EXOTIC SOLUTIONS OF THE CONFORMAL SCALAR CURVATURE EQUATION IN $\mathbb{R}^n$

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Received 13 January 2000

ABSTRACT. – We construct global exotic solutions of the conformal scalar curvature equation
\[ \Delta u + \frac{n(n - 2)}{4} K u^{\frac{n+2}{n-2}} = 0 \]
in $\mathbb{R}^n$, with $K(x)$ approaching 1 near infinity in order as close to the critical exponent as possible. © 2001 Éditions scientifiques et médicales Elsevier SAS

Keywords: Conformal scalar curvature equation; Gluing solutions; Positive scalar curvature
AMS classification: Primary 35J60; Secondary 58J05

1. Introduction

We consider a special class of positive solutions of the conformal scalar curvature equation
\[ \Delta u + \frac{n(n - 2)}{4} K u^{\frac{n+2}{n-2}} = 0 \] in $\mathbb{R}^n$. (1.1)

Here $\Delta$ is the standard Laplacian on $\mathbb{R}^n$ equipped with Euclidean metric $g_0$, $K$ a smooth function on $\mathbb{R}^n$, and $n \geq 3$ an integer. The solutions we construct breach a rather natural lower bound and have peculiar asymptotic property.

Eq. (1.1) is studied extensively by many authors in connection with the prescribed scalar curvature problem on a Riemannian manifold in general and on $\mathbb{R}^n$ and $S^n$ in particular (on $S^2$, the Nirenberg problem; cf. [1,3–5,9,12,14,15,17,20,21,23,24,26] and the references within). As in the case of the Yamabe problem, recent studies indicate that the case when $K$ is strictly positive affords many interesting and subtle developments.

Assume that $K$ is bounded between two positive constants in $\mathbb{R}^n$. An important feature of Eq. (1.1) is the asymptotic behavior of $u(x)$ for large $|x|$ (cf. [2,5–8,10,12,16,18,19,26]).
It is simpler to classify with the help of the Kelvin transformation:

\[
y = \frac{x}{|x|^2} \quad \text{and} \quad w(y) := |y|^{2-n} u(y/|y|^2) \quad \text{for} \; x, y \in \mathbb{R}^n \setminus \{0\}.
\]  

(1.2)

From (1.2), \(w\) satisfies the equation

\[
\Delta w(y) + \frac{n(n - 2)}{4} \tilde{K}(y) w^{\frac{4}{n-2}}(y) = 0 \quad \text{for} \; y \in \mathbb{R}^n \setminus \{0\},
\]

where \(\tilde{K}(y) := K(y/|y|^2)\) for \(y \neq 0\) (see, for instance, [18]). \(w\) (and \(u\)) is said to have fast decay if \(w\) has a removable singularity at the origin. Otherwise, it is called a singular solution. In order to have reasonable control on the geometric and analytic behavior of singular solutions, it is crucial to obtain the upper bound or slow decay

\[
w(y) \leq C_1 |y|^{-(n-2)/2} \quad \text{as} \; y \to 0, \quad \text{i.e.,} \quad u(x) \leq C_1 |x|^{-(n-2)/2} \quad \text{for} \; |x| \gg 1,
\]

(1.4)

where \(C_1\) is a positive constant. The question on slow decay is discussed in depth in [2, 5–8,16,18,19,22] (cf. also [27]; note that our definition of slow decay is slightly different from the one in [5] and [8]). Guided by the case when \(K\) is equal to a positive constant outside a compact subset of \(\mathbb{R}^n\) (see [2,16]), it is natural to ask whether a singular positive solution \(u\) with slow decay also satisfies the lower bound

\[
w(y) \geq C_2 |y|^{-(n-2)/2} \quad \text{as} \; y \to 0, \quad \text{i.e.,} \quad u(x) \geq C_2 |x|^{-(n-2)/2} \quad \text{for} \; |x| \gg 1,
\]

