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HÖLDER CONTINUITY CONDITIONS FOR THE SOLVABILITY OF DIRICHLET PROBLEMS INVOLVING FUNCTIONALS WITH FREE DISCONTINUITIES

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ABSTRACT. – The paper deals with the problem of minimizing a free discontinuity functional under Dirichlet boundary conditions. An existence result was known so far for $C^1(\partial\Omega)$ boundary data $\hat{u}$. We show here that the same result holds for $\hat{u} \in C^{0,\mu}(\partial\Omega)$ if $\mu > \frac{1}{2}$ and it cannot be extended to cover the case $\mu = \frac{1}{2}$. The proof is based on some geometric measure theoretic properties, in part introduced here, which are proved *a priori* to hold for all the possible minimizers. © 2001 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – On savait que le minimum d'une fonctionnelle à discontinuité libre avec condition de Dirichlet sur le bord était atteint quand la donnée de bord $\hat{u}$ est $C^1(\partial\Omega)$. Nous étendons ce résultat à $\hat{u} \in C^{0,\mu}(\partial\Omega)$ si $\mu > \frac{1}{2}$ et montrons qu'il n'est plus vrai pour $\mu = \frac{1}{2}$. Pour cela, nous démontrons des propriétés de théorie de la mesure géométrique *a priori* valides pour tout minimiseur de la fonctionnelle. © 2001 Éditions scientifiques et médicales Elsevier SAS

Introduction

This paper is concerned with a Dirichlet boundary value problem involving a Mumford–Shah functional or, more in general, a functional with "free discontinuities", which is usually studied with Neumann homogenous boundary data. Such functionals can be seen as depending on two variables: a function u and a closed set K which contains the discontinuity points of u. Each one of them can be easily determined in an optimal way when the other one is given, so one can see these functionals as only depending on the function variable u or on the set variable K and considering the other one as implicitly defined.

The existence theory for minima of functionals with free discontinuities has been made by E. De Giorgi and his collaborators in the following steps. A new functional space, denoted by SBV, has been introduced in [10], then a compactness theorem and

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the subsequent existence of a function in SBV which minimizes the functional have been proved by L. Ambrosio in [1] and [2] (see [3] for a more recent proof). A regularity result is then needed in order to prove the closure (modulo a negligible set) of the set of discontinuity points and it has been made by E. De Giorgi, M. Carriero and A. Leaci in [11].

An alternative approach, which uses the set variable and directly finds a closed minimizer, has been proposed in [7] for the case of two dimensions (see [18] and, more recently, [17] for arbitrary dimension). Such an approach works with some *a priori* density estimates on the minimal sets, obtained by a technique called *Excision Method* in [18] which has been more recently extended to the case of a general dimension in [21]. The crucial assumption in order to apply such a method is the Hölder continuity of u out of a suitably small set. The density theorems obtained in this way allow the proof of some semicontinuity results, with respect to the Hausdorff distance, which give the existence directly in the class of closed subsets of Ω, by trivial compactness arguments.

The case of Dirichlet boundary conditions has been treated, following the SBV approach, in [5] for C^1 boundary conditions (the regularity of boundary data will be always assumed with the possible exception of a closed $(N-1)$-negligible set).

The theorems in this paper sharpen the result in [5], by showing that the Hölder continuity of the boundary datum $\hat{u}$ *is the really crucial assumption which makes the difference between the existence of a closed minimizer or not*. We shall make use of the excision method in order to prove that if $\hat{u}$ is Hölder continuous of exponent $\mu > \frac{1}{2}$ then a boundary version of the density theorems holds and the semicontinuity properties with respect to the Hausdorff distance and the subsequent existence theorem can be easily deduced. On the other hand, we shall show with various counterexamples that, if $\hat{u}$ is only $C^{1/2}$, even with an arbitrarily small norm, then the density estimates may be false and even the existence of a closed minimizer may fail.

So the Dirichlet Problem seems to have a theoretical interest, because it shows that the $C^{1/2}$ regularity considered in the excision method has an intrinsic meaning, which does not depend on the particular approach. Beside this, a more applicative perspective relies on the study of boundary value problems originated from the mechanics of materials which undergo fractures and have a prescribed deformation of the boundary.

1. Notation and main results

Let X be a subset of $\mathbb{R}^N$, we shall denote by $\mathcal{H}^\alpha(X)$ the Hausdorff measure of dimension α and by $|X|$ its Lebesgue measure. We shall denote by Ω an open smooth bounded subset of $\mathbb{R}^N$, by g a given measurable function from Ω in $[0,1]$ and by $\hat{u}$ a function defined on $\partial\Omega$ with values in $[0,1]$. For any positive real number $\mu \leqslant 1$, let $||u||_\mu$ denote the Hölder (semi)norm of exponent μ of a real function u (of course such a norm will be assumed to be equal to $+\infty$ if the function is not Hölder continuous) and let $C^{0,\mu}$ represent the space of Hölder continuous functions with exponent μ. We shall denote by b_K the measure of the unit ball of $\mathbb{R}^K$.

