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# Global weak solutions for 1+2 dimensional wave maps into homogeneous spaces

by

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ABSTRACT. – In this paper, we consider the Cauchy problem of wave maps from 1+2 dimensional Minkowski space into a compact, homogeneous Riemannian manifold. We construct a finite energy global weak solution by a "vanishing viscosity" method. © Elsevier, Paris

RÉSUMÉ. – Dans ce travail nous construisons une solution globale faible avec énergie finie, du problème de Cauchy pour des "application d'ondes" de l'espace de Minkowski à valeurs dans une variété compacte homogène riemannienne. © Elsevier, Paris

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## 1. MAIN RESULT

Given a compact Riemannian manifold  $N$ , isometrically embedded in  $R^n$  for some  $n$ , wave maps of 1+2 dimensional Minkowski space into  $N$  are solutions  $u = (u^1, \dots, u^n) : R \times R^2 \rightarrow N \subset R^n$  of the following system of semilinear wave equations

$$(1.1) \quad \square u^i + \sum_{jk} \Gamma_{jk}^i(u) Q(Du^j, Du^k) = 0, \quad i = 1, \dots, n$$

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where  $\square = \partial_t^2 - \Delta$  is the wave operator,  $u_t = \partial_t u$ ,  $\nabla u = (\partial_{x_1} u, \partial_{x_2} u)$ ,  $D = (\partial_t, \nabla)$

$$(1.2) \quad Q(\xi, \eta) = \xi_0 \eta_0 - \sum_{\alpha=1}^2 \xi_\alpha \eta_\alpha$$

and the coefficients  $\Gamma_{jk}^i$  depend smoothly on  $u$ .

We are interested in constructing a global weak solution to equation (1.1) with the following Cauchy data:

$$(1.3) \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x)$$

where  $u_0(x) \in N$ ,  $u_1(x) \in T_{u_0(x)}N$ . Here  $T_u N$  denotes the tangent space to  $N$  at the point  $u$ .

J. Shatah [10] showed the existence of a finite energy global weak solution by penalty method in case  $N = S^{n-1}$ , the sphere. Recently, A. Freire [2] has been able to generalize Shatah’s argument to prove the existence of global weak solution for certain compact homogeneous spaces  $N$ . In this paper, we shall establish the existence of global weak solution in the case that  $N$  is any compact homogeneous space. We construct our solution by a "vanishing viscosity" method. After the first version of this paper was completed, S. Müller & M. Struwe [7] were able to combine the compactness result of A. Freire, S. Müller & M. Struwe [3] with our viscous approximation method to show the global existence of weak solution for any compact manifold  $N$ .

Recall that the nonlinear term in (1.1) satisfies

$$(1.4) \quad \Gamma(u)Q(Du, Du) \perp T_u N.$$

Thus

$$(1.5) \quad \square u \perp T_u N.$$

We shall regularize the equation by asking

$$(1.6) \quad \square u - \varepsilon \Delta u_t \perp T_u N$$

where  $\varepsilon > 0$  is a small parameter. In section 2, we shall prove that the regularized equation has the form

$$(1.7) \quad \square u - \varepsilon T(u) \Delta u_t + \sum_{jk} \Gamma_{jk}(u) Q(Du^j, Du^k) = 0$$

where  $T(u)$  denotes the projection to  $T_u N$ .

We approximate  $u_0, u_1$  by  $u_{0\varepsilon}$  and  $u_{1\varepsilon}$  such that

$$(1.8) \quad \nabla u_{0\varepsilon}, u_{1\varepsilon} \in C_0^\infty(R^2),$$

$$(1.9) \quad u_{0\varepsilon}(x) \in N, u_{1\varepsilon}(x) \in T_{u_{0\varepsilon}(x)}N \quad \forall x,$$

and

$$(1.10) \quad u_{0\varepsilon} \rightarrow u_0$$

strongly in  $L^2_{loc}$ ,

$$(1.11) \quad \nabla u_{0\varepsilon} \rightarrow \nabla u_0,$$

$$(1.12) \quad u_{1\varepsilon} \rightarrow u_1$$

strongly in  $L^2$ . Without loss of generality, we assume moreover

$$(1.13) \quad \|\nabla u_{0\varepsilon}\|_{L^2}^2 + \|u_{1\varepsilon}\|_{L^2}^2 \leq 2(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2) = 4E_0.$$

