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Existence results for mean field equations

by

Weiyue DING ¹, Jürgen JOST ², Jiayu LI ³ and Guofang WANG ⁴

ABSTRACT. – Let Ω be an annulus. We prove that the mean field equation

$$\begin{aligned} -\Delta\psi &= \frac{e^{-\beta\psi}}{\int_{\Omega} e^{-\beta\psi}} && \text{in } \Omega \\ \psi &= 0 && \text{on } \partial\Omega \end{aligned}$$

admits a solution for $\beta \in (-16\pi, -8\pi)$. This is a supercritical case for the Moser-Trudinger inequality. © Elsevier, Paris

RÉSUMÉ. – On montre que l'équation de champ moyen

$$\begin{aligned} -\Delta\psi &= \frac{e^{-\beta\psi}}{\int_{\Omega} e^{-\beta\psi}} && \text{dans } \Omega \\ \psi &= 0 && \text{sur } \partial\Omega, \end{aligned}$$

pour Ω étant un anneau, admet une solution pour $\beta \in (-16\pi, -8\pi)$. Cela représente un cas supercritique pour l'inégalité de Moser-Trudinger. © Elsevier, Paris

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1. INTRODUCTION

Let Ω be a smooth bounded domain in \mathbb{R}^2 . In this paper, we consider the following mean field equation

$$(1.1) \quad \begin{aligned} -\Delta\psi &= \frac{e^{-\beta\psi}}{\int_{\Omega} e^{-\beta\psi}}, & \text{in } \Omega, \\ \psi &= 0, & \text{on } \partial\Omega, \end{aligned}$$

for $\beta \in (-\infty, +\infty)$. (1.1) is the Euler-Lagrange equation of the following functional

$$(1.2) \quad J_{\beta}(\psi) = \frac{1}{2} \int_{\Omega} |\nabla\psi|^2 + \frac{1}{\beta} \log \int_{\Omega} e^{-\beta\psi}$$

in $H_0^{1,2}(\Omega)$. This variational problem arises from Onsager's vortex model for turbulent Euler flows. In that interpretation, ψ is the stream function in the infinite vortex limit, see [12,p256ff]. The corresponding canonical Gibbs measure and partition function are finite precisely if $\beta > -8\pi$. In that situation, Caglioti et al. [4] and Kiessling [9] showed the existence of a minimizer of J_{β} . This is based on the Moser-Trudinger inequality

$$(1.3) \quad \frac{1}{2} \int_{\Omega} |\nabla\psi|^2 \geq \frac{1}{8\pi} \log \int_{\Omega} e^{-8\pi\psi}, \quad \text{for any } \psi \in H_0^{1,2}(\Omega),$$

which implies the relevant compactness and coercivity condition for J_{β} in case $\beta > -8\pi$. For $\beta \leq -8\pi$, the situation becomes different as described in [4]. On the unit disk, solutions blow up if one approaches $\beta = -8\pi$ -the critical case for (1.3)-(see also [5] and [19]), and more generally, on starshaped domains, the Pohozaev identity yields a lower bound on the possible values of β for which solutions exist. On the other hand, for an annulus, [4] constructed radially symmetric solutions for any β , and the construction of Bahri-Coron [2] makes it plausible that solutions on domains with non-trivial topology exist below -8π . Thus, for $\beta \leq -8\pi$, J_{β} is no longer compact and coercive in general, and the existence of solution depends on the geometry of the domain.

In the present paper, we thus consider the supercritical case $\beta < -8\pi$ on domains with non-trivial topology.

THEOREM 1.1. – *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain whose complement contains a bounded region, e.g. Ω an annulus. Then (1.1) has a solution for all $\beta \in (-16\pi, -8\pi)$.*

The solutions we find, however, are not minimizers of J_β -those do not exist in case $\beta < 8\pi$, since J_β has no lower bound-but unstable critical points. Thus, these solutions might not be relevant to the turbulence problem that was at the basis of [4] and [9].

Certainly we can generalize Theorem 1.1 to the following equation

$$\begin{aligned}
 -\Delta\psi &= \frac{K e^{-\beta\psi}}{\int_\Omega K e^{-\beta\psi}}, & \text{in } \Omega, \\
 \psi &= 0, & \text{on } \partial\Omega,
 \end{aligned}$$

which was studied in [5]. Here K is a positive function on $\bar{\Omega}$.