Let us consider the admissible pairs (u, K), where K is any closed subset of $\overline{\Omega}$ and u is a function in $H^1(\Omega \setminus K)$ such that $u = \hat{u}$ on $\partial\Omega \setminus K$. We shall deal with the minimization of the functional

$$E(u, K) = J(u, K) + \mathcal{H}^{N-1}(K) \tag{1.1}$$

defined on the class of admissible pairs (u, K), where $J(u, K)$ denotes the value of the elliptic functional

$$J(u, K) = \int_{\Omega\setminus K} |\nabla u|^2 + \int_{\Omega\setminus K} |g-u|^2. \tag{1.2}$$

If we fix K, we can determine a unique function $u = u(K, \hat{u})$ which minimizes the functional J on the set $\Omega \setminus K$. The function $u(K, \hat{u})$ can be characterized as the unique weak solution in $\Omega \setminus K$ of the Euler–Lagrange equations

$$\begin{cases} -\Delta u + u = g & \text{in } \Omega \setminus K, \\ u = \hat{u} & \text{on } \partial\Omega \setminus K, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } K. \end{cases} \tag{EL}$$

One can consider the function $u(K, \hat{u})$ almost everywhere defined on all of Ω. Indeed, in order to minimize E, we only need to take into account negligible values of K, with respect to Lebesgue measure, since in the other cases $E(K) = +\infty$. By setting

$$J_{\hat{u}}(K) = J\big(u(K, \hat{u}), K\big) = \inf_u J(u, K),$$

$$E_{\hat{u}}(K) = E\big(u(K, \hat{u}), K\big) = \mathcal{H}^{N-1}(K) + J_{\hat{u}}(K),$$

we can regard the functionals E and J as only depending on the set variable K. We shall use the notation $J(K)$, $E(K)$, $u(K)$ when $\hat{u}$ is supposed to be fixed and we do not need to emphasize its role. The letter c will stand for universal constants, unless differently specified. When the letter c will be used to recall a particular constant, it will be affected with the number of the equation where it is introduced. We shall introduce also constants depending on some variables, $c(\Omega, \hat{u}, c_H, \|u\|_\mu, \ldots)$, and sometimes objects as Ω and $\hat{u}$ will be considered fixed and not explicitely mentioned.

Let B be a ball of $\mathbb{R}^N$, with radius R, we set $\hat{B} = B \cap \Omega$. Under suitable regularity assumptions on $\partial\Omega$, there exists a constant c_0, depending on N and on the geometry of $\partial\Omega$, such that for every $R \leqslant 1$

$$\mathcal{H}^{N-1}(\partial\hat{B}) + |\hat{B}| \leqslant c_0 R^{N-1}. \tag{1.3}$$

If K is a minimum of E and $u = u(K, \hat{u})$ then, for every measurable set $A \subset \overline{\Omega}$, the following estimate (see [21, (1.2)]) holds with $c = 1$

$$\int_A |\nabla u|^2 + \int_A |g-u|^2 + \mathcal{H}^{N-1}(K \cap A) \leqslant c\big(\mathcal{H}^{N-1}(\partial A) + |A|\big). \tag{1.4}$$

If u is as above, by (1.4) and (1.3) we have in particular

$$\int_{\hat{B}} |\nabla u|^2 \leqslant \mathcal{H}^{N-1}(\partial\hat{B}) + |\hat{B}| \leqslant c_0 R^{N-1}. \tag{1.5}$$

In the same way we also have from (1.4), for every minimal K,

$$\mathcal{H}^{N-1}(K \cap \hat{B}) \leqslant \mathcal{H}^{N-1}(\partial \hat{B}) + |\hat{B}| \leqslant c_0 R^{N-1}\,, \tag{1.6}$$

if $R \leqslant 1$.

The main results in this paper can be summarized in the following statements.

THEOREM 1.1 (Uniform density property). – *There exists a constant $\beta > 0$, depending on μ, on the $C^{0,\mu}$ norm of $\hat{u}$ on $\partial\Omega \cap B$ and on the regularity of $\partial\Omega$, such that, if K is a minimum of $E_{\hat{u}}$ and B is a given ball centered at a point of K, of radius $R \leqslant 1$, with $\hat{u}$ Hölder continuous on $\partial\Omega \cap B$, with exponent $\mu > \frac{1}{2}$, then*

$$\mathcal{H}^{N-1}(B \cap K) \geqslant \beta R^{N-1}. \tag{1.7}$$

Theorem 1.1 is a corollary of some stronger results which will be formulated later by recurring to the properties described in the next section. Moreover, Theorem 1.1, as its corresponding inner version established in [21], also admits a weak formulation which can be applied to the singular set of a *SBV*-minimum proving its closure (see [18, Lemma 8.11]). Therefore an application of Ambrosio's Compactness Theorem [3] leads to the following result.