(1.5)

where \(C_2\) is a positive constant. If the lower bound holds, then the conformal metric \(u^{4/(n-2)} g_o\) on \(\mathbb{R}^n\) is complete and has bounded (sectional) curvature [8]. The radial Pohozaev number is an essential invariant in the study of equation (1.1) and is given by

\[
P(u) := \lim_{R \to \infty} \int_{B_o(R)} [x \cdot \nabla K(x)] u^{2n/(n-2)}(x) \, dx,
\]

(1.6)

provided the limit exists. Here \(B_o(R)\) is the open ball with center at the origin and radius equal to \(R > 0\). The following result is shown by Chen and Lin in [6] and [8], mindful of the slightly different notations we use.

**Theorem 1.7 (Chen-Lin).** – Let \(u\) be a positive smooth solution of Eq. (1.1). Assume that \(\lim_{|x| \to \infty} K(x)\) exists and is positive, and there exist positive constants \(l \geq (n-2)/2\) and \(C\) such that

\[C^{-1} |x|^{-(l+1)} \leq |\nabla K(x)| \leq C |x|^{-(l+1)} \quad \text{for all} \; |x| \gg 1.
\]

Then \(u\) has slow decay and \(P(u)\) exists and is non-positive. \(u\) has fast decay if and only if \(P(u) = 0\) (the Kazdan–Warner condition). Furthermore, if \(u\) is a singular solution, then we also have the lower bound \(u(x) \geq C_2 |x|^{-(n-2)/2} \) for all \(|x| \gg 1\) and for some positive constant \(C_2\).
More generally, under the condition that \( \lim_{|x| \to \infty} K(x) \) exists and is positive, and \( |\nabla K| \) is bounded in \( \mathbb{R}^n \), for a positive smooth solution \( u \) of Eq. (1.1) with slow decay, we show in [10] (cf. also [5,8]) that \( P(u) \leqslant 0 \) if \( P(u) \) exists. Moreover, \( P(u) = 0 \) if and only if
\[
\lim_{|x| \to \infty} |x|^{\frac{n-2}{2}} u(x) = 0.
\] (1.8)
In the latter case, the assumption on \( K \) is not strong enough to allow us to deduce that \( u \) has fast decay.

**Definition 1.9.** – We call a singular positive solution \( u \) of Eq. (1.1) with slow decay an exotic solution if (1.8) holds for \( u \). That is, we cannot find a positive constant \( C_2 \) such that \( u(x) \geqslant C_2 |x|^{-\left(\frac{n-2}{2}\right)} \) for all \( |x| \gg 1 \).

Then it is necessary that \( P(u) = 0 \) if \( P(u) \) exists. Exotic solutions are rather peculiar because from \( P(u) = 0 \) one would expect \( u \) to have fast decay. Instead, they decay slowly and the conformal metric \( u^{\frac{4}{n-2}} g_o \) remains to be complete, but the (sectional) curvature is unbounded [8]. Theorem 1.7 leads to the observation that there are no exotic solutions if \( |\nabla K| \) decays to zero near infinity fast enough.

(Local) Exotic solutions are first found by Chen and Lin in [8]. By a scaling and the Kelvin transform, we may consider the equation
\[
\Delta u + \tilde{K} u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad B_o(1) \setminus \{0\}.
\] (1.10)
Assume that \( \tilde{K} \) is radial and non-increasing in \((0, 1] \), and is given by
\[
\tilde{K}(r) = 1 - Ar^l + R(r)
\] (1.11)
for \( r > 0 \) close to zero. Here \( A > 0 \) and \( 0 < l < (n-2)/2 \) are constants, and \( R(r) = o(r^l) \) and \( R'(r) = o(r^{l-1}) \) for \( r > 0 \) close to zero. Given a positive number \( \alpha \), let \( u(r, \alpha) \) be the unique solution of the initial value problem
\[
\begin{cases}
  u''(r) + \frac{n+2}{n-2} u'(r) + \tilde{K}(r) u^{\frac{n+2}{n-2}}(r) = 0, \\
  u(0) = \alpha \quad \text{and} \quad u'(0) = 0.
\end{cases}
\]
Chen and Lin [8] show elegantly that there exists a sequence \( \alpha_i \to \infty \) such that \( u(r, \alpha_i) \) converges to an (local) exotic \( C^2 \)-solution of Eq. (1.10) in \( B_o(1) \setminus \{0\} \). Subsequently, Lin [22] obtains characterizations of exotic solutions in terms of the asymptotic expansion of \( \tilde{K} \) near the origin.

