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On the Jäger–Kaul theorem concerning harmonic maps

by

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ABSTRACT. – In 1983, Jäger and Kaul proved that the equator map $u^*(x) = (\frac{x}{|x|}, 0) : B^n \rightarrow S^n$ is unstable for $3 \leq n \leq 6$ and a minimizer for the energy functional $E(u, B^n) = \int_{B^n} |\nabla u|^2 dx$ in the class $H^{1,2}(B^n, S^n)$ with $u = u^*$ on ∂B^n when $n \geq 7$. In this paper, we give a new and elementary proof of this Jäger–Kaul result. We also generalize the Jäger–Kaul result to the case of p -harmonic maps. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – En 1983, Jäger et Kaul ont démontrés que l'application équatoriale $u^*(x) = (\frac{x}{|x|}, 0) : B^n \rightarrow S^n$ n'est pas stable si $3 \leq n \leq 6$ et que c'est une minimisateur pour la fonctionnelle d'énergie $E(u, B^n) = \int_{B^n} |\nabla u|^2 dx$ dans la classe $H^{1,2}(B^n, S^n)$ avec $u = u^*$ sur ∂B^n si $n \geq 7$. Nous donnons une preuve nouvelle, élémentaire de ce résultat de Jäger–Kaul. En plus nous généralisons le résultat de Jäger–Kaul au cas des applications p -harmoniques. © 2000 Éditions scientifiques et médicales Elsevier SAS

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1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold with (possibly empty) boundary ∂M and let (N, h) be another compact Riemannian manifold without boundary. Let u be a map from M to N which belongs to $H^{1,2}(M, N)$. We define the energy of u by

$$E(u, M) = \int_M |du|^2 dM, \quad (1.1)$$

where $|du|$ denotes the Hilbert–Schmidt norm of the differential $du(x)$. The critical point of E is called “harmonic”. In a fundamental paper [4], Eells and Sampson established existence of smoothly harmonic maps from M to N assuming N has nonpositive section curvature. Let $K \geq 0$ be an upper bound for the section curvature of N and $B_\rho(q)$ the open geodesic ball in N with center q and radius ρ . Assuming essentially the size restriction

$$u(\partial M) \subset B_\rho(q), \quad \rho \leq \frac{\pi}{2\sqrt{K}}, \quad (1.2)$$

Hildebrandt et al. [10] showed existence of “small” smooth harmonic maps satisfying the condition (1.2) and also discovered that for $n \geq 3$ the equator map $u^* = (\frac{x}{|x|}, 0): B^n \rightarrow S^n$ is a weakly harmonic maps. The uniqueness of harmonic maps in [10] was later proved by Jäger and Kaul [11]. Many important contributions on the regularity of minimizing harmonic maps have been made since then. Schoen and Uhlenbeck [15, 16] obtained that the minimizer of E in $H^{1,2}(M, N)$ is smooth except for a singular set, where the Hausdorff dimension of the singular set is less than or equal to $n - 3$. Meanwhile, Giaquinta and Giusti in [5,6] also proved this result for the case when the image lies in a coordinate chart (boundary regularity by Jost and Meier [13]). We would also like to mention Simon’s deep works on the structure of singularity of minimizing harmonic maps (e.g., [18]).

Let B^n be the unit ball in \mathbb{R}^n with boundary $\partial B^n = S^{n-1}$, where S^{n-1} is the unit sphere in \mathbb{R}^n .

A map $u: B^n \rightarrow S^n$ is called “weakly harmonic” if $u \in H^{1,2}(B^n, S^n)$ and it is a critical point of the energy $\int_{B^n} |\nabla u|^2 dx$, i.e.,

$$\int_{B^n} \nabla u \cdot \nabla \phi dx = \int_{B^n} |\nabla u|^2 u \cdot \phi dx$$

for all $\phi \in H_0^{1,2}(B^n, \mathbb{R}^{n+1}) \cap L^\infty(B^n, \mathbb{R}^{n+1})$.

