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by

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ABSTRACT. - We study the structure of extremals of a class of second order variational problems without convexity, on intervals in $\mathbb{R}_+$. The problems are related to a model in thermodynamics introduced in [7]. We are interested in properties of the extremals which are independent of the length of the interval, for all sufficiently large intervals. As in [12, 13] the study of these properties is based on the relation between the variational problem on bounded, large intervals and a limiting problem on $\mathbb{R}_+$. Our investigation employs techniques developed in [10, 12, 13] along with turnpike techniques developed in [16, 17]. © Elsevier, Paris

Key words: Turnpike properties, $(f)$-good functions, periodic minimizers.

RéSUMÉ. – On étudie la structure des extrémales d’une classe de problèmes variationnels non convexes du deuxième ordre, sur des intervalles de $\mathbb{R}_+$. Ces problèmes sont reliés à un modèle thermodynamique introduit dans [7]. Nous nous intéressons aux propriétés des extrémales qui ne dépendent pas de la longueur des intervalles, pourvu que ceux-ci soient assez grands. Comme dans [12,13] l’étude de ces propriétés s’appuie sur la relation entre le problème variationnel sur de grands intervalles bornés et un problème limite sur $\mathbb{R}_+$. Notre travail emploie des techniques développées dans [10,12,13] ainsi que dans [16,17]. © Elsevier, Paris

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1. INTRODUCTION

In this paper we investigate the structure of optimal solutions of variational problems associated with the functional

\[ J^f(D; w) = |D|^{-1} \int_D f(w(t), w'(t), w''(t)) \, dt, \quad \forall w \in W^{2,1}(D), \]

where \( D \) is a bounded interval on the real line and \( f \in C(R^3) \) belongs to a space of functions to be described below. Specifically we shall consider the problems,

\[ (P_D) \quad \inf \{ J^f(D; w) : w \in W^{2,1}(D) \} \]

and, for \( D = (T_1, T_2) \),

\[ (P_D^{x,y}) \quad \inf \{ J^f(D; w) : w \in W^{2,1}(D), (w, w')(T_1) = x, (w, w')(T_2) = y \}. \]

In connection with these we shall also study the following problem on the half line:

\[ (P_\infty) \quad \inf \{ J^f(w) : w \in W^{2,1}_{loc}(0, \infty) \}, \]

where

\[ J^f(w) = \lim_{T \to \infty} \inf J^f((0, T); w). \]

This can be seen as a limiting problem for \( (P_D) \) as \( |D| \to \infty \). Variational problems of this type were considered by Leizarowitz and Mizel [10]. Similar constrained problems (involving a mass constraint), were studied by Coleman, Marcus and Mizel [7] and by Marcus [12,13]. The constrained problems were conceived as models for determining the thermodynamical equilibrium states of unidimensional bodies involving ‘second order’ materials (see [7]).

Let \( G = G(p, r) \) be a function in \( C^4(R^2) \) such that

\[ \begin{align*}
\partial^2 G/\partial r^2(p, r) &> 0, \\
G(p, r) &\geq |r|^\gamma - b_1 |p|^\beta - b_0, \quad \forall (p, r) \in R^2,
\end{align*} \]

where \( b_1, b_0 \) are positive constants, \( 1 \leq \beta \leq \gamma \) and \( \gamma > 1 \). In addition assume that,

\[ \max\{|G(p, r)|, |\partial G/\partial r(p, r)|, |\partial G/\partial p(p, r)|\} \leq M(|p|)(1 + |r|^\gamma), \]

where \( M : [0, \infty) \to [0, \infty) \) is a continuous function. A typical example is \( G(p, r) = r^2 - bp^2 \).
Let $\alpha, b_2, b_3$ be positive numbers, with $\alpha > \beta$, and let
\begin{equation}
\mathcal{L} = \mathcal{L}(\alpha, b_2, b_3) = \{\phi \in C^2(R^1) : \phi(t) \geq b_3|t|^\alpha - b_2, \quad \forall t \in R^1\}.
\end{equation}

The space $\mathcal{L}$ will be equipped with the standard topology of $C^2$. Finally denote,
\begin{equation}
\mathcal{L}_G = \mathcal{L}_G(\alpha, b_2, b_3) = \{F_\phi : \phi \in \mathcal{L}(\alpha, b_2, b_3)\},
\end{equation}
where,
\begin{equation}
F_\phi(w, p, r) = \phi(w) + G(p, r), \quad \forall (w, p, r) \in R^3.
\end{equation}

The relation between the minimizers of $(PD)$ (for large $|D|$) and those of $(P_\infty)$ plays a crucial role in our study of their structure. This relation was first investigated by Marcus [12, 13] where it was used in order to derive structural properties of minimizers of problem $(PD)$ and of related constrained problems, in the case $f = r^2 - bp^2 + \phi(w)$. In the present paper we pursue this investigation combining techniques of [12, 13] with turnpike techniques as in Zaslavski [16, 17].

One of our main results is the uniqueness of periodic minimizers of $(P_\infty)$ which is generically valid in a very precise sense.

For every potential $\phi \in \mathcal{L}(\alpha, b_2, b_3)$ there exists a family of arbitrarily small perturbations $\{\phi_s = \phi + s\theta : 0 < s < 1\}$, such that problem $(P_\infty)$ with $f = F_{\phi_s}$ possesses a unique (up to translation) periodic minimizer.

The function $\theta$ can be explicitly constructed in terms of the extremal values of periodic minimizers of $(P_\infty)$ with $f = F_\phi$. Combining this result with a recent result of Zaslavski [18], we show that for each potential $\phi_s$ in this family, the corresponding integrand $F_{\phi_s}$ possesses an asymptotic turnpike property, which involves the behaviour of the limit set of minimizers of $(P_\infty)$. Finally, we show that this asymptotic property can be used in order to derive detailed information on the structure of minimizers of problem $(PD)$ for all sufficiently large intervals $D$. In this last part the results are valid not only in the generic sense, but apply to every $f \in \mathcal{L}_G$.

A brief comparison of the present results with those of [13]: In the present work, as in [13], the structure of minimizers of $(PD)$ is described by observing their behaviour in a 'window' of fixed length (independent of $|D|$) which can be placed anywhere in $D$. The results of [13] apply to every integrand of the form $f = r^2 - bp^2 + \phi(w)$, for a class of potentials $\phi$ which
includes the standard two-well potentials. The behaviour of minimizers of $(P_D)$ in a 'window' is described by integral estimates, involving 'mass' and 'energy'. The present results are in part generic, but they deal with a very large class of integrands and the behaviour of minimizers in a 'window' is described by pointwise estimates which provide considerably more detailed information.

For a precise statement of the results mentioned above we need some additional notation and definitions.

Let $\mu(f)$ denote the infimum in $(P_\infty)$ with $f \in \mathcal{L}_G$. Leizarowitz and Mizel [10] proved that, if $\mu(f) < \inf_{(w, s) \in R^2} f(w, 0, s)$, then $(P_\infty)$ possesses a periodic minimizer. Zaslavski [15] showed that the result remains valid for all $f \in \mathcal{L}_G$.

For $w \in W_{loc}^{2,1}(0, \infty)$ put,

$$
\eta^f(T, w) = (J^f((0, T); w) - \mu(f))T, \quad T \in (0, \infty).
$$

Then, either $\eta^f(T, w) = \infty$ or $-\infty$. Furthermore, if $\eta^f(\cdot, w)$ is bounded then $w$ and $w'$ are bounded [15, Prop. 3.1].

Let $w$ be an $(f)$-minimizer of $(P_\infty)$. We shall say that $w$ is $(f)$-good if $\eta^f(\cdot, w)$ is bounded. Equivalently, $w$ is $(f)$-good if and only if there exists a constant $c(w)$ such that,

$$
|J^f(D; w) - \mu(f)| \leq c(w)/|D|
$$

for every bounded interval $D$.

We shall say that $w$ is optimal on compacts, or briefly $c$-optimal, if $w \in W_{loc}^{2,1}(0, \infty) \cap W^{1,\infty}(0, \infty)$ and, for every bounded interval $D$, the restriction $w|_D$ is a minimizer of $(P_D^{xy})$, where $x, y$ are the values of $(w, w')$ at the end points of the interval. By a result of Marcus [13, Th. 4.2(vi)], if the integrand $f$ is of the form $f(w, p, r) = r^2 - bp^2 + \phi(w)$, then every $c$-optimal minimizer of $(P_\infty)$ is $(f)$-good. In fact the result remains valid for the more general class of integrands studied here, (see Proposition 2.3 below).

For $w \in W_{loc}^{2,1}(0, \infty) \cap W^{1,\infty}(0, \infty)$ let $\Omega(w)$ denote the set of limiting points of $(w, w')$ as $t \to \infty$.

DEFINITION 1.1. – Let $f \in \mathcal{L}_G$. We say that $f$ has the asymptotic turnpike property, or briefly (ATP), if there exists a compact set $H(f) \subset R^2$ such that $\Omega(w) = H(f)$ for every $(f)$-good minimizer $w$.

Clearly, if $f$ has (ATP) and $v$ is a periodic $(f)$-minimizer of $(P_\infty)$ then, $H(f) = \{(v, v')(t) : 0 \leq t < \infty\}$.
The asymptotic turnpike property for optimal control problems was studied in [4, 5]. The more standard turnpike property (for problems on finite intervals) is well known in mathematical economics and several variants of it have been studied (see, e.g. [11] and [6, Ch.4 and 6]). Here we shall consider, besides (ATP), the strong turnpike property, or briefly (STP), which is defined as follows.

**Definition 1.2.** Let \( f \in \mathcal{L}_G \) and let \( w \) be a periodic \((f)\)-minimizer of \((P_\infty)\) with period \( T_w > 0 \). We say that \( f \) has the strong turnpike property if, for every \( \epsilon > 0 \) and every bounded set \( K \subset \mathbb{R}^2 \), there exists \( L > 0 \) such that every minimizer \( v \) of \((P_{(0,T)}^{x,y})\), with \( x, y \in K \) and \( T > T_w + 2L \), satisfies the following:

For every \( a \in [L, T - L - T_w] \) there exists \( \tilde{a} \in [0, T_w) \) such that,

\[
|(v',v')(a + t) - (w',w')((\tilde{a} + t)| \leq \epsilon, \quad \forall t \in [0, T_w].
\]

Note that (STP) implies uniqueness up to translation for periodic minimizers of \((P_\infty)\). Furthermore, if \( f \) has (STP), the structural information contained in (1.8) extends to arbitrary minimizers of the unconstrained problem \((P_{(0,T)})\). More precisely we have,

**Proposition 1.1.** Suppose that \( f \in \mathcal{L}_G \) possesses (STP). Let \( w \) be the (unique) periodic minimizer of \((P_\infty)\) whose period will be denoted by \( T_w \). Then, given \( \epsilon > 0 \), there exists \( L > 0 \) such that every minimizer \( v \) of \((P_{(0,T)}^{x,y})\) with \( T > T_w + 2L \) satisfies (1.8) for every \( a \in [L, T - L - T_w] \) and some \( \tilde{a} \in [0, T_w) \) depending on \( v \) and \( a \).

This is a consequence of the fact that the set of minimizers of \((P_{(0,T)})\) is bounded in \( C^1[0,T] \) by a constant \( A \) independent of \( T \), (see [12, Lemma 2.2]).

Our main results are the following.

**Theorem 1.1.** For \( f \in \mathcal{L}_G \), (STP) holds if and only if (ATP) holds.

**Theorem 1.2.** For every \( \phi \in \mathcal{L} \) there exists a non-negative function \( \theta \in C^\infty(\mathbb{R}^1) \) with \( \theta^{(m)} \in L^\infty(\mathbb{R}^1), \ m = 0,1, \ldots, \) such that for every \( s \in (0,1) \), problem \((P_\infty)\) with \( f = F_{\phi+s\theta} \) possesses a unique (up to translation) periodic minimizer.

**Theorem 1.3.** (i) For every \( \phi \in \mathcal{L} \) there exists a function \( \theta \) as in Theorem 1.2 such that,

\[ F_{\phi+s\theta} \text{ possesses (ATP), } \forall s \in (0,1). \]
(ii) (ATP) holds generically in $\mathcal{L}_G$, in the following sense as well: there exists a countable intersection of open everywhere dense sets in $\mathcal{L}$, say $\mathcal{G}$, such that

$$\phi \in \mathcal{G} \implies F_\phi \text{ possesses (ATP).}$$

A result related to the second part of Theorem 1.3 was obtained by Zaslavski [16], who established the generic validity of (ATP) in a larger space, in a weaker sense.

The proofs of these theorems, in a slightly more general form, are presented in sections 2 (Theorem 1.1) and 3 (Theorems 1.2, 1.3). In addition, in section 3, we establish a number of properties of periodic minimizers of $(P_\infty)$ which apply to every $f \in \mathcal{L}_G$ and may be of independent interest.

2. EQUIVALENCE OF (ATP) AND (STP)

In this section we shall establish Theorem 1.1 for problems involving a larger family of integrands $f$. Put,

$$\mathfrak{A} = \{f \in C(R^3) : |f(x_1, x_2, x_3)| \to \infty \text{ as } |x_3| \to \infty, \text{ uniformly with respect to } (x_1, x_2) \text{ in compact sets}\}.$$ 

$\mathfrak{A}$ will be equipped with the uniformity determined by the base,

$$E(N, \epsilon) = \{(f, g) \in \mathfrak{A} \times \mathfrak{A} :$$

$$|f(x) - g(x)| \leq \epsilon, \, (x = (x_1, x_2, x_3) \in R^3, \,$$

$$|x_i| \leq N, \, i = 1, 2, 3),$$

$$1 - \epsilon \leq (|f(x)| + 1)/(|g(x)| + 1) \leq 1 + \epsilon,$$

$$\forall (x \in R^3, |x_1|, |x_2| \leq N)\}$$

where $N$ and $\epsilon$ are positive numbers. It is easy to verify that the uniform space $\mathfrak{A}$ is metrizable and complete [8].