The exponent \( (n-2)/2 \) is found to be critical. For \( l \geqslant (n-2)/2 \), Theorem 1.7 shows that there are no exotic solutions of Eq. (1.1). In this paper we construct global exotic solutions of Eq. (1.1) in \( \mathbb{R}^n \). As described above, in [8], an abstract existence argument is used to show the existence of (local) exotic solutions. Our construction is explicit by gluing the Delaunay–Fowler-type solutions. Given any positive number \( \delta \), we show that there is an exotic solution of Eq. (1.1) with \( |K - 1| \leqslant \delta^2 \) in \( \mathbb{R}^n \). Moreover, with regard to the critical exponent \( (n-2)/2 \), we show that, given any positive function \( \varphi(r) \) defined
for \( r \gg 1 \) such that
\[
\phi(r) \quad \text{is non-decreasing for } r \gg 1 \quad \text{and} \quad \lim_{r \to \infty} \frac{(n-2)}{2} \phi(r) = \infty,
\]
(1.12)

(for example, \( \phi(r) = r^{-(n-2)/2} \ln(r) \) for \( r \gg 1 \)), we construct an exotic solution of Eq. (1.1) with
\[
|K(x) - 1| \leq C_3 \phi(|x|) \quad \text{for all } |x| \gg 1,
\]
(1.13)

where \( C_3 \) is a positive constant. The analytic property of exotic solutions resides in a neighborhood of infinity, or, by the Kelvin transformation, on a neighborhood of the origin. Our emphasis on the whole \( \mathbb{R}^n \) reflects the geometric viewpoint of conformal deformations of Euclidean space \( (\mathbb{R}^n, g_0) \). We follow the convention of using \( c, C, C', C_1, \ldots \) to denote positive constants, whose actual values may differ from section to section.

## 2. Delaunay–Fowler-type solutions

Introduce polar coordinates \((r, \theta)\) in \( \mathbb{R}^n \), where \( r = |x| \) and \( \theta = x/|x| \) for \( x \in \mathbb{R}^n \setminus \{0\} \).

Let \( t = \ln r \) for \( r > 0 \) and
\[
v(t, \theta) = r^{(n-2)/2} u(r, \theta) \quad \text{for } r > 0 \quad \text{and} \quad \theta \in S^{n-1}.
\]
(2.1)

By the above transformation, Eq. (1.1) can be re-written as
\[
\frac{\partial^2 v}{\partial t^2} + \nabla_\theta v - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} K v^{\frac{n+2}{4}} = 0 \quad \text{in } \mathbb{R} \times S^{n-1}.
\]
(2.2)

Here \( \Delta_\theta \) is the Laplacian on the standard unit sphere in \( \mathbb{R}^n \) and \( K(t, \theta) := K(x) \), where \( |x| = e^t \) and \( x/|x| = \theta \). For the case \( K \equiv 1 \) in \( \mathbb{R} \times S^{n-1} \), consider radial solutions \( v \) of (2.2) and the ODE
\[
v'' - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{\frac{n+2}{4}} = 0 \quad \text{in } \mathbb{R}.
\]
(2.3)

