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On the non-locality of quasiconvexity

by

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ABSTRACT. – It is shown that in the class of smooth real-valued functions on $n \times m$ matrices ($n \geq 3$, $m \geq 2$) there can be no “local condition” which is equivalent to quasiconvexity. © Elsevier, Paris.

Key words: Quasiconvexity, rank-one convexity.


A continuous function $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ is called locally quasiconvex if at every point $X \in \mathbb{R}^{n \times m}$ there exists a neighborhood in which it coincides with a quasiconvex function. In this note we show that a $C^2$-function satisfying a strict Legendre-Hadamard condition at every point is locally quasiconvex. Using Šverák’s (cf. [21]) example of a rank-one convex function which is not quasiconvex we show that in dimensions $n \geq 3$, $m \geq 2$ there are locally quasiconvex functions that are not quasiconvex. Indeed, for any positive number $r > 0$ we give an example of a smooth function, which equals a quasiconvex function on any ball of radius $r$, but which is not itself quasiconvex. As a consequence of this we obtain that in dimensions $n \geq 3$, $m \geq 2$ there is no “local condition” which

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for $C^\infty$-functions is equivalent to quasiconvexity. In particular, we confirm the conjecture of Morrey (cf. [12]) saying that in general there is no condition involving only $f$ and a finite number of its derivatives, which is both necessary and sufficient for quasiconvexity. However, it might still be possible to find a "local condition" which is equivalent to quasiconvexity in e.g. the class of polynomials.

The proof relies heavily on Šverák’s example of a rank-one convex function which is not quasiconvex, and the main contribution here is contained in Lemma 2. Lemma 2 provides an extension result for quasiconvex functions, and is proved by use of Taylor’s formula, a slight extension of Dacorogna’s quasiconvexification formula and the equivalence of rank-one convexity and quasiconvexity for quadratic forms.

In the last part of this note we consider rank-one convexity and quasiconvexity in an abstract setting. We hereby prove that in the class of $C^\infty$-functions, any convexity concept between rank-one convexity and quasiconvexity, which is equivalent to a "local condition" is in fact rank-one convexity.

For convenience of the reader and to fix the notation we recall some definitions. The space of (real) $n \times m$ matrices is denoted by $\mathbb{R}^{n \times m}$. We use the usual Hilbert-Schmidt norm for matrices.

A continuous real-valued function $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ is said to be rank-one convex at $X \in \mathbb{R}^{n \times m}$ if the inequality

$$f(X) \leq tf(Y) + (1 - t)f(Z)$$

holds for all $t \in [0,1]$, $Y, Z \in \mathbb{R}^{n \times m}$ satisfying $\text{rank}(Y - Z) \leq 1$ and $X = tY + (1 - t)Z$. The function $f$ is rank-one convex if it is rank-one convex at each point.

The space of compactly supported $C^\infty$-functions $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is denoted by $D(\mathbb{R}^m; \mathbb{R}^n)$, or briefly, by $D$. The support of $\varphi$ is denoted by $\text{spt}\varphi$, and the gradient of $\varphi$ at $x$, $D\varphi(x)$, is identified in the usual way with a $n \times m$ matrix.

A continuous real-valued function $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ is said to be quasiconvex at $X \in \mathbb{R}^{n \times m}$ if the inequality

$$\int_{\mathbb{R}^m} (f(X + D\varphi(x)) - f(X)) \, dx \geq 0$$

holds for all $\varphi \in D$. The function $f$ is quasiconvex if it is quasiconvex at each point.
If for \( X \in \mathbb{R}^{n \times m} \) there exists a positive number \( \delta = \delta(X) > 0 \), such that the inequality (2) holds for all \( \varphi \in \mathcal{D} \) satisfying \( \sup_x |D\varphi(x)| \leq \delta \), then \( f \) is said to be weakly quasiconvex at \( X \). As above, \( f \) is weakly quasiconvex if it is weakly quasiconvex at each point.

The concepts of quasiconvexity and weak quasiconvexity are due to Morrey [12]. A concept of quasiconvexity relevant for higher order problems has been introduced by Meyers [11] (see also [5]).

It is obvious that quasiconvexity of \( f \) implies weak quasiconvexity of \( f \), and, as shown by Morrey [12], weak quasiconvexity of \( f \) implies rank-one convexity of \( f \). Hence it follows in particular that quasiconvexity of \( f \) implies rank-one convexity of \( f \).