In 1983, Jäger and Kaul [12] proved the following result:

THEOREM (Jäger–Kaul). –

- (i) When $3 \leq n \leq 6$, the equator map $u^* = (\frac{x}{|x|}, 0)$ is unstable.
- (ii) When $n \geq 7$, the equator map u^* is a minimizer of the energy functional $E(u, B^n) = \int_{B^n} |\nabla u|^2$ for all maps $u \in H^{1,2}(B^n, S^n)$ with $u = u^*$ on ∂B^n .

After this theorem, Giaquinta and Soucek [7] and Schoen and Uhlenbeck [17] proved that the Hausdorff dimension of the singular set of minimizing harmonic maps into a hemisphere is less than or equal to $n - 7$.

For any $p \in \mathbb{R}$ with $n > p \geq 2$, we define the p -energy of maps in $H^{1,p}(B^n, S^n)$ by

$$E_p(u, B^n) = \int_{B^n} |\nabla u|^p dx.$$

A map $u \in H^{1,p}(B^n, S^n)$ is called “weakly p -harmonic” if u satisfies

$$\int_{B^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx = \int_{B^n} |\nabla u|^p u \cdot \phi dx$$

for all $\phi \in H_0^{1,p}(B^n, \mathbb{R}^{n+1}) \cap L^\infty(B^n, \mathbb{R}^{n+1})$. It is also easy to check that the equator map u^* is a weakly p -harmonic map from B^n to S^n for $2 \leq p < n$.

In this paper, we first generalize the Jäger–Kaul theorem to the p -energy by the following:

THEOREM A. – Assume that $n > p \geq 2$.

- (i) For $3 \leq n < 2 + p + 2\sqrt{p}$, the equator map $u^* = (\frac{x}{|x|}, 0)$ is an unstable p -harmonic map from B^n to S^n .
- (ii) When $n \geq 2 + p + 2\sqrt{p}$, then u^* is a minimizer of the p -energy E_p in the class $H^{1,p}(B^n, S^n)$ with boundary value u^* on ∂B^n . Moreover, if $n \geq 2 + p + 2\sqrt{p}$, u^* is the unique minimizer.

We present a new proof of Theorem A(ii) and also point out that our proof is different from and much simpler than the original proof by Jäger and Kaul.

Remark a. – When $p = 2$, $n \geq 7 > 4 + 2\sqrt{2}$. Thus the proof of Theorem A also extends the Helein theorem in [9, Theorem 3.1] for $n \geq 9$

to the case when $n \geq 7$, i.e., for $n \geq 7$, we have

$$E(u) - E(u^*) \geq K_n \|u - u^*\|_{H^{1,2}(B^n)}^2,$$

where u is a map in $H^{1,2}(B^n, S^n)$ which agree with u^* on ∂B^n , and K_n is a strict positive constant.

Remark b. – Consider the case of maps from B^n into S^{n-1} . Then $u^* = \frac{x}{|x|}$ is a minimizer of $E(u; B^n)$ for maps from B^n into S^{n-1} . This result has been proved by Brezis et al. [2] for $n = 3$, by Jäger and Kaul [12] for $n \geq 7$ and by Lin [14] for $n \geq 3$. Moreover, Coron and Gulliver proved that $u^* = \frac{x}{|x|}$ is a minimizer of the p -energy functional $E_p(u; B^n)$ for maps from B^n into S^{n-1} for $p \leq n - 1$. Theorem A recovers the partial result of Coron and Gulliver [3] for $n \geq 2 + p + 2\sqrt{p}$. Perhaps, the uniqueness of the minimizer of p -energy functional of maps from B^n into S^{n-1} for $n \geq 2 + p + 2\sqrt{p}$ is also a new result.

Let $M = B^n$ again and let N be an ellipsoid of \mathbb{R}^{n+1} , i.e.,

$$N := \left\{ u = (v, z): |v|^2 + \frac{z^2}{a^2} = 1 \right\} \subset \mathbb{R}^{n+1},$$

where $v \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $a > 0$ is a constant.