Let $a = (a_1, a_2, a_3, a_4) \in R^4, a_i > 0, i = 1, 2, 3, 4$ and let $\alpha, \beta, \gamma$ be real numbers such that $1 \leq \beta < \alpha, \beta \leq \gamma$ and $\gamma > 1$. Denote by $\mathfrak{M} = \mathfrak{M}(\alpha, \beta, \gamma, a)$ the family of functions $\{f\}$ such that

\begin{align}
(i) & \quad f \in \mathfrak{A} \cap C^2(R^3), \quad \partial f/\partial x_2 \in C^2(R^3), \quad \partial f/\partial x_3 \in C^3(R^3), \\
(ii) & \quad \partial^2 f/\partial x_3^2 > 0, \\
(iii) & \quad f(x) \geq a_1|x_1|^{\alpha} - a_2|x_2|^{\beta} + a_3|x_3|^{\gamma} - a_4, \\
(iv) & \quad (|f| + |\nabla f|)(x) \leq M_f(|x_1| + |x_2|)(1 + |x_3|^{\gamma}), \quad \forall x \in R^3,
\end{align}
where \( M_f : [0, \infty) \mapsto [0, \infty) \) is a continuous function depending on \( f \).
Finally, let \( \mathcal{M} \) denote the closure of \( \mathcal{M} \) in \( \mathfrak{A} \). The notations and definitions presented in the introduction with respect to \( f \in \mathcal{L}_G \) apply equally well to \( f \in \mathcal{M} \) and the various statements quoted there remain valid in this context. Put,

\[
I^f(T_1, T_2, w) = \int_{T_1}^{T_2} f(w(t), w'(t), w''(t)) dt
\]

where \( -\infty < T_1 < T_2 < +\infty \), \( w \in W^{2,1}(T_1, T_2) \) and \( f \in \mathcal{M} \).

For \( T > 0 \), \( x, y \in \mathbb{R}^2 \), \( f \in \mathcal{M} \), put

\[
U^f_T(x, y) := \inf \{ I^f(0, T, w) : w \in W^{2,1}(0, T), \quad (w, w')(0) = x, \quad (w, w')(T) = y \}.
\]

Let \( v \in W^{2,1}(D) \) where \( D = (T_1, T_2) \) is a bounded interval. Given \( \delta > 0 \), we shall say that \( v \) is an \( (f, \delta) \)-approximate minimizer in \( D \) if,

\[
I^f(T_1, T_2, v) \leq U^f_{|D|}(X_v(T_1), X_v(T_2)) + \delta,
\]

where \( X_v(t) = (v(t), v'(t)) \), \( t \in D \).

For \( x \in \mathbb{R}^n \), \( B \subset \mathbb{R}^n \) put \( d(x, B) := \inf \{|x - y| : y \in B\} \) (where \( | \cdot | \) is the Euclidean norm) and denote by \( \text{dist}(A, B) \) the distance in the Hausdorff metric between two subsets \( A, B \) of \( \mathbb{R}^n \).

We claim that:

**Lemma 2.1.** Suppose that \( f \in \mathcal{M} \) and that \( v \) is an \( (f) \)-good function.

Then, given \( \delta > 0 \) there exists \( T_\delta > 0 \) such that, for every bounded interval \( (T, T') \) with \( T \geq T_\delta \),

\[
I^f(T, T', v) \leq U^f_{|D|}(X_v(T), X_v(T')) + \delta,
\]

i.e. \( v \) is an \( (f, \delta) \)-approximate minimizer in \( (T, T') \).

**Proof.** If the claim is not valid there exists a sequence of disjoint intervals \( D_n = (T_n, T'_n), \ n = 1, 2, \ldots \) with \( T_n \to \infty \) such that,

\[
I^f(T_n, T'_n, v) - U^f_{|D_n|}(x_n, y_n) \geq \delta, \quad n = 1, 2, \ldots,
\]

where \( x_n = X_v(T_n) \) and \( y_n = X_v(T'_n) \). Let \( h_n \) denote a minimizer of problem \( (P_{D_n}^x, y_n) \) and let \( \tilde{v} \) be the function on \( [0, \infty) \) defined as follows,

\[
\tilde{v}(t) = v(t), \quad t \in [0, \infty) \setminus \bigcup_n D_n, \quad \tilde{v}(t) = h_n(t), \quad t \in D_n, \ n = 1, 2, \ldots.
\]
Then \( \tilde{v} \in W^{2,1}_{\text{loc}}(0, \infty) \) and
\[
\eta^f(T, \tilde{v}) = (I^f(0, T, \tilde{v}) - I^f(0, T, v)) + \eta^f(T, v).
\]

Since \( \eta^f(\cdot, v) \) is bounded, say by \( M \), it follows that,
\[
\eta^f(T_{n}', \tilde{v}) \leq M - \sum_{k=1}^{n} (I^f(T_k, T_{k}', v) - U^f_{T_k-T_k}(x_k, y_k)).
\]

This inequality and (2.5) imply that \( \eta^f(T_{n}', \tilde{v}) \to -\infty \) as \( n \to \infty \). However this is impossible because \( \eta^f(\cdot, w) \) is bounded from below for every \( w \in W^{2,1}_{\text{loc}}(0, \infty) \). \( \square \)

For the next lemma we need the following interpolation inequality (see e.g. Adams [1]):

Assume that \( p > 1 \) and \( \epsilon > 0 \). Then there exists a constant \( C\epsilon(p) \) such that, for every \( T \geq 1 \),
\[
(2.6) \quad \int_0^T |u''|^p \, dt \leq \epsilon \int_0^T |u''|^p \, dt + C\epsilon(p) \int_0^T |u|^p \, dt, \quad \forall u \in W^{2,p}(0, T).
\]

**Lemma 2.2.** – (i) For every \( \tau > 0 \) there exist positive constants \( b_0, b_1, b_2 \) (depending on \( \tau \)) such that, for every \( T \geq \tau \),
\[
(2.7) \quad I^f(0, T, v) \geq \int_0^T \frac{1}{2} (a_3|v''|^\gamma + a_1|v|\alpha) \, dt - b_0T \geq b_1\|v\|_{C^1(0,T)} - b_2T,
\]
for every \( v \in W^{2,1}(0, T) \) and every \( f \in \mathcal{M} \). In particular, for every \( M > 0 \) and \( T \geq \tau \) there exists a constant \( b_\tau(M, T) > 0 \) (depending continuously on \( M, T \)) such that, for every \( f \in \mathcal{M} \),
\[
(2.8) \quad v \in W^{2,1}(0, T), \quad I^f(0, T, v) \leq M \quad \implies \quad v \in W^{2,\gamma}(0, T), \quad \|v\|_{W^{2,\gamma}(0,T)} \leq b_\tau(M, T).
\]

(ii) For every \( f \in \mathcal{M} \): if \( v \in W^{2,1}_{\text{loc}}(0, \infty) \) is an \( (f) \)-good function then,
\[
(2.9) \quad \sup_{T \geq 0} \int_0^{T+1} (|v''|^\gamma + |v|\alpha) \, dt < \infty.
\]

Consequently, \( v \) and \( v' \) are uniformly continuous on \([0, \infty)\).

**Proof.** – (i) In the proof we shall assume that \( \tau = 1 \). For arbitrary \( \tau > 0 \) the result can be obtained by rescaling. By (2.2), every \( f \in \mathcal{M} \) satisfies,
\[
(2.10) \quad f(x) \geq a_1|x_1|^\alpha - a_2|x_2|^\beta + a_3|x_3|^\gamma - a_4.
\]

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Clearly this remains valid for every \( f \in \mathcal{M} \). Note that if \( \beta = 1 \) then \( \gamma' = \min(\alpha, \gamma) > 1 \) and therefore, if \( \beta' \in (1, \gamma') \) we have,

\[
f(x) \geq a_1|a_1|^\alpha - a_2|a_2|^\beta' + a_3|a_3|^\gamma - (a_2 + a_4).
\]

Therefore, without loss of generality, we may assume that \( \beta > 1 \). Hence, by (2.6) with \( p = \beta \) and \( \epsilon = \frac{a_3}{2a_2} \), we find that, for \( f \in \mathcal{M} \) and \( T \geq 1 \)

\[
(2.11) \quad I^f(0, T, v) \geq \int_0^T \frac{1}{2} (a_3|v''|\gamma + a_1|v|\alpha) \, dt - b_0 T, \quad \forall v \in W^{2,1}(0, T)
\]

where

\[
(2.12) \quad b_0 = \frac{\max(a_2 C_\epsilon(\beta) t^{\beta} - a_1 t^{\alpha}/2) + a_4 + a_3/2.}
\]

(In fact, \( v \in C^1[0, T] \). Therefore, by (2.2), \( I^f(0, T; v) \) is finite if \( v'' \in L^\gamma(0, T) \) and \( +\infty \) otherwise.) This proves the first inequality in (2.7). In order to obtain the second inequality in (2.7) observe that,

\[
\int_s^{s+1} (|v''|\gamma + |v|\alpha) \, dt \geq \int_s^{s+1} (|v''|\gamma' + |v|\gamma') \, dt - 1
\]

\[
\geq c_0 \sup_{s \leq t \leq s+1} (|v(t)| + |v'(t)|)^\gamma' - 1,
\]

for every \( s \in [0, T - 1] \), where \( c_0 \) is a constant which depends only on \( \gamma' = \min(\alpha, \gamma) \). Combining this with the first inequality in (2.7) we obtain,

\[
I^f(0, T, v) \geq c_1 \int_0^T (|v''|\gamma + |v|\alpha) \, dt - b_0 T
\]

\[
\geq c_1 (c_0 \sup_{0 \leq t \leq T} (|v(t)| + |v'(t)|)^\gamma' - 1) - b_0 T
\]

\[
\geq c_1 (c_0 \|v\|_{C^1(0, T)} - 2) - b_0 T,
\]

where \( c_1 \) is a constant which depends only on \( a_1, a_3 \). This completes the proof of (2.7). Finally (2.8) follows from (2.7):

\[
\int_0^T |v''|\gamma' \, dt \leq 2(M + T b_0)/a_3, \quad \int_0^T |v|\gamma' \, dt \leq T((M + b_2)/b_1)\gamma,
\]

for every \( v \) as in (2.8).

(ii) Since \( v \) is \((f)\)-good, \((v, v')\) is bounded in \([0, \infty)\). Clearly, \( U_1^f(x, y) \) is bounded for \((x, y)\) in a compact set. Therefore Lemma 2.1 implies that
If \((T, T + 1, v)\) is bounded by a bound independent of \(T \geq 0\). Hence (2.9) follows from (2.7).

Using these lemmas it is easy to verify that,

**Lemma 2.3.** For \(f \in \mathcal{M}\), (STP) implies (ATP).

**Proof.** Assume that \(f\) has (STP) and let \(v\) be an \((f)\)-good function. Pick \(\xi \in \Omega(v)\) and let \(\{t_k\}\) be a sequence tending to \(+\infty\) such that \((v, v')(t_k) \to \xi\). Put \(v_k(t) = v(t + t_k), t \geq -t_k\). By Lemma 2.2, for every bounded interval \(D\),

\[
\sup_k \int_D (|v_k''|^\gamma + |v_k'|^\alpha) \, dt < \infty.
\]

Therefore there exists a subsequence \(v_{k_n}\) which converges weakly, say to \(u\), in \(W^{2,\gamma}_{loc}(R^1)\). In particular \(\{(v_{k_n}, v'_{k_n})\}\) converges uniformly on compact sets. Applying inequality (2.4) to \(v_{k_n}\) and taking the limit, we find that (for every bounded interval \(D = (0, T)\)) \(u|_{D}\) is a minimizer of problem \((P_D)\), where \(x, y\) are the values of \((u, u')\) at the endpoints of \(D\). This is a consequence of the continuity of \(U_f^l(\cdot, \cdot)\) in \(R^2\) and of the weak lower semicontinuity of the functional \(I^f(0, T, \cdot)\) in \(W^{2,\gamma}_2(D)\), (see [3]). Since \(f\) has (STP) it follows that, for every \(\epsilon > 0\), (1.8) holds with \(v\) replaced by an arbitrary translate of \(u\), i.e. \(u(\cdot + \tau), \tau \in R^1\). Consequently, if \(w\) is a periodic minimizer of \((P_\infty)\) then, \(E := \{(u, u')(t) : t \in R^1\} = \Omega(w)\). In particular, \(\xi = (u, u')(0) \in \Omega(w)\) and we conclude that \(\Omega(v) \subset \Omega(w)\). On the other hand \(E \subset \Omega(v)\), so that \(\Omega(v) = \Omega(w)\). Thus \(f\) possesses (ATP).

The fact that (ATP) implies (STP) requires a more delicate argument. Actually we shall prove a more comprehensive result, which will also be used in the proof of Theorem 1.3. Roughly this result states that if \(f \in \mathcal{M}\) has (ATP) then, for every \(\epsilon > 0\) there exists \(\delta > 0\) such that, if \(v\) is an \((f, \delta)\)-approximate minimizer in \((0, T)\) and \(T\) is sufficiently large, then \(v\) satisfies (1.8), which is the condition required for (STP). Furthermore this property persists in a neighborhood of \(f\) in \(\mathcal{M}\). The precise formulation follows.

**Theorem 2.1.** Assume that \(g \in \mathcal{M}\) satisfies (ATP). Let \(w\) be a periodic minimizer of \((P_\infty)\) with integrand \(g\) and let \(T_w > 0\) be a period of \(w\).

