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Global weak solutions for 1+2 dimensional wave maps into homogeneous spaces

by

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ABSTRACT. – In this paper, we consider the Cauchy problem of wave maps from 1+2 dimensional Minkowski space into a compact, homogeneous Riemannian manifold. We construct a finite energy global weak solution by a "vanishing viscosity" method. © Elsevier, Paris

RÉSUMÉ. – Dans ce travail nous construisons une solution globale faible avec énergie finie, du problème de Cauchy pour des "application d'ondes" de l'espace de Minkowski à valeurs dans une variété compacte homogène riemannienne. © Elsevier, Paris

1. MAIN RESULT

Given a compact Riemannian manifold N, isometrically embedded in R^n for some n, wave maps of 1+2 dimensional Minkowski space into N are solutions $u = (u^1, \cdots, u^n) : R \times R^2 \to N \subset R^n$ of the following system of semilinear wave equations

$$\Box u^i + \sum_{jk} \Gamma^i_{jk}(u) Q(Du^j, Du^k) = 0, \quad i = 1, \cdots, n \tag{1.1}$$

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where $\Box = \partial_t^2 - \triangle$ is the wave operator, $u_t = \partial_t u$, $\nabla u = (\partial_{x_1} u, \partial_{x_2} u)$, $D = (\partial_t, \nabla)$

$$Q(\xi, \eta) = \xi_0 \eta_0 - \sum_{\alpha=1}^{2} \xi_\alpha \eta_\alpha \tag{1.2}$$

and the coefficients Γ^i_{jk} depend smoothly on u.

We are interested in constructing a global weak solution to equation (1.1) with the following Cauchy data:

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{1.3}$$

where $u_0(x) \in N$, $u_1(x) \in T_{u_0(x)}N$. Here $T_u N$ denotes the tangent space to N at the point u.

J. Shatah [10] showed the existence of a finite energy global weak solution by penalty method in case $N = S^{n-1}$, the sphere. Recently, A. Freire [2] has been able to generalize Shatah's argument to prove the existence of global weak solution for certain compact homogeneous spaces N. In this paper, we shall establish the existence of global weak solution in the case that N is any compact homogeneous space. We construct our solution by a "vanishing viscosity" method. After the first version of this paper was completed, S. Müller & M. Struwe [7] were able to combine the compactness result of A. Freire, S. Müller & M. Struwe [3] with our viscous approximation method to show the global existence of weak solution for any compact manifold N.

Recall that the nonlinear term in (1.1) satisfies

$$\Gamma(u) Q(Du, Du) \perp T_u N. \tag{1.4}$$

Thus

$$\Box u \perp T_u N. \tag{1.5}$$

We shall regularize the equation by asking

$$\Box u - \varepsilon \triangle u_t \perp T_u N \tag{1.6}$$

where $\varepsilon > 0$ is a small parameter. In section 2, we shall prove that the regularized equation has the form

$$\Box u - \varepsilon T(u) \triangle u_t + \sum_{jk} \Gamma_{jk}(u) Q(Du^j, Du^k) = 0 \tag{1.7}$$

where $T(u)$ denotes the projection to $T_u N$.

We approximate u_0, u_1 by $u_{0\varepsilon}$ and $u_{1\varepsilon}$ such that

$$\nabla u_{0\varepsilon}, u_{1\varepsilon} \in C_0^\infty(R^2), \tag{1.8}$$

$$u_{0\varepsilon}(x) \in N, u_{1\varepsilon}(x) \in T_{u_{0\varepsilon}(x)}N \quad \forall x, \tag{1.9}$$

and

$$u_{0\varepsilon} \to u_0 \tag{1.10}$$

strongly in L^2_{loc},

$$\nabla u_{0\varepsilon} \to \nabla u_0, \tag{1.11}$$

$$u_{1\varepsilon} \to u_1 \tag{1.12}$$

strongly in L^2. Without loss of generality, we assume moreover

$$\|\nabla u_{0\varepsilon}\|^2_{L^2} + \|u_{1\varepsilon}\|^2_{L^2} \leq 2(\|\nabla u_0\|^2_{L^2} + \|u_1\|^2_{L^2}) = 4E_0. \tag{1.13}$$