Baldes [1] in 1984 and Helein [8] in 1988 generalized the work of Jäger and Kaul when N is an ellipsoid:

THEOREM. –

- (i) (Baldes) When $a^2 \geq 1$ and $n \geq 7$, the equator map u^* is the unique minimizer of the energy functional E in $H^{1,2}(B^n; N)$ with boundary value $(x, 0)$.
- (ii) (Helein) If $0 < a < 1$ and $a^2 \geq 4(n-1)/(n-2)^2$, the equator map u^* is the unique minimizer of the energy functional E in $H^{1,2}(B^n; N)$ with boundary value $(x, 0)$.

We would point out that all proofs of Baldes [1] and Helein [8] are variants of the proof of Jäger and Kaul [12]. After the first version of this paper (CMA-Preprint, September 1996), the author was asked whether one could recover and generalize the results of Baldes and Helein using our proof of Theorem A. Here we generalize their results to p -harmonic maps with $n > p \geq 2$ by the following:

THEOREM B. –

- (i) When $a^2 \geq 1$ and $n \geq 2 + p + 2\sqrt{p}$, the equator map u^* is the unique minimizer of the p -energy functional E_p in $H^{1,p}(B^n; N)$ with boundary value $(x, 0)$.
- (ii) If $0 < a < 1$ and $a \geq 4(n - 1)/(n - p)^2$, the equator map u^* is the unique minimizer of the p -energy functional E_p in $H^{1,p}(B^n; N)$ with boundary value $(x, 0)$.

2. PROOF OF THEOREM A

LEMMA 1. – For any $p < n$, we have

$$(n - p)^2 \int_{B^n} |\phi(x)|^2 \frac{1}{|x|^p} dx \leq 4 \int_{B^n} \frac{1}{|x|^{p-2}} \left| \frac{\partial \phi}{\partial r} \right|^2 dx$$

for all $\phi \in H_0^{1,p} \cap L^\infty(B^n, \mathbb{R})$ with $r = |x|$ where that equality occurs only if $\phi = 0$.

Proof. – The case of $p = 2$ was proved in [1]. Integrating by parts and using Cauchy’s inequality, we have

$$\begin{aligned} \int_{B^n} |\phi|^2 \frac{1}{|x|^p} dx &= \int_{|\omega|=1} \int_0^1 \phi^2 r^{n-p-1} dr d\omega \\ &= \frac{2}{n - p} \int_{|\omega|=1} \int_0^1 \phi \frac{\partial \phi}{\partial r} r^{n-p} dr d\omega \\ &\leq \frac{1}{2} \int_{|\omega|=1} \int_0^1 \phi^2 r^{n-p-1} dr d\omega \\ &\quad + \frac{2}{(n - p)^2} \int_{|\omega|=1} \int_0^1 \left(\frac{\partial \phi}{\partial r} \right)^2 r^{n-p+1} dr d\omega \end{aligned}$$

for all $\phi \in H_0^{1,p} \cap L^\infty(B^n, \mathbb{R})$. The above inequality becomes equality iff

$$\phi = \frac{2}{(n - p)} \frac{\partial \phi}{\partial r},$$

this is possible only if $\phi = 0$. This proves our claim. \square

Remark c. – Lemma 1 can be also proved in following way:

$$\begin{aligned} & \inf_{\phi \neq 0, \text{supp } \phi \subset \bar{B}^n \setminus \{0\}} \frac{\int_{B^n} |\nabla \phi|^2 r^{-(p-2)} \, dx}{\int_{B^n} r^{-p} |\phi|^2 \, dx} \\ &= \inf_{\tilde{\phi} \neq 0, \text{supp } \tilde{\phi} \subset (0,1]} \frac{\int_0^1 \left| \frac{\partial \tilde{\phi}}{\partial r} \right|^2 r^{n+1-p} \, dr}{\int_0^1 |\tilde{\phi}|^2 r^{n-1-p} \, dr} \\ &\geq \inf_{\tilde{\phi} \neq 0, \text{supp } \tilde{\phi} \subset (0,\infty)} \frac{\int_0^\infty \left| \frac{\partial \tilde{\phi}}{\partial r} \right|^2 r^{n+1-p} \, dr}{\int_0^\infty |\tilde{\phi}|^2 r^{n-1-p} \, dr} = \frac{(n-p)^2}{4}. \end{aligned}$$

This can be done by modifying a lemma taken from [17, Lemma 1.3].

