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Decay estimates for the critical semilinear wave equation

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ABSTRACT. – In this paper we prove that finite energy solutions (with added regularity) to the critical wave equation $\Box u + u^5 = 0$ on $\mathbb{R}^3$ decay to zero in time. The proof is based on a global space-time estimate and dilation identity. © Elsevier, Paris

RÉSUMÉ. – Dans cet article, on montre que les solutions à énergie finie (avec régularité ajoutée) de l'équation des ondes critique $\Box u + u^5 = 0$, dans $\mathbb{R}^3$ décroissent vers zéro en temps. La démonstration est basée sur une estimée temps-espace globale et une identité de dilatation. © Elsevier, Paris

1. INTRODUCTION

In this note we will show that the solutions of the critical semilinear wave equation with finite energy initial data

$$u_{tt} - \Delta u + u^5 = 0 \qquad \text{on} \quad \mathbb{R}^3 \times \mathbb{R} \tag{1.1}$$

$$u_{|t=0} = u_0, \quad u_{t|t=0} = u_1 \qquad \text{on} \quad \mathbb{R}^3 \tag{1.2}$$

$$(u_0, u_1) \in \dot{H}^1 \times L^2 \tag{1.3}$$

are globally in the space

$$(1.4)\qquad (u,\, u_t) \in C\Big(\mathbb{R},\, \dot H^{1} \times L^2\Big) \cap L^4\Big(\mathbb{R},\, \dot B^{1/2} \times \dot B^{-1/2}\Big)$$

The study of the general semilinear wave equations dates back to the early works of Segal [7], Jörgen [5], Strauss [10]. For a detailed bibliography, see Zuily [16]. For nonlinearities that are subcritical with respect to the H^1 norm, Ginibre and Velo [3] have shown global existence and uniqueness of solutions in the space defined by (1.4), using a subtle improvement of the Strichartz ([13], [14]) estimates for the wave equations. For the critical problem, when the initial data are only of finite energy, Shatah and Struwe [9] have shown global existence and uniqueness of solutions in the space

$$(1.5)\qquad (u,\, u_t) \,\in\, C(\mathbb{R},\, H^1 \times L^2) \,\cap\, L^4_{\text{loc}}\Big(\mathbb{R},\, \dot B^{1/2} \,\times\, \dot B_4^{-1/2}\Big)$$

Their approach hinges on showing that the energy and the Morawetz identity hold for weak solutions, and thus are able to prove non-concentration of the energy of solutions. This identity was used originally to prove the existence of globally smooth solutions by Struwe [15], and Grillakis [3]. In the radial case, Ginibre, Soffer and Velo [1] have shown that the solutions are in the space defined by (1.4). The proof of (1.4) that we present here is a consequence of the decay estimate

$$g(t) = \frac{1}{6}\int_{\mathbb{R}^3} |u(t,\, x)|^6\, dx \underset{|t|\to+\infty}{\longrightarrow} 0$$

which is obtained using the methods of Shatah and Struwe [9]. These decay estimates are used by Bahouri and Gerard [1] to prove scattering of solutions to the above equation with finite energy initial data.

2. STUDY OF THE FUNCTION g

LEMMA 2.1. – *Let u be the solution of the Cauchy problem (1.1), (1.2), (1.3), then*

$$(2.1)\qquad g(t) = \frac{1}{6}\int_{\mathbb{R}^3} |u(t,\, x)|^6\, dx \underset{|t|\to+\infty}{\longrightarrow} 0$$

Proof. – For any $\varepsilon_0 > 0$ we have to show the existence of T_0 such that

$$(2.2)\qquad \forall t > T_0 \quad , \qquad |\, g(t)\,| \,\le\, \varepsilon_0\,.$$

Since the initial data has finite energy, we have for R large enough

$$\int_{|x|\geq R} e(u)\,(0,\,x)\,dx \;\leq\; \frac{\varepsilon_0}{8} \tag{2.3}$$

where

$$e(u) \;=\; \frac{1}{2}\,(|u_t|^2 + |\nabla_x u|^2) \;+\; \frac{1}{6}\,|u|^6 \tag{2.4}$$

denotes the energy density.

The classical energy-conservation law on the exterior of a truncated forward light cone (see Strauss [11]) implies that

$$\int_{|x|>R+t} e(u)\,dx + \frac{1}{\sqrt{2}}\ \text{flux}\ (0,t) \;\leq\; \frac{\varepsilon_0}{8} \tag{2.5}$$

where the flux on the mantle is given by

$$\text{flux}\ (a,b) \stackrel{\text{def}}{=} \int_{M_b^a} \left\{ \frac{1}{2}\left|\frac{x}{|x|}u_t + \nabla u\right|^2 + \frac{|u|^6}{6} \right\} d\sigma \tag{2.6}$$

$$M_a^b \stackrel{\text{def}}{=} \left\{(x,t)\in\mathbb{R}^3\times[a,b]\,;\quad |x| = R+t\right\} \tag{2.7}$$