With the same method, we may also handle the equation

$$(1.4) \quad \Delta u - c + cK e^u = 0, \quad \text{for } 0 \leq c < \infty$$

on a compact Riemann surface Σ of genus at least 1, where K is a positive function. (1.4) can also be considered as a mean field equation because it is the Euler-Lagrange equation of the functional

$$(1.5) \quad J_c(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 + c \int_\Sigma u - c \log \int_\Sigma K e^u.$$

Because of the term $c \int_\Sigma u$, J_c remains invariant under adding a constant to u , and therefore we may normalize u by the condition.

$$\int_\Sigma K e^u = 1$$

which explains the absence of the factor $(\int K e^u)^{-1}$ in (1.4). $c < 8\pi$ again is a subcritical case that can easily be handled with the Moser-Trudinger inequality. The critical case $c = 8\pi$ yields the so-called Kazdan-Warner equation [8] and was treated in [7] and [14] by giving sufficient conditions for the existence of a minimizer of $J_{8\pi}$. Here, we construct again saddle point type critical points to show

THEOREM 1.2. – *Let Σ be a compact Riemann surface of positive genus. Then (1.4) admits a non-minimal solution for $8\pi < c < 16\pi$.*

Now we give a outline of the proof of the Theorems. First from the non-trivial topology of the domain, we can define a minimax value α_β , which is bounded below by an improved Moser-Trudinger inequality, for $\beta \in (-16\pi, -8\pi)$. Using a trick introduced by Struwe in [16] and [17], for a certain dense subset $\Lambda \subset (-16\pi, -8\pi)$ we can overcome the lack of a

coercivity condition and show that α_β is achieved by some u_β for $\beta \in \Lambda$. Next, for any fixed $\bar{\beta} \in (-16\pi, -8\pi)$, considering a sequence $\beta_k \subset \Lambda$ tending to $\bar{\beta}$, with the help of results in [3] and [11] we show that u_{β_k} subconverges strongly to some $u_{\bar{\beta}}$ which achieves $\alpha_{\bar{\beta}}$.

After completing our paper, we were informed that Struwe and Tarantello [18] obtained a non-constant solution of (1.4), when Σ is a flat torus with fundamental cell domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, $K \equiv 1$ and $c \in (8\pi, 4\pi^2)$. In this case, it is easy to check that our solution obtained in Theorem 1.2 is non-constant.

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2. MINIMAX VALUES

Let $\rho = -\beta$ and $u = -\beta\psi$. We rewrite (1.1) as

$$(2.1) \quad \begin{aligned} -\Delta u &= \rho \frac{e^u}{\int_{\Omega} e^u}, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

and (1.2) as

$$(2.2) \quad J_\rho(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \rho \log \int_{\Omega} e^u$$

for $u \in H_0^{1,2}(\Omega)$.

It is easy to see that J_ρ has no lower bound for $\rho \in (8\pi, 16\pi)$. Hence, to get a solution of (1.1) for $\rho \in (8\pi, 16\pi)$, we have to use a minimax method. First, we define a center of mass of u by

$$m_c(u) = \frac{\int_{\Omega} x e^u}{\int_{\Omega} e^u}.$$

Let B be the bounded component of $\mathbb{R}^2 \setminus \Omega$. For simplicity, we assume that B is the unit disk centered at the origin. Then we define a family of functions

$$h : D \rightarrow H_0^{1,2}(\Omega)$$

satisfying

$$(2.3) \quad \lim_{r \rightarrow 1} J_\rho(h(r, \theta)) \rightarrow -\infty$$

and

$$(2.4) \quad \lim_{r \rightarrow 1} m_c(h(r, \theta)) \text{ is a continuous curve enclosing } B.$$

Here $D = \{(r, \theta) | 0 \leq r < 1, \theta \in [0, 2\pi)\}$ is the open unit disk. We denote the set of all such families by \mathcal{D}_ρ . It is easy to check that $\mathcal{D}_\rho \neq \emptyset$. Now we can define a minimax value

$$\alpha_\rho := \inf_{h \in \mathcal{D}_\rho} \sup_{u \in h(D)} J_\rho(u).$$

The following lemma will make crucial use of the non-trivial topology of Ω , more precisely of the fact that the complement of Ω has a bounded component.