THEOREM 1.2. – *If $\hat{u}$ is locally Hölder continuous with exponent $\mu > \frac{1}{2}$ out of a closed set H with null $(N-1)$-dimensional measure, then there exists a closed set K which is minimal for $E_{\hat{u}}$.*

We shall actually give a different proof of this result without using the weak approach, but following the semicontinuity technique established in [17]. Furthermore, we will show that these results are in some sense optimal and that the μ-Hölder continuity of $\hat{u}$, with $\mu > \frac{1}{2}$, is the crucial ingredient for density results. Indeed, we shall devote a section to the proof of the following three counterexamples. Let us remark that in the theorems the condition $\mu = \frac{1}{2}$ is not allowed.

COUNTEREXAMPLE 1.1. – *For every $\beta > 0$ there exists $\hat{u}$ Hölder continuous, with exponent $\frac{1}{2}$ and norm $||\hat{u}||_{\frac{1}{2}}$ arbitrarily small, such that, if K is a minimum of $E_{\hat{u}}$, then we can find a point of $K \cap \partial\Omega$ and a radius $R \leqslant 1$ for which* (1.7) *does not hold.*

COUNTEREXAMPLE 1.2. – *For any open bounded smooth $\Omega \subset \mathbb{R}^N$ and every $\bar{x} \in \partial\Omega$, there exists $\hat{u}$ Hölder continuous with exponent $\frac{1}{2}$ and of class C^∞ on $\partial\Omega \setminus \{\bar{x}\}$, such that every closed minimum K of $E_{\hat{u}}$, containing $\bar{x}$, has zero density in $\bar{x}$.*

We can note that in the above case a closed minimal set always exists because $\hat{u}$ satisfies the hypothesis of the existence Theorem 1.2, since it is C^∞ out of the closed set $\{\bar{x}\}$, which clearly has null $(N-1)$-dimensional measure. Counterexample 1.2 shows that all these conditions, even if they assure the existence of a closed minimum, do not permit to assure the density estimate. Of course $\hat{u}$ cannot be locally C^μ for any $\mu > \frac{1}{2}$, in $\bar{x}$. For the same reasons K has nonzero density in all its other points.

COUNTEREXAMPLE 1.3. – *There exists $\hat{u}$, Hölder continuous with exponent $\frac{1}{2}$, such that $E_{\hat{u}}(K)$ has no minimum in the class of closed subsets of $\overline{\Omega}$.*

Counterexample 1.3 shows that the existence Theorem 1.2 fails if we allow $\mu = \frac{1}{2}$, even if we take $H = \emptyset$ and we assume the regularity conditions globally on $\partial\Omega$.

2. Fine geometric properties of the singular set

In this section we shall introduce some geometric measure theoretic tools, which will play a key role in the sequel. Before introducing the technical definitions, we briefly remark some intuitive geometric facts. The role of the singular set K in this kind of problems relies in allowing the function u to make a jump (inside Ω) or to get free from the constraint to agree with the boundary datum (on $\partial\Omega$). Then, if we take a ball centered on K inside Ω, at a microscale K will split the ball in two almost equal parts between which u makes a jump which is independent on the scale and so it is big with respect to the radius of the ball. On the other side, if we take a ball centered on a point of a part of K which, roughly speaking, runs along the boundary of Ω, at a microscale we can guess that there exists a ball B centered on K, such that the set $\Omega \cap B$ will essentially result to be insulated by K with respect to $\partial\Omega$. Such a description acquires a deeper meaning if one quantifies how low such a microscale should be, so we are led to formulate the following definitions which are going to be settled in a general context.

Before the geometric definitions, we introduce the following property, when u is a L^1_{Loc} function from a set $X \subset \mathbb{R}^N$ in $\mathbb{R}$.

$$\text{For every ball} \quad B \subset \mathbb{R}^N\colon \int_B |\nabla u| \leqslant c|B|^{\frac{2N-1}{2N}} \leqslant \|u\|^*_{2N} R^{\frac{2N-1}{2}}, \tag{WS}$$

where R is the radius of B and ∇u is defined, in the sense of distributions, on $\overset{\circ}{X} \setminus K$ and it is assumed to be extended by zero on the rest of $\mathbb{R}^N$. This is an estimate on ∇u in Morrey Spaces. We are not using the usual notation of Morrey norm and the reader is not required to be acquainted on Morrey spaces, since no result in this area is going to be employed here. Anyway, we are using the notation $\|u\|^*_{2N}$ in order to emphasize that this is a weak case of L^{2N}-summability. When $A \subset \Omega$, K is a minimum of E and $u = u(K)$, (WS) follows from (1.5) and $\|u\|^*_{2N}$ can be easily estimated (see [21]).

Let $A \subset \mathbb{R}^N$, $D \subset A$ and let K be a closed subset of A. The following discussion applies to the present context under the choice $A = \overline{\Omega}$ and $D = \partial\Omega$.