In connection with the study of surfaces of revolution of constant curvature by Delaunay [11] and a class of semilinear differential equations by Fowler [13], positive smooth solutions of Eq. (2.3) are known as Delaunay–Fowler-type solutions. We refer to [16,24,25] for basic properties of the solutions. Eq. (2.3) is autonomous and the Hamiltonian energy
\[
H(v, v') = (v')^2 - \frac{(n-2)^2}{4} [v^2 - v^{2n/(n-2)}]
\]
(2.4)

is constant along solutions of (2.3). For a positive smooth solution \( v \) of (2.3), \( H \) is a non-positive constant in the interval \( [-((n-2)/n)^{n/2}(n-2)/2, 0] \) (see [16]). By shifting the parameter, we may normalize the solution so that
\[
v(0) = \max_{t \in \mathbb{R}} v(t).
\]
(2.5)
Let \( v_o \) be a positive solution of Eq. (2.3) with \( H = 0 \). Under the normalization, we have
\[
v_o(t) = (\cosh t)^{(2-n)/2} \quad \text{for } t \in \mathbb{R}.
\]
(2.6)

We note that, by the transformation in (2.1), \( v_o \) corresponds to
\[
u_o(x) = \left( \frac{2}{1 + |x|^2} \right)^{(n-2)/2} \quad \text{for } x \in \mathbb{R}^n,
\]
(2.7)

which is a solution of Eq. (1.1) when \( K \equiv 1 \) in \( \mathbb{R}^n \). In particular, \( u_o \) is smooth near 0, which corresponds to \( s \to -\infty \) for \( v_o \). The other extreme is when \( H = -[(n - 2)/n]^{n/2}(n - 2)/2 \), and the corresponding solution \( v \) is a constant function given by \( v(t) = [(n - 2)/n]^{(n-2)/4} \) for \( t \in \mathbb{R} \).

For \( H \in (-[(n - 2)/n]^{n/2}(n - 2)/2, 0) \), the solution can be indexed by the parameter \( \varepsilon = \min_{t \in \mathbb{R}} v(t) \), which is called the neck-size of the solution, or the Fowler parameter. We have \( \varepsilon \in (0, [(n - 2)/n]^{(n-2)/4}) \) and
\[
H = H(\varepsilon) = \frac{(n - 2)^2}{4} \left[ \varepsilon^{2n/(n-2)} - \varepsilon^2 \right].
\]
(2.8)

Denote the normalized positive solution by \( v_\varepsilon \), where \( 0 < \varepsilon < [(n - 2)/n]^{(n-2)/4} \). It is known that \( v_\varepsilon \) is periodic with period \( T_\varepsilon \). Moreover, we always have [16]
\[
\varepsilon \leq v_\varepsilon(t) \leq v_\varepsilon(0) < 1 \quad \text{for } t \in \mathbb{R}.
\]
(2.9)

The following result is essentially proved in [24] (cf. also [16]).

**Lemma 2.10.** \( T_\varepsilon \), the period of \( v_\varepsilon \), is monotone in \( \varepsilon \) for \( \varepsilon \in (0, [(n - 2)/n]^{(n-2)/4}) \). We have \( T_\varepsilon \to 2\pi \sqrt{n - 2} \) as \( \varepsilon \to 0 \) \([n - 2]/n]^{(n-2)/4} \) and \( T_\varepsilon \to \infty \) as \( \varepsilon \to 0^+ \). Furthermore, there exists a positive constant \( C \), independent on \( \varepsilon \), such that
\[
-\frac{4}{n - 2} \ln(C\varepsilon) \leq T_\varepsilon \leq -\frac{4}{n - 2} \ln(C^{-1}\varepsilon) \quad \text{as } \varepsilon \to 0^+.
\]
(2.11)