In the special case where \( f \) is a quadratic form the converse is also true. Hence for quadratic forms the notion of rank-one convexity is equivalent to the notion of quasiconvexity (cf. [13]). A famous conjecture of Morrey [12] is that in dimensions \( n \geq 2, m \geq 2 \) there are rank-one convex functions that are not quasiconvex. In dimensions \( n \geq 3, m \geq 2 \) this was confirmed by Šverák in [21] giving a remarkable example of a polynomial of degree four which is rank-one convex, but not quasiconvex. In the remaining non-trivial cases, i.e. \( n = 2, m \geq 2 \), the question remains open. The problem is discussed in [3], [4], and more recently, in [15], [17], [26], [27].

It is not hard to see that for a \( C^2 \)-function \( f : \mathbb{R}^{n \times m} \mapsto \mathbb{R} \) rank-one convexity is equivalent to satisfaction of the Legendre-Hadamard (or ellipticity) condition at every \( X \in \mathbb{R}^{n \times m} \), i.e. for each \( X \in \mathbb{R}^{n \times m} \)

\[
D^2 f(X)(a \otimes b, a \otimes b) \geq 0
\]

for all \( a \in \mathbb{R}^n, b \in \mathbb{R}^m \).

If for some \( X \in \mathbb{R}^{n \times m} \) the inequality (3) holds strictly for all \( a \neq 0, b \neq 0 \), then we say that \( f \) satisfies a strict Legendre-Hadamard (or strong ellipticity) condition at \( X \). This is equivalent to the existence of a positive number \( c = c(X) \), such that

\[
D^2 f(X)(a \otimes b, a \otimes b) \geq c|a|^2|b|^2
\]

(4)

for all \( a \in \mathbb{R}^n, b \in \mathbb{R}^m \). By using the Fourier transformation and the Plancherel theorem it is easily seen that (4) is equivalent to

\[
\int_{\mathcal{B}} D^2 f(X)(D\varphi(x), D\varphi(x)) \, dx \geq c \int_{\mathcal{B}} |D\varphi(x)|^2 \, dx
\]

(5)

for all \( \varphi \in \mathcal{D} \) with \( \text{spt} \varphi \subset \mathcal{B} \), where \( \mathcal{B} := \{ x \in \mathbb{R}^m : |x| < 1 \} \).

By using Taylor’s formula and the equivalence of rank-one convexity and quasiconvexity for quadratic forms it can be proved that a $C^2$-function $f$ satisfying a strict Legendre-Hadamard condition at every point is weakly quasiconvex. The same kind of reasoning was used by Tartar [22] in proving a local form of a conjecture in compensated compactness.

**Definition.** A continuous real-valued function $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ is said to be locally quasiconvex at $X \in \mathbb{R}^{n \times m}$ if there exists a quasiconvex function $g : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$, such that $f = g$ in a neighborhood of $X$.

The function $f$ is locally quasiconvex if it is locally quasiconvex at each point.

One could define a similar concept of local rank-one convexity. However, by using a mollifier argument and the Legendre-Hadamard condition it is easily proved that this concept coincides with the usual concept of rank-one convexity. It is obvious that there is no need for a local concept of weak quasiconvexity.

If $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ is a locally bounded Borel function, then we define its quasiconvexification, $Qf : \mathbb{R}^{n \times m} \rightarrow [-\infty, +\infty]$, as

$$Qf(X) := \sup\{g(X) : g \text{ quasiconvex and } g \leq f\}.$$  

Notice that if at some $X$, $Qf(X) > -\infty$, then $Qf$ is quasiconvex.

The following result is a slight extension of a similar result due to Dacorogna [6]. We refer to [8] for the proof of this and for some extensions along these lines.

**Lemma 1.** Let $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ be a locally bounded Borel function. Then

$$Qf(X) = \inf \left\{ \int_B f(X + D\varphi) \, dx : \varphi \in \mathcal{D} \text{ with } \text{spt}\varphi \subset B \right\}.$$  

For a $C^2$-function $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ we have by Taylor’s formula

$$f(X + Y) = f(X) + Df(X)Y + \frac{1}{2}D^2f(X)(Y;Y) + R(X;Y),$$

where the remainder term $R(X;Y)$ is given by

$$R(X;Y) = \int_0^1 (1 - t)(D^2f(X + tY) - D^2f(X))(Y;Y) \, dt.$$  

For notational reasons it is convenient to introduce an auxiliary function, which essentially is a continuity modulus for the second derivative of $f$.  