We consider the following Cauchy problem for the regularized equation (1.7):

$$u(0,x) = u_{0\varepsilon}, \quad u_t(0,x) = u_{1\varepsilon}. \tag{1.14}$$

The following proposition was proved by S. Müller & M. Struwe [7]:

PROPOSITION 1.1. – *Let N be a compact Riemannian manifold then there exists a global smooth solution to the Cauchy problem (1.7),(1.14), provided that the initial data satisfy (1.8)(1.9).*

The global smooth solution to the regularized equation satisfies the following energy equality:

$$\begin{aligned} \|Du(t,\cdot)\|^2_{L^2} + 2\varepsilon \int_0^t \|\nabla u_s(s,\cdot)\|^2_{L^2} ds \\ = \|\nabla u_{0\varepsilon}\|^2_{L^2} + \|u_{1\varepsilon}\|^2_{L^2} \leq 4E_0. \end{aligned} \tag{1.15}$$

We shall prove that as $\varepsilon \to 0$,the solution of the regularized equation weakly converges to a global weak solution of (1.1). For that purpose, we make use of a geometric idea of Hélein as well as a variant of the well known div-curl Lemma of Murat [8] and Tartar [11]. We shall use the assumption that N is a homogeneous Riemannnian manifold at this point. By this, we mean that the group of isometries of N acts transitively on N, i.e. for any two points $p, q \in N$, there exists an isometry of N that maps p to q. The group of isometries of N is a Lie group, which we denote by Γ. We assume that its Lie algebra γ has some Euclidean structure and consider an orthonormal basis $(e_1, \cdots, e_p)$ of γ. We denote by ρ the representation of γ in the set of smooth sections of the tangent bundle of N. By Lemma 2 of Hélein [5], we know that there exist p smooth tangent vector fields $Y_1, \cdots, Y_p$ of N such that for any tangent vector $V \in T_u N$, we have

$$V = \sum_{\alpha=1}^{p} (V \cdot \rho(e_\alpha)(u)) Y_\alpha(u).$$

In section 2, we shall prove that the identities

$$\sum_{\alpha=1}^{p} (u_t \cdot \rho(e_\alpha)(u)) Y_{\alpha t}(u) = -\sum_{jk} \Gamma_{jk}(u) u_t^j u_t^k \tag{1.16}$$

and

$$\sum_{\alpha=1}^{p} (u_{x_i} \cdot \rho(e_\alpha)(u)) Y_{\alpha x_i}(u) = -\sum_{jk} \Gamma_{jk}(u) u_{x_i}^j u_{x_i}^k, \quad i = 1, 2 \tag{1.17}$$

hold for any u with $Du \in L^\infty([0,\tau]; L^2(R^2))$ and $u(t,x) \in N\ \mathcal{L}^3 a.e.$. By (1.16),(1.17), the regularized equation becomes

(1.18)

$$\Box u - \varepsilon T(u) \Delta u_t = \sum_{\alpha=1}^{p} [(u_t \cdot \rho(e_\alpha)(u)) Y_{\alpha t}(u) - \sum_{i=1}^{2} (u_{x_i} \cdot \rho(e_\alpha)(u)) Y_{\alpha x_i}(u)].$$

We now use the energy equality along with a variant of div-curl Lemma to pass to the weak limit. We shall establish the following

THEOREM 1.2. – *There exits a global finite energy weak solution to the Cauchy problem (1.1),(1.3) of 1+2 dimensional wave maps into a compact, homogeneous Riemannian manifold, provided that the initial energy is bounded. The weak solution satisfies (1.1),(1.3) in the sense of distributions, that is for any test function* $\phi = (\phi^1, \cdots, \phi^n) \in C_0^\infty(R^3)$, *there holds*

$$\sum_i \int_0^\infty \int_{R^2} [\Box \phi^i u^i + \phi^i \sum_{jk} \Gamma_{jk}^i(u) Q(Du^j, Du^k)] dx dt \\ + \int_{R^2} [\phi^i(0,x) u_1(x) - \phi_t^i(0,x) u_0(x)] dx = 0. \tag{1.19}$$