We consider the following Cauchy problem for the regularized equation (1.7):

$$(1.14) \quad u(0, x) = u_{0\varepsilon}, \quad u_t(0, x) = u_{1\varepsilon}.$$

The following proposition was proved by S. Müller & M. Struwe [7]:

PROPOSITION 1.1. – *Let  $N$  be a compact Riemannian manifold then there exists a global smooth solution to the Cauchy problem (1.7),(1.14), provided that the initial data satisfy (1.8)(1.9).*

The global smooth solution to the regularized equation satisfies the following energy equality:

$$(1.15) \quad \begin{aligned} \|Du(t, \cdot)\|_{L^2}^2 + 2\varepsilon \int_0^t \|\nabla u_s(s, \cdot)\|_{L^2}^2 ds \\ = \|\nabla u_{0\varepsilon}\|_{L^2}^2 + \|u_{1\varepsilon}\|_{L^2}^2 \leq 4E_0. \end{aligned}$$

We shall prove that as  $\varepsilon \rightarrow 0$ , the solution of the regularized equation weakly converges to a global weak solution of (1.1). For that purpose, we make use of a geometric idea of Hélein as well as a variant of the well known div-curl Lemma of Murat [8] and Tartar [11]. We shall use the

assumption that  $N$  is a homogeneous Riemannian manifold at this point. By this, we mean that the group of isometries of  $N$  acts transitively on  $N$ , i.e. for any two points  $p, q \in N$ , there exists an isometry of  $N$  that maps  $p$  to  $q$ . The group of isometries of  $N$  is a Lie group, which we denote by  $\Gamma$ . We assume that its Lie algebra  $\gamma$  has some Euclidean structure and consider an orthonormal basis  $(e_1, \dots, e_p)$  of  $\gamma$ . We denote by  $\rho$  the representation of  $\gamma$  in the set of smooth sections of the tangent bundle of  $N$ . By Lemma 2 of Hélein [5], we know that there exist  $p$  smooth tangent vector fields  $Y_1, \dots, Y_p$  of  $N$  such that for any tangent vector  $V \in T_u N$ , we have

$$V = \sum_{\alpha=1}^p (V \cdot \rho(e_\alpha)(u)) Y_\alpha(u).$$

In section 2, we shall prove that the identities

$$(1.16) \quad \sum_{\alpha=1}^p (u_t \cdot \rho(e_\alpha)(u)) Y_{\alpha t}(u) = - \sum_{jk} \Gamma_{jk}(u) u_t^j u_t^k$$

and

$$(1.17) \quad \sum_{\alpha=1}^p (u_{x_i} \cdot \rho(e_\alpha)(u)) Y_{\alpha x_i}(u) = - \sum_{jk} \Gamma_{jk}(u) u_{x_i}^j u_{x_i}^k, \quad i = 1, 2$$

hold for any  $u$  with  $Du \in L^\infty([0, \tau]; L^2(\mathbb{R}^2))$  and  $u(t, x) \in N$   $\mathcal{L}^3$ a.e.. By (1.16),(1.17), the regularized equation becomes

$$(1.18) \quad \square u - \varepsilon T(u) \Delta u_t = \sum_{\alpha=1}^p [(u_t \cdot \rho(e_\alpha)(u)) Y_{\alpha t}(u) - \sum_{i=1}^2 (u_{x_i} \cdot \rho(e_\alpha)(u)) Y_{\alpha x_i}(u)].$$

We now use the energy equality along with a variant of div-curl Lemma to pass to the weak limit. We shall establish the following

**THEOREM 1.2.** – *There exists a global finite energy weak solution to the Cauchy problem (1.1),(1.3) of 1+2 dimensional wave maps into a compact, homogeneous Riemannian manifold, provided that the initial energy is bounded. The weak solution satisfies (1.1),(1.3) in the sense of distributions, that is for any test function  $\phi = (\phi^1, \dots, \phi^n) \in C_0^\infty(\mathbb{R}^3)$ , there holds*

$$(1.19) \quad \sum_i \int_0^\infty \int_{\mathbb{R}^2} [\square \phi^i u^i + \phi^i \sum_{jk} \Gamma_{jk}^i(u) Q(Du^j, Du^k)] dx dt + \int_{\mathbb{R}^2} [\phi^i(0, x) u_1(x) - \phi_t^i(0, x) u_0(x)] dx = 0.$$