If B is a given ball contained in A, of radius R and ε is any given positive number, then we say that B is ε-*split* by K, if there exists a function $u : B \to \mathbb{R}$, which satisfies (WS) on $B \setminus K$, with $\|u\|^*_{2N} = 1$ and such that B does not contain any subset $\tilde{B}$ for which

$$|\tilde{B}| \geqslant \left(\tfrac{1}{2} + \varepsilon\right)|B| \tag{2.8}$$

and

$$\operatorname*{\mathit{osc}}_{\tilde{B}} u \leqslant \varepsilon^{-1} R^{1/2}, \tag{2.9}$$

where $\mathit{osc}_{\tilde{B}}\, u = \sup_{\tilde{B}} u - \inf_{\tilde{B}} u$. We will refer to any subset $\tilde{B}$ satisfying (2.8) and (2.9) as *region with ordinary oscillation*. In other words, for every u such that $\|u\|^*_{2N} = 1$, we can find a subset with ordinary oscillation $\tilde{B}_u$ in a ball B if and only if B is not ε-*split*. Moreover, condition (WS) allows to show (see the next section) that u is locally Hölder continuous out of K and so, on every ε-*split* ball the trace of K must be remarkable, if ε is sufficiently small with respect to $(\|u\|^*_{2N})^{-1}$. Let B be any ball, not necessarily contained in A, with radius R. By denoting with ρ the supremum among the radii of the ε-*split* balls contained in $B \cap A$, then we shall call the ratio $\nu_b = \rho/R$ *bisection factor*. It represents the scale transition needed to reach a ε-*split* ball in B; obviously $\nu_b = \nu_b(B, K, \varepsilon)$.

DEFINITION 2.1. – *We shall say that K satisfies the Bisection property when for every $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that, for every ball B centered on K, with radius $R \leqslant 1$, the lower estimate $\nu_b(B, K, \varepsilon) \geqslant \alpha(\varepsilon)$ holds.*

Let B be a ball centered on D, with radius R, and π_B be a disk of B. Given $\zeta \in \pi_B$, let $L(\zeta)$ be the set of the segments contained in the chord C_ζ of B perpendicular in ζ to π_B and contained in $A \cap B \setminus K$. Given $\varepsilon > 0$, we set

$$\pi_B^\varepsilon = \{\zeta \in \pi_B \mid \exists l \in L(\zeta), \mathcal{H}^1(l) > \varepsilon R, l \cap D \neq \emptyset\}.$$

DEFINITION 2.2. – *We define ε-insulated by K the balls such that every disk π_B satisfies the condition*

$$\mathcal{H}^{N-1}(\pi_B^\varepsilon) < \varepsilon \mathcal{H}^{N-1}(\pi_B).$$

The geometric picture corresponding to an ε-*insulated* ball can be visualized by thinking to a ball B centered on $D = \partial\Omega$ in which almost every segment starting from $D \cap B$ meets the set K after a very short length, as shown in Fig. 1.

Also, for a given ball B with radius R, we define the *insulation factor* $\nu_i = \nu_i(B, K, \varepsilon)$ as the ratio between the supremum among the radii of the ε-*insulated* balls contained in B and R.

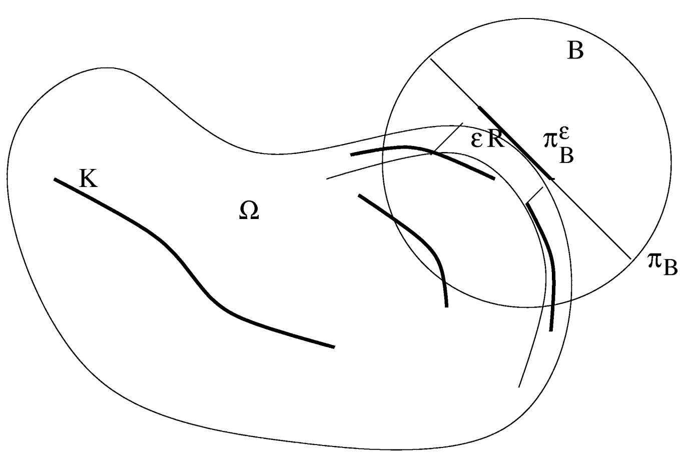

Fig. 1. The set π_B^ε is given by the projection of the segments of lenght εR, orthogonal to π_B, which start on $\partial\Omega$ and do not meet K. Such points of $\partial\Omega$ are not ε-insulated by K.

DEFINITION 2.3. – *We shall say that K satisfies the Insulation property when for every $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that, for every ball B centered on K, with radius $R \leqslant 1$, the lower estimate $\nu_i(B, K, \varepsilon) \geqslant \alpha(\varepsilon)$ holds.*

Trivially such a property will never hold unless $K \subset D$, because otherwise we fix B such that $B \cap D = \emptyset$, which implies $\nu_i = 0$.

As we have remarked above, at a microscale the geometric characterizations of the singular set are quite trivial. The meaning of such two properties consists in a quantification, for given $\varepsilon > 0$, of the scale transition needed to reach an *ε-split* or *ε-insulated* ball inside a ball $B_R(x)$ centered in a point $x \in K$. This corresponds to get a lower bound either on ν_b or ν_i. Such a bound depends on ε, but it must be uniform with respect to $B_R(x)$ for every $x \in K$ and for every $R \leqslant 1$.