Therefore to prove the lemma, it suffices to show the existence of T_0 such that

$$\forall t \;>\; T_0 \quad , \quad \frac{1}{6}\int_{|x|\leq R+t} |u(t,\,x)|^6\,dx \;\leq\; \frac{\varepsilon_0}{2}$$

and by translating time $t \longrightarrow t + R$ it is sufficient to prove

$$\forall t \;>\; T_0 \quad , \quad \frac{1}{6}\int_{|x|\leq t} |u(t,\,x)|^6\,dx \;\leq\; \frac{\varepsilon_0}{2}. \tag{2.8}$$

Proceeding exactly as in Shatah and Struwe [8], multiply equation (1.1) by $tu_t + x\cdot\nabla u + u$ to obtain the identity

$$\partial_t(tQ_0 \;+\; u_t u) - \;\text{div}\;(tP_0) \;+\; R_0 \;=\; 0 \tag{2.9}$$

where

$$Q_0 = e \;+\; u_t\Big(\frac{x}{t}\cdot\nabla u\Big),$$

$$P_o = \frac{x}{t}\left(\frac{|u_t|^2 - |\nabla u|^2}{2} \;-\; \frac{|u|^6}{6}\right) \;+\; \nabla u\Big(u_t \;+\; \frac{x}{t}\;\cdot\;\nabla u \;+\; \frac{u}{t}\Big)\,.$$

$$R_0 = \frac{|u|^6}{3}$$

Integrating equation (2.9) over the truncated cone $K_{T_1}^{T_2} = \{z = (x,\, t) \big| |x| < t, T_1 \le t \le T_2\}$, where $0 < T_1 < T_2$, we obtain by Stokes formula

$$\text{(2.10)} \int_{D(T_2)} (T_2\, Q_0 \,+\, u_t u)\, dx \;-\; \int_{D(T_1)} (T_1\, Q_0 \,+\, u_t u)\, dx$$
$$- \frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} \left(t\, Q_0 \,+\, u_t u \,+\, tP_0 \cdot \frac{x}{|x|} \right) d\sigma \;+\; \int_{K_{T_1}^{T_2}} \frac{|u|^6}{3} dx\, dt$$
$$= \mathrm{I} \,+\, \mathrm{II} \,+\, \mathrm{III} \,+\, \mathrm{IV} \;=\; 0$$

where

$$D(T_i) \;=\; \left\{ x \,\in\, \mathbb{R}^3 \big| |x| \;\le\; T_i \right\}$$

denotes space-like sections for $i \in \{1,\, 2\}$, and $M_{T_1}^{T_2}$ denotes the truncated mantle. On $M_{T_1}^{T_2}$, we have $|x| = t$, therefore we can rewrite the term III using spherical coordinates

$$\mathrm{III} \;=\; -\frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} \left\{ r(u_t \,+\, u_r)^2 \;+\; u(u_t \,+\, u_r) \right\} d\sigma$$

Parametrizing $M_{T_1}^{T_2}$ via $y \longrightarrow (|y|,\, y)$, and setting $v(y) = u(|y|,\, y)$ we find

$$\mathrm{III} = - \int_{T_1}^{T_2} \int_{S^2} r\left(v_r \,+\, \frac{v}{r} \right)^2 r^2\, dr\, d\omega$$
$$+ \int_{T_1}^{T_2} \int_{S^2} \frac{1}{2} \left(r^2\, v^2 \right)_r dr\, d\omega$$

Integrating by parts, we obtain

$$\text{(2.11)} \quad \mathrm{III} = - \int_{T_1}^{T_2} \int_{S^2} r\left(v_r \,+\, \frac{v}{r} \right)^2 r^2\, dr\, d\omega$$
$$+ \frac{1}{2} \int_{S^2} T_2^2\, v^2\, (T_2\, \omega)\, d\omega \;-\; \frac{1}{2} \int_{S^2} T_1^2 v^2\, (T_1\, \omega)\, d\omega$$

To estimate the first term we have

$$\mathrm{I} = \int_{D(T_2)} \left\{ T_2 \left(\frac{|u_t|^2}{2} \,+\, \frac{1}{2} \left(u_r \,+\, \frac{1}{r} u \right)^2 \,+\, \frac{1}{2r^2}\, |\nabla_\omega\, u|^2 \right.\right.$$
$$\left.\left. +\, \frac{1}{6} |u|^6 \right) + r \left(u_r \,+\, \frac{1}{r} u \right) u_t \right\} dx$$
$$- \frac{1}{2} \int_0^{T_2} \int_{S^2} T_2 \left(r u^2 \right)_r dr\, d\omega$$

Integrating by parts in the last term of the expression for I yields

$$(2.12)\ \mathrm{I} = \int_{D(T_2)} \Big\{ T_2 \left(\frac{|u_t|^2}{2} + \frac{1}{2}\Big(u_r + \frac{1}{r}u\Big)^2 + \frac{1}{2r^2}\,|\nabla_\omega u|^2 + \frac{1}{6}|u|^6 \right) + r\Big(u_r + \frac{1}{r}u\Big)\,u_t \Big\}\,dx - \frac{1}{2}\int_{S^2} T_2^2\ v^2\,(T_2\omega)\,d\omega$$

