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## Existence results for mean field equations

by

Weiyue DING <sup>1</sup>, Jürgen JOST <sup>2</sup>, Jiayu LI <sup>3</sup> and Guofang WANG <sup>4</sup>

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ABSTRACT. – Let  $\Omega$  be an annulus. We prove that the mean field equation

$$\begin{aligned} -\Delta\psi &= \frac{e^{-\beta\psi}}{\int_{\Omega} e^{-\beta\psi}} && \text{in } \Omega \\ \psi &= 0 && \text{on } \partial\Omega \end{aligned}$$

admits a solution for  $\beta \in (-16\pi, -8\pi)$ . This is a supercritical case for the Moser-Trudinger inequality. © Elsevier, Paris

RÉSUMÉ. – On montre que l'équation de champ moyen

$$\begin{aligned} -\Delta\psi &= \frac{e^{-\beta\psi}}{\int_{\Omega} e^{-\beta\psi}} && \text{dans } \Omega \\ \psi &= 0 && \text{sur } \partial\Omega, \end{aligned}$$

pour  $\Omega$  étant un anneau, admet une solution pour  $\beta \in (-16\pi, -8\pi)$ . Cela représente un cas supercritique pour l'inégalité de Moser-Trudinger. © Elsevier, Paris

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## 1. INTRODUCTION

Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^2$ . In this paper, we consider the following mean field equation

$$(1.1) \quad \begin{aligned} -\Delta\psi &= \frac{e^{-\beta\psi}}{\int_{\Omega} e^{-\beta\psi}}, & \text{in } \Omega, \\ \psi &= 0, & \text{on } \partial\Omega, \end{aligned}$$

for  $\beta \in (-\infty, +\infty)$ . (1.1) is the Euler-Lagrange equation of the following functional

$$(1.2) \quad J_{\beta}(\psi) = \frac{1}{2} \int_{\Omega} |\nabla\psi|^2 + \frac{1}{\beta} \log \int_{\Omega} e^{-\beta\psi}$$

in  $H_0^{1,2}(\Omega)$ . This variational problem arises from Onsager's vortex model for turbulent Euler flows. In that interpretation,  $\psi$  is the stream function in the infinite vortex limit, see [12,p256ff]. The corresponding canonical Gibbs measure and partition function are finite precisely if  $\beta > -8\pi$ . In that situation, Caglioti et al. [4] and Kiessling [9] showed the existence of a minimizer of  $J_{\beta}$ . This is based on the Moser-Trudinger inequality

$$(1.3) \quad \frac{1}{2} \int_{\Omega} |\nabla\psi|^2 \geq \frac{1}{8\pi} \log \int_{\Omega} e^{-8\pi\psi}, \quad \text{for any } \psi \in H_0^{1,2}(\Omega),$$

which implies the relevant compactness and coercivity condition for  $J_{\beta}$  in case  $\beta > -8\pi$ . For  $\beta \leq -8\pi$ , the situation becomes different as described in [4]. On the unit disk, solutions blow up if one approaches  $\beta = -8\pi$  -the critical case for (1.3)-(see also [5] and [19]), and more generally, on starshaped domains, the Pohozaev identity yields a lower bound on the possible values of  $\beta$  for which solutions exist. On the other hand, for an annulus, [4] constructed radially symmetric solutions for any  $\beta$ , and the construction of Bahri-Coron [2] makes it plausible that solutions on domains with non-trivial topology exist below  $-8\pi$ . Thus, for  $\beta \leq -8\pi$ ,  $J_{\beta}$  is no longer compact and coercive in general, and the existence of solution depends on the geometry of the domain.

In the present paper, we thus consider the supercritical case  $\beta < -8\pi$  on domains with non-trivial topology.

**THEOREM 1.1.** – *Let  $\Omega \subset \mathbb{R}^2$  be a smooth, bounded domain whose complement contains a bounded region, e.g.  $\Omega$  an annulus. Then (1.1) has a solution for all  $\beta \in (-16\pi, -8\pi)$ .*

The solutions we find, however, are not minimizers of  $J_\beta$ -those do not exist in case  $\beta < 8\pi$ , since  $J_\beta$  has no lower bound-but unstable critical points. Thus, these solutions might not be relevant to the turbulence problem that was at the basis of [4] and [9].