As far as only balls contained in Ω are concerned, the bisection property is satisfied for any minimal K of the functional E. Indeed, this is just a matter of inner regularity and it is proved in [21, Theorem 3]. The presence of boundary conditions is inessential in this case. We are led again to the former case when we deal with balls, not completely contained in Ω, in which there are points of K sufficiently inner to Ω. This occurs, for example, when the ball is centered at a point in which K meets $\partial\Omega$ with an appreciable incidence angle. Otherwise, if this is not the case, we expect the insulation property will be satisfied by minimal sets K when the boundary datum $\hat{u}$ is suitably regular. So, in conclusion, for any ball, at least one of the factors ν_b and ν_i should be estimable. This leads to formulate the following *Bisection-Insulation* property which seems to be the natural regularity property expectable for the minima of E.

DEFINITION 2.4. – *We shall say that K satisfies the Bisection-Insulation property when for every $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that, for every ball B centered on K, with radius $R \leqslant 1$, the lower estimate* $\max(\nu_i(B, K, \varepsilon), \nu_b(B, K, \varepsilon)) \geqslant \alpha(\varepsilon)$ *holds.*

DEFINITION 2.5. – *Given a ball B centered on K, with a radius R, we say that the set K is ε-concentrated on B, if the mean density of K on B is bigger than $1 - \varepsilon$, namely*

$$\mathcal{H}^{N-1}(K \cap B) > (1 - \varepsilon) b_{N-1} R^{N-1}.$$

Therefore, given any ball B (not necessarily contained in A), with radius R, we denote by $\nu_c = \nu_c(B, K, \varepsilon)$ the *concentration factor*, that is the ratio between the supremum among the radii of the balls contained in B on which K results *ε-concentrated* and R.

DEFINITION 2.6. – *We shall say that K satisfies the Concentration property when for every $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that, for every ball centered on K, with radius $R \leqslant 1$, it results $\nu_c(B, K, \varepsilon) \geqslant \alpha(\varepsilon)$.*

In these properties the threshold $R \leqslant 1$ is merely conventional, a different choice leads at most to a different value of $\alpha(\varepsilon)$. For this reason, in proving such properties, one can work with balls with suitably small radius. Therefore, we limit ourself to show the regularity properties, assumed in the proof of these estimates, are satisfied for balls B with a suitably small radius. *Bisection* and *Concentration* properties are related. More precisely, in [16] it is shown that for every $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that $\nu_c(\varepsilon, \Omega) \geqslant \nu_b(\varepsilon')$, therefore *Bisection* implies *Concentration*.

Moreover, if D is a smooth closed manifold, one can trivially check that the same relation also holds between ν_c and ν_i on small enough balls, namely on balls with radius less or equal to a suitable constant $\rho(\varepsilon) \leqslant 1$ which depends on ε and on the regularity of D. Then also *Insulation* implies *Concentration*. Combining the two facts we see that for every $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that $\nu_c(\varepsilon, \Omega) \geqslant \sup(\nu_b(\varepsilon'), \rho(\varepsilon)\nu_i(\varepsilon'))$ so *Concentration* follows by *Bisection-Insulation*.

We recall that the concentration property has a meaningful application when we deal with the Hausdorff measure of the limit with respect to the Hausdorff distance. Indeed, with the notation introduced in this section, Propositions 10.10 and 10.14 of [18] can be stated as follows. Let K_n, K be closed subsets of $\mathbb{R}^N$, such that $K_n \to K$ by the Hausdorff distance $\mathbf{d}$ and assume that a Vitali covering $\hat{\mathcal{B}}$ of K consisting of balls, can be found in such a way that for every $\varepsilon > 0$ and for every $B \in \hat{\mathcal{B}}$, $\liminf_n \nu_c(B, K_n, \varepsilon) \geqslant \alpha(\varepsilon) > 0$. By [18, Proposition 10.10], $(K_n)_{n\in\mathbb{N}}$ has a subsequence which satisfies uniformly the concentration property. Consequently, by [18, Proposition 10.14], we have

$$\mathcal{H}^{N-1}(K) \leqslant \liminf_n \mathcal{H}^{N-1}(K_n). \tag{2.10}$$

We conclude the section by stating some further terminology, useful for situations which are never going to happen for small balls in the case of a minimal K, but will nevertheless need to be considered in the forthcoming arguments. When a ball $B \subset \mathbb{R}^N$ contains a too big quantity of K (with a threshold fixed as $c_0 + 1$ times the measure of the corresponding sphere, where c_0 is a suitable constant depending on the regularity of Ω) we shall call it *overfull*, i.e. B is overfull if

$$\mathcal{H}^{N-1}(K \cap B) \geqslant (c_0 + 1)\mathcal{H}^{N-1}(\partial B). \tag{2.11}$$

Analogously as before, given any ball B of radius R, we define the *overfullness factor* ν_0 as the ratio between the supremum among the radii of the overfull balls contained in B and R.

The chief aim of this paper relies in proving the Bisection-Insulation property which, as we have just remarked, implies the Concentration property. This allows, in particular, to deduce Theorem 1.1 and to get Theorem 1.2 through semicontinuity techniques exposed in Section 5. The scale transition estimates, which constitute the Bisection-Insulation property, are instead directly provided by the Excision Method, developed in [21], which will be introduced in Section 4. The differences of construction required by this context, with respect to the inner case discussed in [21], will be treated in Sections 7 and 8. The use of the Excision theorem to provide bisected-insulated balls requires the knowledge of Hölder continuity properties of u out of such balls. The proof of such properties is the aim of the next section.