Given \(\epsilon, M > 0\) there exists a neighbourhood of \(g\) in \(\mathcal{M}\), say \(\mathcal{U}_g\), and positive numbers \(\delta, \ell\) such that the following statement holds:

Let \(f \in \mathcal{U}_g\) and let \(T \geq T_w + 2\ell\). If \(v \in W^{2,1}(0, T)\) satisfies,

\[
|X_v(0)| \leq M, \quad |X_v(T)| \leq M, \quad I^f(0, T, v) \leq U_f^l(X_v(0), X_v(T)) + \delta,
\]

\[\text{(2.13)}\]
then, for each \( s \in [\ell, T - T_w - \ell] \) there exists \( \xi \in [0, T_w] \) such that,

\[
|X_v(s + t) - X_v(\xi + t)| \leq \varepsilon, \quad \forall t \in [0, T_w].
\]

**Remark.** - The conclusion of the theorem can be slightly strengthened as follows:

There exist \( \tau_1 \in [0, \ell] \) and \( \tau_2 \in [T - \ell, T] \) such that, for every \( s \in [\tau_1, \tau_2 - T_w] \) there exists \( \xi \in [0, T_w] \) such that (2.14) holds. Furthermore, if

\[
d(X_v(0), \Omega(w)) \leq \delta, \text{ (respectively } d(X_v(T), \Omega(w)) \leq \delta),
\]

the statement holds with \( \tau_1 = 0 \), (respectively \( \tau_2 = T \)).

The proof of the theorem will be based on several lemmas. One of the key ingredients in this proof is provided by the following result due to Leizarowitz and Mizel [10, Sec. 4]. (See also Leizarowitz [9] for a similar result in the context of a discrete model.)

**Proposition 2.1.** - Let \( f \in \mathcal{M} \). Then there exist a continuous function \( \pi^f : \mathbb{R}^2 \rightarrow \mathbb{R}^1 \) given by,

\[
\pi^f(x) = \inf \{ \liminf_{T \to \infty} [I^f(0, T, w) - T \mu(f)] : w \in W^{2,1}_{loc}(0, \infty), \; X_w(0) = x \},
\]

for every \( x \in \mathbb{R}^2 \)

and a continuous nonnegative function \( (T, x, y) \rightarrow \theta^f_T(x, y) \) defined for \( T > 0 \) and \( x, y \in \mathbb{R}^2 \) such that,

\[
U^f_T(x, y) = T \mu(f) + \pi^f(x) - \pi^f(y) + \theta^f_T(x, y)
\]

for all \( x, y, T \) as above. Furthermore, for every \( T > 0 \) and every \( x \in \mathbb{R}^2 \) there is \( y \in \mathbb{R}^2 \) such that \( \theta^f_T(x, y) = 0 \).

Let \( f \in \mathcal{M} \). For \( D = (T_1, T_2) \) and \( v \in W^{2,1}(D) \) put,

\[
\Theta^f(D; v) = \theta^f_{T_2 - T_1}(X_v(T_1), X_v(T_2)),
\]

\[
\Gamma^f(D; v) = I^f(T_1, T_2, v) - (T_2 - T_1) \mu(f) + \pi^f(X_v(T_2)) - \pi^f(X_v(T_1)).
\]

From (2.15) and Proposition 2.1 it follows that

\[
\Gamma^f(D; v) \geq \Theta^f(D; v) \geq 0.
\]

Clearly, if \( v \) is a minimizer of \((P^x_v, y), x = X_v(T_1), y = X_v(T_2)\) then \( \Gamma^f(D; v) = \Theta^f(D; v) \). However \( \Gamma^f(D; v) \) may be positive even in this
case. Note that, in the present notation, a function $v \in W^{2,1}(D)$ is an $(f, \delta)$–approximate minimizer in $D$ (see (2.3b)), iff

$$\Gamma^f(D; v) - \Theta^f(D; v) \leq \delta.$$ 

In this context we introduce the following additional terminology: Let $v$ be a minimizer of $(P_{\infty})$. We shall say that $v$ is $(f)$–perfect if

$$(2.15b) \quad \Gamma^f(D, u) = 0 \text{ for every bounded interval } D.$$ 

If $v \in W^{2,1}_{\text{loc}}(R) \cap W^{1,\infty}(R)$ and $v$ satisfies (2.15b), then $v$ is a minimizer of $(P_{\infty})$ and hence it is $(f)$–perfect. This is an immediate consequence of the definition of $\Gamma^f$ and the fact that $\pi^f$ is continuous.

Obviously every $(f)$–perfect minimizer is $c$–optimal. Using this fact, it can be shown that every $(f)$–perfect minimizer is $(f)$–good (see Proposition 2.3 below). Clearly the converse does not hold, but a partial converse is provided by the following result.

**Lemma 2.4.** –Let $f \in \mathcal{M}$ and suppose that $v$ is $(f)$–good. Then, for every $\delta > 0$ there exists $\mathcal{T}(\delta)$ such that, for $D = (T_1, T_2)$,

$$(2.16) \quad \Gamma^f(D; v) \leq \delta, \quad \forall T_1 \geq T(\delta).$$

In particular every periodic minimizer of $(P_{\infty})$ is $(f)$–perfect.

**Proof.** – Since $\pi^f$ is continuous, if $v$ is an $(f)$–good function then $\Gamma^f(D; v)$ is bounded. Furthermore, since $D \rightarrow \Gamma^f(D; v)$ is an additive, non-negative set function, it follows that for every $\delta > 0$ there exists $T(\delta) > 0$ such that (2.16) holds. The last statement of the lemma is a consequence of this inequality. 

The next result shows that every $(f)$–good function generates a family of perfect minimizers.

**Lemma 2.5.** –Let $f \in \mathcal{M}$ and let $v \in W^{2,1}_{\text{loc}}(0, \infty)$ be an $(f)$–good function. Then, given $\xi \in \Omega(v)$, there exists $u \in W^{2,1}_{\text{loc}}(R^1)$ such that

$$(*) \quad \{(u, u')(t) : t \in R^1\} \subset \Omega(v) \text{ and } (u, u')(0) = \xi,$$

and $u$ is an $(f)$–perfect minimizer.

**Proof.** – Let $u$ be constructed as in Lemma 2.3. Then, $u$ satisfies $(*)$ and, in the notation of that lemma,

$$\Gamma^f(D, u) \leq \liminf_{k \rightarrow \infty} \Gamma^f(D + t_{nk}, v).$$
This follows from the growth conditions on \( f \) (see (2.2)), and the fact that \( v_{nk} \to u \) weakly in \( W^{2,\gamma}(D) \). However, by Lemma 2.4, \( \Gamma^f(D+\tau, v) \to 0 \) as \( \tau \to \infty \). Therefore \( u \) satisfies (2.15b) and consequently, since \( u \in W^{1,\infty}(R) \), it follows that it is \((f)\)-perfect.

Another useful ingredient in our proof is the following result for which we refer the reader to [10] (proof of Proposition 4.4) and [16].

**Proposition 2.2.** Let \( f \in \mathcal{M} \). For every \( M_1, M_2, c > 0 \) there exists a positive number \( A = A_f(M_1, M_2, c) \) such that the following statement holds for every \( T \geq c \). If

\[
v \in W^{2,1}(0, T), \quad |X_v(0)| \leq M_1, \quad |X_v(T)| \leq M_1,
\]

and if \( v \) is an \((f, M_2)\)-approximate minimizer in \((0, T)\) (see (2.3b)) then,

\[
|X_v(t)| \leq A, \quad \forall t \in [0, T].
\]

Furthermore, for every \( g \in \mathcal{M} \) there is a neighbourhood \( \mathcal{U}_g \) in \( \mathcal{M} \) such that \( A_f(M_1, M_2, c) \) can be chosen uniformly with respect to \( f \) in \( \mathcal{U}_g \).

We also need the following lemma.

**Lemma 2.6.** Let \( f \in \mathcal{M} \). Then, for every compact set \( E \) there exists a constant \( M = M(E) > 0 \) such that, for every \( T \geq 1 \),

\[
(2.17) \quad U_T^f(x, y) \leq T \mu(f) + M, \quad \forall x, y \in E.
\]

**Proof.** Let \( w \) be a periodic minimizer of \((P_\infty)\) with period \( T_w > 0 \). Clearly, for every \( A > 0 \),

\[
\sup\{U_T^f(x, y) : x, y \in E, \ 1 \leq T \leq A\} < \infty.
\]

Therefore, it is sufficient to show that there exists \( M \) such that (2.17) holds for \( T \geq 4T_w \). Put \( D = (0, T) \). Let \( \tau \) be the largest integer which does not exceed \( T/T_w \) and put \( l = 2^{-1}(T - (\tau - 1)T_w) \). Let \( D' = (l, T - l) \) so that \( |D'| = (\tau - 1)T_w \).

Given \( x, y \in E \) let \( v_1 \) (resp. \( v_2 \)) be a minimizer of problem \((P_1^{x,y})\) with \( z = (w, w')(l) \) (resp. \((P_1^{\zeta,y})\) with \( \zeta = (w, w')(T - l) \)). Let \( v \in W^{2,\gamma}(D) \) be the function given by,

\[
v(t) = \begin{cases} 
v_1(t), & t \in (0, l) \\
w(t), & t \in D' \\
v_2(t - T + l), & t \in (T - l, T) \end{cases}
\]
Since \( w \) and \( w' \) are bounded and \( T_w/2 \leq l \leq T_w \) it follows that there exists a constant \( M_1 \) (independent of \( x, y, T \)) such that,
\[
U^f_l(x, z) = I^f(0, l, v_1) \leq M_1 \quad \text{and} \quad U^f_l(\zeta, y) = I^f(0, l, v_2) \leq M_1.
\]
Since 
\[
I^f(l, T - l, w) = (T - 2l)\mu(f)
\]
it follows that,
\[
U^f_l(x, y) \leq I^f(0, T, v) \leq (T - 2l)\mu(f) + 2M_1,
\]
which implies (2.17).

Using these results we can establish the following relation between approximate minimizers and \((f)\)-good functions.

**Proposition 2.3.** – Let \( f \in \mathcal{M} \) and \( M > 0 \). Denote by \( \mathcal{A}(f, M) \) the family of minimizers \( v \) of \((P_\infty)\) such that \( v \) is an \((f, M)\)-approximate minimizer in every bounded interval \( D \subset \mathbb{R}^+ \) such that \( |D| \geq 1 \). Then
\[
v \in \mathcal{A}(f, M) \implies v \text{ is } (f)\text{-good}.
\]

In particular, every c-optimal function is \((f)\)-good. Furthermore, the family of periodic minimizers is uniformly bounded in the norm \( \|v\|_{(1)} := \sup_{\mathbb{R}^+} |X_v| \).

**Proof.** – Let \( v \) be a minimizer of \((P_\infty)\). Then, for every \( T > 0 \),
\[
limit_{T' \to -\infty} \frac{1}{T' - T} I^f(T, T', v) = \mu(f).
\]
Hence there exists \( T_0 > T \) such that
\[
I^f(T_0, T_0 + 1, v) \leq M := \mu(f) + 1.
\]
Consequently there exists a monotone sequence \( \{T_n\} \) tending to \(+\infty\) such that,
\[
I^f(T_n, T_n + 1, v) \leq M, \quad n = 1, 2, \ldots.
\]
By Lemma 2.2 there exists a constant \( M_1 \) (independent of \( v \)) such that,
\[
(*) \quad \sup\{|X_v(t)|: T_n \leq t \leq T_n + 1\} \leq M_1, \quad n = 1, 2, \ldots.
\]
Now suppose that, \( v \in \mathcal{A}(f, M) \). Then inequality (*) and Proposition 2.2 imply that there exists a constant \( M_2 \) (independent of \( v \)) such that,
\[
(**) \quad \sup\{|X_v(t)|: T_1 \leq t\} \leq M_2, \quad n = 1, 2, \ldots.
\]
Thus \( |X_v| \in L^\infty(\mathbb{R}^+) \). (Note that in general \( T_1 \) depends on \( v \) so that \( \sup_{\mathbb{R}^+} |X_v| \) may not be uniformly bounded relative to \( v \in \mathcal{A}(f, M) \).)
Further, inequality (2.3b), the boundedness of $X_v$ and Lemma 2.6 imply that,

$$I_f^f(0, T, v) \leq U_f^f(X_v(0), X_v(T)) + M \leq T \mu(f) + M + M', \quad \forall T \geq 1,$$

where $M' = M(E)$ is as in (2.17) with $E = \text{cl}\{X_v(t) : t \in R_+\}$. Thus $\eta^f(\cdot, v)$ is bounded on $(1, \infty)$ and hence on $R_+$, i.e. $v$ is (f)-good.

If $v$ is a c-optimal function then, by definition, $X_v$ is bounded and therefore, by the previous part of the proof, $v$ is (f)-good.

Finally, if $v$ is a periodic minimizer then inequality (**) implies that $\sup_{R_+} |X_v| \leq M_2$, which proves the last assertion of the proposition.

The next lemma will be needed in order to establish the stability of (ATP).

**Lemma 2.7.** Let $g \in \mathcal{M}$ and let $D = (0, T)$. For $M > 0$ put,

$$\mathcal{V}_M(D) = \{v \in W^{2,1}(D) : \int_0^T (|v''|^2 + |v|^a) \, dt \leq M\}.$$

Then for every $\epsilon, M > 0$ there exists a neighbourhood $\mathcal{N}_g$ of $g$ in $\mathcal{M}$ such that, for every $f \in \mathcal{M}$,

$$|I^f(0, T, v) - I^g(0, T, v)| < \epsilon, \quad \forall v \in \mathcal{V}_M(D),$$

and

$$x, y \in R^2, \ |x|, |y| < M \implies |U^f_T(x, y) - U^g_T(x, y)| < \epsilon.$$

The neighborhood $\mathcal{N}_g$ can be chosen independently of $T$ for $T$ in compact sets of $(0, \infty)$.