Certainly we can generalize Theorem 1.1 to the following equation

$$\begin{aligned} -\Delta\psi &= \frac{Ke^{-\beta\psi}}{\int_\Omega Ke^{-\beta\psi}}, & \text{in } \Omega, \\ \psi &= 0, & \text{on } \partial\Omega, \end{aligned}$$

which was studied in [5]. Here  $K$  is a positive function on  $\bar{\Omega}$ .

With the same method, we may also handle the equation

$$(1.4) \quad \Delta u - c + cKe^u = 0, \quad \text{for } 0 \leq c < \infty$$

on a compact Riemann surface  $\Sigma$  of genus at least 1, where  $K$  is a positive function. (1.4) can also be considered as a mean field equation because it is the Euler-Lagrange equation of the functional

$$(1.5) \quad J_c(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 + c \int_\Sigma u - c \log \int_\Sigma Ke^u.$$

Because of the term  $c \int_\Sigma u$ ,  $J_c$  remains invariant under adding a constant to  $u$ , and therefore we may normalize  $u$  by the condition.

$$\int_\Sigma Ke^u = 1$$

which explains the absence of the factor  $(\int Ke^u)^{-1}$  in (1.4).  $c < 8\pi$  again is a subcritical case that can easily be handled with the Moser-Trudinger inequality. The critical case  $c = 8\pi$  yields the so-called Kazdan-Warner equation [8] and was treated in [7] and [14] by giving sufficient conditions for the existence of a minimizer of  $J_{8\pi}$ . Here, we construct again saddle point type critical points to show

**THEOREM 1.2.** *– Let  $\Sigma$  be a compact Riemann surface of positive genus. Then (1.4) admits a non-minimal solution for  $8\pi < c < 16\pi$ .*

Now we give a outline of the proof of the Theorems. First from the non-trivial topology of the domain, we can define a minimax value  $\alpha_\beta$ , which is bounded below by an improved Moser-Trudinger inequality, for  $\beta \in (-16\pi, -8\pi)$ . Using a trick introduced by Struwe in [16] and [17], for a certain dense subset  $\Lambda \subset (-16\pi, -8\pi)$  we can overcome the lack of a

coercivity condition and show that  $\alpha_\beta$  is achieved by some  $u_\beta$  for  $\beta \in \Lambda$ . Next, for any fixed  $\bar{\beta} \in (-16\pi, -8\pi)$ , considering a sequence  $\beta_k \subset \Lambda$  tending to  $\bar{\beta}$ , with the help of results in [3] and [11] we show that  $u_{\beta_k}$  subconverges strongly to some  $u_{\bar{\beta}}$  which achieves  $\alpha_{\bar{\beta}}$ .

After completing our paper, we were informed that Struwe and Tarantello [18] obtained a non-constant solution of (1.4), when  $\Sigma$  is a flat torus with fundamental cell domain  $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ ,  $K \equiv 1$  and  $c \in (8\pi, 4\pi^2)$ . In this case, it is easy to check that our solution obtained in Theorem 1.2 is non-constant.

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## 2. MINIMAX VALUES

Let  $\rho = -\beta$  and  $u = -\beta\psi$ . We rewrite (1.1) as

$$(2.1) \quad \begin{aligned} -\Delta u &= \rho \frac{e^u}{\int_{\Omega} e^u}, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

and (1.2) as

$$(2.2) \quad J_\rho(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \rho \log \int_{\Omega} e^u$$

for  $u \in H_0^{1,2}(\Omega)$ .