LEMMA 2.1. – For any $\rho \in (8\pi, 16\pi)$ $\alpha_\rho > -\infty$.

Remark. – It is an interesting question whether $\alpha_{16\pi} = -\infty$.

To prove Lemma 2.1, we use the improved Moser-Trudinger inequality of [6] (see also [1]). Here we have to modify a little bit.

LEMMA 2.2. – Let S_1 and S_2 be two subsets of $\bar{\Omega}$ satisfying $\text{dist}(S_1, S_2) \geq \delta_0 > 0$ and $\gamma_0 \in (0, 1/2)$. For any $\epsilon > 0$, there exists a constant $c = c(\epsilon, \delta_0, \gamma_0) > 0$ such that

$$\int_\Omega e^u \leq c \exp\left\{\frac{1}{32\pi - \epsilon} \int_\Omega |\nabla u|^2 + c\right\}$$

holds for all $u \in H_0^{1,2}(\Omega)$ satisfying

$$(2.5) \quad \frac{\int_{S_1} e^u}{\int_\Omega e^u} \geq \gamma_0 \quad \text{and} \quad \frac{\int_{S_2} e^u}{\int_\Omega e^u} \geq \gamma_0.$$

Proof. – The Lemma follows from the argument in [6] and the following Moser-Trudinger inequality

$$(*) \quad \frac{1}{2} \int_\Omega |\nabla u|^2 - 8\pi \log \int_\Omega e^u \geq c$$

for any $u \in H_0^{1,2}(\Omega)$, where c is a constant independent of $u \in H_0^{1,2}(\Omega)$. \square

We will discuss the inequality (*) and its application in another paper.

Proof of Lemma 2.1. – For fixed $\rho \in (8\pi, 16\pi)$ we claim that there exists a constant c_ρ such that

$$(2.6) \quad \sup_{u \in h(D)} J_\rho(u) \geq c_\rho, \quad \text{for any } h \in \mathcal{D}_\rho.$$

Clearly (2.6) implies the Lemma. By the definition of h , for any $h \in \mathcal{D}_\rho$, there exists $u \in h(D)$ such that

$$m_c(u) = 0.$$

We choose $\epsilon > 0$ so small that $\rho < 16\pi - 2\epsilon$. Assume (2.6) does not hold. Then we have sequences $\{h_i\} \subset \mathcal{D}_\rho$ and $\{u_i\} \subset H_0^{1,2}(\Omega)$ such that $u_i \in h_i(D)$ and

$$(2.7) \quad m_c(u_i) = 0$$

$$(2.8) \quad \lim_{i \rightarrow \infty} J(u_i) = -\infty.$$

We have the following Lemma.

LEMMA 2.3. – *There exists $x_0 \in \bar{\Omega}$ such that*

$$(2.9) \quad \lim_{i \rightarrow \infty} \frac{\int_{B_{1/2}(x_0) \cap \Omega} e^{u_i}}{\int_{\Omega} e^{u_i}} \rightarrow 1.$$

Proof. – Set

$$A(x) := \lim_{i \rightarrow \infty} \frac{\int_{B_{1/4}(x) \cap \Omega} e^{u_i}}{\int_{\Omega} e^{u_i}}.$$

Assume that the Lemma were false, then there exists $x_0 \in \bar{\Omega}$ such that

$$A(x_0) < 1 \quad \text{and} \quad A(x_0) \geq A(x) \quad \text{for any } x \in \bar{\Omega}.$$

It is easy to check $A(x_0) > 0$, since Ω can be covered by finite many balls of radius $1/4$. Let $\gamma_0 = A(x_0)/2$. Recalling (2.8) and applying lemma 2.2, we obtain

$$(2.10) \quad \frac{\int_{\Omega \setminus B_{1/2}(x_0)} e^{u_i}}{\int_{\Omega} e^{u_i}} \rightarrow 0$$

as $i \rightarrow \infty$, which implies (2.9). □

Now we continue to prove Lemma 2.1. (2.9) implies

$$\begin{aligned} \frac{\int_{\Omega} x e^{u_i}}{\int_{\Omega} e^{u_i}} - x_0 &= \frac{\int_{\Omega} (x - x_0) e^{u_i}}{\int_{\Omega} e^{u_i}} \\ &= \frac{\int_{B_{1/2}(x_0)} (x - x_0) e^{u_i}}{\int_{\Omega} e^{u_i}} + o(1) \end{aligned}$$

which, in turn, implies that $|m_c(u_i) - x_0| < 2/3$. This contradicts (2.7). \square

LEMMA 2.4. – α_{ρ}/ρ is non-increasing in $(8\pi, 16\pi)$.