2.1. Proof of Theorem A(ii)

Let $u^* = (\frac{x}{|x|}, 0)$ be the “equator map” from $B^n \rightarrow S^n$. It is easy to see

$$|\nabla u^*|^2 = \frac{n-1}{r^2}.$$

Let $w \in H^{1,2}(B^n, S^n)$ be any function with boundary value $w = u^*$ on ∂B^n .

By Lemma 1, we obtain

$$\int_{B^n} |\nabla u^*|^{p-2} |\nabla \phi|^2 \, dx \geq \frac{(n-p)^2}{4(n-1)} \int_{B^n} |\nabla u^*|^p \phi^2 \, dx.$$

When $n \geq 2 + p + \sqrt{4p}$, we have

$$\frac{(n-p)^2}{4(n-1)} \geq 1.$$

When $n \geq 2 + p + \sqrt{4p}$, we get

$$\int_{B^n} |\nabla u^*|^{p-2} |\nabla(u^* - w)|^2 \, dx \geq \int_{B^n} |\nabla u^*|^p (u^* - w)^2 \, dx \tag{2.1}$$

for all $w \in H^{1,p}(B^n, S^n)$ with $w = u^*$ on ∂B^n . Moreover, we know

$$\int_{B^n} |\nabla u^*|^{p-2} |\nabla(u^* - w)|^2 \, dx$$

$$\begin{aligned}
 &= \int_{B^n} |\nabla u^*|^p dx - 2 \int_{B^n} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla w dx \\
 &\quad + \int_{B^n} |\nabla u^*|^{p-2} |\nabla w|^2 dx
 \end{aligned} \tag{2.2}$$

and

$$\int_{B^n} |\nabla u^*|^p (u^* - w)^2 dx = \int_{B^n} |\nabla u^*|^p (2 - 2u^* \cdot w) dx. \tag{2.3}$$

Notice that u^* is a weakly p -harmonic map, i.e.,

$$\int_{B^n} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla \phi dx = \int_{B^n} |\nabla u^*|^p u^* \cdot \phi dx$$

for all $\phi \in H_0^{1,p}(B^n, \mathbb{R}^{n+1})$. By taking $\phi = u^* - w$, we have

$$\int_{B^n} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla w dx = \int_{B^n} |\nabla u^*|^p u^* \cdot w dx. \tag{2.4}$$

From (2.1)–(2.4), we get for $n \geq 2 + p + \sqrt{4p}$

$$\int_{B^n} |\nabla u^*|^{p-2} |\nabla w|^2 dx \geq \int_{B^n} |\nabla u^*|^p dx.$$

By the Hölder inequality, we have

$$\int_{B^n} |\nabla u^*|^{p-2} |\nabla w|^2 dx \leq \left(\int_{B^n} |\nabla u^*|^p dx \right)^{(p-2)/p} \left(\int_{B^n} |\nabla w|^p dx \right)^{2/p}.$$

Combing two above inequalities, we have

$$\int_{B^n} |\nabla u^*|^p dx \leq \int_{B^n} |\nabla w|^p dx$$

for all $w \in H^{1,p}(B^n, S^n)$ with boundary value $w = u^*$ on ∂B^n when $n \geq 2 + p + 2\sqrt{p}$.

From Lemma 1, we know that (2.1) become equality only if $w = u^*$. If $n \geq 2 + p + 2\sqrt{p}$, we have

$$\int_{B^n} |\nabla u^*|^p dx < \int_{B^n} |\nabla w|^p dx$$

for all $w \in H^{1,p}(B^n, S^n)$ with boundary value $w = u^*$ on ∂B^n and $w \neq u^*$. It means that u^* is the unique minimizer for $n \geq 2 + p + 2\sqrt{p}$. This proves Theorem A(ii).