3. Partial Hölder continuity properties

The aim of this section is to show some partial Hölder continuity properties in order to apply the Excision method in Section 4 which generalizes [21, Theorem 4.6]. Given u and K as in the previous section, $u \in H^1(\Omega \setminus K)$, we are going to settle some local partial Hölder continuity results for u under assumption (WS), by modifying some results of [21]. Therefore, we shall assume (WS) holds throughout this section with a given value of $\|u\|_{2N}^{*}$. In [21], given a ball $B \subset \Omega$, some estimates on a neighborhood V of $K \cap B$, such that u is Hölder continuous on $B \setminus V$, are obtained. The essential difference with the variant which we are going to establish relies in the circumstance that, in our case, $B \not\subset \Omega$ and we shall check the Hölder continuity on $(\hat{B} \setminus V) \cup (B \cap \partial\Omega)$ of a different function u^*, defined as u on $\hat{B} \setminus V$ and as $\hat{u}$ on $B \cap \partial\Omega$. Note that u and $\hat{u}$ may be different on $\partial\Omega \cap K$. This means, roughly speaking, that we shall prove the Hölder continuity of u according with $\hat{u}$. The Hölder continuity of u would not be enough for the application of the abstract approach developed in the last sections. More precisely, we shall prove the following statement.

THEOREM 3.1. – *Given a ball B of suitably small radius R, centered in $\overline{\Omega}$, a closed subset $K \subset \overline{\Omega}$ and ε sufficiently small, there exists $V \subset \hat{B}$ and a positive constant $c(\varepsilon, \Omega)$, depending on ε and on Ω, such that, for every $u : \hat{B} \to \mathbb{R}$ which satisfies* (WS), *one can find a positive constant c_H depending on ε, $||u||_{2N}^{*}$, $||\hat{u}||_{\frac{1}{2}}$ and on Ω, such that*

$$u^* \text{ is Hölder continuous with exponent } \tfrac{1}{2} \text{ and norm } c_H \text{ on } \hat{B} \setminus V, \tag{H1}$$

$$\mathcal{H}^{N-1}(\partial V) \leqslant c(\varepsilon, \Omega)\mathcal{H}^{N-1}(K \cap B), \tag{H2}$$

$$|V| \leqslant c(\varepsilon, \Omega) \sup(\nu_b, \nu_i, \nu_0) R\mathcal{H}^{N-1}(K \cap B), \tag{H3}$$

where ν_i, ν_b, ν_o are computed in (B, K, ε).

The remaining part of this section is devoted to the proof of Theorem 3.1. We shall come back to deal with the main course of the paper in the next section. Analogously to the previous section, we shall make the construction of V by working in a general context. So, let A be a subset of $\mathbb{R}^N$ such that (1.3) holds with $\hat{B} = B \cap A$ and satisfying, for some given constant c_R, the condition

For every x, y in A there exists a ball $B \subset A$ of radius r such that

$$d(x, B), d(y, B) \leqslant c_R r \leqslant c_R^2 d(x, y) \tag{R}$$

and such that the convex hulls of the sets $\{x\} \cup B$ and $\{y\} \cup B$ are contained in A.

The constants c_0 and c_R quantify the regularity of A required in this section. Specifically, the geometry of the set A affects the estimates by involving c_0 and c_R in the determination of the constants. Property (R) obviously holds in the case $A = \Omega \cap B$, when Ω is sufficiently smooth and the radius R of B is suitably small.

Let K be a closed subset of A and let $u : A \to \mathbb{R}$ be continuous on $A \setminus K$. Let $D \subset A$ and $\hat{u} : D \to \mathbb{R}$ such that $\hat{u} \in C^{0,1/2}(D)$. We will assume that $u = \hat{u}$ on $D \setminus K$. Let us define

$$u^* = \begin{cases} u & \text{on } A \setminus D, \\ \hat{u} & \text{on } D. \end{cases}$$

Now we are going to recall or to establish some notation which will be employed in the following. For any given ball $B \subset \mathbb{R}^N$, λB will denote the ball with the same center of B

and the radius multiplied by λ. Moreover $B' = (c_R + 1)B$ and $B'' = (2c_R(c_R + 2) + 1)B$. Let us note that B' and B'' depend on c_R and so on the shape of A. By [21, (6.13)] it follows that if ε is sufficiently small and B is ε-*split*, then the trace of K on B has a measure greater than or equal to $c\mathcal{H}^{N-1}(\partial B)$. This is the first requirement about the smallness of ε. From now on, we shall assume that $\varepsilon > 0$ has been consequently fixed and it will not be always mentioned in the following notation. Furthermore we shall assume to have taken $\varepsilon < 1/\sqrt{c_R + 1}$.