In the same manner, the second term can be written

$$(2.13)\ \mathrm{II} = -\int_{D(T_1)} \Big\{ T_1 \left(\frac{|u_t|^2}{2} + \frac{1}{2}\Big(u_r + \frac{1}{r}u\Big)^2 + \frac{1}{2r^2}\,|\nabla_\omega u|^2 + \frac{1}{6}|u|^6 \right) + r\Big(u_r + \frac{1}{r}u\Big)\,u_t \Big\}\,dx + \frac{1}{2}\int_{S^2} T_1^2\ v^2\,(T_1\omega)\,d\omega$$

Let $T_2 = T > 0$, and $T_1 = \varepsilon T$ for some $0 < \varepsilon < 1$. Substituting equations (2.11), (2.12), and (.213) into equation (2.10), and using Hardy's inequality

$$(2.14)\qquad \int \frac{|u|^2}{|x|^2}\,dx \ \leq\ C \int |\nabla u|^2 dx$$

we deduce

$$(2.15)\ T\int_{D(T)} \frac{|u|^6}{6}\,dx \ \leq\ C\,\varepsilon\,TE \ +\ \int_{\varepsilon T}^{T}\int_{S^2} T\Big(v_r + \frac{v}{r}\Big)^2 r^2\,dr\,d\omega$$

where C is a constant and

$$E = \int_{\mathbb{R}^3} e(u)(t,\,x)\,dx$$

denotes the energy.

Dividing by T, we obtain

$$\int_{D(T)} \frac{|u|^6}{6}\,dx \ \leq\ C\,\varepsilon\,E \ +\ \int_{\varepsilon T}^{\infty}\int_{S^2} \Big(v_r + \frac{v}{r}\Big)^2 r^2\,dr\,d\omega$$

Choose ε such that

$$C\,\varepsilon\,E \ =\ \frac{\varepsilon_0}{4}$$

From Hardy's inequality and the energy inequality (2.5) there exists a T_0 such that

$$\int_{\varepsilon T_0}^{\infty}\int_{S^2} \Big(v_r + \frac{v}{r}\Big)^2 r^2\,dr\,d\omega \ \leq\ 2\,\mathrm{flux}\,(\varepsilon T_0, \infty) \ <\ \frac{\varepsilon_0}{4}.$$

This proves inequality (2.8) which implies the L^6 norm decay of solutions.

3. STATEMENT AND PROOF OF THE THEOREM

THEOREM 3.1. – *The Cauchy problem (1.1), (1.2), (1.3) has a unique global solution u in the space*

$$(u,\, u_t) \;\in\; C\big(\mathbb{R},\, H^1 \,\times\, L^2\big) \;\cap\; L^4\Big(\mathbb{R},\, \dot{B}_4^{1/2} \,\times\, \dot{B}_4^{-1/2}\Big)$$

Proof. – We have to show that $u \in L^4\Big([T_0,\, \infty\,[\,,\; \dot{B}_4^{1/2}\Big)$, for some T_0. Fix $\varepsilon_0 > 0$ sufficiently small and choose T_0 such that (2.2) is satisfied.

Following the proof of the proposition 1.4 of Shatah and Struwe [8], we find, for $T > T_0$

$$\begin{aligned}(3.1)\qquad ||u||_{L^4\left([T_0,\,T];\; \dot{B}_4^{1/2}(\mathbb{R}^3)\right)} \;\leq\; C\Big\{ & E^{1/2} \\ & + ||u||^3_{L^4\left([T_0,\,T];\; \dot{B}_4^{1/2}(\mathbb{R}^3)\right)} \sup_{T_0 \,\leq\, t \,\leq\, T} ||u(t,\,\cdot)||^2_{L^6(\mathbb{R}^3)}\Big\}\end{aligned}$$

where E denotes the energy and C is a constant independent of T. Using the L^6 decay of solutions we obtain for arbitrary small ε_0

$$(3.2)\quad ||u||_{L^4\left([T_0,\,T];\; \dot{B}_4^{1/2}(\mathbb{R}^3)\right)} \;\leq\; CE^{1/2} + \varepsilon_0^{1/3}\; ||u||^3_{L^4\left([T_0,\,T];\; \dot{B}_4^{1/2}(\mathbb{R}^3)\right)}$$

Choose ε_0 sufficiently small, then the above inequality implies

$$(3.3)\qquad ||u||_{L^4\left([T_0,\,T];\; \dot{B}_4^{1/2}(\mathbb{R}^3)\right)} \;\leq\; 2\,C\,E^{1/2}$$

for all $T > T_0$, and letting $T \to \infty$ finishes the proof of the theorem.

Remark 3.1. – The same result holds in n dimensions. The proof is identical and uses the n dimensional result of Shatah and Struwe [9].

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