**Proof.** Put $M_0(T) = \sup\{||v||_{C^1[0,T]} : v \in \mathcal{V}_M(D)\}$. By Lemma 2.2, if $T \in (T_1, T_2)$, with $0 < T_1 < T_2 < \infty$, then $M_1 = \sup_{T \in [T_1, T_2]} M_0(T) < \infty$. For every $N, \delta > 0$ let $B_g(N, \delta) = \{f \in \mathcal{M} : (f, g) \in E(N, \delta)\}$ (see (2.1)). Now, given $\delta > 0$ choose $N > 2M_1$ sufficiently large so that, for every $f \in B_g(N, \delta)$,

$$x \in R^3, \ |x_1|, |x_2| \leq M_1, \ |x_3| \geq N \implies g(x) > 0,$$

$$1 - 2\delta < f(x)/g(x) < 1 + 2\delta.$$

Assume that $f \in B_g(N, \delta)$ and $v \in \mathcal{V}_M(D)$. Then,

$$|I^f(0, T, v) - I^g(0, T, v)| \leq \int_{E(v,N)} |(f - g)(v, v', v'')| \, dt$$

$$+ \int_{E'(v,N)} |(f - g)(v, v', v'')| \, dt.$$

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where \( E(v, N) = \{ t \in D : |v''(t)| \leq N \} \) and \( E'(v, N) = D \setminus E(v, N) \).

The first term on the right is bounded by \( T \delta \) and the second by \( 2\delta \int_D |g(v, v', v'')| \). The last integral is uniformly bounded for \( v \in \mathcal{V}_M(D) \).

This follows from the inequality,

\[
|f|(x) \leq M_f(|x_1| + |x_2|)(1 + |x_3|^\gamma), \quad \forall x \in \mathbb{R}^3,
\]

which, by (2.2), holds for \( f \in \mathcal{M} \) and remains valid also for \( f \in \overline{\mathcal{M}} \).

Therefore, choosing \( \delta \) sufficiently small so that the right hand side of (2.21) is smaller than \( \epsilon \) and then choosing \( N \) sufficiently large as indicated before, we obtain (2.18).

Finally, (2.19) is a consequence of (2.18) and the fact that (by Proposition 2.2) the family of minimizers of \( (P^x,y)_D \), \( |x|, |y| \leq M \), is bounded by a bound independent of \( f \) for \( f \) in a neighbourhood of \( g \).

The next lemma plays an important role in the proof of Theorem 2.1 and the results following it.

**Lemma 2.8.** Let \( f \in \mathcal{M} \) and let \( D = (T_1, T_2) \) be a bounded interval. Suppose that \( w_1, w_2 \in W^{2,1}(D) \) and that \( \Gamma^f(D, w_1) = \Gamma^f(D, w_2) = 0 \). If there exists \( \tau \in (T_1, T_2) \) such that \( (w_1, w_1')(\tau) = (w_2, w_2')(\tau) \) then \( w_1 = w_2 \) everywhere in \( D \).

**Proof.** Put

\[
u(t) = w_1(t), \quad t \in [T_1, \tau], \quad u(t) = w_2(t), \quad t \in (\tau, T_2].
\]

Evidently \( u \in W^{2,1}(D) \) and \( \Gamma^f(D, u) = 0 \). Since \( u, w_1, w_2 \) satisfy the Euler-Lagrange equation we conclude that \( u = w_1, w_2 \) everywhere in \( D \).

To complete the proof of Theorem 2.1 we need two more auxiliary results, stated below as Lemmas A and B. The proofs of these lemmas, which are more technical than the previous ones, will be given in Appendixes A and B respectively. In both of these lemmas we consider an integrand \( f \) possessing (ATP) and study the relation between a fixed periodic minimizer of \( (P_\infty) \), say \( w \), and approximate minimizers of \( (P_{(0,T)}) \). In Lemma A it is shown that (given \( \epsilon, M > 0 \)) there exists \( \ell > T_w = \) (period of \( w \)) such that every \( (f, M) \)-approximate minimizer in \( (0, T) \), \( T > \ell \), whose endvalues are bounded by \( M \), is intermittently close to \( w \) in the following sense. Every interval \( D \subset (0, T) \), \( |D| = \ell \), contains a subinterval \( D^* \) of length \( T_w \) such that \( \sup_{D^*} |X_v - X_w| < \epsilon \) where \( w^* \) is a translate of \( w \). In Lemma B it is shown that if in addition to the above, the endvalues of \( v \) are sufficiently close to \( \Omega(w) \) (=the limit set of \( w \)), and if \( M \) is sufficiently small, then the relation described above holds in...
every subinterval $D^*$ of length $T_w$. (In general the translate $w^*$ will depend on $D^*$.) Finally, these properties persist in a neighborhood of the given integrand. The precise formulation follows.

**Lemma A.** Suppose that $g \in \mathcal{M}$ possesses (ATP). Let $w$ be a periodic minimizer of $(P_\infty)$ with integrand $g$ and let $T_w > 0$ be a period of $w$. Given $M_0, M_1, \epsilon > 0$ there exists an integer $q_1 \geq 1$ and a neighborhood $\mathcal{U}$ of $g$ in $\mathcal{M}$ such that the following statement holds.

Let $f \in \mathcal{U}$ and $T \geq q_1 T_w$. If $v \in W^{2,1}(0, T)$ satisfies

\begin{equation}
|X_v(t)| \leq M_0 \text{ for } t = 0, T,
\end{equation}

then, for every $\tau \in [0, T - q_1 T_w]$ there exist $\xi \in [0, T_w)$ and $s \in [\tau, \tau + (q_1 - 1)T_w]$ such that

\begin{equation}
|X_v(s + t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w].
\end{equation}

**Lemma B.** Let $g, w, T_w$ be as in Lemma A. Given $\epsilon > 0$ there exist $\delta \in (0, 1)$ and $Q_0 > T_w$, such that for every $Q > Q_0$ there exists a neighborhood $\mathcal{U}_Q$ of $g$ in $\mathcal{M}$ such that the following statement holds.

Let $f \in \mathcal{U}_Q$ and $\tau \in [Q_0, Q]$. If $v \in W^{2,1}(0, \tau)$ satisfies,

\begin{equation}
d(X_v(t), \Omega_w(t)) \leq \delta \text{ for } t = 0, \tau,
\end{equation}

then, for every $s \in [0, \tau - T_w]$ there exists $\xi \in [0, T_w)$ such that (2.25) holds.

**Proof of Theorem 2.1.** It is sufficient to prove the theorem for all sufficiently large $M$. Therefore we may assume that

\begin{equation}
M > 2||X_w||_{L^\infty(R)} + 8.
\end{equation}

By Proposition 2.2 there exist a neighborhood of $g$ in $\mathcal{M}$, say $\mathcal{N}(M)$, and a number $S > M + 1$ such that for each $f \in \mathcal{N}(M)$ and each $T \geq \inf\{1, T_w\}$:

\begin{equation}
v \in W^{2,1}(0, T), |X_v(0)|, \quad |X_v(T)|
\end{equation}

\begin{equation}
\leq M + 1, \quad I^f(0, T, v) \leq U_T^f(X_v(0), X_v(T)) + 4
\end{equation}

implies that,

\begin{equation}
|X_v(t)| \leq S, \quad t \in [0, T].
\end{equation}

Given $\epsilon$ as in the theorem, there exist $\delta \in (0, 1)$ and $Q_0 > T_w$ such that the statement of Lemma B holds.
By Lemma A there exist a positive integer $q_1$ and a neighborhood of $g$ in $\mathfrak{M}$, say $\mathfrak{N}(S, \delta)$, such that for each $f$ in this neighborhood and each $T \geq q_1 T_w$:

\begin{equation}
(2.29) \quad v \in W^{2,1}(0, T), \quad |X_v(t)| \leq S + 1 \quad \text{for} \quad t = 0, T,
\end{equation}

\begin{equation}
I^f(0, T, v) \leq U^f_T(X_v(0), X_v(T)) + 4
\end{equation}

implies that for every $\tau \in [0, T - q_1 T_w]$ there exist $\xi \in [0, T_w)$ and $s \in [\tau, \tau + (q_1 - 1) T_w]$ such that

\begin{equation}
|X_v(s + t) - X_w(\xi + t)| \leq \delta, \quad t \in [0, T_w].
\end{equation}

Choose

\begin{equation}
(2.31) \quad Q_1 > 8(Q_0 + q_1 T_w).
\end{equation}

By Lemma B there exists a neighborhood of $g$ in $\mathfrak{M}$, say $\mathfrak{N}_e$ such that for each $f \in \mathfrak{N}_e$ and each $\tau \in [Q_0, Q_1]$:

If $v \in W^{2,1}(0, \tau)$ satisfies (2.26) then for every $s \in [0, \tau - T_w]$ there is $\xi \in [0, T_w)$ such that,

\begin{equation}
|X_v(s + t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w].
\end{equation}

We claim that the statement of the theorem holds with $\mathfrak{U}_g = \mathfrak{N}(M) \cap \mathfrak{N}(S, \delta) \cap \mathfrak{N}_e$, with $\delta$ as above and $\ell = 2q_1 T_w + 4(Q_1 + 4)$.

Assume that $f \in \mathfrak{U}_g$, $T \geq 2\ell + T_w$ and $v$ satisfies (2.13). Then $v$ satisfies (2.28) and consequently (2.29). Therefore, for each $\tau \in [0, T - q_1 T_w]$ there exist $\xi \in [0, T_w)$ and $s \in [\tau, \tau + (q_1 - 1) T_w]$ such that (2.30) holds. Let $m$ be the largest integer such that $(m + 1)q_1 T_w \leq T$. Put $\tau_k = kq_1 T_w$, $k = 0, \ldots, m + 1$. Then, for $k = 0, \ldots, m$, $\tau_k$ is in $[0, T - q_1 T_w]$ and consequently there exists $\xi_k \in [0, T_w)$ and $s_k \in [\tau_k, \tau_k + (q_1 - 1) T_w] \subset [\tau_k, \tau_{k+1}]$ such that,

\begin{equation}
|X_v(s_k + t) - X_w(\xi_k + t)| \leq \delta, \quad t \in [0, T_w], \quad k = 0, \ldots, m.
\end{equation}

This implies,

\begin{equation}
(2.34) \quad d(X_v(s_k), \Omega(w)) \leq \delta, \quad k = 0, \ldots, m.
\end{equation}

Let $\nu_0$ be the smallest integer such that $\nu_0 \geq Q_0/(q_1 T_w)$ and let $\nu_1$ be the largest integer such that $\nu_1 \leq Q_1/(q_1 T_w)$. Since $Q_1 - Q_0 > 8q_1 T_w$ we...
have \( \nu_1 - \nu_0 > 6 \). an interval \( D_{j,k} := [s_j, s_k] \) where \( 0 \leq j < k \leq m \) and observe that if \( \nu_0 + 1 < k - j \leq \nu_1 - 1 \) then,

\[
Q_0 \leq \nu_0 q_1 T_w < \tau_k - \tau_{j+1} \leq |D_{j,k}| \leq \tau_{k+1} - \tau_j \leq \nu_1 q_1 T_w \leq Q_1.
\]

Further observe that the last inequality in (2.13) implies that,

\[
I^f(s_j, s_k, v) \leq U_f^f(X_v(s_j), X_v(s_k)) + \delta.
\]

Indeed this holds for every subinterval of \([0, T]\) because,

\[
I^f(a, b, v) \text{ is additive and } U_{b-a}^f(X_v(a), X_v(b)) \text{ is subadditive}
\]
on finite partitions of \((0, T)\) consisting of subintervals and because

\[
I^f(a, b, v) \geq U_{b-a}^f(X_v(a), X_v(b)).
\]

Therefore we may apply Lemma B to the function \( v \) restricted to \( D_{j,k} \) where \( \nu_0 + 1 < k - j \leq \nu_1 - 1 \), and conclude that for every \( s \in [s_j, s_k - T_w] \) there exists \( \xi \in [0, T_w) \) such that (2.32) holds. Finally this implies that for every \( s \in [s_0, s_m - T_w] \) there exists \( \xi \in [0, T_w) \) such that (2.32) holds. Since \( s_0 \leq q_1 T_w \) and \( T - s_m \geq 2q_1 T_w \), we find that the theorem holds with \( \ell \) as above.

The following result is an immediate consequence of Theorem 2.1, Proposition 2.2 and Lemma 2.1. Roughly it states that if \( f \) has \((ATP)\) and \( w \) is a periodic minimizer of \((P_\infty)\) then every \((f)\)-good function is eventually 'close' to \( w \).

**Theorem 2.2.** – Assume that \( g \in \mathfrak{M} \) has \((ATP)\) and \( w \in W^{2,1}_{loc}(R^1) \) is a periodic \((g)\)-minimizer with a period \( T_w > 0 \). Then, for every \( \epsilon > 0 \), there exists a neighborhood \( \mathcal{U} \) of \( g \) in \( \mathfrak{M} \) such that for each \( f \in \mathcal{U} \):

If \( v \) is an \((f)\)-good function, there exists \( t_\epsilon \) (depending on \( \epsilon, v \)) such that, for every \( s \geq t_\epsilon \), there exists \( \xi \in [0, T_w) \) such that,

\[
|X_v(s + t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w].
\]

**Corollary 2.1.** – If \( f \in \mathfrak{M} \) has \((ATP)\) then problem \((P_\infty)\) possesses a unique (up to translation) periodic minimizer.