Let U be the union of the balls B' corresponding to all the ε-*split* balls B contained in A. By [21, Lemma 6.14] (the argument is reported in the first part of Lemma 3.1 below) we get that every function u which satisfies (WS) on $A \setminus K$ with $||u||^*_{2N} = 1$ is Hölder continuous with a norm which only depends on c_R and ε, on $A \setminus (U \cup K)$. Furthermore, we can easily estimate $|U|$ by a Vitali type argument. Indeed, by [18, Lemma 7.1] we can take a family of disjoint balls B_i', with B_i ε-*split* with radius r_i such that the balls $5B_i' = 5(c_R + 1)B_i$ cover U. Then, if (WS) holds,

$$\begin{aligned}|U| &\leqslant \sum_i |5(c_R + 1)B_i| \leqslant c{c_R}^N \sum_i |B_i| \\ &\leqslant {c_R}^N c \sum_i r_i \mathcal{H}^{N-1}(\partial B_i) \\ &\leqslant {c_R}^N c \sum_i r_i \mathcal{H}^{N-1}(K \cap B_i) \\ &\leqslant {c_R}^N c (\sup_i r_i) \mathcal{H}^{N-1}(K). \end{aligned} \tag{3.12}$$

If B is any ball, we can apply locally the above construction by considering only the ε-*split* balls which are contained in B'', getting a set U_B. The same proof as before shows that u is Hölder continuous on $(B \cap A) \setminus (U_B \cup K)$ and (3.12) leads to the following estimate.

$$|U_B| \leqslant c\nu_b(B'') R \mathcal{H}^{N-1}(K \cap B''), \tag{3.13}$$

where R denotes the radius of B and $\nu_b(B'')$ is the bisection factor of B'' (under our choice of K and ε). Note that every point of U_B belongs, by construction, to some other ball having a bisection factor greater or equal to $(c_R + 1)^{-1}$.

Let $\mathcal{B}_1$ be the set of the balls B centered in A, such that B'' is *overfull*. We put $V_1 = \bigcup_{B \in \mathcal{B}_1} B''$. If $B \notin \mathcal{B}_1$, (3.13) gives

$$|U_B| \leqslant c(c_0, c_R)\nu_b(B'')|B|. \tag{3.14}$$

Of course in the above estimate we can take $c \geqslant 1$.

Let $\mathcal{B}_2$ be the set of the balls such that $\nu_b(B'') \geqslant \frac{\varepsilon^2}{2c_{(3.14)}}$. If $B \notin \mathcal{B}_1 \cup \mathcal{B}_2$, then (3.14) implies

$$|U_B| \leqslant \frac{\varepsilon^2}{2}|B|. \tag{3.15}$$

We set $V_2 = \bigcup_{B \in \mathcal{B}_2} B''$. Note that if $x \notin V_2$ and B is any ball contained in A such that $x \in B'$, B cannot be ε-*split*. Indeed, otherwise, setting $B_1 = (2c_R(c_R + 2) + 1)^{-1}B'$, B_1

must belong to $\mathcal{B}_2$ since $\nu_b(B_1'') = \nu_b(B') \geqslant \frac{1}{c_R+1} \geqslant \varepsilon^2 \geqslant \frac{\varepsilon^2}{2c_{(3.14)}}$. Then $x \in B' = B_1'' \subset V_2$, a contradiction. Therefore, $U \subset V_2$ and, if B is any ball contained in A, then $U_B \subset V_2$. Finally, we call $\mathcal{B}_3$ the set of the ε-*insulated* balls and we set $V_3 = \bigcup_{B \in \mathcal{B}_3} B$. Now we call $\mathcal{B}$ the set of the balls B'' obtained for $B \in \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ and we take, by Vitali Covering Lemma (see [18, Lemma 7.14]) a set of disjoint balls $\tilde{\mathcal{B}} \subset \mathcal{B}$ such that, setting

$$V = K \cup \bigcup_{B \in \tilde{\mathcal{B}}} (5B \cap A), \tag{3.16}$$

we have

$$(V_1 \cup V_2 \cup V_3) \cap A \subset V. \tag{3.17}$$

We shall show that V satisfies the properties required in Theorem 3.1. The following lemma shows (H1).

LEMMA 3.1. – *If u satisfies* (WS), *then u^* is Hölder continuous on $A \setminus V$, with exponent $\frac{1}{2}$ and norm $c_H = c_H(\varepsilon, c_R, ||\hat{u}||_{\frac{1}{2}}, ||u||_{2N}^*)$.*

Proof. – Of course, we can normalize u and assume that $||u||_{2N}^* = 1$. Let x and $y \in (A \setminus V) \cup D$. If x and $y \in A \setminus V$ we can use the same proof of Lemma 6.14 in [21], which we briefly recall, in order to compute c_H. Let us take a ball B as in (R), since $x \in B'$ and $x \notin V_2$, as we have already observed, B cannot be ε-*split*. So B must contain a region with ordinary oscillation $\tilde{B}$, for which (2.8) and (2.9) hold.