Moreover, the energy inequality is satisfied

$$\|\nabla u(t, \cdot)\|_{L^2}^2 + \|u_t(t, \cdot)\|_{L^2}^2 \leq 2E_0.$$

### 2. REGULARIZED EQUATION

We first write down the regularized nonlinear equation. Let  $T$  denote the projector to the tangent space and let  $P$  denote the projector to the normal space. Then, for any tangent vector  $Y$ , we have

$$(2.1) \quad P(Y_{x_i}) = Y_{x_i} - TY_{x_i} = - \sum_{jk} \Gamma_{jk}(u) u_{x_i}^j Y^k.$$

Thus

$$P(\square u) = - \sum_{jk} \Gamma_{jk}(u) Q(Du^j, Du^k),$$

so the regularized equation is

$$(2.2) \quad \square u - \varepsilon T \Delta u_t + \sum_{jk} \Gamma_{jk}(u) Q(Du^j, Du^k) = 0.$$

We now make use of the assumption that  $N$  is a homogeneous space. We denote the group of isometries of  $N$  by  $\Gamma$ , and its Lie algebra by  $\gamma$ . Let  $e_1, \dots, e_p$  be an orthonormal basis of  $\gamma$  and let  $\rho(e_1)(u), \dots, \rho(e_p)(u)$  be its representation in the set of smooth section of tangent bundle of  $N$ . By Lemma 2 of Hélein [5], we know that there exist  $p$  smooth tangent vector fields  $Y_1, \dots, Y_p$  of  $N$  such that for any tangent vector  $V \in T_u N$ , we have

$$(2.3) \quad V = \sum_{\alpha=1}^p (V \cdot \rho(e_\alpha)(u)) Y_\alpha(u).$$

In the following, we shall prove that the identities

$$(2.4) \quad \sum_{\alpha=1}^p (V \cdot \rho(e_\alpha)(u)) Y_{\alpha x_i}(u)(t, x) = - \sum_{jk} \Gamma_{jk}(u) u_{x_i}^j V^k(t, x), \mathcal{L}^3 a.e. i = 0, 1, 2,$$

hold for any  $u, V$  with  $Du, V \in L^\infty((0, \tau); L^2(\mathbb{R}^2))$  and  $u(t, x) \in N$   $\mathcal{L}^3 a.e.$

We first prove that (2.4) holds for any smooth function  $u$  and smooth vector field  $V$ . Noting that  $\rho(e_\alpha)(u)$  is a Killing vector field, we know that its covariant derivatives vanish, namely

$$(2.5) \quad T \partial_{x_i} (\rho(e_\alpha)(u)) = 0 \quad i = 0, 1, 2.$$

By (2.3), we have

$$\begin{aligned}
 (2.6) \quad TV_{x_i} &= \sum_{\alpha=1}^p (TV_{x_i} \cdot \rho(e_\alpha)(u)) Y_\alpha(u) \\
 &= \sum_{\alpha=1}^p (V_{x_i} \cdot \rho(e_\alpha)(u)) Y_\alpha(u) \quad i = 0, 1, 2;
 \end{aligned}$$

and

$$\begin{aligned}
 (2.7) \quad V_{x_i} &= \sum_{\alpha=1}^p (V \cdot \rho(e_\alpha)(u)) Y_{\alpha x_i}(u) \\
 &+ \sum_{\alpha=1}^p (V_{x_i} \cdot \rho(e_\alpha)(u)) Y_\alpha(u) + \sum_{\alpha=1}^p (V \cdot (\rho(e_\alpha)(u))_{x_i}) Y_\alpha(u) \\
 &= \sum_{\alpha=1}^p (V \cdot \rho(e_\alpha)(u)) Y_{\alpha x_i}(u) + TV_{x_i} \quad i = 0, 1, 2.
 \end{aligned}$$

Thus, (2.4) holds for smooth  $u$  and  $V$ .