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For each \( r \in (0, +\infty) \) define \( \Omega_r : (0, +\infty) \mapsto [0, +\infty) \) as (the norm being the usual one for bilinear mappings)

\[
\Omega_r(t) := \sup \{|D^2 f(X + Y) - D^2 f(X)| : |X| \leq r, |Y| < t\}.
\]

Obviously, \( \Omega_r \) is non-decreasing and continuous, and since \( D^2 f \) is uniformly continuous on compact sets, \( \Omega_r(t) \to 0 \) as \( t \to 0^+ \). Furthermore, we notice that if \( |X| \leq r \), then

\[
|R(X; Y)| \leq \frac{1}{2} \Omega_r(|Y|)|Y|^2
\]

for all \( Y \in \mathbb{R}^{n \times m} \).

**Lemma 2.** Let \( f : \mathbb{R}^{n \times m} \to \mathbb{R} \) be a \( C^2 \)-function, and assume that there exist numbers \( c, r > 0 \), such that

\[
\int_{\mathcal{B}} D^2 f(X)(D\varphi, D\varphi) \, dx \geq c \int_{\mathcal{B}} |D\varphi|^2 \, dx
\]

for \( |X| \leq r \) and \( \varphi \in \mathcal{D} \) with \( \text{spt}\varphi \subseteq \mathcal{B} \). Put \( \delta := (1/2) \sup\{t \in (0, r) : c \geq \Omega_r(t)\} \). Then there exists a quasiconvex function \( g : \mathbb{R}^{n \times m} \to \mathbb{R} \) of at most quadratic growth, such that

\[
f(X) = g(X) \quad \text{whenever} \quad |X| \leq \delta.
\]

**Remark.** Being quasiconvex \( g \) is necessarily locally Lipschitz continuous (cf. [6]), however, I do not know whether it is possible to obtain a quasiconvex extension \( g \) of \( f \) which is as regular as \( f \) is.

**Proof.** Define the function \( g := QG \), where

\[
G(X) := \begin{cases} 
  f(X) & \text{if } |X| \leq \delta, \\
  \sup_{|Y| \leq \delta} (f(Y) + Df(Y)(X - Y)) + \frac{1}{2} D^2 f(Y)(X - Y, X - Y) & \text{otherwise}.
\end{cases}
\]

Then obviously \( g \) is quasiconvex, of at most quadratic growth and \( g(X) \leq f(X) \) for \( |X| \leq \delta \). We claim that \( g(X) = f(X) \) for \( |X| \leq \delta \). Fix \( X \) with \( |X| < \delta \). Let \( \varepsilon > 0 \) and find \( \varphi = \varphi_\varepsilon \in \mathcal{D} \), such that

\[
|\mathcal{B}|(g(X) + \varepsilon) > \int_{\mathcal{B}} G(X + D\varphi) \, dx.
\]
Using Taylor’s formula, (6) and (7) we obtain

\[ |B|(g(X) + \varepsilon) > \int_{B \cap \{|X+D\varphi| \leq \delta\}} f(X + D\varphi) \, dx + \int_{B \cap \{|X+D\varphi| > \delta\}} \left( f(X) + Df(X)D\varphi + \frac{1}{2} D^2 f(X)(D\varphi, D\varphi) \right) \, dx \]

\[ = \int_{B \cap \{|X+D\varphi| \leq \delta\}} R(X, D\varphi) \, dx + \int_{B \cap \{|X+D\varphi| > \delta\}} \left( f(X) + Df(X)(D\varphi) + \frac{1}{2} D^2 f(X)(D\varphi, D\varphi) \right) \, dx \]

\[ \geq |B|f(X) + \frac{1}{2} \int_{B \cap \{|X+D\varphi| \leq \delta\}} |D\varphi|^2 (c - \Omega_r(|D\varphi|)) \, dx \geq |B|f(X), \]

where the last inequality follows from the definition of \( \delta \).

\[ \square \]

**PROPOSITION 1.** Let \( f : \mathbb{R}^{n \times m} \to \mathbb{R} \) be a \( C^2 \)-function satisfying a strict Legendre-Hadamard condition at every point. Then \( f \) is locally quasiconvex.