Finally we observe that Theorem 1.1 can be easily deduced from Theorem 2.1. Suppose that \( G \) satisfies (1.1) and (1.2) and let \( \mathfrak{L}(\alpha, b_2, b_3) \) and \( \mathfrak{L}_G(\alpha, b_2, b_3) \) be defined as in (1.3),(1.4). Clearly, for \( G \) and \( \mathfrak{L} \) as in (1.1)–(1.4) and an appropriate choice of \( a \),

\[
\mathfrak{L}_G(\alpha, b_2, b_3) \subset \mathfrak{M}(\alpha, \beta, \gamma, a),
\]

and the operator

\[
\phi \rightarrow F_{\phi} \in \mathfrak{M}(\alpha, \beta, \gamma, a), \quad \phi \in \mathfrak{L}(\alpha, b_2, b_3)
\]
is continuous. Therefore Theorem 2.1 implies Theorem 1.1.

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3. PROOF OF THEOREMS 1.2, 1.3

First we establish a more general version of Theorem 1.2:

**Theorem 3.1.** Let \( f \in \mathcal{M} \). Then there exists a nonnegative function \( \phi \in C^\infty(\mathbb{R}^1) \) such that \( \phi(t) > 0 \) for all large \( |t| \), \( \phi^{(m)} \) is bounded for every \( m \geq 0 \), and the following statement holds.

Denote

\[
(3.1) \quad f_\rho(x_1, x_2, x_3) = f(x_1, x_2, x_3) + \rho \phi(x_1), \quad (\rho, x_1, x_2, x_3) \in \mathbb{R}^4.
\]

Then for each \( \rho \in (0, 1) \), \( f_\rho \in \mathcal{M} \), \( \mu(f_\rho) = \mu(f) \) and problem \((P_\infty)\) with \( f = f_\rho \) possesses a unique (up to translation) periodic minimizer.

We start with a brief description of the strategy of the proof, which will be presented through several lemmas. Given \( f \in \mathcal{M} \), denote by \( \mathcal{E}(f) \) the set of all periodic \((f)\)-minimizers of \((P_\infty)\). If \( w \in \mathcal{E}(f) \) is not a constant, we denote by \( \tau(w) \) the minimal period of \( w \). In the first lemma we show that every non-constant periodic minimizer \( w \) has precisely two extremal points in each interval \([a, a + \tau(w))\) and is strictly monotone between two consecutive extremal points. Using this fact we show that if \( \mu(f) < \inf\{f(t, 0, 0) : t \in \mathbb{R}^1\} \), then the set \( \{\tau(w) : w \in \mathcal{E}(f)\} \) is bounded. Next we show that there exists \( w^* \in \mathcal{E}(f) \) whose range \( D_{w^*} \) is minimal in the sense that it is either disjoint from or strictly contained in the range of any other element \( w \in \mathcal{E}(f) \), unless \( w \) is a translate of \( w^* \). Finally we observe that if there exists \( \phi \in C^\infty(R) \) which vanishes on \( D_{w^*} \) and is positive everywhere else, then the assertion of Theorem 3.1 holds. Since \( D_{w^*} \) is a closed bounded interval, such a function is easily constructed.

**Lemma 3.1.** Assume that \( w \in \mathcal{E}(f) \) and \( w \) is not constant. Applying an appropriate translation we may assume that \( w(0) = \min_{\mathbb{R}^1} w \). Then there exists \( \tau \in (0, \tau(w)) \) such that \( w \) is strictly increasing in \([0, \tau]\) and strictly decreasing in \([\tau, \tau(w)]\).

**Remark.** In the special case \( f(v, v', v'') = |v''|^2 - q|v'|^2 - (v^2 - 1)^2 \), this lemma was independently established by Mizel, Peletier, Troy [14]. Their proof uses the special symmetries of the integrand.

**Proof.** Let \( E = \{\tau \in [0, \infty) : w'(\tau) = 0\} \). We claim that \( E \cap [0, \tau(w)] \) is a finite set. Otherwise there exists a sequence of positive numbers \( \{t_n\} \) converging to a point \( t^* \in [0, \tau(w)] \), such that \( w'(t_n) = 0, \ n = 1, 2, \ldots \). By the mean value theorem, this implies that for \( m = 1, \ldots, 4 \), there exists a sequence \( \{t_{m,n}\}_{n=1}^{\infty} \) converging to \( t^* \), such that \( w^{(m)}(t_{m,n}) = 0 \) for
all \( n \). Therefore \( w^{(m)}(t^*) = 0, \ m = 1, \ldots, 4 \). Since \( w \) satisfies the Euler–Lagrange equation corresponding to our variational problem this implies that \( w \) is a constant, contrary to our assumption. (Note that, for \( f \in \mathcal{M} \) the Euler–Lagrange equation is a regular, fourth order equation.)

Put,

\[ \tau_1 = \sup \{ \tau \in E \cap [0, \tau(w)]: w'(t) \geq 0, \ \forall t \in [0, \tau] \} \]

Clearly \( \tau_1 \in (0, \tau(w)) \) and \( w \) is strictly increasing in \((0, \tau_1)\). Similarly we define

\[ \tau_2 = \sup \{ \tau \in E \cap (\tau_1, \tau(w)]: w'(t) \leq 0, \ \forall t \in [\tau_1, \tau] \} \]

Proceeding in this manner we obtain a strictly increasing sequence \( \{ \tau_j : j = 0, \ldots, k \} \) such that \( \tau_0 = 0, \ \tau_k = \tau(w), \ w'(\tau_j) = 0, \ j = 0, \ldots, k \) and \( w' \) does not change sign in each of the intervals \( D_j = [\tau_j, \tau_{j+1}) \), \( j = 0, \ldots, k - 1 \). More precisely, \( w \) is strictly increasing in \( D_j \), if \( j \) is even, and strictly decreasing in \( D_j \), if \( j \) is odd. Obviously \( k \) is even.

Let \( D^*_j \) denote the interval \([w(\tau_j), w(\tau_{j+1})]\) (resp. \([w(\tau_{j+1}), w(\tau_j)]\)) when \( j \) is even (resp. odd).

Evidently, for each integer \( j \), \( 0 \leq j \leq k \) the function \( t \rightarrow w(t), t \in D_j \) is invertible. Composing the inverse function thus obtained with the function \( t \rightarrow w'(t), t \in D_j \), we obtain a function \( h_j \in C(D^*_j) \) such that \( w'(t) = h_j(w(t)) \) for every \( t \in D_j \).

Now we claim that for \( i < j \), \( w(\tau_j) \neq w(\tau_i) \), unless \( i = 0 \) and \( j = k \). Suppose that there exists \( (i, j) \neq (0, k) \) such that \( 0 \leq i < j \leq k \) and \( w(\tau_j) = w(\tau_i) \). Then let \( u \) be the periodic function, with period \( \tau_j - \tau_i \), such that \( u(t) = w(t), t \in [\tau_i, \tau_j] \). Recall that \( w'(\tau_m) = 0 \) for \( m = 0, \ldots, k \). Hence \( u \in W^{2,1}_{\text{loc}}(\mathbb{R}^1) \). Furthermore, by Lemma 2.4, \( \Gamma^f(D; u) = \Gamma^f(D; w) = 0 \) in every bounded interval \( D \). (Recall that the function \( D \rightarrow \Gamma^f(D; v) \) is additive.) Therefore by Lemma 2.8, \( u \equiv w \), which contradicts the assumption that the period of \( u \) is strictly smaller than \( \tau(w) \).

Next, we claim that, if \( k > 2 \) then \( D^*_j \subset D^*_j \) for \( j = 1, \ldots, k \). We verify this claim by induction. For \( j = 1 \), we have \( w(0) = w(\tau_2) < w(\tau_1) \). (Recall that \( w(0) \) is the minimum of \( w \).) Furthermore, since \( k > 2 \), the previous argument yields \( w(0) < w(\tau_2) < w(\tau_1) \). Now suppose that the claim holds for \( j = 1, \ldots, m - 1 \). To fix ideas assume that \( m \) is even. Then we know that \( w \) is strictly increasing in \( D_m \) so that \( w(\tau_{m+1}) > w(\tau_m) \). We must show that \( w(\tau_{m+1}) < w(\tau_m) \). Suppose the contrary. Since, by assumption, \( D^*_m \subset D^*_{m-2} \) it follows that,

\[ w(\tau_{m-2}) < w(\tau_m) < w(\tau_{m-1}) < w(\tau_{m+1}). \]
Therefore the functions $h_{m-2}$ and $h_m$ defined in $D_{m-2}^*$ and $D_m^*$ respectively must intersect somewhere in $[w(\tau_m), w(\tau_{m-1})]$. (Recall that both functions are non-negative in their intervals of definition and vanish at the end points of these intervals.) This means that there exist $s_1 \in D_{m-2}$ and $s_2 \in D_m$ such that $(w, w')(s_1) = (w, w')(s_2)$. However, applying once again Lemma 2.8, the argument used before shows that this is impossible and proves our claim.

Combining the last two claims we conclude that, if $k > 2$, the inclusion $D_j^* \subset D_{j-1}^*$, $j = 1, \cdots, k$ is strict. But this is impossible because $w(\tau_0) = w(\tau_k)$.

**Corollary 3.1.** Suppose that $f \in \mathfrak{M}$ and that $f(x_1, x_2, x_3) = f(x_1, -x_2, x_3)$, for every $x \in \mathbb{R}^3$. Let $w$ and $\bar{\tau}$ be as in the statement of the lemma. Then $w' > 0$ in $(0, \bar{\tau})$ and $w' < 0$ in $(\tau, \tau(w))$. Furthermore, $\bar{\tau} = \tau(w)/2$ and $w$ is even.

**Proof.** Since $f$ is even in the second argument, it follows that the function $\bar{w}$ given by $\bar{w}(t) = w(-t)$ is also a periodic minimizer. Recall that we assume that $w(0) = \min_{\mathbb{R}} w$ so that $w'(0) = 0$. Consequently, $X_w(0) = X_{\bar{w}}(0)$. Hence, by Lemma 2.8, $w \equiv \bar{w}$ i.e. $w$ is even. Further this implies that $w(t) = w(\tau(w) - t)$ for every real $t$. Now suppose that $s \in (0, \tau(w))$ and $w'(s) = 0$. Then $X_w(s) = X_w(\tau(w) - s)$. Using again Lemma 2.8 we deduce that $w(t) = w(t + 2s - \tau(w))$, for every $t \in \mathbb{R}^1$. Thus $2s - \tau(w)$ is a period of $w$ and therefore it must be equal to $k\tau(w)$ for some integer $k$. Since $s \in (0, \tau(w))$ it follows that $k = 0$. This proves our assertion.

**Lemma 3.2.** Assume that $f \in \mathfrak{M}$ satisfies the condition,

\[(3.2) \quad \mu(f) < \inf \{ f(t, 0, 0) : t \in \mathbb{R}^1 \}.\]

Then no element of $\mathcal{C}(f)$ is constant and

\[(3.3) \quad \sup \{ \tau(w) : w \in \mathcal{C}(f) \} < \infty.\]

**Remark.** This result was established by Marcus [13] in the special case $f(v, v', v'') = |v''|^2 - \mu|v'|^2 + \psi(v)$, for a large class of potentials $\psi$. \vspace{.5cm}

**Proof.** Step 1 – Suppose that $\{T_i\}_{i=0}^\infty$ is a sequence of positive numbers tending to infinity, and that $\{w_i : w_i \in W^{2,1}(0, T_i)\}$ is a sequence of functions such that,

\[(3.4) \quad (i) \quad I(f(0, T_i, w_i) = T_i \mu(f) + \pi f(X_{w_i}(0)) - \pi f(X_{w_i}(T_i)), \quad i = 0, 1, 2, \ldots, \]

\[(ii) \quad \{ |X_{w_i}(0)| \}_{i=0}^\infty \text{ and } \{ |X_{w_i}(T_i)| \}_{i=0}^\infty, \text{ are bounded}, \]

\[(iii) \quad w_i'(t) \geq 0, \quad t \in (0, T_i), \quad i = 0, 1, 2, \ldots. \]
We claim that,

$$\mu(f) = \inf \{ f(z, 0, 0) : z \in R^1 \}. \tag{3.5}$$

The same conclusion holds if in (3.4), the condition "\(w'_i(t) \geq 0\)" is replaced by the condition "\(w'_i(t) \leq 0\)."

Assumption (3.4)(i) implies that \(I^f(0, T_i, w_i) = U^f_{T_i}(X_{w_i}(0), X_{w_i}(T_i))\) and consequently, Proposition 2.2 and assumption (3.4)(ii) imply that there exists \(M > 0\) such that,

$$\sup_{t \in [0, T_i]} |X_{w_i}(t)| \leq M, \quad i = 0, 1, 2, \ldots, \tag{3.6}$$

and

$$\|w_i\|_{W^{2,\gamma}(T_i, T_{i+1})} \leq M, \quad \forall T \in (0, T_i - 1), \quad i = 0, 1, 2, \ldots. \tag{3.7}$$

Therefore there exists a subsequence (which we shall continue to denote by \(\{w_i\}\)) and a function \(v \in W^{2,\gamma}_{loc}(0, \infty)\) such that, for every \(T \geq 1,\)

$$w_i \rightarrow v \text{ weakly in } W^{2,\gamma}(0, T) \text{ as } i \rightarrow \infty.$$  

By the lower semicontinuity of integral functionals [3] and Proposition 2.1,

$$I^f(0, T, v) = T\mu(f) + \pi^f(X_v(0)) - \pi^f(X_v(T)), \quad \forall T \geq 1. \tag{3.8}$$

By (3.7),

$$\|v\|_{W^{2,\gamma}(T, T+1)} \leq M, \quad \forall T \in (1, \infty)$$

and by (3.4)(iii), \(v' \geq 0\) in \((0, \infty)\). Consequently \(v(t)\) possesses a finite limit, say \(d_0\), and \(v'(t) \rightarrow 0\) as \(t \rightarrow \infty\).