Assume that $p = 2$. When $n \geq 7 > 4 + 2\sqrt{2}$, one sees $(n - 2)^2 / (4(n - 1)) > 1$. By Lemma 1, we know

$$\int_{B^n} |\nabla \phi|^2 dx \geq \left(1 - \frac{4(n - 1)}{(n - 2)^2}\right) \int_{B^n} |\nabla \phi|^2 dx + \int_{B^n} |\nabla u^*|^2 \phi^2 dx$$

for all $\phi \in H_0^{1,2}(B^n, \mathbb{R}^{n+1})$. Taking $\phi = w - u^*$, we have

$$\int_{B^n} |\nabla w|^2 dx - \int_{B^n} |\nabla u^*|^2 dx \geq \left(1 - \frac{4(n - 1)}{(n - 2)^2}\right) \int_{B^n} |\nabla(w - u^*)|^2 dx,$$

where $w \in H^{1,2}(B^n, S^n)$ agrees with u^* on ∂B^n . This proves our claim in Remark 1.

For t small we define $u_t : B^n \rightarrow S^n$ by setting

$$u_t(x) = \frac{\left(\frac{x}{|x|}, t\phi(x)\right)}{(1 + t^2\phi^2)^{1/2}}$$

for a smooth function ϕ on B^n vanishing near 0 and ∂B^n . A simple calculation gives

$$\frac{\partial \nabla u_t(x)}{\partial t} \Big|_{t=0} = (0, \dots, 0, \nabla \phi(x)), \quad \frac{\partial}{\partial t} |\nabla u_t|^2 \Big|_{t=0} = 0$$

and

$$\frac{\partial^2 \nabla u_t(x)}{\partial^2 t} \Big|_{t=0} = \phi^2 \left(\nabla \frac{x}{|x|}, 0 \right).$$

Now

$$\frac{1}{2} \frac{\partial^2}{\partial^2 t} |\nabla u_t|^2 \Big|_{t=0} = \left| \frac{\partial \nabla u_t(x)}{\partial t} \Big|_{t=0} \right|^2 + \left(\frac{\partial^2 \nabla u_t(x)}{\partial^2 t}, \nabla u_t \right) \Big|_{t=0}.$$

Then

$$\left. \frac{d^2}{dt^2} E_p(u_t) \right|_{t=0} = \int_{B^n} |\nabla u^*|^{p-2} [|\nabla \phi|^2 - \phi^2 |\nabla u^*|^2] dx.$$

It is easy to check that $u^* = (\frac{x}{|x|}, 0)$ is a weakly p -harmonic map from B^n into S^n . If u^* is stable, we have

$$\int_{B^n} |\nabla u^*|^{p-2} [|\nabla \phi|^2 - \phi^2 |\nabla u^*|^2] dx \geq 0$$

for all smooth ϕ vanishing near 0 and ∂B^n .

2.2. Proof of Theorem A(i)

Let us consider the following equation:

$$\begin{cases} \phi''(r) + \frac{n-p+1}{r} \phi'(r) + \frac{n-1-\varepsilon}{r^2} \phi(r) = 0, \\ \phi(r_0) = \phi(1) = 0. \end{cases} \tag{2.5}$$

for $0 < r_0 < 1$.

By setting $\xi(t) = \phi(e^t)$, Eq. (2.5) becomes

$$\xi''(t) + (n-p)\xi'(t) + (n-1-\varepsilon)\xi(t) = 0.$$

Let

$$\nu := \nu(\varepsilon) = \frac{1}{4} [n^2 - 2(p+2)n + p^2] + 1 + \varepsilon.$$