It is also known that \( v_\varepsilon \) converges uniformly in compact subsets of \( \mathbb{R} \) to the constant solution as \( \varepsilon \to [(n - 2)/n]^{(n-2)/4} \), and to \( v_o(t) = (\cosh t)^{(2-n)/2} \) as \( \varepsilon \to 0^+ \) [16]. For applications in Section 3, we study the order of the latter convergence in more detail. As \( H \) is constant along solutions, we have
\[
H(v_\varepsilon, v_\varepsilon') = -\frac{(n - 2)^2}{4} \varepsilon^2 - \frac{(n - 2)^2}{4} \varepsilon^{2n/(n-2)}(0) \varepsilon^{2n/(n-2)}(0)
\]
for \( \varepsilon \in (0, [(n - 2)/n]^{(n-2)/4}) \). Thus we obtain
\[
v_\varepsilon^2(0) [1 - v_\varepsilon^4/(n-2)](0) = \varepsilon^2 (1 - \varepsilon^{4/(n-2)}) = -\frac{4H}{(n - 2)^2}.
\]
(2.12)

As \( v_\varepsilon(0) > \varepsilon \) when \( \varepsilon \to 0^+ \), it follows from (2.12) that \( v_\varepsilon(0) \to 1 \) and \( \varepsilon \to 0^+ \). Furthermore,
\[
1 - v_\varepsilon^4/(n-2)(0) = O(\varepsilon^2).
\]
We have
\[ v_\varepsilon(0) = \left[1 + O(\varepsilon^2)\right]^{(n-2)/4} = 1 + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+. \tag{2.13} \]
Hence there exists a positive constant \( C_n \) which depends on \( n \) only, such that
\[ |v_\varepsilon(0) - 1| \leq C_n \varepsilon^2 \quad \text{for } \varepsilon > 0 \text{ small}. \tag{2.14} \]
We use the following well-known inequalities a number of times; they can be derived by simple integration methods. For positive constants \( c \) and \( \alpha \geq 1 \), we have
\[ |x^{\alpha} - y^{\alpha}| \leq C|x - y| \quad \text{for } 0 \leq x, y \leq c, \tag{2.15} \]
where \( C = C(\alpha, c) \) is a positive constant; moreover, for \( \beta > 0 \),
\[ (1 + z)^\beta = 1 + O(|z|) \quad \text{as } z \to 0. \tag{2.16} \]
With \( v_o \) given by (2.6), it follows from (2.9) and (2.15) that
\[ |v_\varepsilon(t) - v_o(t)| \leq C_n |v_\varepsilon(t) - v_o(t)|, \tag{2.17} \]
where \( C_n \) is a positive constant depending on \( n \) only. Using Eq. (2.3) we have
\[ |v_\varepsilon''(t) - v_o''(t)| \leq \frac{(n - 2)^2}{4} |v_\varepsilon(t) - v_o(t)| + \frac{n(n - 2)}{4} |v_\varepsilon^{\alpha+2}(t) - v_o^{\alpha+2}(t)| \]
\[ \leq \left[ \frac{(n - 2)^2}{4} + \frac{n(n - 2)}{4} c_n \right] |v_\varepsilon(t) - v_o(t)| \]
\[ = \tilde{C}_n |v_\varepsilon(t) - v_o(t)|, \tag{2.18} \]
where \( \tilde{C}_n \) is the positive constant defined in the formula. We claim that
\[ |v_\varepsilon''(t) - v_o''(t)| \leq 2C_n \tilde{C}_n \varepsilon^2 \quad \text{for } t \in [0, \rho], \tag{2.19} \]
where \( \rho := 1/(2C_n \tilde{C}_n) \). Here \( C_n \) and \( C_n' \) are the positive constants in (2.14) and (2.18), respectively. Without loss of generality, we may assume that \( \rho < C_n \). By (2.14) and (2.18), the bound holds on a neighborhood of 0. Suppose that it holds on \([0, \sigma]\) for some positive number \( \sigma \) less than \( \rho \). As \( v_\varepsilon'(0) = v_o'(0) = 0 \), we have
\[ |v_\varepsilon'(t) - v_o'(t)| \leq 2C_n \tilde{C}_n \varepsilon^2 \sigma \leq \varepsilon^2 \quad \text{for } t \in [0, \sigma]. \]
Hence
\[ |v_\varepsilon(t) - v_o(t)| \leq (C_n + \sigma) \varepsilon^2 < 2C_n \varepsilon^2 \quad \text{for } t \in [0, \sigma]. \tag{2.20} \]
By (2.18) we have
\[ |v_\varepsilon''(\sigma) - v_o''(\sigma)| < 2C_n \tilde{C}_n \varepsilon^2. \]
Using an connectedness argument, we obtain (2.19) as claimed. A similar bound holds in \([-\rho, 0]\). Upon integration we obtain the following lemma.
LEMMA 2.21. – Let $v_ε$ and $v_o$ be the solutions of Eq. (2.3) discussed above. There exists positive constants $ρ$ and $C_ο$ which depend on $n$ but not on (small enough positive) $ε$, such that