Moreover, the energy inequality is satisfied

$$\|\nabla u(t,\cdot)\|_{L^2}^2 + \|u_t(t,\cdot)\|_{L^2}^2 \le 2E_0.$$

2. REGULARIZED EQUATION

We first write down the regularized nonlinear equation. Let T denote the projector to the tangent space and let P denote the projector to the normal space. Then, for any tangent vector Y, we have

$$P(Y_{x_i}) = Y_{x_i} - TY_{x_i} = -\sum_{jk} \Gamma_{jk}(u) u^j_{x_i} Y^k. \tag{2.1}$$

Thus

$$P(\Box u) = -\sum_{jk} \Gamma_{jk}(u) Q(Du^j, Du^k),$$

so the regularized equation is

$$\Box u - \varepsilon T \Delta u_t + \sum_{jk} \Gamma_{jk}(u) Q(Du^j, Du^k) = 0. \tag{2.2}$$

We now make use of the assumption that N is a homogeneous space. We denote the group of isometries of N by Γ, and its Lie algebra by γ. Let $e_1, \cdots, e_p$ be an orthonormal basis of γ and let $\rho(e_1)(u), \cdots, \rho(e_p)(u)$ be its reprensetation in the set of smooth section of tangent bundle of N. By Lemma 2 of Hélein [5], we know that there exist p smooth tangent vector fields $Y_1, \cdots, Y_p$ of N such that for any tangent vector $V \in T_u N$, we have

$$V = \sum_{\alpha=1}^{p} (V \cdot \rho(e_\alpha)(u)) Y_\alpha(u). \tag{2.3}$$

In the following,we shall prove that the identities

(2.4)

$$\sum_{\alpha=1}^{p} (V \cdot \rho(e_\alpha)(u)) Y_{\alpha x_i}(u)(t,x) = -\sum_{jk} \Gamma_{jk}(u) u^j_{x_i} V^k(t,x), \mathcal{L}^3 a.e. i = 0,1,2,$$

hold for any u, V with $Du, V \in L^\infty((0,\tau); L^2(R^2))$ and $u(t,x) \in N$ $\mathcal{L}^3 a.e.$.

We first prove that (2.4) holds for any smooth function u and smooth vector field V. Noting that $\rho(e_\alpha)(u)$ is a Killing vector field, we know that its covariant derivatives vanish, namely

$$T\partial_{x_i}(\rho(e_\alpha)(u)) = 0 \quad i = 0,1,2. \tag{2.5}$$

By (2.3), we have

$$(2.6)\qquad \begin{aligned} TV_{x_i} &= \sum_{\alpha=1}^{p}(TV_{x_i}\cdot\rho(e_\alpha)(u))Y_\alpha(u)\\ &= \sum_{\alpha=1}^{p}(V_{x_i}\cdot\rho(e_\alpha)(u))Y_\alpha(u)\quad i=0,1,2; \end{aligned}$$

and

$$(2.7)\qquad \begin{aligned} V_{x_i} &= \sum_{\alpha=1}^{p}(V\cdot\rho(e_\alpha)(u))Y_{\alpha x_i}(u)\\ &+\sum_{\alpha=1}^{p}(V_{x_i}\cdot\rho(e_\alpha)(u))Y_\alpha(u)+\sum_{\alpha=1}^{p}(V\cdot(\rho(e_\alpha)(u))_{x_i})Y_\alpha(u)\\ &= \sum_{\alpha=1}^{p}(V\cdot\rho(e_\alpha)(u))Y_{\alpha x_i}(u)+TV_{x_i}\quad i=0,1,2. \end{aligned}$$

Thus, (2.4) holds for smooth u and V.