Proof. – We first observe that if $J(u) \leq 0$, then $\log \int_{\Omega} e^u > 0$ which implies that

$$J_{\rho}(u) \geq J_{\rho'}(u) \quad \text{for } \rho' \geq \rho.$$

Hence $\mathcal{D}_{\rho} \subset \mathcal{D}_{\rho'}$ for any $16\pi > \rho' \geq \rho > 8\pi$. On the other hand, it is clear that

$$\frac{J_{\rho}}{\rho} - \frac{J_{\rho'}}{\rho'} = \frac{1}{2} \left(\frac{1}{\rho} - \frac{1}{\rho'} \right) \int_{\Omega} |\nabla u|^2 \geq 0,$$

if $\rho' \geq \rho$. Hence we have

$$\frac{\alpha_{\rho}}{\rho} \geq \frac{\alpha_{\rho'}}{\rho'}$$

for $16\pi > \rho' \geq \rho > 8\pi$. \square

3. EXISTENCE FOR A DENSE SET

In this section we show that α_{ρ} is achieved if ρ belongs to a certain dense subset of $(8\pi, 16\pi)$ defined below.

The crucial problem for our functional is the lack of a coercivity condition, *i.e.* for a Palais-Smale sequence u_i for J_{ρ} , we do not know whether $\int_{\Omega} |\nabla u_i|^2$ is bounded.

We first have the following lemma.

LEMMA 3.1. – *Let u_i be a Palais-Smale sequence for J_{ρ} , *i.e.* u_i satisfies*

$$(3.1) \quad |J_{\rho}(u_i)| \leq c < \infty$$

and

$$(3.2) \quad dJ_\rho(u_i) \rightarrow 0 \text{ strongly in } H^{-1,2}(\Omega).$$

If, in addition, we have

$$(3.3) \quad \int_\Omega |\nabla u_i|^2 \leq c_0, \quad \text{for } i = 1, 2, \dots$$

for a constant c_0 independent of i , then u_i subconverges to a critical point u_0 for J_ρ strongly in $H_0^{1,2}(\Omega)$.

Proof. – The proof is standard, but we provide it here for convenience of the reader.

Since $\int_\Omega |\nabla u_i|^2$ is bounded, there exists $u_0 \in H_0^{1,2}(\Omega)$ such that

- (i) u_i converges to u_0 weakly in $H_0^{1,2}(\Omega)$,
- (ii) u_i converges to u_0 strongly in $L^p(\Omega)$ for any $p > 1$ and almost everywhere,
- (iii) e^{u_i} converges to e^{u_0} strongly in $L^p(\Omega)$ for any $p \geq 1$.

From (i)-(iii), we can show that $dJ(u_0) = 0$, i.e. u_0 satisfies

$$-\Delta u_0 = \rho \frac{e^{u_0}}{\int_\Omega e^{u_0}}.$$

Testing dJ_ρ with $u_i - u_0$, we obtain

$$\begin{aligned} o(1) &= \langle dJ_\rho(u_i) - dJ_\rho(u), u_i - u_0 \rangle \\ &= \int_\Omega |\nabla(u_i - u_0)|^2 - \rho \int_\Omega \left(\frac{e^{u_i}}{\int_\Omega e^{u_i}} - \frac{e^{u_0}}{\int_\Omega e^{u_0}} \right) (u_i - u_0) \\ &= \int_\Omega |\nabla(u_i - u_0)|^2 + o(1), \end{aligned}$$

by (i)-(iii). Hence u_i converges to u_0 strongly in $H_0^{1,2}(\Omega)$. □

Since by Lemma 2.4 $\rho \rightarrow \alpha_\rho/\rho$ is non-increasing in $(8\pi, 16\pi)$, $\rho \rightarrow \alpha_\rho/\rho$ is a.e. differentiable. Set

$$(3.4) \quad \Lambda := \{ \rho \in (8\pi, 16\pi) \mid \alpha_\rho/\rho \text{ is differentiable at } \rho \}.$$

$\bar{\Lambda} = [8\pi, 16\pi]$, see [16]. Let $\rho \in \Lambda$ and choose $\rho_k \nearrow \rho$ such that

$$(3.5) \quad 0 \leq \lim_{k \rightarrow \infty} -\frac{1}{(\rho - \rho_k)} \left(\frac{\alpha_\rho}{\rho} - \frac{\alpha_{\rho_k}}{\rho_k} \right) \leq c_1$$

for some constant c_1 independent of k .