$$|v_ε(t) - v_o(t)| \leq C_οε^2, \quad |v_ε'(t) - v_o'(t)| \leq C_οε^2 \quad \text{and} \quad v_ε(t) \geq 1/2$$

for $t \in [-ρ, ρ]$ and $ε > 0$ close to 0.

3. Gluing solutions

We follow the notations used in Section 2 and consider (2.1) and Eq. (2.2). Slow decay for a positive smooth solution $u$ of equation (1.1) corresponds to $v(s, θ) \leq C$ for $s \gg 1, θ \in S^{m-1}$ and a positive constant $C$. Moreover, $u$ is an (global) exotic solution if and only if there exists a sequence $\{(s_i, θ_i)\} \subset \mathbb{R} \times S^{m-1}$ such that $\lim_{s_i \to \infty} s_i = \infty$ and $\lim_{s_i \to \infty} v(s_i, θ_i) = 0$, and, when the variable $t$ is changed into $r$ via $t = \ln r$, $u$ is smooth across the origin. Let $φ_1$ be a smooth function on $\mathbb{R}$ such that $0 \leq φ_1 \leq 1$ in $\mathbb{R}$ and

$$φ_1(t) = \begin{cases} 1 & \text{for } t \leq -ρ, \\ 0 & \text{for } t \geq ρ. \end{cases}$$

We also require that

$$|φ_1'(t)| \leq 2/ρ \quad \text{and} \quad |φ_1''(t)| \leq 2/ρ^2 \quad \text{for } t \in (-ρ, ρ).$$

Let $φ_2 = 1 - φ_1$ in $\mathbb{R}$. Define

$$v = φ_1 v_o + φ_2 v_ε \quad \text{in } \mathbb{R},$$

where $ε > 0$ is close to zero. It follows that

$$-v''(t) + \frac{(n-2)^2}{4} v(t) = \frac{n(n-2)}{4} [φ_1 v_o^{\frac{n+2}{n-2}}(t) + φ_2 v_ε^{\frac{n+2}{n-2}}(t)] + φ_1'(t) [v_o'(t) - v_o'(t)]$$

for $t \in \mathbb{R}$. We also have

$$φ_1(t)v_o^{\frac{n+2}{n-2}}(t) + φ_2(t)v_ε^{\frac{n+2}{n-2}}(t)$$

$$= φ_1(t)v_o^{\frac{n+2}{n-2}}(t) + φ_2(t)v_ε^{\frac{n+2}{n-2}}(t) + φ_2(t) [v_ε^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)]$$

$$= φ_1(t)v_o(t) + φ_2(t)v_ε(t) [v_ε^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)]$$

$$= \left\{ v(t) + φ_2(t) [v_o(t) - v_ε(t)] \right\}^{\frac{n+2}{n-2}} + φ_2(t) [v_ε^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)]$$

for $t \in [-ρ, ρ]$. We obtain

$$\left| -v''(t) + \frac{(n-2)^2}{4} v(t) \right| \left[ \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}}(t) \right]^{-1} - 1$$
Denote the period of \( v \) by \( \varepsilon_i \).