It is easy to see that  $J_\rho$  has no lower bound for  $\rho \in (8\pi, 16\pi)$ . Hence, to get a solution of (1.1) for  $\rho \in (8\pi, 16\pi)$ , we have to use a minimax method. First, we define a center of mass of  $u$  by

$$m_c(u) = \frac{\int_{\Omega} x e^u}{\int_{\Omega} e^u}.$$

Let  $B$  be the bounded component of  $\mathbb{R}^2 \setminus \Omega$ . For simplicity, we assume that  $B$  is the unit disk centered at the origin. Then we define a family of functions

$$h : D \rightarrow H_0^{1,2}(\Omega)$$

satisfying

$$(2.3) \quad \lim_{r \rightarrow 1} J_\rho(h(r, \theta)) \rightarrow -\infty$$

and

$$(2.4) \quad \lim_{r \rightarrow 1} m_c(h(r, \theta)) \text{ is a continuous curve enclosing } B.$$

Here  $D = \{(r, \theta) | 0 \leq r < 1, \theta \in [0, 2\pi)\}$  is the open unit disk. We denote the set of all such families by  $\mathcal{D}_\rho$ . It is easy to check that  $\mathcal{D}_\rho \neq \emptyset$ . Now we can define a minimax value

$$\alpha_\rho := \inf_{h \in \mathcal{D}_\rho} \sup_{u \in h(D)} J_\rho(u).$$

The following lemma will make crucial use of the non-trivial topology of  $\Omega$ , more precisely of the fact that the complement of  $\Omega$  has a bounded component.

LEMMA 2.1. – For any  $\rho \in (8\pi, 16\pi)$   $\alpha_\rho > -\infty$ .

Remark. – It is an interesting question whether  $\alpha_{16\pi} = -\infty$ .

To prove Lemma 2.1, we use the improved Moser-Trudinger inequality of [6] (see also [1]). Here we have to modify a little bit.

LEMMA 2.2. – Let  $S_1$  and  $S_2$  be two subsets of  $\bar{\Omega}$  satisfying  $\text{dist}(S_1, S_2) \geq \delta_0 > 0$  and  $\gamma_0 \in (0, 1/2)$ . For any  $\epsilon > 0$ , there exists a constant  $c = c(\epsilon, \delta_0, \gamma_0) > 0$  such that

$$\int_\Omega e^u \leq c \exp\left\{\frac{1}{32\pi - \epsilon} \int_\Omega |\nabla u|^2 + c\right\}$$

holds for all  $u \in H_0^{1,2}(\Omega)$  satisfying

$$(2.5) \quad \frac{\int_{S_1} e^u}{\int_\Omega e^u} \geq \gamma_0 \quad \text{and} \quad \frac{\int_{S_2} e^u}{\int_\Omega e^u} \geq \gamma_0.$$

Proof. – The Lemma follows from the argument in [6] and the following Moser-Trudinger inequality

$$(*) \quad \frac{1}{2} \int_\Omega |\nabla u|^2 - 8\pi \log \int_\Omega e^u \geq c$$

for any  $u \in H_0^{1,2}(\Omega)$ , where  $c$  is a constant independent of  $u \in H_0^{1,2}(\Omega)$ .  $\square$

We will discuss the inequality (\*) and its application in another paper.

*Proof of Lemma 2.1.* – For fixed  $\rho \in (8\pi, 16\pi)$  we claim that there exists a constant  $c_\rho$  such that

$$(2.6) \quad \sup_{u \in h(D)} J_\rho(u) \geq c_\rho, \quad \text{for any } h \in \mathcal{D}_\rho.$$

Clearly (2.6) implies the Lemma. By the definition of  $h$ , for any  $h \in \mathcal{D}_\rho$ , there exists  $u \in h(D)$  such that

$$m_c(u) = 0.$$

We choose  $\epsilon > 0$  so small that  $\rho < 16\pi - 2\epsilon$ . Assume (2.6) does not hold. Then we have sequences  $\{h_i\} \subset \mathcal{D}_\rho$  and  $\{u_i\} \subset H_0^{1,2}(\Omega)$  such that  $u_i \in h_i(D)$  and

$$(2.7) \quad m_c(u_i) = 0$$

$$(2.8) \quad \lim_{i \rightarrow \infty} J(u_i) = -\infty.$$

We have the following Lemma.