Next, we prove that (2.4) hold for any  $u$  and  $V$  with

$$Du, V \in L^\infty((0, \tau), C^\infty(\mathbb{R}^2)) \cap L^\infty((0, \tau), L^2(\mathbb{R}^2)).$$

We regularize them by  $u_\kappa = \pi(J_\kappa u)$  and  $V_\kappa = T(u_\kappa)J_\kappa V$ , where  $J_\kappa = J_\kappa(t)$  is the Friedrich's mollifier and  $\pi$  denotes the nearest point projection to  $N$ . We shall prove that  $Du_\kappa \rightarrow Du$  strongly in  $L^2_{loc}((0, \tau) \times \mathbb{R}^2)$ ,  $V_\kappa \rightarrow V$  strongly in  $L^2_{loc}((0, \tau) \times \mathbb{R}^2)$  and

$$(2.8) \quad u_\kappa(t, x) \rightarrow u(t, x) \quad \text{strongly in } L^\infty_{loc}((0, \tau) \times \mathbb{R}^2).$$

It is quite easy to show strong convergence in  $L^2$  once we established (2.8). We have

$$\begin{aligned}
 (2.9) \quad |u_\kappa(t, x) - u(t, x)| &\leq |u(t, x) - J_\kappa u(t, x)| + |u_\kappa(t, x) - J_\kappa u(t, x)| \\
 &\leq 2|u(t, x) - J_\kappa u(t, x)| \leq C\kappa |u_t|_{L^\infty((0, \tau) \times \mathbb{R}^2)}
 \end{aligned}$$

Thus, (2.8) hold. Passing to the limit in

$$(2.10) \quad \sum_{\alpha=1}^p (V_\kappa \cdot \rho(e_\alpha)(u_\kappa)) Y_{\alpha x_i}(u_\kappa) = - \sum_{jk} \Gamma_{jk}(u_\kappa) u_{\kappa x_i}^j V_\kappa^k, \quad i = 0, 1, 2,$$

we get (2.4).

Finally, we prove (2.4) for  $Du, V \in L^\infty((0, \tau), L^2(R^2))$ . We regularize them by  $u_\varepsilon = \pi(J_\varepsilon u)$  and  $V_\varepsilon = T(u_\varepsilon)J_\varepsilon V$ , where  $J_\varepsilon = J_\varepsilon(x)$  is the Friedrich's mollifier. We shall prove that  $Du_\varepsilon \rightarrow Du$  strongly in  $L^2_{loc}((0, \tau) \times R^2)$ ,  $V_\varepsilon \rightarrow V$  strongly in  $L^2_{loc}((0, \tau) \times R^2)$ . For that purpose, we denote by  $\Omega$  a compact set in  $R^2$  and define

$$(2.11) \quad G_\varepsilon(t) = \sup_{x \in \Omega} \int_{B_\varepsilon(x)} |\nabla u(t, y)|^2 dy,$$

where  $B_\varepsilon$  is a ball of radius  $\varepsilon$  in  $R^2$  centered at  $x$ . By Schoen & Uhlenbeck [9], section 4,  $G_\varepsilon(t)$  converges to zero as  $\varepsilon$  goes to zero for any fixed  $t$ . Moreover

$$(2.12) \quad |u_\varepsilon(t, x) - J_\varepsilon u(t, x)| \leq C G_\varepsilon^{\frac{1}{2}}(t), \quad \forall x \in \Omega.$$

Thus

$$(2.13) \quad \int_0^\tau \|J_\varepsilon u(s, \cdot) - u_\varepsilon(s, \cdot)\|_{L^\infty(\Omega)}^2 ds \leq C \int_0^\tau G_\varepsilon(s) ds.$$

We have

$$(2.14) \quad G_\varepsilon(s) \leq \int_{R^2} |\nabla u(s, x)|^2 dx,$$

so by dominant convergence theorem, we get

$$(2.15) \quad \lim_{\varepsilon \rightarrow 0} \int_0^\tau \|J_\varepsilon u(s, \cdot) - u_\varepsilon(s, \cdot)\|_{L^\infty(\Omega)}^2 ds \leq C \lim_{\varepsilon \rightarrow 0} \int_0^\tau G_\varepsilon(s) ds = C \int_0^\tau \lim_{\varepsilon \rightarrow 0} G_\varepsilon(s) ds = 0.$$