By Lemma 2.21, (2.16), (2.17), (3.1) and (3.4) that \( v \) satisfies the equation

\[
\frac{d^2}{dt^2} v - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} K \frac{v^2}{v^{n-2}} = 0 \quad \text{in } \mathbb{R},
\]

where \( K \) is a smooth function on \( \mathbb{R} \) such that

\[
|K(t) - 1| = \left| \frac{d^2}{dt^2} v - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{n-2} \right|^{1/2} \leq C_1 \varepsilon^2.
\]

for \( t \in [-\rho, \rho] \), and \( K \equiv 1 \) in \( \mathbb{R} \setminus [-\rho, \rho] \). Here \( C_1 \) is a positive constant that depends on \( n \) only, so far as \( \varepsilon > 0 \) is close to zero.

Let \( \{ \varepsilon_i \} \) be a decreasing sequence of small positive numbers such that \( \lim_{i \to \infty} \varepsilon_i = 0 \).

Denote the period of \( v_{\varepsilon_i} \) by \( T_{\varepsilon_i} \) for \( i = 1, 2, \ldots \).

With \( \varepsilon_1 \) small enough, we may assume that \( T_{\varepsilon_i} \gg \rho \).

We construct a positive smooth function by first gluing \( v_o \) and \( v_{\varepsilon_i} \) on \([-\rho, \rho]\) as described above and call the resulting positive smooth function \( v_1 \). Note that \( v_1 = v_{\varepsilon_1} \) in \( \mathbb{R} \setminus (0, \rho) \). As \( v_{\varepsilon_i}(t + T_{\varepsilon_i}) = v_{\varepsilon_i}(t) \) for \( t \in \mathbb{R} \) and \( v_{\varepsilon_i} \) and \( v_1 \) are close to \( v_o \) near \([-\rho, \rho] \), we let

\[
v_{\varepsilon_2}(t) = v_{\varepsilon_2}(t - T_{\varepsilon_1}) \quad \text{for } t \in \mathbb{R},
\]

and glue \( v_{\varepsilon_2} \) and \( v_1 \) (that is, \( v_{\varepsilon_2} \)) on \( [T_{\varepsilon_1} - \rho, T_{\varepsilon_1} + \rho] \) in a process similar to the one described above. Call the resulting function \( v_2 \). We continue to glue the solutions on the intervals

\[
[T_{\varepsilon_1} + T_{\varepsilon_2} - \rho, T_{\varepsilon_1} + T_{\varepsilon_2} + \rho], \ldots, \left[ \sum_{k=1}^i T_{\varepsilon_k} - \rho, \sum_{k=1}^i T_{\varepsilon_k} + \rho \right], \ldots
\]

by \( v_o, v_{\varepsilon_1}, v_{\varepsilon_2}, \ldots \), respectively, after shifting appropriately. In particular, in the \((i + 1)\)th step, let

\[
v_{\varepsilon_{i+1}}(t) = v_{\varepsilon_{i+1}} \left( t - \sum_{k=1}^{i+1} T_{\varepsilon_k} \right) \quad \text{and} \quad v_{\varepsilon_i}(t) = v_{\varepsilon_i} \left( t - \sum_{k=1}^i T_{\varepsilon_k} \right) \quad \text{for } t \in \mathbb{R},
\]

and glue \( v_{\varepsilon_{i+1}} \) with \( v_{\varepsilon_i} \) on the interval \( [\sum_{k=1}^i T_{\varepsilon_k} - \rho, \sum_{k=1}^i T_{\varepsilon_k} + \rho] \). Finally we obtain a positive smooth function \( v \) on \( \mathbb{R} \) which satisfies the equation

\[
v'' - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} K v^{n-2} = 0 \quad \text{in } \mathbb{R}
\]
for some smooth function $K$ such that