LEMMA 2.3. – *There exists  $x_0 \in \bar{\Omega}$  such that*

$$(2.9) \quad \lim_{i \rightarrow \infty} \frac{\int_{B_{1/2}(x_0) \cap \Omega} e^{u_i}}{\int_{\Omega} e^{u_i}} \rightarrow 1.$$

*Proof.* – Set

$$A(x) := \lim_{i \rightarrow \infty} \frac{\int_{B_{1/4}(x) \cap \Omega} e^{u_i}}{\int_{\Omega} e^{u_i}}.$$

Assume that the Lemma were false, then there exists  $x_0 \in \bar{\Omega}$  such that

$$A(x_0) < 1 \quad \text{and} \quad A(x_0) \geq A(x) \quad \text{for any } x \in \bar{\Omega}.$$

It is easy to check  $A(x_0) > 0$ , since  $\Omega$  can be covered by finite many balls of radius  $1/4$ . Let  $\gamma_0 = A(x_0)/2$ . Recalling (2.8) and applying lemma 2.2, we obtain

$$(2.10) \quad \frac{\int_{\Omega \setminus B_{1/2}(x_0)} e^{u_i}}{\int_{\Omega} e^{u_i}} \rightarrow 0$$

as  $i \rightarrow \infty$ , which implies (2.9). □

Now we continue to prove Lemma 2.1. (2.9) implies

$$\begin{aligned} \frac{\int_{\Omega} x e^{u_i}}{\int_{\Omega} e^{u_i}} - x_0 &= \frac{\int_{\Omega} (x - x_0) e^{u_i}}{\int_{\Omega} e^{u_i}} \\ &= \frac{\int_{B_{1/2}(x_0)} (x - x_0) e^{u_i}}{\int_{\Omega} e^{u_i}} + o(1) \end{aligned}$$

which, in turn, implies that  $|m_c(u_i) - x_0| < 2/3$ . This contradicts (2.7).  $\square$

LEMMA 2.4. –  $\alpha_{\rho}/\rho$  is non-increasing in  $(8\pi, 16\pi)$ .

*Proof.* – We first observe that if  $J(u) \leq 0$ , then  $\log \int_{\Omega} e^u > 0$  which implies that

$$J_{\rho}(u) \geq J_{\rho'}(u) \quad \text{for } \rho' \geq \rho.$$

Hence  $\mathcal{D}_{\rho} \subset \mathcal{D}_{\rho'}$  for any  $16\pi > \rho' \geq \rho > 8\pi$ . On the other hand, it is clear that

$$\frac{J_{\rho}}{\rho} - \frac{J_{\rho'}}{\rho'} = \frac{1}{2} \left( \frac{1}{\rho} - \frac{1}{\rho'} \right) \int_{\Omega} |\nabla u|^2 \geq 0,$$

if  $\rho' \geq \rho$ . Hence we have

$$\frac{\alpha_{\rho}}{\rho} \geq \frac{\alpha_{\rho'}}{\rho'}$$

for  $16\pi > \rho' \geq \rho > 8\pi$ .  $\square$

### 3. EXISTENCE FOR A DENSE SET

In this section we show that  $\alpha_{\rho}$  is achieved if  $\rho$  belongs to a certain dense subset of  $(8\pi, 16\pi)$  defined below.

The crucial problem for our functional is the lack of a coercivity condition, *i.e.* for a Palais-Smale sequence  $u_i$  for  $J_{\rho}$ , we do not know whether  $\int_{\Omega} |\nabla u_i|^2$  is bounded.

We first have the following lemma.

LEMMA 3.1. – *Let  $u_i$  be a Palais-Smale sequence for  $J_{\rho}$ , *i.e.*  $u_i$  satisfies*

$$(3.1) \quad |J_{\rho}(u_i)| \leq c < \infty$$



and

$$(3.2) \quad dJ_\rho(u_i) \rightarrow 0 \text{ strongly in } H^{-1,2}(\Omega).$$

If, in addition, we have

$$(3.3) \quad \int_\Omega |\nabla u_i|^2 \leq c_0, \quad \text{for } i = 1, 2, \dots$$

for a constant  $c_0$  independent of  $i$ , then  $u_i$  subconverges to a critical point  $u_0$  for  $J_\rho$  strongly in  $H_0^{1,2}(\Omega)$ .