By (2.15), it is very easy to prove strong convergence  $Du_\varepsilon \rightarrow Du$  in  $L^2$ ,  $V_\varepsilon \rightarrow V$  in  $L^2$ . By the conclusion of the last step, we have

$$(2.16) \quad \sum_{\alpha=1}^p (V_\varepsilon \cdot \rho(e_\alpha)(u_\varepsilon)) Y_{\alpha x_i}(u_\varepsilon) = - \sum_{jk} \Gamma_{jk}(u_\varepsilon) u_{\varepsilon x_i}^j V_\varepsilon^k, \quad i = 0, 1, 2.$$

Passing to the limit, we get (2.4). Identities (1.16) and (1.17) are easy consequences of (2.4).

We end this section by writing down the term  $T(u)\Delta u_t$  explicitly. For that purpose, it is convenient to assume that  $N$  is parallelizable. However,



we emphasize that the explicit expression of  $T(u)\Delta u_t$  will never be used in our proofs. Therefore, we do not assume  $N$  is parallelizable in our theorems.

We have

$$(2.17) \quad T\Delta u_t = \Delta u_t - P\Delta u_t$$

$$P\Delta u_t = P(T\Delta u)_t + P(P\Delta u)_t$$

$$T\Delta u = \Delta u + \sum_{lm} \Gamma_{lm}(u) \nabla u^l \cdot \nabla u^m$$

$$P\Delta u = - \sum_{lm} \Gamma_{lm}(u) \nabla u^l \cdot \nabla u^m$$

and

$$P(T\Delta u)_t = - \sum_{jk} \Gamma_{jk}(u) u_t^j \Delta u^k - \sum_{jklm} \Gamma_{jm}(u) \Gamma_{kl}^m(u) u_t^j \nabla u^k \cdot \nabla u^l,$$

$$(P\Delta u)_t = -2 \sum_{jk} \Gamma_{jk}(u) \nabla u_t^j \cdot \nabla u^k - \sum_{jkl} \frac{\partial \Gamma_{lk}(u)}{\partial u^j} u_t^j \nabla u^l \cdot \nabla u^k.$$

To calculate  $P(P\Delta u)_t = (P\Delta u)_t - T(P\Delta u)_t$ , we make use of the assumption that  $N$  is parallelizable. Then there exists a complete set of orthonormal tangent vector fields  $Z_1(u), \dots, Z_K(u)$ . We thus get

$$T(P\Delta u)_t = - \sum_{\alpha=1}^K \sum_{jklm} \left( \frac{\partial \Gamma_{lk}^m(u)}{\partial u^j} u_t^j \nabla u^l \cdot \nabla u^k Z_\alpha^m(u) \right) Z_\alpha(u).$$

Therefore,

$$(2.18) \quad -P\Delta u_t = \sum_{jk} \Gamma_{jk}(u) (u_t^j \Delta u^k + 2 \nabla u_t^j \nabla u^k) + \sum_{jkl} C_{jkl}(u) u_t^j \nabla u^k \cdot \nabla u^l$$

where

$$(2.19) \quad C_{jkl}^i(u) = \sum_m \Gamma_{jm}^i(u) \Gamma_{kl}^m(u) - \sum_{\alpha=1}^K \sum_m \left( \frac{\partial \Gamma_{lk}^m(u)}{\partial u^j} Z_\alpha^m(u) Z_\alpha^i(u) \right) - \frac{\partial \Gamma_{lk}^i(u)}{\partial u^j}.$$

### 3. WEAK LIMIT

In this section, we shall prove Theorem 1.2. By proposition 1.1, there exists a global smooth solution  $u_\varepsilon$  to the Cauchy problem (1.7),(1.14) of the regularized equation. The smooth solution satisfies the energy inequality

$$(3.1) \quad \|\nabla u_\varepsilon(t, \cdot)\|_{L^2}^2 + \|u_{\varepsilon t}(t, \cdot)\|_{L^2}^2 \leq 2E_0 \quad \forall t,$$

so there exists a subsequence, still denoted by  $u_\varepsilon$  for convenience, and a function  $u$  such that  $\nabla u_\varepsilon \rightharpoonup \nabla u$ ,  $u_{\varepsilon t} \rightharpoonup u_t$  weakly in  $L^\infty([0, \tau], L^2(\mathbb{R}^2))$  and  $u_\varepsilon \rightharpoonup u$  weakly  $*$  in  $L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ . By passing to the weak limit,  $u$  still satisfies the energy inequality

$$(3.2) \quad \|\nabla u(t, \cdot)\|_{L^2}^2 + \|u_t(t, \cdot)\|_{L^2}^2 \leq 2E_0 \quad \forall t.$$