Let \(v_j, \ j = 0, 1, 2, \ldots\) be the function defined in \([0, 1]\) by \(v_j(t) = v(j + t)\). By (3.8) the sequence \(\{v_j\}\) is bounded in \(W^{2,\gamma}(0, 1)\) and therefore a subsequence will converge weakly in this space to a function \(u\). Clearly \(u\) is the constant function \(u \equiv d_0\). Since \(I^f(0, 1, v_j) = \mu(f) + \pi^f(X_v(j)) - \pi^f(X_v(j + 1))\) and \(X_v(j)\) converges, we conclude (by the lower semicontinuity of integral functionals) that \(I^f(0, 1, u) = \mu(f)\). This implies (3.5). It is obvious that the conclusion remains valid if the sign in (3.4)(iii) is inverted.

**Step 2**. – Assume that the assertion of the lemma is not valid. Then there exists a sequence \(\{w_i\}_{i=1}^\infty\) in \(\mathcal{E}(f)\) such that

$$\tau(w_i) \rightarrow \infty \text{ as } i \rightarrow \infty. \tag{3.9}$$
Without loss of generality we may assume that \( w_i(0) = \min_R w_i, i = 1, 2, \ldots \).

By Lemma 3.1, for each integer \( i \geq 1 \) there exists a number \( \tilde{\tau}_i \in (0, \tau(w_i)) \) such that \( w_i \) is strictly increasing in \([0, \tilde{\tau}_i]\) and strictly decreasing in \([\tilde{\tau}_i, \tau(w_i)]\). In view of (3.9) either \( \tilde{\tau}_i \to 0 \) or \( \tau(w_i) - \tilde{\tau}_i \to \infty \) or both. In the first case put \( T_i = \tilde{\tau}_i \) and \( v_i = w_i|_{[0,\tilde{\tau}_i]} \); in the second case put \( T_i = \tau(w_i) - \tilde{\tau}_i \) and define \( v_i \) in \([0,T_i]\) by, \( v_i(t) = w_i(t + \tilde{\tau}_i) \) for \( i = 1, 2, \ldots \). Then the sequence \( \{T_i\} \) tends to infinity and the sequence \( \{v_i\} \) satisfies conditions (i), (iii) of Step 1, possibly with a negative sign in (iii). Furthermore, by Proposition 2.3 there exists a number \( S > 0 \) such that

\[
(3.10) \quad \sup \{|X_v(t)| : t \in R^1, v \in \mathcal{E}(f)\} \leq S.
\]

Thus the sequence \( \{v_i\} \) satisfies also condition (ii).

Consequently, the statement established in Step 1 implies that (3.5) holds, which contradicts the assumptions of the lemma.

**Lemma 3.3.** Let \( f \in \mathcal{M} \). If \( w_1, w_2 \in \mathcal{E}(f) \) then the sets

\[
D_i := \{w_i(t) : t \in R\}, \quad i = 1, 2
\]

are either disjoint or one of them is contained in the other. Furthermore if, say, \( D_1 \subseteq D_2 \) then either \( w_1 \) is a translate of \( w_2 \) or \( D_1 \) is contained in the interior of \( D_2 \).

**Proof.** We may assume that \( w_1(0) = \min_R w_1, i = 1, 2 \). By Lemmas 2.8 and 2.4, if \( w_1 \not\equiv w_2 \), then for any two points \( s_i \in (0, \tau(w_i)), i = 1, 2 \) we have \((w_1, w'_1)(s_1) \neq (w_2, w'_2)(s_2)\). Therefore, if one of the two functions (say \( w_1 \)) is a constant, then the value of this constant must be different from both the minimum and the maximum of \( w_2 \) so that our claim holds. Thus we assume that neither of the two functions is a constant. Hence, by Lemma 3.1, there exists exactly one point \( \tilde{\tau}_i \) in \((0, \tau(w_i))\) such that \( w_i \) is strictly increasing in \([0, \tilde{\tau}_i]\) and strictly decreasing in \([\tilde{\tau}_i, \tau(w_i)]\). Consequently the function \( w_i, i = 1, 2 \) is represented in the phase plane \((w, w')\) by a simple closed curve \( \Lambda_i \) consisting of two branches stretching between the points \((w_1(0), 0)\) and \((w_1(\tilde{\tau}_i), 0)\) and \( \Lambda_1 \cap \Lambda_2 = \emptyset \). Since \( D_i = \{w_i(0), w_i(\tilde{\tau}_i)\} \) this proves our claim.

Define

\[
(3.11) \quad \mathcal{D} = \{\{w(t) : t \in R^1\} : w \in \mathcal{E}(f)\}.
\]

**Lemma 3.4.** Let \( f \in \mathcal{M} \). The set \( \mathcal{D} \), ordered according to set inclusion, possesses a minimal element \( D_0 \) such that, for every \( D \in \mathcal{D} \) either \( D_0 \subseteq D \) or \( D_0 \cap D = \emptyset \).
Furthermore, if

\[ \mu(f) < \inf \{ f(z, 0, 0) : z \in \mathbb{R}^1 \}, \]

then \( \mathcal{D} \) possesses only finitely many minimal elements.

Proof. – If \( \mu(f) = \inf \{ f(z, 0, 0) : z \in \mathbb{R}^1 \} \) then there exists a periodic minimizer which is a constant so that \( \mathcal{D} \) contains an element \( D_0 \) consisting of one point. Obviously \( D_0 \) is a minimal element of \( \mathcal{D} \). Therefore we may assume that (3.12) is valid. We claim that under this assumption,

\[ \alpha := \inf \{|D| : D \in \mathcal{D} \} > 0, \]

and that there exists \( v \in \mathcal{E}(f) \) such that \( \max v - \min v = \alpha \).

Let \( \{w_n\} \) be a sequence in \( \mathcal{E}(f) \) such that \( \alpha_n := \max w_n - \inf w_n \rightarrow \alpha \). We may assume that each function \( w_n \) attains its minimum at zero. Put \( b_n := \min_R w_n, \ c_n := \max_R w_n \) and \( \tau_n := \tau(w_n) \). By Lemma 3.2 the sequence of periods \( \{\tau(w_n)\} \) is bounded and, by Proposition 2.3, the set \( \mathcal{E}(f) \) is uniformly bounded. Therefore, by taking a subsequence if necessary, we may assume that \( \{b_n\}, \ \{c_n\} \) and \( \{\tau_n\} \) converge. We denote their limits by \( b^*, c^*, \tau^* \) respectively. By Lemma 2.2, \( \{w_n\} \) is bounded in \( W^{2,\gamma}_\text{loc}(\mathbb{R}) \) and consequently there exists a subsequence \( \{w_{n_j}\} \) which converges weakly in \( W^{2,\gamma}(0, T) \) and strongly in \( C^1(0, T) \), for any \( T > 0 \). Its limit \( v \) satisfies \( b^* = v(0) = \min_{\mathbb{R}^+} v \) and \( c^* = \max_{\mathbb{R}^+} v \). By the weak lower semicontinuity of the functionals, \( v \) is \( (f) \)-perfect (see (2.15b)). If \( \tau^* = 0 \) then \( b^* = c^* \), i.e. \( v \) is a constant. However, by (3.12), this is impossible. Thus \( \tau^* > 0 \) and \( v \) is a periodic minimizer with period \( \tau^* \). Hence \( D^* = [b^*, c^*] \in \mathcal{D} \) and \( c^* - b^* = \alpha \). Since \( v \) is not a constant \( \alpha > 0 \). Therefore (3.13) holds and our claim is proved. In view of Lemma 3.3 this implies that \( D^* \) is a minimal element.

In order to verify the last statement of the lemma, observe that if \( D_1, D_2 \) are two distinct minimal elements of \( \mathcal{D} \) then, by Lemma 3.3, \( D_1 \cap D_2 = \emptyset \). Therefore, the uniform boundedness of \( \mathcal{E}(f) \) and (3.13) imply that the number of minimal elements is finite. \( \square \)

Proof of Theorem 3.1. – Let \( w_0 \) be a function in \( \mathcal{E}(f) \) such that

\[ [b, c] = \{w_0(t) : t \in \mathbb{R}\} \]

is a minimal element of \( \mathcal{D} \). Let \( \phi \) be a function in \( C^\infty(\mathbb{R}) \) such that,

\[ \phi(x) = 0, \forall x \in [b, c], \quad \phi(x) > 0, \forall x \in \mathbb{R} \setminus [b, c], \]

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and $\phi^{(m)} \in L^\infty(R)$, $m = 0, 1, 2, \ldots$. In the present case such a function is easily constructed. In a more general context the existence of such functions was established in [2, Ch. 2, Sec.3].

With $\phi$ as above, let $f_\rho$ be defined as in the statement of the theorem. Then

$$J^f_\rho(v) \geq J^f(v), \quad \forall v \in W^{2,1}_{loc}(0, \infty).$$

If $v$ is a periodic function, equality holds in (3.14) if and only if

$$\{v(t) : t \in [0, \infty)\} \subseteq [b, c].$$

Hence

$$\mu(f_\rho) \geq \mu(f) = J^f(w_0) = J^f_\rho(w_0) \geq \mu(f_\rho).$$

Consequently, $\mu(f) = \mu(f_\rho)$ and $w_0$ is a minimizer of $(P_\infty)$ with integrand $f_\rho$. We claim that $w_0$ is the unique (up to translation) periodic minimizer of this problem. Indeed, if $w$ is another periodic minimizer of this problem then, by (3.14), (3.15), $w \in \mathcal{E}(f)$ and $\{w(t) : t \in R\} \subseteq [b, c]$. Since $[b, c]$ is a minimal element of $\mathcal{D}$ it follows that $\{w(t) : t \in R\} = [b, c]$. However, by Lemma 3.3, this implies that $w$ is a translate of $w_0$.

Next we prove a slightly stronger formulation of Theorem 1.3 (i):

**Theorem 3.2.** Let $f \in \mathcal{M}$. If $\phi \in C^\infty(R)$ and $f_\rho$ are as in Theorem 3.1 then, for each $\rho \in (0, 1)$, $f_\rho$ possesses (ATP).

**Proof.** First suppose that $\mu(f) < \inf_R f(\cdot, 0, 0)$. In this case the statement of the theorem is an immediate consequence of Theorem 3.1 and the following result of Zaslavski [18]:

Assume that $h \in \mathcal{M}$ and that $\mu(h) < \inf_R h(\cdot, 0, 0)$. Then $h$ has (ATP) if and only if there exists a unique (up to translation) periodic $(h)$-minimizer.

Next suppose that $\mu(f) = \inf_R f(\cdot, 0, 0)$. Then there exists $\xi_0 \in R^1$ such that $f(\xi_0, 0, 0) = \mu(f)$ and $\phi$ is positive everywhere except at $\xi_0$. By Theorem 3.1, for every $\rho \in (0, 1)$, problem $(P_\infty)$ with integrand $f_\rho$ has a unique periodic minimizer, namely the constant function with value $\xi_0$. In order to prove that $(f_\rho)$ possesses (ATP) we must prove that,

$$v \in W^{2,1}_{loc}(0, \infty) \quad \text{and} \quad v \text{ is } (f_\rho)-\text{good} \implies \lim_{t \to \infty} (v, v')(t) = (\xi_0, 0).$$

Let $v$ satisfy the assumptions of (3.16) for some $\rho \in (0, 1)$. Then, in view of (3.14), $J^f(v) = \mu(f)$. Since

$$0 \leq \eta^f_\rho(T, v) - \eta^f(T, v) = \int_0^T \rho \phi(v(t)) \, dt,$$
and \( \eta^f(\cdot, v) \) is bounded on \((0, \infty)\) it follows that \( \eta^f(\cdot, v) \) is bounded, i.e. \( v \) is an \((f)\)-good function, and \( \lim_{T \to \infty} \int_0^T \phi(v(t)) \, dt < \infty \). We claim that
\[
\lim_{t \to \infty} v(t) = \xi_0.
\]
Indeed by Lemma 2.2 \( v \) and \( v' \) are uniformly continuous on \((0, \infty)\). Therefore, if there exists a sequence \( \{t_n\} \) tending to infinity such that \( v(t_n) \to \xi_1 \neq \xi_0 \) then there exists a positive \( \delta \) such that
\[
\liminf_{n \to \infty} \text{dist}(\xi_0, \{v(t) : t_n - \delta \leq t \leq t_n + \delta\}) > 0.
\]
Since \( v \) is bounded and \( \phi \) is positive except at \( \xi_0 \) this contradicts the integrability of \( \phi(v(\cdot)) \) on \((0, \infty)\).