LEMMA 3.2. – α_ρ is achieved by a critical point u_ρ for J_ρ provided that $\rho \in \Lambda$.

Proof. – Assume, by contradiction, that the Lemma were false. From Lemma 3.1, there exists $\delta > 0$ such that

$$(3.6) \quad \|dJ_\rho(u)\|_{H^{-1,2}(\Omega)} \geq 2\delta$$

in

$$N_\delta := \{u \in H_0^{1,2}(\Omega) \mid \int_\Omega |\nabla u|^2 \leq c_2, |J_\rho(u) - \alpha_\rho| < \delta\}.$$

Here, c_2 is any fixed constant such that $N_\delta \neq \emptyset$. Let $X_\rho : N_\delta \rightarrow H_0^{1,2}(\Omega)$ be a pseudo-gradient vector field for J_ρ in N_δ , i.e. a locally Lipschitz vector field of norm $\|X_\rho\|_{H_0^{1,2}} \leq 1$ with

$$(3.7) \quad \langle dJ_\rho(u), X_\rho(u) \rangle < -\delta.$$

See [15] for the construction of X_ρ .

Since

$$\begin{aligned} \|dJ_\rho(u) - dJ_{\rho_k}(u)\| &= \|dJ_\rho - \frac{\rho}{\rho_k} dJ_{\rho_k}(u)\| + \|(1 - \frac{\rho}{\rho_k})dJ_{\rho_k}(u)\| \\ &\leq \frac{1}{2}(1 - \frac{\rho}{\rho_k}) \int |\nabla u|^2 + c(1 - \frac{\rho}{\rho_k}) \int_\Omega |\nabla u|^2 \rightarrow 0 \end{aligned}$$

uniformly in $\{u \mid \int_\Omega |\nabla u|^2 \leq c_2\}$, X_ρ is also a pseudo-gradient vector field for J_{ρ_k} in N_δ with

$$(3.8) \quad \langle dJ_{\rho_k}(u), X_\rho(u) \rangle < -\delta/2,$$

for $u \in N_\delta$, provided that k is sufficiently large.

For any sequence $\{h_k\}$, $h_k \in \mathcal{D}_{\rho_k} \subset \mathcal{D}_\rho$ such that

$$(3.9) \quad \sup_{u \in h_k(D)} J_{\rho_k}(u) \leq \alpha_{\rho_k} + \rho - \rho_k$$

and all $u \in h_k(D)$ such that

$$(3.10) \quad J_\rho(u) \geq \alpha_\rho - (\rho - \rho_k),$$

we have the following estimate

$$\begin{aligned} (3.11) \quad \frac{1}{2} \int_\Omega |\nabla u|^2 &= \rho \cdot \rho_k \frac{\frac{J_{\rho_k}(u)}{\rho_k} - \frac{J_\rho(u)}{\rho}}{\rho - \rho_k} \\ &\leq \rho \cdot \rho_k \frac{\frac{\alpha_{\rho_k}}{\rho_k} - \frac{\alpha_\rho}{\rho}}{\rho - \rho_k} + (\rho + \rho_k) \\ &\leq C \end{aligned}$$

by (3.5), (3.9) and (3.10), where $C = (16\pi)^2 c_1 + 32\pi$.

Now we consider in N_δ the following pseudo-gradient flow for J_ρ . First choose a Lipschitz continuous cut-off function η such that $0 \leq \eta \leq 1$, $\eta = 0$ outside N_δ , $\eta = 1$ in $N_{\delta/2}$. Then consider the following flow in $H_0^{1,2}(\Omega)$ generated by ηX_ρ

$$\begin{aligned} \frac{\partial \phi}{\partial t}(u, t) &= \eta(\phi(u, t))X_\rho(\phi(u, t)) \\ \phi(u, 0) &= u. \end{aligned}$$

By (3.7) and (3.8), for $u \in N_{\delta/2}$, we have

$$(3.12) \quad \frac{d}{dt} J_\rho(\phi(u, t))|_{t=0} \leq -\delta$$

and

$$(3.13) \quad \frac{d}{dt} J_{\rho_k}(\phi(u, t))|_{t=0} \leq -\delta/2$$

for large k .

