta, \delta)$ given by lemma 3.3 (in fact $s_n - t_n \to +\infty$). Moreover $|u_n(t)| \leq \delta$ for $t \in [t_n, s_n]$. Setting $\epsilon_n = \max\{\frac{1}{n}, \varphi(u_n) - \varphi(v_1) - \varphi(v_2)\}$ we can choose $\bar{u}_n \in H^1([-\bar{T}, \bar{T}]; \mathbf{R}^2)$ such that $|\bar{u}_n(t)| \leq \delta$ for any $t \in [-\bar{T}, \bar{T}]$, $\bar{u}_n(\pm\bar{T}) = x_\pm$ and

$$\int_{-\bar{T}}^{\bar{T}} \left(\frac{1}{2}|\dot{\bar{u}}_n|^2 - V(\bar{u}_n)\right) dt \leq m_{\bar{T}}[x_-, x_+] + \epsilon_n.$$

Finally we put

$$v_n(t) = \begin{cases} u_n(t) & \text{for } t < t_n \\ \bar{u}_n(t - t_n - \bar{T}) & \text{for } t_n \leq t \leq t_n + 2\bar{T} \\ u_n(t - t_n - 2\bar{T} + s_n) & \text{for } t > t_n + 2\bar{T}. \end{cases}$$

Then, for $n \in \mathbf{N}$ large enough, $v_n \in \Lambda_{k_1+k_2}$ and

$$\begin{aligned}\varphi(v_n) &= \varphi(u_n) - \int_{-\bar{T}}^{\bar{T}} \left(\frac{1}{2}|\dot{\bar{u}}_n|^2 - V(\bar{u}_n)\right) dt + \int_{t_n}^{s_n} \left(\frac{1}{2}|\dot{v}_n|^2 - V(v_n)\right) dt \\ &\leq \varphi(v_1) + \varphi(v_2) + 2\,\epsilon_n + m_{\bar{T}}[x_-, x_+] - m_{s_n - t_n}[x_-, x_+].\end{aligned}$$

Since $\epsilon_n \longrightarrow 0$ and $s_n - t_n \longrightarrow +\infty$, using lemma 3.3, we infer that for some $n \in \mathbf{N}$ large enough $\varphi(v_n) < \varphi(v_1) + \varphi(v_2)$. □

Remark 3.5. – Let us suppose that there is $v \in \Lambda_1 \cap K(c_1)$ such that $v(s) \cdot v(t) > 0$ for $s < -T$ and $t > T$. Then the argument used in lemma 3.4 can be applied to construct $u \in H^1([-T_1, T_1], \mathbf{R}^2)$ such that $u(t) = v(t)$ for $|t| \leq T \leq T_1$, $u(-T_1) = u(T_1)$, $\text{ind}_\xi(u) = 1$ and $\int_{-T_1}^{T_1} (\frac{1}{2}|\dot{u}|^2 - V(u))\, dt < \varphi(v) = c_1$. The presence of this closed curve u is precisely the additional assumption made in [R] to get $c_k < kc_1$ for some $k > 1$ sufficiently large and hence the existence of a second homoclinic solution with winding number k. Actually in our case we get the result for $k = 2$.

Finally we easily get the following compactness result.

LEMMA 3.6. – *If* (h_{k-1}) *holds, then* $c_k < c_{k_1} + \cdots + c_{k_l}$ *whenever* $l > 1$ *and* $k_1, \ldots, k_l \in \mathbf{Z} \setminus \{0\}$ *verify* $k_1 + \cdots + k_l = k$.

Proof. – Firstly note that if there exists $j \in \{1, \ldots, l\}$ such that $|k_j| \geq k$ we get immediately that $c_k \leq c_{k_j} < c_{k_1} + \ldots + c_{k_l}$. Therefore we can assume that $|k_j| < k$ for any $j \in \{1, \ldots, l\}$ and in fact $k_j \in \{1, \ldots, k-1\}$, since $c_m = c_{-m}$ for any $m \in \mathbf{Z}$.

Then for every $j = 1, \ldots, l$ we take $v_j \in \Lambda_{k_j}$ according to the assumption (h_{k-1}). Noting that $l > 1$, we can apply lemma 3.4 to the pair v_1 and v_2 and we find $v \in \Lambda_{k_1+k_2}$ such that $\varphi(v) < \varphi(v_1) + \varphi(v_2)$. Then we take sequences $(t_n^2), \ldots, (t_n^l) \subset \mathbf{R}$ such that $|t_n^i - t_n^j| \to \infty$ as $n \to \infty$ for $i \neq j$. Defining $u_n = v(\cdot - t_n^2) + \sum_{j>2} v_j(\cdot - t_n^j)$, we get that for $n \in \mathbf{N}$ large enough $u_n \in \Lambda_k$ and $c_k \leq \lim \varphi(u_n) = \varphi(v) + \sum_{j>2} \varphi(v_j) < c_{k_1} + \cdots + c_{k_l}$. □

Then the multiplicity result plainly follows.

