Decay estimates for the critical semilinear wave equation


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ABSTRACT. – In this paper we prove that finite energy solutions (with added regularity) to the critical wave equation \( \Box u + u^5 = 0 \) on \( \mathbb{R}^3 \) decay to zero in time. The proof is based on a global space-time estimate and dilation identity. © Elsevier, Paris

RÉSUMÉ. – Dans cet article, on montre que les solutions à énergie finie (avec régularité ajoutée) de l’équation des ondes critique \( \Box u + u^5 = 0 \), dans \( \mathbb{R}^3 \) décroissent vers zéro en temps. La démonstration est basée sur une estimée temps-espice globale et une identité de dilatation. © Elsevier, Paris

1. INTRODUCTION

In this note we will show that the solutions of the critical semilinear wave equation with finite energy initial data

\begin{align}
(1.1) \quad & u_{tt} - \Delta u + u^5 = 0 & \text{on } \mathbb{R}^3 \times \mathbb{R} \\
(1.2) \quad & u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1 & \text{on } \mathbb{R}^3 \\
(1.3) \quad & (u_0, u_1) \in \dot{H}^1 \times L^2
\end{align}
are globally in the space
\begin{equation}
(u, u_t) \in C\left( \mathbb{R}, H^1 \times L^2 \right) \cap L^4 \left( \mathbb{R}, \dot{B}^{1/2} \times \dot{B}^{-1/2} \right)
\end{equation}

The study of the general semilinear wave equations dates back to the early works of Segal [7], Jörgen [5], Strauss [10]. For a detailed bibliography, see Zuily [16]. For nonlinearities that are subcritical with respect to the $H^1$ norm, Ginibre and Velo [3] have shown global existence and uniqueness of solutions in the space defined by (1.4), using a subtle improvement of the Strichartz ([13], [14]) estimates for the wave equations. For the critical problem, when the initial data are only of finite energy, Shatah and Struwe [9] have shown global existence and uniqueness of solutions in the space
\begin{equation}
(u, u_t) \in C\left( \mathbb{R}, H^1 \times L^2 \right) \cap L^4_{\text{loc}} \left( \mathbb{R}, \dot{B}^{1/2} \times \dot{B}^{-1/2} \right)
\end{equation}

Their approach hinges on showing that the energy and the Morawetz identity hold for weak solutions, and thus are able to prove non-concentration of the energy of solutions. This identity was used originally to prove the existence of globally smooth solutions by Struwe [15], and Grillakis [3]. In the radial case, Ginibre, Soffer and Velo [1] have shown that the solutions are in the space defined by (1.4). The proof of (1.4) that we present here is a consequence of the decay estimate
\[
g(t) = \frac{1}{6} \int_{\mathbb{R}^3} |u(t, x)|^6 \, dx \quad \rightarrow \quad 0
\]

which is obtained using the methods of Shatah and Struwe [9]. These decay estimates are used by Bahouri and Gerard [1] to prove scattering of solutions to the above equation with finite energy initial data.

2. STUDY OF THE FUNCTION $g$

LEMMMA 2.1. – Let $u$ be the solution of the Cauchy problem (1.1), (1.2), (1.3), then
\begin{equation}
g(t) = \frac{1}{6} \int_{\mathbb{R}^3} |u(t, x)|^6 \, dx \quad \rightarrow \quad 0
\end{equation}

Proof. – For any $\varepsilon_0 > 0$ we have to show the existence of $T_0$ such that
\begin{equation}
\forall t > T_0 \quad , \quad |g(t)| \leq \varepsilon_0.
\end{equation}
Since the initial data has finite energy, we have for $R$ large enough

\begin{equation}
\int_{|x| \geq R} e(u)(0, x) \, dx \leq \frac{\varepsilon_0}{8}
\end{equation}

where

\begin{equation}
e(u) = \frac{1}{2} (|u_t|^2 + |\nabla_x u|^2) + \frac{1}{6} |u|^6
\end{equation}

denotes the energy density.

The classical energy-conservation law on the exterior of a truncated forward light cone (see Strauss [11]) implies that

\begin{equation}
\int_{|x| > R + t} e(u) \, dx + \frac{1}{\sqrt{2}} \text{flux}(0, t) \leq \frac{\varepsilon_0}{8}
\end{equation}

where the flux on the mantle is given by

\begin{equation}
\text{flux}(a, b) \overset{\text{def}}{=} \int_{M_a^b} \left\{ \frac{1}{2} \frac{|x|}{|x|} u_t + \nabla u \right\}^2 + \frac{|u|^6}{6} \, d\sigma
\end{equation}

\begin{equation}
M_{a}^{b} = \{(x, t) \in \mathbb{R}^3 \times [a, b] \mid |x| = R + t\}
\end{equation}