Next, we prove that (2.4) hold for any u and V with

$$Du, V\in L^\infty((0,\tau),C^\infty(R^2))\cap L^\infty((0,\tau),L^2(R^2)).$$

We regularize them by $u_\kappa = \pi(J_\kappa u)$ and $V_\kappa = T(u_\kappa)J_\kappa V$, where $J_\kappa = J_\kappa(t)$ is the Friedrich's mollifier and π denotes the nearest point projection to N. We shall prove that $Du_\kappa \to Du$ strongly in $L^2_{loc}((0,\tau)\times R^2)$, $V_\kappa \to V$ strongly in $L^2_{loc}((0,\tau)\times R^2)$ and

$$(2.8)\qquad u_\kappa(t,x)\to u(t,x)\quad \textit{strongly in}\quad L^\infty_{loc}((0,\tau)\times R^2).$$

It is quite easy to show strong convergence in L^2 once we established (2.8). We have

$$(2.9)\quad \begin{aligned} |u_\kappa(t,x)-u(t,x)| &\le |u(t,x)-J_\kappa u(t,x)|+|u_\kappa(t,x)-J_\kappa u(t,x)|\\ &\le 2|u(t,x)-J_\kappa u(t,x)|\le C\kappa|u_t|_{L^\infty((0,\tau)\times R^2))} \end{aligned}$$

Thus, (2.8) hold. Passing to the limit in

$$(2.10)\quad \sum_{\alpha=1}^{p}(V_\kappa\cdot\rho(e_\alpha)(u_\kappa))Y_{\alpha x_i}(u_\kappa) = -\sum_{jk}\Gamma_{jk}(u_\kappa)u^j_{\kappa x_i}V^k_\kappa,\quad i=0,1,2,$$

we get (2.4).

Finally, we prove (2.4) for $Du, V \in L^\infty((0,\tau), L^2(R^2))$. We regularize them by $u_\varepsilon = \pi(J_\varepsilon u)$ and $V_\varepsilon = T(u_\varepsilon)J_\varepsilon V$, where $J_\varepsilon = J_\varepsilon(x)$ is the Friedrich's mollifier. We shall prove that $Du_\varepsilon \to Du$ strongly in $L^2_{loc}((0,\tau)\times R^2)$, $V_\varepsilon \to V$ strongly in $L^2_{loc}((0,\tau)\times R^2)$. For that purpose, we denote by Ω a compact set in R^2 and define

$$G_\varepsilon(t) = sup_{x\in\Omega}\int_{B_\varepsilon(x)} |\nabla u(t,y)|^2 dy, \tag{2.11}$$

where B_ε is a ball of radius ε in R^2 centered at x. By Schoen & Uhlenbeck [9], section 4, $G_\varepsilon(t)$ converges to zero as ε goes to zero for any fixed t. Moreover

$$|u_\varepsilon(t,x) - J_\varepsilon u(t,x)| \le CG_\varepsilon^{\frac{1}{2}}(t), \quad \forall x \in \Omega. \tag{2.12}$$

Thus

$$\int_0^\tau |J_\varepsilon u(s,\cdot) - u_\varepsilon(s,\cdot)|^2_{L^\infty(\Omega)} ds \le C\int_0^\tau G_\varepsilon(s)ds. \tag{2.13}$$

We have

$$G_\varepsilon(s) \le \int_{R^2} |\nabla u(s,x)|^2 dx, \tag{2.14}$$

so by dominant convergence theorem, we get

$$\lim_{\varepsilon\to 0}\int_0^\tau |J_\varepsilon u(s,\cdot) - u_\varepsilon(s,\cdot)|^2_{L^\infty(\Omega)} ds \le \tag{2.15}$$

$$C\lim_{\varepsilon\to 0}\int_0^\tau G_\varepsilon(s)ds = C\int_0^\tau \lim_{\varepsilon\to 0} G_\varepsilon(s)ds = 0.$$