Proof of theorem 2.5. – We argue as in the proof of theorem 2.3. By lemma 3.1 (ii) the set $\Lambda_k \cap \{\varphi \leq c_k + 1\}$ is closed in $H^1(\mathbf{R}, \mathbf{R}^2)$. Then, for the Ekeland principle, there is a PS sequence $(u_n) \subset \Lambda_k$ at the level c_k. Consequently, for the lemma 3.2, we have $u_n = w_1(\cdot - t_n^1) + \ldots + w_l(\cdot - t_n^l) + y_n$ where $l \in \mathbf{N}$, $w_1, \ldots, w_l \in K$, $\|y_n\| \to 0$ and

$\varphi(u_n) = \varphi(w_1) + \ldots + \varphi(w_l)$. Calling J the set of indices $j \in \{1, \ldots, l\}$ such that $\text{ind}_\xi(w_j) \neq 0$, then $\sum_{j\in J} \text{ind}_\xi(w_j) = k$ and, if $l > 1$, by lemma 3.2 (ii) J contains at least two elements. In this case we can apply lemma 3.6 and we get $c_k < \sum_{j\in J} \varphi(w_j) \leq \sum_{j=1}^{l} \varphi(w_j) = c_k$, a contradiction. Hence $l = 1$ and the conclusion follows as in the proof of theorem 2.3. □

Proof of theorem 2.4. – By theorem 2.3, there is $v \in \Lambda_1(\xi) \cap \overline{K(c_1(\xi))}$ for some $\xi \in S$. Let $\Omega(v)$ be the unbounded component of $\mathbf{R}^2 \setminus \overline{\text{range}\,(v)}$. We claim that range $(v') \subset \overline{\Omega(v)}$ where $v'(t) = Rv(t)$ and R is the rotation matrix of an angle $\frac{2\pi}{m}$ given by (V6). Otherwise there are at least two intervals $I = (s_1, t_1)$ and $J = (s_2, t_2)$ with $-\infty \leq s_i < t_i \leq +\infty$, such that the closure of $\{v(t) : t \in J\} \cup \{v'(t) : t \in I\}$ defines a closed curve in $\mathbf{R}^2 \setminus \Omega(v)$ and $\int_I (\frac{1}{2}|\dot{v}'|^2 - V(v'))\, dt > \int_J (\frac{1}{2}|\dot{v}|^2 - V(v))\, dt$. We consider the function $w \in \Lambda$ defined by

$$w(t) = \begin{cases} v(t - t_1 + t_2 - s_2 + s_1) & \text{for } t \leq t_1 - t_2 + s_2 \\ R^{-1}v(t - t_1 + t_2) & \text{for } t_1 - t_2 + s_2 < t < t_1 \\ v(t) & \text{for } t \geq t_1. \end{cases}$$

We note that w is obtained substituting $R^{-1}v|_J$ to $v|_I = R^{-1}v'|_I$, up to reparametrizations of the time. By the definition of v', $\int_{\mathbf{R}\setminus J}(\frac{1}{2}|\dot{v}|^2 - V(v))\, dt \geq \int_I (\frac{1}{2}|\dot{v}'|^2 - V(v'))\, dt$ and it holds that $w \in \Lambda_1(\xi)$ and $\varphi(w) < \varphi(v)$, a contradiction. Then, for $m \geq 5$,

$$\liminf_{\substack{s\to-\infty \\ t\to+\infty}} \frac{v(s) \cdot v(t)}{|v(s)|\,|v(t)|} \geq \cos(2\pi/m) > 0,$$

that is (h_1) holds. An analogous argument works to prove (h_k) for $k > 1$. Then the conclusion follows by applying theorem 2.5. □

Proof of corollary 2.8. – Let $v_j \in \Lambda_j$ be a homoclinic orbit such that $\varphi(v_j) = c_j(\xi)$. By remark 2.7, the condition (2.6) reduces to $x_i^- \cdot x_j^+ > 0$ where $x_j^\pm = \lim_{t\to\pm\infty} \frac{v_j(t)}{|v_j(t)|}$. Proving that $x_j^\pm \in \text{span}_+\{x_1, x_2\}$ we get the thesis. Arguing by contradiction, let us suppose that $x_j^- \notin \text{span}_+\{x_1, x_2\}$. Then there is $T \in \mathbf{R}$ such that $v_j(t) \notin \text{span}_+\{x_1, x_2\}$ for any $t \leq T$. If (V6)' holds, since $V(x) = V_0(\,|x|\,)$ for $x \in \mathbf{R}^2 \setminus \text{span}_+\{x_1, x_2\}$, by the conservation of the angular momentum and of the energy we infer that $v_j(t) = |v_j(t)|x_j^-$ for any $t \in \mathbf{R}$, contrary to the fact that $v_j \in \Lambda$.

If (V6)″ holds, then, setting $\bar{v}_j(t) = p(v_j(t))$, we get that $\bar{v}_j \in \Lambda_j$, $\varphi(\bar{v}_j) = c_j(\xi)$ and range $\bar{v}_j \subset \text{span}_+\{x_1, x_2\}$. Hence we get the thesis. □