It remains to prove that  $u$  satisfies (1.1) in the sense of distributions. For that purpose, we recall equation (1.18)

$$(3.3) \quad \square u_\varepsilon - \varepsilon T \Delta u_{\varepsilon t} = \sum_{\alpha=1}^p [(u_{\varepsilon t} \cdot \rho(e_\alpha)(u_\varepsilon)) Y_{\alpha t}(u_\varepsilon) - (\nabla u_\varepsilon \cdot \rho(e_\alpha)(u_\varepsilon)) \nabla Y_\alpha(u_\varepsilon)].$$

We test the equation by smooth functions  $\phi = (\phi_1, \dots, \phi_n)$  supported in  $[-\tau, \tau] \times \mathbb{R}^2$  and then integrate by parts to get

$$(3.4) \quad \int_0^\infty \int_{\mathbb{R}^2} [\square \phi \cdot u_\varepsilon - \varepsilon \phi \cdot T \Delta u_{\varepsilon t}] dx dt$$

$$= \int_0^\infty \int_{\mathbb{R}^2} \phi \cdot \sum_{\alpha=1}^p [(u_{\varepsilon t} \cdot \rho(e_\alpha)(u_\varepsilon)) Y_{\alpha t}(u_\varepsilon) - (\nabla u_\varepsilon \cdot \rho(e_\alpha)(u_\varepsilon)) \nabla Y_\alpha(u_\varepsilon)] dx dt$$

$$- \int_{\mathbb{R}^2} [\phi(0, x) \cdot u_{1\varepsilon}(x) - \phi_t(0, x) \cdot u_{0\varepsilon}(x)] dx = 0.$$

We first prove that  $\varepsilon T \Delta u_{\varepsilon t} \rightharpoonup 0$  in the sense of distributions. In fact, we have

$$(3.5) \quad \varepsilon \int_0^\infty \int_{\mathbb{R}^2} (\phi \cdot T \Delta u_{\varepsilon t}) dx dt = \varepsilon \int_0^\infty \int_{\mathbb{R}^2} (T \phi) \cdot \Delta u_{\varepsilon t} dx dt$$

$$= -\varepsilon \int_0^\infty \int_{\mathbb{R}^2} \nabla(T \phi) \cdot \nabla u_{\varepsilon t} dx dt$$

$$\leq \varepsilon \sqrt{\tau} \|\nabla u_{\varepsilon t}\|_{L^2([0, \tau] \times \mathbb{R}^2)} \sup_t \|\nabla(T \phi)(t, \cdot)\|_{L^2(\mathbb{R}^2)}.$$

Noting that

$$(3.6) \quad T\phi = \sum_{\alpha} (\phi \cdot \rho(e_{\alpha})(u_{\varepsilon})) Y_{\alpha}(u_{\varepsilon})$$

and using the energy inequality, we immediately get

$$(3.7) \quad |\varepsilon \int_0^{\infty} \int_{\mathbb{R}^2} (\phi \cdot T \Delta u_{\varepsilon t}) dx dt| \leq C \sqrt{\varepsilon \tau} \rightarrow 0.$$

We now prove

$$(3.8) \quad (u_{\varepsilon t} \cdot \rho(e_{\alpha})(u_{\varepsilon})) Y_{\alpha t}(u_{\varepsilon}) - (\nabla u_{\varepsilon} \cdot \rho(e_{\alpha})(u_{\varepsilon})) \nabla Y_{\alpha}(u_{\varepsilon}) \rightharpoonup \\ (u_t \cdot \rho(e_{\alpha})(u)) Y_{\alpha t}(u) - (\nabla u \cdot \rho(e_{\alpha})(u)) \nabla Y_{\alpha}(u)$$

in the sense of distributions. For that purpose, we use the following variant of div-curl Lemma of Murat [8] and Tartar [11].