*Proof.* – The proof is standard, but we provide it here for convenience of the reader.

Since  $\int_\Omega |\nabla u_i|^2$  is bounded, there exists  $u_0 \in H_0^{1,2}(\Omega)$  such that

- (i)  $u_i$  converges to  $u_0$  weakly in  $H_0^{1,2}(\Omega)$ ,
- (ii)  $u_i$  converges to  $u_0$  strongly in  $L^p(\Omega)$  for any  $p > 1$  and almost everywhere,
- (iii)  $e^{u_i}$  converges to  $e^{u_0}$  strongly in  $L^p(\Omega)$  for any  $p \geq 1$ .

From (i)-(iii), we can show that  $dJ(u_0) = 0$ , i.e.  $u_0$  satisfies

$$-\Delta u_0 = \rho \frac{e^{u_0}}{\int_\Omega e^{u_0}}.$$

Testing  $dJ_\rho$  with  $u_i - u_0$ , we obtain

$$\begin{aligned} o(1) &= \langle dJ_\rho(u_i) - dJ_\rho(u), u_i - u_0 \rangle \\ &= \int_\Omega |\nabla(u_i - u_0)|^2 - \rho \int_\Omega \left( \frac{e^{u_i}}{\int_\Omega e^{u_i}} - \frac{e^{u_0}}{\int_\Omega e^{u_0}} \right) (u_i - u_0) \\ &= \int_\Omega |\nabla(u_i - u_0)|^2 + o(1), \end{aligned}$$

by (i)-(iii). Hence  $u_i$  converges to  $u_0$  strongly in  $H_0^{1,2}(\Omega)$ . □

Since by Lemma 2.4  $\rho \rightarrow \alpha_\rho/\rho$  is non-increasing in  $(8\pi, 16\pi)$ ,  $\rho \rightarrow \alpha_\rho/\rho$  is a.e. differentiable. Set

$$(3.4) \quad \Lambda := \{ \rho \in (8\pi, 16\pi) \mid \alpha_\rho/\rho \text{ is differentiable at } \rho \}.$$

$\bar{\Lambda} = [8\pi, 16\pi]$ , see [16]. Let  $\rho \in \Lambda$  and choose  $\rho_k \nearrow \rho$  such that

$$(3.5) \quad 0 \leq \lim_{k \rightarrow \infty} -\frac{1}{(\rho - \rho_k)} \left( \frac{\alpha_\rho}{\rho} - \frac{\alpha_{\rho_k}}{\rho_k} \right) \leq c_1$$

for some constant  $c_1$  independent of  $k$ .

LEMMA 3.2. –  $\alpha_\rho$  is achieved by a critical point  $u_\rho$  for  $J_\rho$  provided that  $\rho \in \Lambda$ .

*Proof.* – Assume, by contradiction, that the Lemma were false. From Lemma 3.1, there exists  $\delta > 0$  such that

$$(3.6) \quad \|dJ_\rho(u)\|_{H^{-1,2}(\Omega)} \geq 2\delta$$

in

$$N_\delta := \{u \in H_0^{1,2}(\Omega) \mid \int_\Omega |\nabla u|^2 \leq c_2, |J_\rho(u) - \alpha_\rho| < \delta\}.$$

Here,  $c_2$  is any fixed constant such that  $N_\delta \neq \emptyset$ . Let  $X_\rho : N_\delta \rightarrow H_0^{1,2}(\Omega)$  be a pseudo-gradient vector field for  $J_\rho$  in  $N_\delta$ , i.e. a locally Lipschitz vector field of norm  $\|X_\rho\|_{H_0^{1,2}} \leq 1$  with

$$(3.7) \quad \langle dJ_\rho(u), X_\rho(u) \rangle < -\delta.$$

See [15] for the construction of  $X_\rho$ .