$$|K(t) - 1| \leq C_2 \varepsilon_1^2 \quad \text{for } t \in \mathbb{R},$$

where $C_2$ is a positive constant depending on $n$ only. We may choose $\varepsilon_1 > 0$ as small as we like. We also have

$$v\left(\sum_{k=1}^{i} T_{\varepsilon_k} - T_{\varepsilon_i}/2\right) = v_i(T_{\varepsilon_i}/2) = \varepsilon_i \to 0 \quad \text{and}$$

$$v\left(\sum_{k=1}^{i} T_{\varepsilon_k}\right) \to 1^- \quad \text{as } i \to \infty.$$  \hfill (3.9)

As $v(t) = v_0(t)$ for $t \leq -\rho$, by (2.6) and (2.7), the corresponding solution $u$ related to $v$ by (2.1) is smooth across the origin. Thus $v$ corresponds to an exotic solution $u$ of Eq. (1.1) through (2.1).

Given a positive function $\varphi(r)$ defined for $r \gg 1$ which satisfies (1.12), let $\psi(t) = \varphi(e^t)$. It follows that $\psi$ is defined for $t \gg 1$ and

$$e^{(n-2)t/2}\psi(t) \quad \text{(3.10)}$$

is non-decreasing for $t \gg 1$ and unbounded from above. Let

$$\sigma(t) = \ln\left[e^{(n-2)t/2}\psi(t)\right] \quad \text{for } t \gg 1.$$ \hfill (3.11)

We have $\lim_{t \to \infty} \sigma(t) = \infty$. Choose a decreasing sequence of numbers $\{\varepsilon_i\}$ such that $\varepsilon_1$ is small enough and the corresponding periods $T_{\varepsilon_i}$ of $v_{\varepsilon_i}$ satisfy the relation

$$\sigma(T_{\varepsilon_i}) \geq \frac{n - 2}{2} \sum_{k=1}^{i-1} T_{\varepsilon_k} \quad \text{for } i = 2, 3, \ldots.$$ \hfill (3.12)

By gluing the solutions $v_0, v_{\varepsilon_i}, i = 1, 2, \ldots$, as described above, we obtain a positive smooth function $v$ which satisfies Eq. (3.7) for a smooth function $K$. Suppose that

$$t \notin [-\rho, \rho] \cup [T_{\varepsilon_1} - \rho, T_{\varepsilon_1} + \rho] \cup \cdots \cup \left[\sum_{k=1}^{i} T_{\varepsilon_k} - \rho, \sum_{k=1}^{i} T_{\varepsilon_k} + \rho\right] \cup \cdots,$$

then $K(t) = 1$. Suppose that

$$t \in \left[\sum_{k=1}^{i} T_{\varepsilon_k} - \rho, \sum_{k=1}^{i} T_{\varepsilon_k} + \rho\right] \quad \text{for some } i \in \mathbb{N}.$$  

According to the construction above and Lemma 2.10, we have

$$|K(t) - 1| \leq C_3 \varepsilon_i^2 \leq C_4 \exp\left(-\frac{n - 2}{2} T_{\varepsilon_i}\right)$$
\[ C_3 \exp \left( -\frac{n-2}{2} T_{\varepsilon} \right) \leq C_3 \exp \left( -\frac{n-2}{2} \sum_{k=1}^{i} T_{\varepsilon} \right) \leq C_4 \exp \left( \frac{n-2}{2} \right) \psi(t), \]

where \( C_3 \) and \( C_4 \) are positive constants that depend on \( n \) only. Hence we obtain \( |K(t) - 1| \leq C_5 \psi(t) \) for \( t \gg 1 \) and for a positive constant \( C_5 \). The corresponding solution \( u \) is an exotic solution of Eq. (1.1) which satisfies (1.13). We note that \( K(t) \) in this case is not monotonic for large \( t \).

REFERENCES