LEMMA 2.1. – *Let  $\Omega$  be an open set in  $\mathbb{R}^d$  and let  $u_{\varepsilon} \rightharpoonup u$  weakly in  $H_{loc}^1(\Omega)$ ,  $u_{\varepsilon} \rightharpoonup u$  weakly  $*$  in  $L^{\infty}(\Omega)$  and  $v_{\varepsilon} \rightharpoonup v$  weakly in  $(L_{loc}^2(\Omega))^d$ . Suppose that*

$$(3.9) \quad \operatorname{div} v_{\varepsilon} = f_{\varepsilon} + g_{\varepsilon}$$

where  $f_{\varepsilon} \rightarrow 0$  strongly in  $H^{-1}$  and  $g_{\varepsilon} \rightarrow 0$  strongly in  $L_{loc}^1$ , then

$$(3.10) \quad \operatorname{grad} u_{\varepsilon} \cdot v_{\varepsilon} \rightharpoonup \operatorname{grad} u \cdot v$$

in the sense of distributions.

Noting (2.5), we have

$$(3.11) \quad \partial_t (u_{\varepsilon t} \cdot \rho(e_{\alpha})(u_{\varepsilon})) - \nabla (\nabla u_{\varepsilon} \cdot \rho(e_{\alpha})(u_{\varepsilon})) = \square u_{\varepsilon} \cdot \rho(e_{\alpha})(u_{\varepsilon}) \\ = \varepsilon \Delta u_{\varepsilon t} \cdot \rho(e_{\alpha})(u_{\varepsilon}) = \varepsilon \nabla (\nabla u_{\varepsilon t} \cdot \rho(e_{\alpha})(u_{\varepsilon})) - \varepsilon \nabla u_{\varepsilon t} \cdot \nabla (\rho(e_{\alpha})(u_{\varepsilon})).$$

By the energy equality, the term

$$\sqrt{\varepsilon} (\nabla u_{\varepsilon t} \cdot \rho(e_{\alpha})(u_{\varepsilon}))$$

is uniformly bounded in  $L_{loc}^2$  and the term

$$\sqrt{\varepsilon} \nabla u_{\varepsilon t} \cdot \nabla (\rho(e_{\alpha})(u_{\varepsilon}))$$

is uniformly bounded in  $L^1_{loc}(R^+ \times R^2)$ , so the conditions of Lemma 2.1 is verified. Therefore, we proved that  $u$  satisfied

$$\int_0^\infty \int_{R^2} \left[ \square \phi \cdot u + \phi \cdot \sum_{\alpha=1}^p ((u_t \cdot \rho(e_\alpha)(u))Y_{\alpha t}(u) - (\nabla u \cdot \rho(e_\alpha)(u))\nabla Y_\alpha(u)) \right] + \int_{R^2} [\phi(0, x) \cdot u_1(x) - \phi_t(0, x) \cdot u_0(x)] dx = 0.$$

By (1.16) and (1.17), we thus complete the proof of Theorem 1.3.

It remains to prove Lemma 2.1. Let  $\phi \in C^\infty_0(\Omega)$  be a test function, then

$$\int \phi(\text{grad}u_\varepsilon \cdot v_\varepsilon - \text{grad}u \cdot v) dx = \int \phi \text{grad}(u_\varepsilon - u) \cdot v_\varepsilon + \int \phi \text{grad}u \cdot (v_\varepsilon - v) dx.$$

The second term tends to 0 in view of weak convergence. Integrating by parts in the first term yields

$$- \int [\phi(u_\varepsilon - u)f_\varepsilon + \phi(u_\varepsilon - u)g_\varepsilon + (u_\varepsilon - u)\text{grad}\phi \cdot v_\varepsilon] dx.$$

The first term in the above expression tends to 0 because  $\phi(u_\varepsilon - u)$  is bounded in  $H^1$  while  $f_\varepsilon \rightarrow 0$  strongly in  $H^{-1}$ , the second term in the above expression tends to 0 because  $\phi(u_\varepsilon - u)$  is bounded in  $L^\infty$  while  $g_\varepsilon \rightarrow 0$  strongly in  $L^1_{loc}$  and the third term tends to 0 because  $u_\varepsilon - u \rightarrow 0$  strongly in  $L^2_{loc}$  by Rellich's compactness theorem. Thus, we finished the proof of Lemma 2.1.

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