Since

$$\begin{aligned} \|dJ_\rho(u) - dJ_{\rho_k}(u)\| &= \|dJ_\rho - \frac{\rho}{\rho_k} dJ_{\rho_k}(u)\| + \|(1 - \frac{\rho}{\rho_k})dJ_{\rho_k}(u)\| \\ &\leq \frac{1}{2}(1 - \frac{\rho}{\rho_k}) \int |\nabla u|^2 + c(1 - \frac{\rho}{\rho_k}) \int_\Omega |\nabla u|^2 \rightarrow 0 \end{aligned}$$

uniformly in  $\{u \mid \int_\Omega |\nabla u|^2 \leq c_2\}$ ,  $X_\rho$  is also a pseudo-gradient vector field for  $J_{\rho_k}$  in  $N_\delta$  with

$$(3.8) \quad \langle dJ_{\rho_k}(u), X_\rho(u) \rangle < -\delta/2,$$

for  $u \in N_\delta$ , provided that  $k$  is sufficiently large.

For any sequence  $\{h_k\}$ ,  $h_k \in \mathcal{D}_{\rho_k} \subset \mathcal{D}_\rho$  such that

$$(3.9) \quad \sup_{u \in h_k(D)} J_{\rho_k}(u) \leq \alpha_{\rho_k} + \rho - \rho_k$$

and all  $u \in h_k(D)$  such that

$$(3.10) \quad J_\rho(u) \geq \alpha_\rho - (\rho - \rho_k),$$

we have the following estimate

$$\begin{aligned} (3.11) \quad \frac{1}{2} \int_\Omega |\nabla u|^2 &= \rho \cdot \rho_k \frac{\frac{J_{\rho_k}(u)}{\rho_k} - \frac{J_\rho(u)}{\rho}}{\rho - \rho_k} \\ &\leq \rho \cdot \rho_k \frac{\frac{\alpha_{\rho_k}}{\rho_k} - \frac{\alpha_\rho}{\rho}}{\rho - \rho_k} + (\rho + \rho_k) \\ &\leq C \end{aligned}$$

by (3.5), (3.9) and (3.10), where  $C = (16\pi)^2 c_1 + 32\pi$ .

Now we consider in  $N_\delta$  the following pseudo-gradient flow for  $J_\rho$ . First choose a Lipschitz continuous cut-off function  $\eta$  such that  $0 \leq \eta \leq 1$ ,  $\eta = 0$  outside  $N_\delta$ ,  $\eta = 1$  in  $N_{\delta/2}$ . Then consider the following flow in  $H_0^{1,2}(\Omega)$  generated by  $\eta X_\rho$

$$\begin{aligned} \frac{\partial \phi}{\partial t}(u, t) &= \eta(\phi(u, t))X_\rho(\phi(u, t)) \\ \phi(u, 0) &= u. \end{aligned}$$

By (3.7) and (3.8), for  $u \in N_{\delta/2}$ , we have

$$(3.12) \quad \frac{d}{dt} J_\rho(\phi(u, t))|_{t=0} \leq -\delta$$

and

$$(3.13) \quad \frac{d}{dt} J_{\rho_k}(\phi(u, t))|_{t=0} \leq -\delta/2$$

for large  $k$ .

It is clear that for any  $h \in \mathcal{D}_{\rho_k}$   $h(r, \theta) \notin N_\delta$  for  $r$  close to 1. Hence  $\phi(h, t) \in \mathcal{D}_{\rho_k}$  for any  $t > 0$ . In particular,  $\phi(\cdot, t)$  preserves the class of  $h_k \in \mathcal{D}_{\rho_k}$  with condition (3.9). On the other hand, for any  $h \in \mathcal{D}_\rho$  by definition

$$\sup_{u \in h(D)} J_\rho(u) \geq \alpha_\rho.$$

Hence for any  $h_k \in \mathcal{D}_{\rho_k}$  with condition (3.9),  $\sup_{u \in \phi(h(D), t)} J_\rho(u)$  is achieved in  $N_{\delta/2}$ , provided that  $k$  is large enough. Consequently, by (3.12), we have