It is clear that for any $h \in \mathcal{D}_{\rho_k}$ $h(r, \theta) \notin N_\delta$ for r close to 1. Hence $\phi(h, t) \in \mathcal{D}_{\rho_k}$ for any $t > 0$. In particular, $\phi(\cdot, t)$ preserves the class of $h_k \in \mathcal{D}_{\rho_k}$ with condition (3.9). On the other hand, for any $h \in \mathcal{D}_\rho$ by definition

$$\sup_{u \in h(D)} J_\rho(u) \geq \alpha_\rho.$$

Hence for any $h_k \in \mathcal{D}_{\rho_k}$ with condition (3.9), $\sup_{u \in \phi(h(D), t)} J_\rho(u)$ is achieved in $N_{\delta/2}$, provided that k is large enough. Consequently, by (3.12), we have

$$\frac{d}{dt} \sup\{J_\rho(u) | u \in \phi(h(D), t)\} \leq -\delta$$

for all $t \geq 0$, which is a contradiction. □

4. PROOF OF THEOREM 1.1

From section 3, we know that for any $\bar{\rho} \in (8\pi, 16\pi)$ there exists a sequence $\rho_k \nearrow \bar{\rho}$ such that α_{ρ_k} is achieved by u_k . Consequently u_k satisfies

$$(4.1) \quad \begin{aligned} -\Delta u_k &= \rho_k \frac{e^{u_k}}{\int_\Omega e^{u_i}}, & \text{in } \Omega, \\ u_k &= 0, & \text{on } \partial\Omega. \end{aligned}$$

From Lemma 2.4, we have

$$(4.2) \quad J_{\bar{\rho}}(u_k) = \alpha_{\rho_k} \leq c_0,$$

for some constant $c_0 > 0$ which is independent of k . Let $v_k = u_k - \log \int_{\Omega} e^{u_k}$. Then v_k satisfies

$$(4.3) \quad -\Delta v_k = \rho_k e^{v_k}$$

with

$$(4.4) \quad \int_{\Omega} e^{v_k} = 1.$$

By results of Brezis-Merle [3] and Li-Shafrir [11] we have

LEMMA 4.1 ([3], [11]). *—There exists a subsequence (also denoted by v_k) satisfying one of the following alternatives:*

- (i) $\{v_k\}$ is bounded in $L_{loc}^{\infty}(\Omega)$;
- (ii) $v_k \rightarrow -\infty$ uniformly on any compact subset of Ω ;
- (iii) there exists a finite blow-up set $\Sigma = \{a_1, \dots, a_m\} \subset \Omega$ such that, for any $1 \leq i \leq m$, there exists $\{x_k\} \subset \Omega$, $x_k \rightarrow a_i$, $u_k(x_k) \rightarrow \infty$, and $v_k(x) \rightarrow -\infty$ uniformly on any compact subset of $\Omega \setminus \Sigma$. Moreover,

$$(4.5) \quad \rho_k \int_{\Omega} e^{v_k} \rightarrow \sum_{i=1}^m 8\pi n_i$$

where n_i is positive integer.

For our special functions v_k , we can improve Lemma 4.1 as follows

LEMMA 4.2. *—There exists a subsequence (also denoted by v_k) satisfying one of the following alternatives:*

- (i) $\{v_k\}$ is bounded in $L_{loc}^{\infty}(\Omega)$;
- (ii) $v_k \rightarrow -\infty$ uniformly on $\bar{\Omega}$;
- (iii) there exists a finite blow-up set $\Sigma = \{a_1, \dots, a_m\} \subset \bar{\Omega}$ such that, for any $1 \leq i \leq m$, there exists $\{x_k\} \subset \Omega$, $x_k \rightarrow a_i$, $u_k(x_k) \rightarrow \infty$, and $v_k(x) \rightarrow -\infty$ uniformly on any compact subset of $\bar{\Omega} \setminus \Sigma$. Moreover, (4.5) holds.