Therefore to prove the lemma, it suffices to show the existence of $T_0$, such that

\[ \forall t > T_0, \quad \frac{1}{6} \int_{|x| \leq R + t} |u(t, x)|^6 \, dx \leq \frac{\varepsilon_0}{2} \]

and by translating time $t \longrightarrow t + R$ it is sufficient to prove

\begin{equation}
\forall t > T_0, \quad \frac{1}{6} \int_{|x| \leq t} |u(t, x)|^6 \, dx \leq \frac{\varepsilon_0}{2}.
\end{equation}

Proceeding exactly as in Shatah and Struwe [8], multiply equation (1.1) by $t u_t + x \cdot \nabla u + u$ to obtain the identity

\begin{equation}
\partial_t (t Q_0 + u_t u) - \text{div} (t P_0) + R_0 = 0
\end{equation}

where

\begin{align*}
Q_0 &= e + u_t \left( \frac{x}{t} \cdot \nabla u \right), \\
P_0 &= \frac{x}{t} \left( \frac{|u_t|^2 - |
abla u|^2}{2} - \frac{|u|^6}{6} \right) + \nabla u \left( u_t + \frac{x}{t} \cdot \nabla u + \frac{u}{t} \right), \\
R_0 &= \frac{|u|^6}{3}
\end{align*}
Integrating equation (2.9) over the truncated cone \( K_{T_1}^{T_2} = \{ z = (x, t) \mid |x| < t, T_1 \leq t \leq T_2 \} \), where \( 0 < T_1 < T_2 \), we obtain by Stokes formula

\[
(2.10) \quad \int_{D(T_2)} (T_2 Q_0 + u_t u) \, dx - \int_{D(T_1)} (T_1 Q_0 + u_t u) \, dx \\
- \frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} \left( t Q_0 + u_t u + t P_0 \cdot \frac{x}{|x|} \right) \, d\sigma + \int_{K_{T_1}^{T_2}} \frac{|u|^6}{3} \, dx \, dt \\
= I + II + III + IV = 0
\]

where

\[
D(T_i) = \{ x \in \mathbb{R}^3 \mid |x| \leq T_i \}
\]
denotes space-like sections for \( i \in \{1, 2\} \), and \( M_{T_1}^{T_2} \) denotes the truncated mantle. On \( M_{T_1}^{T_2} \), we have \( |x| = t \), therefore we can rewrite the term III using spherical coordinates

\[
III = -\frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} \left\{ r(u_t + u_r)^2 + u(u_t + u_r) \right\} d\sigma
\]

Parametrizing \( M_{T_1}^{T_2} \) via \( y \to (|y|, y) \), and setting \( v(y) = u(|y|, y) \) we find

\[
III = -\int_{T_1}^{T_2} \int_{S^2} r \left( v_r + \frac{v}{r} \right)^2 r^2 \, dr \, d\omega \\
+ \int_{T_1}^{T_2} \int_{S^2} \frac{1}{2} (r^2 v^2) \, dr \, d\omega
\]

Integrating by parts, we obtain

\[
(2.11) \quad III = -\int_{T_1}^{T_2} \int_{S^2} r \left( v_r + \frac{v}{r} \right)^2 r^2 \, dr \, d\omega \\
+ \frac{1}{2} \int_{S^2} T_2^2 v^2 (T_2 \omega) \, d\omega - \frac{1}{2} \int_{S^2} T_1^2 v^2 (T_1 \omega) \, d\omega
\]

To estimate the first term we have

\[
I = \int_{D(T_2)} \left\{ T_2 \left( \frac{|u_t|^2}{2} + \frac{1}{2} \left( u_r + \frac{1}{r} u \right)^2 \right) + \frac{1}{2r^2} |\nabla \omega u|^2 \\
+ \frac{1}{6} |u|^6 \right\} + r \left( u_r + \frac{1}{r} u \right) \, dx \\
- \frac{1}{2} \int_0^{T_2} \int_{S^2} T_2 (ru^2) \, dr \, d\omega
\]
Integrating by parts in the last term of the expression for $I$ yields

\begin{equation}
(2.12) \quad I = \int_{D(T_2)} \left\{ T_2 \left( \frac{|u_t|^2}{2} + \frac{1}{2} \left( u_r + \frac{1}{r} u \right)^2 + \frac{1}{2r^2} |\nabla u|^2 \right.ight.
\nonumber
\left. + \frac{1}{6} |u|^6 \right) + r \left( u_r + \frac{1}{r} u \right) u_t \right\} \, dx
\end{equation}

\nonumber

\begin{equation}
- \frac{1}{2} \int_{S^2} T_2^2 \, v^2 (T_2 \omega) \, d\omega
\end{equation}