$$\frac{d}{dt} \sup\{J_\rho(u) | u \in \phi(h(D), t)\} \leq -\delta$$

for all  $t \geq 0$ , which is a contradiction. □

### 4. PROOF OF THEOREM 1.1

From section 3, we know that for any  $\bar{\rho} \in (8\pi, 16\pi)$  there exists a sequence  $\rho_k \nearrow \bar{\rho}$  such that  $\alpha_{\rho_k}$  is achieved by  $u_k$ . Consequently  $u_k$  satisfies

$$(4.1) \quad \begin{aligned} -\Delta u_k &= \rho_k \frac{e^{u_k}}{\int_\Omega e^{u_i}}, & \text{in } \Omega, \\ u_k &= 0, & \text{on } \partial\Omega. \end{aligned}$$

From Lemma 2.4, we have

$$(4.2) \quad J_{\bar{\rho}}(u_k) = \alpha_{\rho_k} \leq c_0,$$

for some constant  $c_0 > 0$  which is independent of  $k$ . Let  $v_k = u_k - \log \int_{\Omega} e^{u_k}$ . Then  $v_k$  satisfies

$$(4.3) \quad -\Delta v_k = \rho_k e^{v_k}$$

with

$$(4.4) \quad \int_{\Omega} e^{v_k} = 1.$$

By results of Brezis-Merle [3] and Li-Shafrir [11] we have

LEMMA 4.1 ([3], [11]). *—There exists a subsequence (also denoted by  $v_k$ ) satisfying one of the following alternatives:*

- (i)  $\{v_k\}$  is bounded in  $L_{loc}^{\infty}(\Omega)$ ;
- (ii)  $v_k \rightarrow -\infty$  uniformly on any compact subset of  $\Omega$ ;
- (iii) there exists a finite blow-up set  $\Sigma = \{a_1, \dots, a_m\} \subset \Omega$  such that, for any  $1 \leq i \leq m$ , there exists  $\{x_k\} \subset \Omega$ ,  $x_k \rightarrow a_i$ ,  $u_k(x_k) \rightarrow \infty$ , and  $v_k(x) \rightarrow -\infty$  uniformly on any compact subset of  $\Omega \setminus \Sigma$ . Moreover,

$$(4.5) \quad \rho_k \int_{\Omega} e^{v_k} \rightarrow \sum_{i=1}^m 8\pi n_i$$

where  $n_i$  is positive integer.

For our special functions  $v_k$ , we can improve Lemma 4.1 as follows

LEMMA 4.2. *—There exists a subsequence (also denoted by  $v_k$ ) satisfying one of the following alternatives:*

- (i)  $\{v_k\}$  is bounded in  $L_{loc}^{\infty}(\Omega)$ ;
- (ii)  $v_k \rightarrow -\infty$  uniformly on  $\bar{\Omega}$ ;
- (iii) there exists a finite blow-up set  $\Sigma = \{a_1, \dots, a_m\} \subset \bar{\Omega}$  such that, for any  $1 \leq i \leq m$ , there exists  $\{x_k\} \subset \Omega$ ,  $x_k \rightarrow a_i$ ,  $u_k(x_k) \rightarrow \infty$ , and  $v_k(x) \rightarrow -\infty$  uniformly on any compact subset of  $\bar{\Omega} \setminus \Sigma$ . Moreover, (4.5) holds.

*Proof.* — From Lemma 4.1, we only have to consider one more case in which blow-up points are in the boundary of  $\Omega$ . There are two possibilities: One is bubbling too fast such that after rescaling we obtain a solution of  $-\Delta u = e^u$  in a half plane; Another is bubbling slow such that after

rescaling we obtain a solution of  $-\Delta u = e^u$  in  $\mathbb{R}^2$ . One can exclude the first case. In the second case, one can follow the idea in [11] to show that (4.5) holds. See also [10].  $\square$

*Proof of Theorem 1.1.* – (4.4), (4.5) and  $\bar{\rho} \in (8\pi, 16\pi)$  imply that cases (ii) and (iii) in Lemma 4.2 does not occur. Consequently  $\{v_k\}$  is bounded in  $L^\infty_{loc}(\Omega)$ . Now we can again apply Lemma 2.2 as follows.