When $3 \leq n < 2 + p + 2\sqrt{p}$, there exists a small ε such that $\nu < 0$ and we choose r_0 : $0 < r_0 < 1$ such that $\sqrt{-\nu} \ln r_0$ is a multiple of 2π . Then it is easily checked (see [12]) that the function

$$\phi(r) = \begin{cases} r^{(p-n)/2} \sin(\sqrt{-\nu} \cdot \ln r), & \text{for } r_0 < r \leq 1, \\ 0, & \text{for } r \leq r_0. \end{cases}$$

solves Eq. (2.5). This means that for $3 \leq n < 2 + p + 2\sqrt{p}$, there exists ε small and a non-zero $\phi(r)$ on $[r_0, 1]$, $\phi(1) = \phi(r_0) = 0$, such that

$$\int_0^1 r^{2-p} \left[\phi'(r)^2 - \frac{(n-1)}{r^2} \phi^2 \right] r^{n-1} dr = - \int_0^1 \frac{\varepsilon}{r^2} r^{2-p} \phi^2 r^{n-1} dr < 0.$$

In other words, we see that u^* is unstable for $3 \leq n < 2 + p + 2\sqrt{p}$. This proves Theorem A(i).

From the proof of Theorem A, we have

COROLLARY 2. – Assume that u is a stable p -harmonic map from B^n into S^n and the values of u are on the equator $(S^{n-1}, 0)$ of S^n . Then u is a local minimizer of the energy functional E_p in $H^{1,p}(B^n, S^n)$.

3. PROOF OF THEOREM B

In this section, let N be the ellipsoid of \mathbb{R}^{n+1} defined in Section 1 and suppose that $p \in \mathbb{R}$ with $n > p \geq 2$. We define the p -energy of maps in $H^{1,p}(B^n, N)$ by

$$E_p(v, z; B^n) = \int_{B^n} (|\nabla v|^2 + |\nabla z|^2)^{p/2} dx.$$

We write $u = (v, z)$ with $v \in \mathbb{R}^n, z \in \mathbb{R}$. A map $u = (v, z) \in H^{1,p}(B^n, N)$ is called “weakly p -harmonic” if u satisfies in the sense of distributions the following equations:

$$\begin{aligned} \operatorname{div}(|\nabla u|^{p-2} \nabla v) + |\nabla u|^{p-2} \lambda v &= 0, \\ \operatorname{div}(|\nabla u|^{p-2} \nabla z) + |\nabla u|^{p-2} \lambda \frac{z}{a^2} &= 0, \end{aligned}$$

where

$$\lambda = \left(|\nabla v|^2 + \frac{|\nabla z|^2}{a^2} \right) \frac{a^4}{a^4 v^2 + z^2}.$$

Proof of Theorem B. – Let $u = (v, z)$ be any function in $H^{1,p}(B^n, N)$ with boundary values $(x, 0)$ on ∂B^n . Using Lemma 1 again, we have

$$\int_{B^n} \left| \nabla \frac{x}{|x|} \right|^{p-2} \left| \nabla \left(\frac{x}{|x|} - v \right) \right|^2 dx$$

$$\geq \frac{(n-p)^2}{4(n-1)} \int_{B^n} \left| \nabla \frac{x}{|x|} \right|^p \left| \frac{x}{|x|} - v \right|^2 dx \tag{3.1}$$

and

$$\int_{B^n} \left| \nabla \frac{x}{|x|} \right|^{p-2} |\nabla z|^2 dx \geq \frac{(n-p)^2 a^2}{4(n-1)} \int_{B^n} \left| \nabla \frac{x}{|x|} \right|^p \frac{|z|^2}{a^2} dx. \tag{3.2}$$

(i) Assume that $a \geq 1$ and $n \geq 2 + p + 2\sqrt{p}$. Note the fact $v^2 + \frac{z^2}{a^2} = 1$. Thus using (3.1) and (3.2), the proof of Theorem A(ii) yields

$$\int_{B^n} \left| \nabla \frac{x}{|x|} \right|^{p-2} |\nabla u|^2 dx \geq \frac{(n-p)^2}{4(n-1)} \int_{B^n} \left| \nabla \frac{x}{|x|} \right|^p dx$$

for all $u = (v, z)$ with same boundary values $(x, 0)$. The same argument in the proof of Theorem A(ii) gives that $(\frac{x}{|x|}, 0)$ is the unique minimizer of E_p if $n \geq 2 + p + 2\sqrt{p}$.