By (2.15), it it very easy to prove strong convergence $Du_\varepsilon \to Du$ in L^2, $V_\varepsilon \to V$ in L^2. By the the conclusion of the last step, we have

(2.16)

$$\sum_{\alpha=1}^{p}(V_\varepsilon\cdot\rho(e_\alpha)(u_\varepsilon))Y_{\alpha x_i}(u_\varepsilon) = -\sum_{jk}\Gamma_{jk}(u_\varepsilon)u^j_{\varepsilon x_i}V^k_\varepsilon, \quad i = 0,1,2.$$

Passing to the limit, we get (2.4). Identities (1.16) and (1.17) are easy consequences of (2.4).

We end this section by writting down the term $T(u)\triangle u_t$ explicitly. For that purpose, it is convenient to assume that N is parallelizable. However, we emphasis that the explicit expression of $T(u)\triangle u_t$ will never be used in our proofs. Therefore, we do not assume N is parallelizable in our theorems.

We have

$$T\triangle u_t = \triangle u_t - P\triangle u_t \tag{2.17}$$

$$P\triangle u_t = P(T\triangle u)_t + P(P\triangle u)_t$$

$$T\triangle u = \triangle u + \sum_{lm} \Gamma_{lm}(u)\nabla u^l \cdot \nabla u^m$$

$$P\triangle u = -\sum_{lm} \Gamma_{lm}(u)\nabla u^l \cdot \nabla u^m$$

and

$$P(T\triangle u)_t = -\sum_{jk} \Gamma_{jk}(u)u_t^j \triangle u^k - \sum_{jklm} \Gamma_{jm}(u)\Gamma_{kl}^m(u)u_t^j \nabla u^k \cdot \nabla u^l,$$

$$(P\triangle u)_t = -2\sum_{jk} \Gamma_{jk}(u)\nabla u_t^j \cdot \nabla u^k - \sum_{jkl} \frac{\partial \Gamma_{lk}(u)}{\partial u^j} u_t^j \nabla u^l \cdot \nabla u^k.$$

To calculate $P(P\triangle u)_t = (P\triangle u)_t - T(P\triangle u)_t$, we make use of the assumption that N is parallelizable. Then there exists a complete set of orthonormal tangent vector fields $Z_1(u), \cdots, Z_K(u)$. We thus get

$$T(P\triangle u)_t = -\sum_{\alpha=1}^{K} \sum_{jklm} \left(\frac{\partial \Gamma_{lk}^m(u)}{\partial u^j} u_t^j \nabla u^l \cdot \nabla u^k Z_\alpha^m(u) \right) Z_\alpha(u).$$

Therefore,

(2.18)

$$-P\triangle u_t = \sum_{jk} \Gamma_{jk}(u)(u_t^j \triangle u^k + 2\nabla u_t^j \nabla u^k) + \sum_{jkl} C_{jkl}(u) u_t^j \nabla u^k \cdot \nabla u^l$$

where

(2.19)

$$C_{jkl}^i(u) = \sum_m \Gamma_{jm}^i(u)\Gamma_{kl}^m(u) - \sum_{\alpha=1}^{K} \sum_m \left(\frac{\partial \Gamma_{lk}^m(u)}{\partial u^j} Z_\alpha^m(u) Z_\alpha^i(u) \right) - \frac{\partial \Gamma_{lk}^i(u)}{\partial u^j}.$$

3. WEAK LIMIT

In this section, we shall prove Theorem 1.2. By proposition 1.1, there exists a global smooth solution u_ε to the Cauchy problem (1.7),(1.14) of the regularized equation. The smooth solution satisfies the energy inequality

$$\|\nabla u_\varepsilon(t,\cdot)\|_{L^2}^2 + \|u_{\varepsilon t}(t,\cdot)\|_{L^2}^2 \leq 2E_0 \quad \forall t, \tag{3.1}$$

so there exists a subsequence, still denoted by u_ε for convenience, and a function u such that $\nabla u_\varepsilon \to \nabla u$, $u_{\varepsilon t} \to u_t$ weakly in $L^\infty([0,\tau], L^2(R^2))$ and $u_\varepsilon \to u$ weakly $*$ in $L^\infty(R^+ \times R^2)$.By passing to the weak limit, u still satisfies the energy inequality

$$\|\nabla u(t,\cdot)\|_{L^2}^2 + \|u_t(t,\cdot)\|_{L^2}^2 \leq 2E_0 \quad \forall t. \tag{3.2}$$