Let  $S_1$  and  $S_2$  be two disjoint compact subdomains of  $\Omega$ . Since  $\{v_k\}$  is bounded in  $L^\infty_{loc}(\Omega)$ , we have

$$\frac{\int_{S_i} e^{u_k}}{\int_{\Omega} e^{u_k}} = \int_{S_i} e^{v_k} \geq c_0, \quad i = 1, 2$$

for a constant  $c_0 = c_0(S_1, S_2, \Omega) > 0$  independent of  $k$ . Choosing  $\epsilon$  such that  $16\pi - \bar{\rho} > 2\epsilon$  and applying Lemma 2.2, with the help of (4.2), we obtain

$$\begin{aligned} c &\geq J_{\rho_k}(u_k) = \frac{1}{2} \int_{\Omega} |\nabla u_k|^2 - \rho_k \log \int_{\Omega} e^{u_k} \\ &\geq \frac{1}{2} \left(1 - \frac{\rho_k}{16\pi - \epsilon/2}\right) \int_{\Omega} |\nabla u|^2 \\ &\geq \frac{1}{2} \left(1 - \frac{\bar{\rho}}{16\pi - \epsilon/2}\right) \int_{\Omega} |\nabla u|^2 \end{aligned}$$

which implies that  $\int_{\Omega} |\nabla u_k|^2$  is bounded. Now by the same argument in the proof of Lemma 3.1,  $u_k$  subconverges to  $u_{\bar{\rho}}$  strongly in  $H^{1,2}_0(\Omega)$  and  $u_{\bar{\rho}}$  is a critical point of  $J_{\bar{\rho}}$ . Clearly,  $u_{\bar{\rho}}$  achieves  $\alpha_{\bar{\rho}}$ . This finishes the proof of Theorem 1.1.  $\square$

*Proof of Theorem 1.2.* – Since the proof is very similar to one presented above, we only give a sketch of the proof of Theorem 1.2. Let  $\Sigma$  be a Riemann surface of positive genus. We embed  $X : \Sigma \rightarrow \mathbb{R}^N$  for some  $N \geq 3$  and define the center of mass for a function  $u \in H^{1,2}(\Sigma)$  by

$$m_c(u) = \frac{\int_{\Sigma} X e^u}{\int_{\Sigma} e^u}.$$

Since  $\Sigma$  is of positive genus, we can choose a Jordan curve  $\Gamma^1$  on  $\Sigma$  and a closed curve  $\Gamma^2$  in  $\mathbb{R}^N \setminus \Sigma$  such that  $\Gamma^1$  links  $\Gamma^2$ . We know that  $\inf_{u \in H^{1,2}(\Sigma)} J_c(u)$  is finite if and only if  $c \in [0, 8\pi]$  (see [7]). Now define a family of functions  $h : D \rightarrow H^{1,2}(\Sigma)$  (as in section 2) satisfying

$$\lim_{r \rightarrow 1} J_{\rho}(h(r, \theta)) \rightarrow -\infty$$

and

$\lim_{r \rightarrow 1} m_c(h(r, \theta))$  as a map from  $S^1 \rightarrow \Gamma^1$  is of degree 1.

Let  $\mathcal{D}_c$  denote the set of all such families. It is also easy to check that  $\mathcal{D}_c \neq \emptyset$ . Set

$$\alpha_c := \inf_{h \in \mathcal{D}_c} \sup_{u \in h(D)} J_c(u).$$

We first have

$$\alpha_c > -\infty,$$

using the fact that  $\Gamma^1$  links  $\Gamma^2$  and Lemma 2.2. Then by the same method as presented above, we can prove that  $\alpha_c$  is achieved by some  $u_c \in H^{1,2}(\Sigma)$ , which is a solution of (1.4), for  $c \in (8\pi, 16\pi)$ .  $\square$

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