**Proof.** This follows easily by applying Lemma 2 to the functions \( f_X(Y) := f(X + Y), Y \in \mathbb{R}^{n \times m} \), where \( X \in \mathbb{R}^{n \times m} \) is fixed. \[ \square \]

According to Šverák [21] there exists a polynomial \( p \) of degree four on \( \mathbb{R}^{3 \times 2} \), which is rank-one convex but not quasiconvex. A closer inspection of the proof in [21] reveals that we may take \( p \) so that it additionally satisfies a strict Legendre-Hadamard condition at every point, hence by the above result \( p \) is locally quasiconvex.

Recall that a continuous function \( f \) is polyconvex if \( f(X) \) can be written as a convex function of the minors of \( X \). A polyconvex function is quasiconvex, but not conversely (cf. Ball [2], and [1], [20], [24], [25]). If one defines a concept of local polyconvexity as done above for quasiconvexity it is possible to prove that there are locally polyconvex functions on \( \mathbb{R}^{n \times m} \) \( (n, m \geq 2) \) that are not polyconvex. In higher dimensions, i.e. \( n \geq 3, m \geq 2, \) there are locally polyconvex functions on \( \mathbb{R}^{n \times m} \) that are not quasiconvex (cf. [9]).

**PROPOSITION 2.** Assume that \( n \geq 3, m \geq 2 \). For any \( r > 0 \) there exists a \( C^\infty \)-function \( f_r : \mathbb{R}^{n \times m} \to \mathbb{R} \) with the following two properties:

(I) \( f_r \) is not quasiconvex;

(II) for all \( X \in \mathbb{R}^{n \times m} \) there exists a quasiconvex function \( g_X \), such that \( g_X(Y) = f_r(Y) \) holds for \( |Y - X| < r \).

In particular, local quasiconvexity does not imply quasiconvexity.
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Proof. Let \( p : \mathbb{R}^{n \times m} \to \mathbb{R} \) be a polynomial of degree four which is rank-one convex, but not quasiconvex (cf. Šverák [21]). Take for each \( s > 1 \) two auxiliary functions \( \zeta_s, \xi_s \in C^\infty(\mathbb{R}) \) verifying

\[
\zeta_s(t) = \begin{cases} 
1 & \text{if } t < s \\
0 & \text{if } t > s + 1,
\end{cases}
\]

\[
\xi_s(t) = \begin{cases} 
0 & \text{if } t < s - 1 \\
t^2 & \text{if } t > s + 1,
\end{cases}
\]

and \( \xi_s \) non-decreasing, convex and \( \xi''_s(t) > 0 \) for \( t \in (s - 1, s + 1) \).

It is not hard to see that we may find \( s > 1 \) and \( k > 0 \), such that

\[
p(X)\zeta_s(|X|) + k\xi_s(|X|)
\]

is rank-one convex, but not quasiconvex (cf. Šverák [19] remark 3.4 and [20]). Next take \( \varepsilon > 0 \), so that

\[
g(X) := p(X)\zeta_s(|X|) + k\xi_s(|X|) + \varepsilon|X|^2
\]

is not quasiconvex. Notice that \( g \) satisfies a uniform Legendre-Hadamard condition:

\[
\int_B D^2 g(X)(D\varphi, D\varphi) \, dx \geq \varepsilon \int_B |D\varphi|^2 \, dx
\]

for all \( X \in \mathbb{R}^{n \times m} \) and all \( \varphi \in D \) with \( \text{spt}\varphi \subset B \).

Notice also that if \( R(X, Y) \) denotes the remainder term in the Taylor expansion of \( g \) about \( X \), then for some constant \( C > 0 \)

\[
|R(X, Y)| \leq 3 \int_0^1 (1 - t)^2 \sum_{|\alpha|=3} |\partial^\alpha g(X + tY) Y_\alpha| \, dt \leq C|Y|^3
\]

for all \( X, Y \in \mathbb{R}^{n \times m} \). In the notation of Lemma 2 (see (6)) this corresponds to \( \Omega_r(t) = 2Ct, \, t > 0 \), independent of \( r > 0 \).