(ii) Assume that $0 < a < 1$ and $a^2 \geq 4(n-1)/(n-p)^2$. Thus using (3.1) and (3.2) again, we get

$$\int_{B^n} \left| \nabla \frac{x}{|x|} \right|^{p-2} |\nabla u|^2 dx > \frac{(n-p)^2 a^2}{4(n-1)} \int_{B^n} \left| \nabla \frac{x}{|x|} \right|^p dx$$

for all $u = (v, z) \neq (\frac{x}{|x|}, 0)$ with same boundary values. The same argument in the proof of Theorem A(ii) gives that $(\frac{x}{|x|}, 0)$ is the unique minimizer of E_p . This proves Theorem B. \square

Remark d. – It is obvious from the proof of Theorem A(i) to see that $(\frac{x}{|x|}, 0)$ is unstable for E_p when $a^2 < 4(n-p)/(n-2)^2$.

REFERENCES

[1] A. BALDES, Stability and uniqueness properties of the equator map from a ball into an ellipsoid, *Math. Z.* 185 (1984) 505–516.
 [2] H. BREZIS, J.-M. CORON and E.H. LIEB, Harmonic maps with defects, *Comm. Math. Phys.* 107 (1986) 649–705.
 [3] J.-M. CORON and R. GULLIVER, Minimizing p -harmonic maps into spheres, *J. Reine Angew. Math.* 401 (1989) 82–100.
 [4] J. EELLS and J.H. SAMPSON, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* 86 (1964) 109–160.

- [5] M. GIAQUINTA and E. GIUSTI, On the regularity of the minima of variational integrals, *Acta Math.* 148 (1982) 31–46.
- [6] M. GIAQUINTA and E. GIUSTI, The singular set of the minima of certain quadratic functionals, *Ann. Scuola Norm. Sup. Pisa* (4) 11 (1984) 45–55.
- [7] M. GIAQUINTA and J. SOUCEK, Harmonic maps into a hemisphere, *Ann. Scuola Norm. Sup. Pisa* (4) 12 (1985) 81–90.
- [8] F. HELEIN, Regularity and uniqueness of harmonic maps into an ellipsoid, *Manuscripta Math.* 60 (1988) 235–257.
- [9] F. HELEIN, Approximations of Sobolev maps between an open set and an Euclidean sphere, boundary data and singularities, *Math. Ann.* 285 (1989) 125–140.
- [10] S. HILDEBRANDT, H. KAUL and K.-O. WIDMAN, An existence theorem for harmonic mappings of Riemannian manifolds, *Acta Math.* 138 (1977) 1–16.
- [11] W. JÄGER and H. KAUL, Uniqueness and stability of harmonic maps and their Jacobi fields, *Manuscripta Math.* 28 (1979) 269–291.
- [12] W. JÄGER and H. KAUL, Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems, *J. Reine Angew. Math.* 343 (1983) 146–161.
- [13] J. JOST and M. MEIER, Boundary regularity for minima of certain quadratic functionals, *Math. Ann.* 262 (1983) 549–561.
- [14] F.-H. LIN, Une remarque sur l’application $x/|x|$, *C. R. Acad. Sci. Paris* 305 (1987) 529–531.
- [15] R. SCHOEN and K. UHLENBECK, A regularity theory for harmonic maps, *J. Differential Geom.* 17 (1982) 307–335.
- [16] R. SCHOEN and K. UHLENBECK, Boundary theory and the Dirichlet problem for harmonic maps, *J. Differential Geom.* 18 (1983) 253–268.
- [17] R. SCHOEN and K. UHLENBECK, Regularity of minimizing harmonic maps into the sphere, *Invent. Math.* 78 (1984) 89–100.
- [18] L. SIMON, *Theorems on Regularity and Singularity of Energy Minimizing Maps*, Lecture Notes in Mathematics, ETH Zürich, Birkhäuser, Basel, 1996.