It remains to prove that u satisfies (1.1) in the sense of distributions. For that purpose, we recall equation (1.18)
(3.3)

$$\Box u_\varepsilon - \varepsilon T \triangle u_{\varepsilon t} = \sum_{\alpha=1}^{p} [(u_{\varepsilon t} \cdot \rho(e_\alpha)(u_\varepsilon)) Y_{\alpha t}(u_\varepsilon) - (\nabla u_\varepsilon \cdot \rho(e_\alpha)(u_\varepsilon)) \nabla Y_\alpha(u_\varepsilon)].$$

We test the equation by smooth functions $\phi = (\phi_1, \cdots, \phi_n)$ supported in $[-\tau, \tau] \times R^2$ and then integrate by parts to get

$$\int_0^\infty \int_{R^2} [\Box\phi \cdot u_\varepsilon - \varepsilon\phi \cdot T\triangle u_{\varepsilon t}] dxdt \tag{3.4}$$

$$= \int_0^\infty \int_{R^2} \phi \cdot \sum_{\alpha=1}^{p} [(u_{\varepsilon t} \cdot \rho(e_\alpha)(u_\varepsilon)) Y_{\alpha t}(u_\varepsilon) - (\nabla u_\varepsilon \cdot \rho(e_\alpha)(u_\varepsilon)) \nabla Y_\alpha(u_\varepsilon)] dxdt$$

$$- \int_{R^2} [\phi(0,x) \cdot u_{1\varepsilon}(x) - \phi_t(0,x) \cdot u_{0\varepsilon}(x)] dx = 0.$$

We first prove that $\varepsilon T \triangle u_{\varepsilon t} \rightharpoonup 0$ in the sense of distributions. In fact, we have

$$\varepsilon \int_0^\infty \int_{R^2} (\phi \cdot T\triangle u_{\varepsilon t}) dxdt = \varepsilon \int_0^\infty \int_{R^2} (T\phi) \cdot \triangle u_{\varepsilon t} dxdt \tag{3.5}$$

$$= -\varepsilon \int_0^\infty \int_{R^2} \nabla(T\phi) \cdot \nabla u_{\varepsilon t} dxdt$$

$$\leq \varepsilon\sqrt{\tau} \|\nabla u_{\varepsilon t}\|_{L^2([0,\tau]\times R^2)} sup_t \|\nabla(T\phi)(t,\cdot)\|_{L^2(R^2)}.$$

Noting that

$$T\phi = \sum_{\alpha} (\phi \cdot \rho(e_\alpha)(u_\varepsilon)) Y_\alpha(u_\varepsilon) \tag{3.6}$$

and using the energy inequality, we immediately get

$$|\varepsilon \int_0^\infty \int_{R^2} (\phi \cdot T\triangle u_{\varepsilon t}) dx dt| \leq C\sqrt{\varepsilon\tau} \to 0. \tag{3.7}$$

We now prove

$$(u_{\varepsilon t} \cdot \rho(e_\alpha)(u_\varepsilon)) Y_{\alpha t}(u_\varepsilon) - (\nabla u_\varepsilon \cdot \rho(e_\alpha)(u_\varepsilon)) \nabla Y_\alpha(u_\varepsilon) \rightharpoonup \tag{3.8}$$

$$(u_t \cdot \rho(e_\alpha)(u)) Y_{\alpha t}(u) - (\nabla u \cdot \rho(e_\alpha)(u)) \nabla Y_\alpha(u)$$

in the sense of distributions. For that purpose, we use the following variant of div-curl Lemma of Murat [8] and Tartar [11].