Fix \( X_0 \in \mathbb{R}^{n \times m} \). We claim that there exists a quasiconvex extension of \( g \) from the closed ball \( |X - X_0| \leq \varepsilon/(4C) \). Indeed, define \( g_{X_0}(X) := g(X_0 + X) \) and notice that by Lemma 2 we may find a quasiconvex function \( G_{X_0} \), such that \( g(X + X_0) = g_{X_0}(X) = G_{X_0}(X) \) for \( |X| \leq \varepsilon/(4C) \), or equivalently, such that

\[
g(X) = G_{X_0}(X - X_0) \quad \text{for} \quad |X - X_0| \leq \frac{\varepsilon}{4C}.
\]
This proves the claim. Finally we define the function \( f_r \) as

\[
 f_r(X) := g \left( \frac{4C}{\varepsilon r} X \right), \quad X \in \mathbb{R}^{n \times m}.
\]

This finishes the proof. \( \square \)

Let \( C^\infty(\mathbb{R}^{n \times m}) \) denote the space of all real-valued \( C^\infty \)-functions \( f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R} \) and let \( \mathcal{F} \) denote the space of all extended real-valued functions \( F : \mathbb{R}^{n \times m} \rightarrow [-\infty, +\infty] \).

If we define the operator \( \mathcal{P}_{rc} : C^\infty(\mathbb{R}^{n \times m}) \rightarrow \mathcal{F} \) as

\[
 \mathcal{P}_{rc}(f)(X) := \inf \left\{ D^2 f(X)(a \otimes b, a \otimes b) : a \in \mathbb{R}^n, b \in \mathbb{R}^m \right\}, \quad X \in \mathbb{R}^{n \times m},
\]

then \( f \in C^\infty(\mathbb{R}^{n \times m}) \) is rank-one convex if and only if \( \mathcal{P}_{rc}(f) = 0 \). Furthermore, the operator \( \mathcal{P}_{rc} \) is local in the sense that if \( f, g \in C^\infty(\mathbb{R}^{n \times m}) \) are equal in a neighborhood of \( X \), then also \( \mathcal{P}_{rc}(f) \) equals \( \mathcal{P}_{rc}(g) \) in a neighborhood of \( X \). Thus:

\[
 f = g \text{ in a neighborhood of } X \Rightarrow \mathcal{P}_{rc}(f) = \mathcal{P}_{rc}(g) \text{ in a neighborhood of } X.
\]

It would be interesting if one could find a similar condition for quasiconvexity. That is, a local operator \( \mathcal{P}_{qc} : C^\infty(\mathbb{R}^{n \times m}) \rightarrow \mathcal{F} \) with the property

\[
 (*) \quad \mathcal{P}_{qc}(f) = 0 \iff f \text{ is quasiconvex}
\]

for \( f \in C^\infty(\mathbb{R}^{n \times m}) \).

**Theorem 1.** – In dimensions \( n \geq 3, m \geq 2 \) there does not exist a local operator

\[
 \mathcal{P} : C^\infty(\mathbb{R}^{n \times m}) \rightarrow \mathcal{F}
\]

with the property \((*)\).

**Remark.** – The proof will show that the operator \( \mathcal{P} \) cannot satisfy \((*)\) and the following locality-type condition: There exists a number \( r > 0 \), such that for \( f, g \in C^\infty(\mathbb{R}^{n \times m}) \) and \( X \in \mathbb{R}^{n \times m} \)

\[
 f(Y) = g(Y) \text{ for } |Y - X| \leq r \Rightarrow \mathcal{P}(f)(X) = \mathcal{P}(g)(X).
\]

**Proof.** – We argue by contradiction and assume that it is possible to find a local operator with the property \((*)\).
By Proposition 2 we may find a $C^\infty$-function $f : \mathbb{R}^{n \times m} \mapsto \mathbb{R}$ which is not quasiconvex, but agrees with quasiconvex functions on all balls of, say, radius one.

Let $\Phi_\varepsilon \in C^\infty$, $\varepsilon > 0$, be a non-negative mollifier with support contained in $\{X : |X| \leq \varepsilon\}$. Put $f_\varepsilon := f \ast \Phi_\varepsilon$, i.e. the convolution of $f$ and $\Phi_\varepsilon$.