Next we claim that \( \lim_{t \to \infty} v'(t) = 0 \). If not, assume for instance that \( \limsup_{t \to \infty} v'(t) = \zeta > 0 \). Then, because of the uniform continuity of \( v' \), it follows that there exists a sequence \( \{t_n\} \) tending to infinity and a positive \( \delta \) such that \( \inf \{v'(t) : t_n - \delta \leq t \leq t_n + \delta\} > \zeta / 2 \) for all sufficiently large \( n \). Therefore \( v(t_n + \delta) - v(t_n) > \delta \zeta / 2 \) for all sufficiently large \( n \), which contradicts (3.17). Thus \( \lim_{t \to \infty} (v, v')(t) = (\xi_0, 0) \) and (3.16) is proved. \( \square \)

Finally we turn to,

Proof of Theorem 1.3 (ii). – Denote by \( E \) the set of all functions \( \phi \in \mathcal{L}(\alpha, b_2, b_3) \) such that \( F_{\phi} \) has (ATP). By Theorem 3.2 the set \( E \) is everywhere dense in \( \mathcal{L}(\alpha, b_2, b_3) \). For each \( \phi \in E \) there exist \( v_\phi \in W^{2,1}_{loc}(R^1), T_\phi > 0 \) such that
\[
(3.18) \quad v_\phi(t + T_\phi) = v_\phi(t), \quad t \in R^1, \quad I^{F_{\phi}}(0, T_\phi, v_\phi) = \mu(F_{\phi}) T_\phi.
\]
Let \( \phi \in E, n \geq 1 \) be an integer. By (3.18), the definition of the set \( E \), the continuity of the operator
\[
\phi \to F_{\phi}, \quad \phi \in \mathcal{L}(\alpha, b_2, b_3).
\]
and Theorem 2.2 there exist an open neighborhood \( U(\phi, n) \) of \( \phi \) in \( \mathcal{L}(\alpha, b_2, b_3) \) such that for each \( \psi \in U(\phi, n) \) and each \((F_{\psi})\)-good function \( w \in W^{2,1}_{loc}(0, \infty) \)
\[
(3.19) \quad \text{dist}(\Omega(w), \{X_{v_\phi}(t) : t \in R^1\}) \leq (2n)^{-1}.
\]
Define
\[
\mathcal{F} = \bigcap_{n=1}^{\infty} \cup \{U(\phi, n) : \phi \in E\}.
\]
Let \( h \in \mathcal{F} \), \( w_1, w_2 \) be \((F_h)\)-good functions. To complete the proof of the theorem it is sufficient to show that \( \Omega(w_1) = \Omega(w_2) \). Let \( \epsilon \in (0, 1) \). There exist an integer \( n \geq 8\epsilon^{-1} \) and \( \phi \in E \) such that \( h \in U(\phi, n) \). It follows from the definition of \( U(\phi, n) \) that
\[
\text{dist}(\Omega(w_i), \{X_{v_\phi}(t) : t \in R^1\}) \leq (2n)^{-1}, \quad i = 1, 2, \quad \text{dist}(\Omega(w_1), \Omega(w_2)) \leq \epsilon.
\]
This completes the proof of the theorem. \( \square \)
This appendix is devoted to the proof of Lemma A, which will be based on several additional lemmas.

**Lemma A.1.** Let \( \epsilon, M > 0 \). Then there exist \( \delta > 0 \) and an integer \( q_1 \geq 1 \) such that for each \( v \in W^{2,1}(0, q_1 T_w) \) which satisfy

\[
I^g(0, q_1 T_w, v) \leq q_1 T_w \mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(q_1 T_w)) + \delta
\]

there exist \( \xi \in [0, T_w) \), \( \tau \in [0, (q_1 - 1)T_w] \) such that

\[
|X_v(\tau + t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w].
\]

**Proof.** Let us assume the converse. Then for each integer \( p \geq 1 \) there exists \( v_p \in W^{2,1}(0, p T_w) \) such that

\[
I^g(0, p T_w, v_p) \leq p T_w \mu(g) + \pi^g(X_{v_p}(0)) - \pi^g(X_{v_p}(p T_w)) + 2^{-p}
\]

and for each \( \xi \in [0, T_w) \), each \( \tau \in [0, (p - 1)T_w] \)

\[
\sup\{|X_{v_p}(\tau + t) - X_w(\xi + t)| : t \in [0, T_w]\} > \epsilon.
\]

By (A.2) and Proposition 2.2 there exists \( M_1 > 0 \) such that for each integer \( p \geq 1 \)

\[
|X_{v_p}(t)| \leq M_1, \quad t \in [0, p T_w].
\]

(A.2), (A.4) and (2.2) imply that for any integer \( n \geq 1 \) the sequence \( \{v_{p_k}\}_{k=1}^{\infty} \) is bounded in \( L^\gamma[0, n T_w] \). It is easy to verify that there are \( v \in W^{2,\gamma}_{loc}(0, \infty) \) and a strictly increasing subsequence of natural numbers \( \{p_k\}_{k=1}^{\infty} \) such that for every integer \( n \geq 1 \)

\[
v_{p_k} \rightharpoonup v \text{ as } k \to \infty \text{ weakly in } W^{2,\gamma}(0, n T_w).
\]

By (A.2) and the lower semicontinuity of integral functionals [3] for each integer \( n \geq 1 \)

\[
I^g(0, n T_w, v) = n T_w \mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(n T_w)).
\]
Clearly

\begin{equation}
|X_v(t)| \leq M_1, \quad t \in [0, \infty).
\end{equation}

It follows from (A.5) and the definition of \( \{v_p\}_{p=1}^{\infty} \) (see (A.2), (A.3)) that for each \( \tau \in [0, \infty) \) and each \( \xi \in [0, T_w) \)
\begin{equation}
\sup\{|X_v(\tau + t) - X_w(\xi + t)| : t \in [0, T_w]\} > 2^{-1} \epsilon.
\end{equation}

(A.6) and (A.7) imply that the function \( v \) is \((g)\)-good. Then
\begin{equation}
\Omega(v) = \Omega(w).
\end{equation}

There exists a sequence of numbers \( \{t_j\}_{j=1}^{\infty} \subset (0, \infty) \) such that
\begin{equation}
t_1 \geq 8T_w + 8, \quad t_{j+1} - t_j \geq 8T_w, \quad j = 1, 2, \ldots, \quad X_v(t_j) \to X_w(0) \text{ as } j \to \infty.
\end{equation}

For each integer \( j \geq 1 \) we define \( u_j \in W^{2,1}(-4T_w, 4T_w) \) as follows
\begin{equation}
u_j(t) = v(t_j + t), \quad t \in [-4T_w, 4T_w].
\end{equation}

By (2.2), (A.11), (A.6) and (A.7) the sequence \( \{u''_j\}_{j=1}^{\infty} \) is bounded in \( L^\gamma[-4T_w, 4T_w] \). It is easy to verify that there are \( u \in W^{2,1}(-4T_w, 4T_w) \) and a strictly increasing subsequence of natural numbers \( \{j_p\}_{p=1}^{\infty} \) such that
\begin{equation}
u_{j_p}(t) \to u(t), \quad u'_{j_p}(t) \to u'(t) \text{ as } p \to \infty \text{ uniformly in } [-4T_w, 4T_w],
\end{equation}
\begin{equation}u''_{j_p} \to u'' \text{ as } p \to \infty \text{ weakly in } L^\gamma[-4T_w, 4T_w].
\end{equation}

By (A.6) and the lower semicontinuity of integral functionals [3]
\begin{equation}I^g(-4T_w, 4T_w, u) = 8T_w \mu(g) + \pi^g(X_u(-4T_w)) + \pi^g(X_u(4T_w)).
\end{equation}

Clearly
\begin{equation}
X_u(0) = X_w(0).
\end{equation}

It follows from (A.11), (A.12) and (A.8) which holds for each \( \tau \in [0, \infty) \) and each \( \xi \in [0, T_w) \), that
\begin{equation}
\sup\{|X_u(t) - X_w(t)| : t \in [0, T_w]\} > 4^{-1} \epsilon.
\end{equation}

On the other hand (A.13), (A.14) and Lemma 2.8 imply that \( u(t) = w(t) \) for all \( t \in [-4T_w, 4T_w] \). The obtained contradiction proves the lemma. \( \square \)
Lemma A.2. Let \( M_0, M_1, \epsilon > 0 \). Then there exists an integer \( q \geq 1 \) such that for each \( v \in W^{2,1}(0, qT_w) \) which satisfies
\[
|X_v(s)| \leq M_0, \quad s = 0, qT_w, \quad I^g(0, qT_w, v) \leq U^g_{qT_w}(X_v(0), X_v(qT_w)) + M_1
\]
there exist \( \xi \in [0, T_w), \tau \in [0, (q - 1)T_w] \) such that
\[
|X_v(\tau + t) - X_v(\xi + t)| \leq \epsilon, \quad t \in [0, T_w].
\]

Proof. By Proposition 2.2 there is \( S_0 > M_0 + M_1 + 2 \) such that for each \( \tau \geq 2^{-1}\inf\{T_w, 1\} \), each \( v \in W^{2,1}(0, \tau) \) which satisfies
\[
|X_v(0)|, |X_v(\tau)| \leq M_0, \quad I^g(0, \tau, v) \leq U^g_{\tau}(X_v(0), X_v(\tau)) + M_1 + 1
\]
the following relation holds
\[
|X_v(t)| \leq S_0, \quad t \in [0, \tau].
\]

By Lemma A.1 there exists an integer \( q_1 \geq 1 \) and a number \( \delta > 0 \) such that for each \( v \in W^{2,1}(0, q_1T_w) \) which satisfies
\[
|X_v(t)| \leq S_0, \quad t = 0, q_1T_w,
\]
\[
I^g(0, q_1T_w, v) \leq q_1T_w\mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(q_1T_w)) + \delta
\]
there exist \( \xi \in [0, T_w), \tau \in [0, (q_1 - 1)T_w] \) such that (A.16) holds. By Lemma 2.6 there exists \( K_0 > 0 \) such that for each \( \tau \geq 4T_w \), each \( x, y \in R^2 \) satisfying \( |x|, |y| \leq M_0 + S_0 + 1 \) the following relation holds
\[
U^g_{\tau}(x, y) \leq \tau\mu(g) + \pi^g(x) - \pi^g(y) + K_0.
\]
Here we use the fact that \( \pi^g \) is bounded on compact sets. Fix an integer
\[
q > [(M_1 + K_0 + 1)\delta^{-1} + 4]q_1.
\]
Assume that \( v \in W^{2,1}(0, qT_w) \) and (A.15) holds. It follows from (A.15) and the definition of \( K_0 \) (see (A.19)) that
\[
I^g(0, qT_w, v) \leq U^g_{qT_w}(X_v(0), X_v(qT_w)) + M_1
\]
\[
\leq qT_w\mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(qT_w)) + M_1 + K_0.
\]
By the definition of \( S_0 \) (see (A.17)) and (A.15)
\[
|X_v(t)| \leq S_0, \quad t \in [0, qT_w].
\]
There exists a sequence \( \{t_i\}_{i=0}^s \subset [0, qT_w] \) such that

(A.23) \( t_0 = 0, \ t_{i+1} = t_i + qT_w \) if \( 0 \leq i \leq s-1, \ t_s \in [qT_w - q_1 T_w, qT_w] \).

Clearly

(A.24) \( s \geq q_1^{-1} - 1 \geq 3 + \delta^{-1}(M_1 + K_0 + 1) \).

Together with (A.21) this implies that there is \( j \in \{0, \ldots s - 1\} \) for which

(A.25) \( I^g(t_j, t_{j+1}, v) \leq (t_{j+1} - t_j)\mu(g) + \pi^g(X_v(t_j)) - \pi^g(X_v(t_{j+1})) + \delta. \)

It follows from this relation, (A.22), (A.23) and the definition of \( \delta, q_1 \) (see (A.18)) that there exist \( \xi \in [0, T_w], \tau \in [t_j, t_{j+1} - T_w] \) such that (A.16) holds. This completes the proof of the lemma. 

Proof of Lemma A. – By Proposition 2.2 there are a neighborhood \( \mathcal{U}_1 \) of \( g \) in \( \mathcal{M} \) and a number \( M_2 > M_0 + M_1 \) such that for each \( f \in \mathcal{U}_1 \), each \( T \geq \inf \{T_w, 1\} \) and each \( v \in W^{2,1}(0, T) \) satisfying (2.24) the following relation holds

(A.26) \( |X_v(t)| \leq M_2, \quad t \in [0, T]. \)

By Lemma A.2 there exists an integer \( q_1 \geq 1 \) such that for each \( v \in W^{2,1}(0, q_1 T_w) \) which satisfies

(A.27) \( |X_v(0)|, |X_v(q_1 T_w)| \leq M_2, \)

\( I^g(0, q_1 T_w, v) \leq U^g_{q_1 T_w}(X_v(0), X_v(q_1 T_w)) + 2M_1 + 8 \)

there exist \( \xi \in [0, T_w], s \in [0, (q_1 - 1)T_w] \) such that (2.25) holds.

There exists a number \( \Gamma_0 > 0 \) for which

(A.28) \( \sup\{ |U^g_{q_1 T_w}(x, y)| : x, y \in R^2, |x|, |y| \leq M_2 \} \leq \Gamma_0. \)

By Lemma 2.7 there exists a neighborhood \( \mathcal{U}_2 \) of \( g \) in \( \mathcal{M} \) such that for each \( f \in \mathcal{U}_2 \), each \( x, y \in R^2 \) satisfying \( |x|, |y| \leq M_2 \) the relation \( |U^f_{q_1 T_w}(x, y) - U^g_{q_1 T_w}(x, y)| \leq 2^{-1} \) holds.