Proof. — From Lemma 4.1, we only have to consider one more case in which blow-up points are in the boundary of Ω . There are two possibilities: One is bubbling too fast such that after rescaling we obtain a solution of $-\Delta u = e^u$ in a half plane; Another is bubbling slow such that after

rescaling we obtain a solution of $-\Delta u = e^u$ in \mathbb{R}^2 . One can exclude the first case. In the second case, one can follow the idea in [11] to show that (4.5) holds. See also [10]. \square

Proof of Theorem 1.1. – (4.4), (4.5) and $\bar{\rho} \in (8\pi, 16\pi)$ imply that cases (ii) and (iii) in Lemma 4.2 does not occur. Consequently $\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$. Now we can again apply Lemma 2.2 as follows.

Let S_1 and S_2 be two disjoint compact subdomains of Ω . Since $\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$, we have

$$\frac{\int_{S_i} e^{u_k}}{\int_{\Omega} e^{u_k}} = \int_{S_i} e^{v_k} \geq c_0, \quad i = 1, 2$$

for a constant $c_0 = c_0(S_1, S_2, \Omega) > 0$ independent of k . Choosing ϵ such that $16\pi - \bar{\rho} > 2\epsilon$ and applying Lemma 2.2, with the help of (4.2), we obtain

$$\begin{aligned} c &\geq J_{\rho_k}(u_k) = \frac{1}{2} \int_{\Omega} |\nabla u_k|^2 - \rho_k \log \int_{\Omega} e^{u_k} \\ &\geq \frac{1}{2} \left(1 - \frac{\rho_k}{16\pi - \epsilon/2}\right) \int_{\Omega} |\nabla u|^2 \\ &\geq \frac{1}{2} \left(1 - \frac{\bar{\rho}}{16\pi - \epsilon/2}\right) \int_{\Omega} |\nabla u|^2 \end{aligned}$$

which implies that $\int_{\Omega} |\nabla u_k|^2$ is bounded. Now by the same argument in the proof of Lemma 3.1, u_k subconverges to $u_{\bar{\rho}}$ strongly in $H^{1,2}_0(\Omega)$ and $u_{\bar{\rho}}$ is a critical point of $J_{\bar{\rho}}$. Clearly, $u_{\bar{\rho}}$ achieves $\alpha_{\bar{\rho}}$. This finishes the proof of Theorem 1.1. \square

Proof of Theorem 1.2. – Since the proof is very similar to one presented above, we only give a sketch of the proof of Theorem 1.2. Let Σ be a Riemann surface of positive genus. We embed $X : \Sigma \rightarrow \mathbb{R}^N$ for some $N \geq 3$ and define the center of mass for a function $u \in H^{1,2}(\Sigma)$ by

$$m_c(u) = \frac{\int_{\Sigma} X e^u}{\int_{\Sigma} e^u}.$$

Since Σ is of positive genus, we can choose a Jordan curve Γ^1 on Σ and a closed curve Γ^2 in $\mathbb{R}^N \setminus \Sigma$ such that Γ^1 links Γ^2 . We know that $\inf_{u \in H^{1,2}(\Sigma)} J_c(u)$ is finite if and only if $c \in [0, 8\pi]$ (see [7]). Now define a family of functions $h : D \rightarrow H^{1,2}(\Sigma)$ (as in section 2) satisfying

$$\lim_{r \rightarrow 1} J_{\rho}(h(r, \theta)) \rightarrow -\infty$$

and

$\lim_{r \rightarrow 1} m_c(h(r, \theta))$ as a map from $S^1 \rightarrow \Gamma^1$ is of degree 1.

Let \mathcal{D}_c denote the set of all such families. It is also easy to check that $\mathcal{D}_c \neq \emptyset$. Set

$$\alpha_c := \inf_{h \in \mathcal{D}_c} \sup_{u \in h(D)} J_c(u).$$

We first have

$$\alpha_c > -\infty,$$

using the fact that Γ^1 links Γ^2 and Lemma 2.2. Then by the same method as presented above, we can prove that α_c is achieved by some $u_c \in H^{1,2}(\Sigma)$, which is a solution of (1.4), for $c \in (8\pi, 16\pi)$. \square

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