We claim that if $\varepsilon \in (0, 1/2)$, then $f_\varepsilon$ is quasiconvex.

Fix $X \in \mathbb{R}^{n \times m}$. By the assumption on $f$ we may find a quasiconvex function $g_X : \mathbb{R}^{n \times m} \mapsto \mathbb{R}$, such that

$$f(Y) = g_X(Y) \text{ whenever } |Y - X| \leq 1.$$ 

Now if $g_{X,\varepsilon} := g_X \ast \Phi_\varepsilon$, then $g_{X,\varepsilon}$ is a quasiconvex $C^\infty$-function.

Furthermore, if $|Y - X| < 1/2$, then

$$g_{X,\varepsilon}(Y) = \int_{|Z-Y| \leq \varepsilon} \Phi_\varepsilon(Y-Z)g_X(Z) dZ = f_\varepsilon(Y),$$

hence by the locality of $\mathcal{P}$ and the quasiconvexity of $g_{X,\varepsilon}$

$$\mathcal{P}(f_\varepsilon)(X) = \mathcal{P}(g_{X,\varepsilon})(X) = 0.$$ 

Therefore it follows from the assumption that $f_\varepsilon$ is quasiconvex if $\varepsilon < 1/2$. If we let $\varepsilon$ tend to zero we get a contradiction. \hfill \Box

Before we state the next result we need some additional terminology. Let the space of continuous real-valued functions, be endowed with the usual metric making it a Fréchet space. The dual space, $C(\mathbb{R}^{n \times m})'$, is identified with, $\mathcal{M}_{comp}(\mathbb{R}^{n \times m})$, the space of compactly supported Radon measures. The space $\mathcal{M}_{comp}(\mathbb{R}^{n \times m})$ is endowed with the weak* topology.

Let $\Lambda$ be a non-empty set of compactly supported probabilities on $\mathbb{R}^{n \times m}$ all of which have center of mass at 0. Then we say that a continuous real-valued function $f : \mathbb{R}^{n \times m} \mapsto \mathbb{R}$ is $\Lambda$-convex if

$$\int f(X + Y) d\mu(Y) \geq f(X)$$

for all $\mu \in \Lambda$ and all $X \in \mathbb{R}^{n \times m}$.

Obviously, $\Lambda$-convexity is equivalent to $\operatorname{co}\Lambda$-convexity, where $\operatorname{co}\Lambda$ denotes the closed convex hull of $\Lambda$ in $\mathcal{M}_{comp}(\mathbb{R}^{n \times m})$.

This convexity concept also captures the concept of directional convexity (cf. [10], [14], [18], [23]).
Let $V$ be a non-empty subset of $C^0(\mathbb{R}^{n \times m})$. We say that the concept of $\Lambda$-convexity is local on $V$ if there exists a local operator $\mathcal{P} : V \mapsto \mathcal{F}$, such that for $f \in V$ we have

$$f \text{ is } \Lambda\text{-convex } \iff \mathcal{P}(f) = 0.$$ 

Let denote the set of probabilities $\mu$ of the form

$$\int \Phi \, d\mu := \sum_{i=1}^{N} t_i \Phi(X_i), \, \Phi \in C^0(\mathbb{R}^{n \times m}),$$

where $t_i \in [0, 1]$, $X_i \in \mathbb{R}^{n \times m}$ satisfy the $(H_N)$ condition and $\sum_{i=1}^{N} t_i X_i = 0$. We refer to Dacorogna (cf. [6]) for the definition of the $(H_N)$ condition.

We notice that $\Lambda_{rc}$-convexity is rank-one convexity.

Let $\Lambda_{qc}$ be the set of probabilities $\nu$ of the form

$$\int \Phi \, d\nu := \int_{B} \Phi(D\varphi(x)) \, dx, \, \Phi \in C^0(\mathbb{R}^{n \times m}),$$

for some $\varphi \in \mathcal{D}$ with $\text{spt} \varphi \subset B$.

We notice that $\Lambda_{qc}$-convexity is quasiconvexity.

The probabilities in $\text{co} \Lambda_{rc}$ and $\text{co} \Lambda_{qc}$ can be interpreted as certain homogeneous Young measures (cf. Kinderlehrer and Pedregal [7] and [16]). However, we shall not use this viewpoint here.