In the same manner, the second term can be written

\begin{equation}
(2.13) \quad II = - \int_{D(T_1)} \left\{ T_1 \left( \frac{|u_t|^2}{2} + \frac{1}{2} \left( u_r + \frac{1}{r} u \right)^2 + \frac{1}{2r^2} |\nabla u|^2 \right.ight.
\nonumber
\left. + \frac{1}{6} |u|^6 \right) + r \left( u_r + \frac{1}{r} u \right) u_t \right\} \, dx
\end{equation}

\nonumber

\begin{equation}
+ \frac{1}{2} \int_{S^2} T_1^2 \, v^2 (T_1 \omega) \, d\omega
\end{equation}

Let $T_2 = T > 0$, and $T_1 = \varepsilon T$ for some $0 < \varepsilon < 1$. Substituting equations (2.11), (2.12), and (2.13) into equation (2.10), and using Hardy’s inequality

\begin{equation}
(2.14) \quad \int_{D(T)} \frac{|u|^2}{|x|^2} \, dx \leq C \int |\nabla u|^2 \, dx
\end{equation}

we deduce

\begin{equation}
(2.15) \quad T \int_{D(T)} \frac{|u|^6}{6} \, dx \leq C \varepsilon T E + \int_{\varepsilon T}^T \int_{S^2} T \left( v_r + \frac{v}{r} \right)^2 r^2 \, d\omega \, dr
\end{equation}

where $C$ is a constant and

\begin{equation}
E = \int_{\mathbb{R}^3} e(u)(t, x) \, dx
\end{equation}

denotes the energy.

Dividing by $T$, we obtain

\begin{equation}
\int_{D(T)} \frac{|u|^6}{6} \, dx \leq C \varepsilon E + \int_{\varepsilon T}^\infty \int_{S^2} \left( v_r + \frac{v}{r} \right)^2 r^2 \, d\omega \, dr
\end{equation}

Choose $\varepsilon$ such that

\begin{equation}
C \varepsilon E = \frac{\varepsilon_0}{4}
\end{equation}

From Hardy’s inequality and the energy inequality (2.5) there exists a $T_0$ such that

\begin{equation}
\int_{T_0}^{\infty} \int_{S^2} \left( v_r + \frac{v}{r} \right)^2 r^2 \, d\omega \, dr \leq 2 \text{ flux} \left( \varepsilon T_0, \infty \right) < \frac{\varepsilon_0}{4}.
\end{equation}

This proves inequality (2.8) which implies the $L^6$ norm decay of solutions.
3. STATEMENT AND PROOF OF THE THEOREM

THEOREM 3.1. – The Cauchy problem (1.1), (1.2), (1.3) has a unique global solution \( u \) in the space

\[
(u, u_t) \in C(\mathbb{R}, H^1 \times L^2) \cap L^4(\mathbb{R}, \dot{B}^{1/2}_4 \times \dot{B}^{-1/2}_4)
\]

Proof. – We have to show that \( u \in L^4([T_0, \infty[, \dot{B}^{1/2}_4) \), for some \( T_0 \).

Fix \( \varepsilon_0 > 0 \) sufficiently small and choose \( T_0 \) such that (2.2) is satisfied.

Following the proof of the proposition 1.4 of Shatah and Struwe [8], we find, for \( T > T_0 \)

\[
\|u\|_{L^4([T_0, T]; \dot{B}^{1/2}_4(\mathbb{R}^3))} \leq C \left\{ E^{1/2} + \|u\|_{L^4([T_0, T]; \dot{B}^{1/2}_4(\mathbb{R}^3))}^{3/2} \sup_{T_0 \leq t \leq T} \|u(t, \cdot)\|_{L^6(\mathbb{R}^3)}^2 \right\}
\]

where \( E \) denotes the energy and \( C \) is a constant independent of \( T \). Using the \( L^6 \) decay of solutions we obtain for arbitrary small \( \varepsilon_0 \)

\[
\|u\|_{L^4([T_0, T]; \dot{B}^{1/2}_4(\mathbb{R}^3))} \leq CE^{1/2} + \varepsilon_0^{1/3} \|u\|_{L^4([T_0, T]; \dot{B}^{1/2}_4(\mathbb{R}^3))}^{3/2}
\]

Choose \( \varepsilon_0 \) sufficiently small, then the above inequality implies

\[
\|u\|_{L^4([T_0, T]; \dot{B}^{1/2}_4(\mathbb{R}^3))} \leq 2CE^{1/2}
\]

for all \( T > T_0 \), and letting \( T \to \infty \) finishes the proof of the theorem.

Remark 3.1. – The same result holds in \( n \) dimensions. The proof is identical and uses the \( n \) dimensional result of Shatah and Struwe [9].

REFERENCES


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