By Lemma 2.7 there exists a neighborhood \( \mathcal{U}_3 \) of \( g \) in \( \mathcal{M} \) such that for each \( f \in \mathcal{U}_3 \), each \( v \in W^{2,1}(0, q_1 T_w) \) satisfying

\( \inf\{ I^f(0, q_1 T_w, v), I^g(0, q_1 T_w, v) \} \leq 2\Gamma_0 + 2M_1 + 4 \)

the relation \( |I^f(0, q_1 T_w, v) - I^g(0, q_1 T_w, v)| \leq 2^{-1} \) holds. Set \( \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3. \)
Assume that \( f \in \mathcal{U}, T \geq q_1T_w, v \in W^{2,1}(0, T) \) satisfies (2.24) and \( \tau \in [0, T - q_1T_w] \). By the definition of \( \mathcal{U}_1 \) and \( M_2 \) relation (A.26) holds. It follows from (2.24), (A.26), the definition of \( \mathcal{U}_2 \) and (A.28) that

\[
I^f(\tau, \tau + q_1T_w, v) \leq U_{q_1T_w}^f(X_v(\tau), X_v(\tau + q_1T_w)) + M_1 \\
\leq U_{q_1T_w}^g(X_v(\tau), X_v(\tau + q_1T_w)) + 2^{-1} + M_1 \leq \Gamma_0 + 2^{-1} + M_1.
\]

By this relation and the definition of \( \mathcal{U}_3 \)

\[
|I^f(\tau, \tau + q_1T_w, v) - I^g(\tau, \tau + q_1T_w, v)| \leq 2^{-1},
\]

\[
I^g(\tau, \tau + q_1T_w, v) \leq U_{q_1T_w}^g(X_v(\tau), X_v(\tau + q_1T_w)) + 1 + M_1.
\]

It follows from this relation, (A.26) and the definition of \( q_1 \) (see (A.27)) that there exist \( \xi \in [0, T_w) \), \( s \in [\tau, \tau + q_1T_w - T_w] \) such that (2.25) holds. The lemma is proved.

**APPENDIX B**

Here we establish Lemma B whose proof is based on several auxiliary results.

The following lemma shows that given \( \varepsilon > 0 \) and a \((g)\)-good function \( v \), for sufficiently large \( T \) the restriction of \((v, v')\) to \([T, T + T_w]\) is within \( \varepsilon \) of a translation of \((w, w')\).

**LEMMA B.1.** Assume that \( v \in W^{2,1}_{loc}(0, \infty) \) is a \((g)\)-good function and \( \varepsilon > 0 \). Then there exists \( T(\varepsilon) > 0 \) such that for each \( T \geq T(\varepsilon) \) there is \( \xi \in [0, T_w) \) such that

\[
|X_v(T + t) - X_w(\xi + t)| \leq \varepsilon, \quad t \in [0, T_w].
\]

**Proof.** Since \( v \) is a \((g)\)-good function for each \( \delta > 0 \) there exists \( T(\delta) > 0 \) such that

\[
I^g(\tau_1, \tau_2, v) \leq (\tau_2 - \tau_1)\mu(g) + \pi^g(X_v(\tau_1)) - \pi^g(X_v(\tau_2)) + \delta
\]

for each \( \tau_1 \geq T(\delta) \) and each \( \tau_2 > \tau_1 \) (see Lemma 2.4).

Assume that the lemma is wrong. Then there exists a sequence of numbers \( \{t_i\}_{i=1}^{\infty} \subset (0, \infty) \) such that

\[
t_i \geq T(2^{-i}) + 2i + 2, \quad i = 1, 2, \ldots
\]

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and for each integer \( i \geq 1 \) and each \( \xi \in [0, T_w) \)

\[(B.3) \quad \sup \{ |X_v(t_i + t) - X_w(\xi + t)| : t \in [0, T_w] \} > \epsilon.\]

For each integer \( i \geq 1 \) we define \( u_i \in W^{2,1}_{loc}(-t_i, \infty) \) as follows

\[(B.4) \quad u_i(t) = v(t_i + t), \quad t \in [-t_i, \infty).\]

It follows from the definition of \( T(\delta), \delta > 0 \) (see (B.1)), (B.2), (B.4) and (2.2) that for any integer \( n \geq 1 \) the sequence \( \{u^{(i)}_i\}_{i=n}^{\infty} \) is bounded in \( L^\gamma[-n, n] \).

It is easy to see that there exist \( u \in W^{2,\gamma}_{loc}(R^1) \) and a strictly increasing subsequence of natural numbers \( \{i_p\}_{p=1}^{\infty} \) such that for every integer \( n \geq 1 \)

\[(B.5) \quad u_{i_p} \rightharpoonup u \quad \text{as} \quad p \to \infty \quad \text{weakly in} \quad W^{2,\gamma}(-n, n).\]

By the definition of \( T(\delta), \delta > 0 \) (see (B.1)), (B.2), (B.4), (B.5) and the lower semicontinuity of integral functionals [3]

\[(B.6) \quad I^g(\tau_1, \tau_2, u) = (\tau_2 - \tau_1)\mu(g) + \pi^g(X_u(\tau_1)) - \pi^g(X_u(\tau_2))\]

for each \( \tau_1 \in R^1, \tau_2 > \tau_1 \).

It is easy to see that for each \( t \in R^1 \)

\[X_u(t) \in \Omega(v) = \{X_w(s) : s \in R^1\}.\]

Together with (B.6), Lemma 2.8 this implies that there exists \( \xi_0 \in [0, T_w) \) such that \( u(t) = w(t + \xi_0), t \in R^1 \). It follows from this relation and (B.5), (B.4) that there exists an integer \( p_0 \geq 1 \) such that for each integer \( p \geq p_0 \)

\[|X_v(t_{i_p} + t) - X_w(\xi_0 + t)| \leq 2^{-1}\epsilon, \quad t \in [0, T_w].\]

This is contradictory to the definition of \( \{t_i\}_{i=1}^{\infty} \) (see (B.3)). The obtained contradiction proves the lemma.

**Lemma B.2.** Let \( \epsilon > 0 \). Then there exists \( \delta > 0 \) such that for each \( \tau \geq T_w \) and each \( s \in [0, \tau - T_w] \), if \( v \) is a function in \( W^{2,1}(0, \tau) \) such that

\[(B.7) \quad d(X_v(s), \{X_w(t) : t \in R^1\}) \leq \delta, \quad s = 0, \tau,\]

\[I^g(0, \tau, v) \leq \tau \mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(\tau)) + \delta\]

then there is \( \xi \in [0, T_w) \) for which

\[(B.8) \quad |X_v(s + t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w].\]
Proof. – By Proposition 2.1 and the continuity of $\pi^g$, $U^g_{T_w}$ for each integer $i \geq 1$ there exists $\delta_i \in (0, 4^{-i})$ such that for each $x, y \in \mathbb{R}^2$ satisfying $|x - y| \leq \delta_i$, $d(x, \{X_w(t) : t \in R^1\}) \leq \delta_i$ the following relation holds

\begin{equation}
U^g_{T_w}(x, y) \leq \pi^g(x) - \pi^g(y) + T_w\mu(g) + 2^{-i}.
\end{equation}

Assume that the lemma is wrong. Then for each integer $i \geq 1$ there exist $\tau_i \geq T_w$, $v_i \in W^{2,1}(0, \tau_i)$ such that

\begin{equation}
d(X_v(s), \{X_w(t) : t \in R^1\}) \leq \delta_i, \quad s = 0, \tau_i,
\end{equation}

\begin{equation}
I^g(0, \tau_i, v_i) \leq \tau_i\mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(\tau_i)) + \delta_i
\end{equation}

and there exists $s_i \in [0, \tau_i - T_w]$ such that for each $\xi \in [0, T_w)$

\begin{equation}
\sup\{|X_v(s_i + t) - X_w(\xi + t)| : t \in [0, T_w]\} > \epsilon.
\end{equation}

For each integer $i \geq 1$ there exist $\xi^1_i, \xi^2_i \in [0, T_w)$ such that

\begin{equation}
|X_v(0) - X_w(\xi^1_i)|, |X_v(\tau_i) - X_w(\xi^2_i)| \leq \delta_i.
\end{equation}

For each integer $i \geq 1$ there exists a function $u_i \in W^{2,1}(0, \tau_i + 2T_w)$ such that

\begin{equation}
X_u(0) = X_w(\xi^1_i), \quad u_i(t) = v_i(t - T_w), \quad t \in [T_w, T_w + \tau_i], \quad X_u(\tau_i + 2T_w) = X_w(\xi^2_i),
\end{equation}

\begin{equation}
I^g(s, s + T_w, u_i) = U^g_{T_w}(X_u(s), X_u(s + T_w)), \quad s = 0, \tau_i + T_w.
\end{equation}

It follows from (B.13), (B.12) and the definition of $\{\delta_i\}_{i=1}^\infty$ (see (B.9)) that for each integer $i \geq 1$

\begin{equation}
I^g(s, s + T_w, u_i) \leq T_w\mu(g) + \pi^g(X_u(s))
\end{equation}

\begin{equation}
-\pi^g(X_u(s + T_w)) + 2^{-i}, \quad s = 0, \tau_i + T_w.
\end{equation}

Together with (B.13), (B.10) this implies that for each integer $i \geq 1$

\begin{equation}
I^g(0, \tau_i + 2T_w, u_i) \leq (\tau_i + 2T_w)\mu(g) + \pi^g(X_u(0)) - \pi^g(X_u(\tau_i + 2T_w)) + 3 \cdot 2^{-i}.
\end{equation}

For each integer $i \geq 1$ there exists $\xi^3_i \in [T_w, 2T_w]$ such that

\begin{equation}
T_w^{-1}[\xi^2_i + \xi^3_i - \xi^1_{i+1}] \text{ is an integer.}
\end{equation}

We define sequences of numbers $\{b_i\}_{i=1}^\infty$, $\{c_i\}_{i=1}^\infty$ as follows

\begin{equation}
b_1 = 0, \quad c_i = b_i + \tau_i + 2T_w, \quad b_{i+1} = c_i + \xi^3_i, \quad i = 1, 2, \ldots
\end{equation}
It is easy to verify that there exists $u \in W^{2,1}_{\text{loc}}(0, \infty)$ such that for each integer $i \geq 1$

\begin{equation}
\tag{B.17}
 u(b_i + t) = u_i(t), \quad t \in [0, \tau_i + 2T_w], \quad u(c_i + t) = w(\xi_i^2 + t), \quad t \in [0, \xi_i^3].
\end{equation}

For each integer $i \geq 1$ we set

$$s_i^0 = b_i + T_w + s_i.$$ 

It follows from (B.16), (B.17), (B.13), (B.11) that for each integer $i \geq 1$, for each $\xi \in [0, T_w)$

\begin{equation}
\tag{B.18}
\sup\{ |X_u(s_i^0 + t) - X_w(\xi + t)| : t \in [0, T_w] \} > \epsilon.
\end{equation}

(B.17), (B.14), (B.16) imply that $u$ is a $(g)$-good function. By Lemma B.1 there exists a number $T^* > 0$ such that for each $T > T^*$ there is $\xi \in [0, T_w)$ such that

$$|X_u(T + t) - X_w(\xi + t)| \leq 2^{-1}\epsilon, \quad t \in [0, T_w].$$

This is contradictory to (B.18) which holds for each integer $i \geq 1$ and each $\xi \in [0, T_w)$. The obtained contradiction proves the lemma.

Analogously to Lemma 3.7 in [17] we can establish the following result.

**Lemma B.3.** -Let $f \in \mathfrak{M}$, $w \in W^{2,1}_{\text{loc}}(R^1)$, $T > 0$, $w(t + T) = w(t)$, $t \in R^1$, $I^f(0, T, w) = T\mu(f)$, $\epsilon > 0$. Then there exists an integer $q \geq 1$ such that for any $\xi \in [0, T)$ there is a function $v \in W^{2,1}(0, qT)$ such that $X_v(0) = X_w(0)$, $X_v(qT) = X_w(\xi)$, $I^f(0, qT, v) \leq qT\mu(f) + \pi^f(X_w(0)) - \pi^f(X_w(\xi)) + \epsilon$.

Lemma B.3 implies the following result.

**Lemma B.4.** -Let $\epsilon > 0$. Then there exists a number $q(\epsilon) > 0$ such that for each $\tau \geq q(\epsilon)$, each $\xi_1, \xi_2 \in [0, T_w)$ there exists $v \in W^{2,1}(0, \tau)$ which satisfies $X_v(0) = X_w(\xi_1)$, $X_v(\tau) = X_w(\xi_2)$,

$$I^g(0, \tau, v) \leq \tau\mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(\tau)) + \epsilon.$$ 

Lemma B.4, Proposition 2.1 and the continuity of $\pi^g$ and $U_1^\rho$ imply the following extension of Lemma B.3.

**Lemma B.5.** -Let $\epsilon > 0$. Then there exist numbers $\delta, q(\epsilon) > 0$ such that for each $\tau \geq q(\epsilon)$, each $x, y \in R^2$ satisfying

\begin{equation}
\tag{B.19}
d(x, \{X_w(t) : t \in R^1\}) \leq \delta, \quad d(y, \{X_w(t) : t \in R^1\}) \leq \delta
\end{equation}

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there exists $v \in W^{2,1}(0, \tau)$ which satisfies

$$X_v(0) = x, \ X_v(\tau) = y, \ I^g(0, \tau, v) \leq \tau \mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(\tau)) + \epsilon.$$ 

**Corollary B.1.** Let $\epsilon > 0$ and let $\delta, q(\epsilon) > 0$ be as guaranteed in Lemma B.5. Then for each $\tau \geq q(\epsilon)$, each $x, y \in \mathbb{R}^2$ satisfying (8.19) the following relation holds

$$U^g(\tau)(x, y) \leq \tau \mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(\tau)) + \epsilon.$$

Corollary B.1 and Lemma B.3 imply the following result.

**Lemma B.6.** Let $\epsilon > 0$. Then there exist $b > 0$, $Q > T_w$ such that for each $\tau \geq Q$, each $v \in W^{2,1}(0, \tau)$ which satisfies $d(X_v(s), \{X_w(t) : t \in \mathbb{R}^1\}) \leq \delta$, $s = 0, \tau$, $I^g(0, \tau, v) \leq U^g(\tau)(X_v(0), X_v(\tau)) + \delta$ and each $s \in [0, \tau - T_w]$ there is $\xi \in [0, T_w)$ for which

$$|X_v(s + t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w].$$

Lemmas B.6 and 2.6 imply Lemma B.

**REFERENCES**


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