**Theorem 2.** Let $\Lambda$ be a set of compactly supported probabilities with center of mass at 0. Assume that

$$\text{co} \Lambda_{rc} \subseteq \text{co} \Lambda \subseteq \text{co} \Lambda_{qc}.$$ 

If $\Lambda$-convexity is local on $C^\infty(\mathbb{R}^{n \times m})$, then $\text{co} \Lambda = \text{co} \Lambda_{rc}$.

For the proof of Theorem 2 we need the following result which is essentially contained in [7], [16]. We outline the proof for the convenience of the reader.

**Lemma 3.** Let $\mu$ be a compactly supported probability measure on $\mathbb{R}^{n \times m}$ with center of mass $\bar{\mu} = 0$. If for all rank-one convex $C^\infty$-functions $f : \mathbb{R}^{n \times m} \mapsto \mathbb{R}$ with $\sup_X |D^3 f(X)| \leq 1$ the inequality

$$\int f \, d\mu \geq f(0)$$

holds, then $\mu \in \text{co} \Lambda_{rc}$. 

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Proof. – It is easily seen that if $f$ is a rank-one convex function, then it follows from (8) that also

$$\int f \, d\mu \geq f(0). \quad (9)$$

Let $T$ be a weakly* continuous linear functional on $\mathcal{M}_{\text{comp}}(\mathbb{R}^{n \times m})$ satisfying

$$T(\nu) \geq \alpha \quad (10)$$

for all $\nu \in c\Lambda_{rc}$, where $\alpha \in \mathbb{R}$. By Hahn-Banach’s separation theorem it is enough to show that also $T(\mu) \geq \alpha$. A weakly* continuous linear functional is an evaluation functional. Hence

$$T(\nu) = \int \Phi \, d\nu, \quad \nu \in \mathcal{M}_{\text{comp}}(\mathbb{R}^{n \times m}),$$

for some $\Phi \in C^0(\mathbb{R}^{n \times m})$. Now (10) gives that

$$R\Phi(0) = \inf \left\{ \int \Phi \, d\nu : \nu \in c\Lambda_{rc} \right\} \geq \alpha,$$

where $R\Phi$ is the rank-one convexification of $\Phi$ (cf. Dacorogna [6] and [8]). We end the proof by applying (9) with $f = R\Phi$. □

Proof (of Theorem 2). – Let $P : C^\infty(\mathbb{R}^{n \times m}) \mapsto \mathcal{F}$ denote the local operator detecting $\Lambda$-convexity. Let $\mu \in \Lambda$, and fix a rank-one convex $C^\infty$-function $f$ with $\sup_X |D^3f(X)| \leq 1$. For $\gamma > 0$, put $f_{\gamma}(X) := f(X) + \gamma|X|^2$, $X \in \mathbb{R}^{n \times m}$. Notice that

$$\int_B D^2f(X)(D\varphi, D\varphi) \, dx \geq \gamma \int_B |D\varphi|^2 \, dx$$

for all $\varphi \in \mathcal{D}$ with $\text{spt}\varphi \subset B$, and that $\sup_X |D^3f_{\gamma}(X)| \leq 1$. Hence by Lemma 2 $f_{\gamma}$ coincides with quasiconvex functions on balls of radius $\gamma/4$. Take $\varepsilon \in (0, \gamma/8)$, put $f_{\gamma,\varepsilon} := f_{\gamma} * \Phi_{\varepsilon}$. Here $\Phi_{\varepsilon}$ is the mollifier from the proof of Theorem 2. Obviously, $f_{\gamma,\varepsilon}$ equals quasiconvex $C^\infty$-functions on balls of radius $\gamma/8$. Consequently, by the locality of the operator $P$, $P(f_{\gamma,\varepsilon}) = 0$, and therefore by the assumption, $f_{\gamma,\varepsilon}$ is $\Lambda$-convex. In particular,

$$\int f_{\gamma,\varepsilon} \, d\mu \geq f_{\gamma,\varepsilon}(0)$$

for $\gamma > 0, \varepsilon \in (0, \gamma/8)$. Now let $\gamma$ tend to zero and apply Lemma 3 to finish the proof. □
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REFERENCES


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