LEMMA 2.1. – *Let Ω be an open set in R^d and let $u_\varepsilon \rightharpoonup u$ weakly in $H^1_{loc}(\Omega)$, $u_\varepsilon \rightharpoonup u$ weakly $*$ in $L^\infty(\Omega)$ and $v_\varepsilon \rightharpoonup v$ weakly in $(L^2_{loc}(\Omega))^d$. Suppose that*

$$div v_\varepsilon = f_\varepsilon + g_\varepsilon \tag{3.9}$$

where $f_\varepsilon \to 0$ strongly in H^{-1} and $g_\varepsilon \to 0$ strongly in L^1_{loc}, then

$$grad u_\varepsilon \cdot v_\varepsilon \rightharpoonup grad u \cdot v \tag{3.10}$$

in the sense of distributions.

Noting (2.5), we have

$$\partial_t (u_{\varepsilon t} \cdot \rho(e_\alpha)(u_\varepsilon)) - \nabla (\nabla u_\varepsilon \cdot \rho(e_\alpha)(u_\varepsilon)) = \Box u_\varepsilon \cdot \rho(e_\alpha)(u_\varepsilon) \tag{3.11}$$

$$= \varepsilon \triangle u_{\varepsilon t} \cdot \rho(e_\alpha)(u_\varepsilon) = \varepsilon \nabla (\nabla u_{\varepsilon t} \cdot \rho(e_\alpha)(u_\varepsilon)) - \varepsilon \nabla u_{\varepsilon t} \cdot \nabla(\rho(e_\alpha)(u_\varepsilon)).$$

By the energy equality, the term

$$\sqrt{\varepsilon}(\nabla u_{\varepsilon t} \cdot \rho(e_\alpha)(u_\varepsilon))$$

is uniformly bounded in L^2_{loc} and the term

$$\sqrt{\varepsilon} \nabla u_{\varepsilon t} \cdot \nabla (\rho(e_\alpha)(u_\varepsilon))$$

is uniformly bounded in $L^1_{loc}(R^+ \times R^2)$, so the conditions of Lemma 2.1 is verified. Therefore, we proved that u satisfied

$$\int_0^\infty \int_{R^2} \left[\Box\phi \cdot u + \phi \cdot \sum_{\alpha=1}^{p} ((u_t \cdot \rho(e_\alpha)(u))Y_{\alpha t}(u) - (\nabla u \cdot \rho(e_\alpha)(u))\nabla Y_\alpha(u) \right]$$

$$+ \int_{R^2} [\phi(0,x) \cdot u_1(x) - \phi_t(0,x) \cdot u_0(x)]dx = 0.$$

By (1.16) and (1.17), we thus complete the proof of Theorem 1.3.

It remains to prove Lemma 2.1. Let $\phi \in C_0^\infty(\Omega)$ be a test function, then

$$\int \phi(gradu_\varepsilon \cdot v_\varepsilon - gradu \cdot v)dx = \int \phi grad(u_\varepsilon - u) \cdot v_\varepsilon + \int \phi gradu \cdot (v_\varepsilon - v)dx.$$

The second term tends to 0 in view of weak convergence. Integrating by parts in the first term yields

$$-\int [\phi(u_\varepsilon - u)f_\varepsilon + \phi(u_\varepsilon - u)g_\varepsilon + (u_\varepsilon - u)grad\phi \cdot v_\varepsilon]dx.$$

The first term in the above expression tends to 0 because $\phi(u_\varepsilon - u)$ is bounded in H^1 while $f_\varepsilon \to 0$ strongly in H^{-1}, the second term in the above expression tends to 0 because $\phi(u_\varepsilon - u)$ is bounded in L^∞ while $g_\varepsilon \to 0$ strongly in L^1_{loc} and the third term tends to 0 because $u_\varepsilon - u \to 0$ strongly in L^2_{loc} by Rellich's compactness theorem. Thus, we finished the proof of Lemma 2.1.

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