For a given ε, we take $\lambda < 1$ and consider a sequence of balls B_n given by the sets $\{\lambda^n \xi + (1 - \lambda^n)x\}$, for $\xi \in B$. Since, for every natural n, $x \in B_n'$ and $x \notin V_2$, then there exists $\tilde{B}_n \subset B_n'$ for which (2.8) and (2.9) hold. If we take λ close enough to 1, for every n we get by (2.8), $\tilde{B}_{n-1} \cap \tilde{B}_n \neq \emptyset$, so we can take $z_n \in \tilde{B}_{n-1} \cap \tilde{B}_n$. Thus we shall assume $\lambda = \lambda(\varepsilon)$ be fixed as above and we shall regard it as a function of ε. Note that the sequence z_n converges to x. Let z be a given point of $\tilde{B}$. Then, by (2.9), (R) and by the triangular inequality, since x is not a singular point of u, we obtain

$$\left|u(x) - u(z)\right| \leqslant c_* d(x, y)^{1/2},$$

with $c_* = c_R^{1/2} \varepsilon^{-1} \sum_{i=0}^{+\infty} (\lambda^{1/2})^i = c_R^{1/2} \varepsilon^{-1} (1 - \lambda^{1/2}(\varepsilon))^{-1}$. By applying the same argument to y instead of x and from the triangular inequality, we get the desired estimate.

If x and $y \in D$, then we just have to use the Hölder continuity of $\hat{u}$.

Finally we consider the case $x \in D$ and $y \in A \setminus V$ and we take the ball B centered in x, with radius $r = 2d(x, y)$. Since $y \notin V_1 \cup V_2$, then $B \notin \mathcal{B}_1 \cup \mathcal{B}_2$ and (3.15) holds. Moreover $y \notin U_B$ because $U_B \subset V_2$. Furthermore, since $x \notin V_3$ and so $B \notin \mathcal{B}_3$, we can fix a disk π_B such that $\mathcal{H}^{N-1}(\pi_B^\varepsilon) \geqslant \varepsilon \mathcal{H}^{N-1}(\pi_B)$. We consider the set P_1 of the points $\zeta \in \pi_B^\varepsilon$ such that $\mathcal{H}^1(C_\zeta \cap U_B) \geqslant \varepsilon r$. Since $|U_B| \leqslant \frac{\varepsilon^2}{2}|B|$, we deduce that $\mathcal{H}^{N-1}(P_1) \leqslant \frac{\varepsilon}{2}\mathcal{H}^{N-1}(\pi_B)$. By (WS) we have that

$$\int_{\pi_B^\varepsilon \setminus P_1} \left(\int_{C_\zeta} |\nabla u| \right) \leqslant \int_B |\nabla u| \leqslant r^{N-\frac{1}{2}}.$$

So, we can find $\zeta \in \pi_B^\varepsilon \setminus P_1$ such that $\int_{C_\zeta} |\nabla u| \leqslant \varepsilon^{-1}\sqrt{r}$. Since $\zeta \in \pi_B^\varepsilon$, we can find $l \in L(\zeta)$ satisfying $\mathcal{H}^1(l) \geqslant \varepsilon r$ and, since $\zeta \notin P_1$, we deduce that $l \not\subset U_B$. Therefore, by taking $x' \in l \cap D$ and $z \in l \setminus U_B$, we obtain:

$$\begin{aligned}
\left|u^*(x) - u^*(y)\right| &\leqslant \left|\hat{u}(x) - \hat{u}(x')\right| + \left|u^*(x') - u^*(z)\right| + \left|u^*(z) - u^*(y)\right| \\
&\leqslant \|\hat{u}\|_{\frac{1}{2}} d(x, x')^{1/2} + \int_l |\nabla u| + 2c_* d(x, y)^{1/2} \\
&\leqslant \left(\|\hat{u}\|_{\frac{1}{2}} + 2\varepsilon^{-1} + 2c_*\right) d(x, y)^{1/2},
\end{aligned}$$

which gives the estimate

$$c_H \leqslant \|u\|_{2N}^* \left(\|\hat{u}\|_{\frac{1}{2}} + 2\varepsilon^{-1} + 2c_R^{1/2}\varepsilon^{-1}\left(1 - \lambda^{1/2}(\varepsilon)\right)^{-1}\right). \qquad \square$$

If $B \in \mathcal{B}_1 \cup \mathcal{B}_2$, the inequality $\mathcal{H}^{N-1}(K \cap B'') \geqslant c\varepsilon^2 \mathcal{H}^{N-1}(\partial B)$ trivially holds. If we also assume that D is a smooth $(N-1)$-dimensional manifold (as in Theorem 3.1), then the same property will also satisfied by any $B \in \mathcal{B}_3$ and so by any $B \in \mathcal{B}$.

In order to prove (H2) and (H3) we are going to estimate the set V. Let d be the supremum among the radii of $B \in \tilde{\mathcal{B}}$, then the following estimates hold.

$$|V| \leqslant \sum_{B \in \tilde{\mathcal{B}}} |5B| \leqslant cd \sum_{B \in \tilde{\mathcal{B}}} \mathcal{H}^{N-1}(\partial B) \leqslant cd\varepsilon^{-2}\mathcal{H}^{N-1}(K) \tag{3.18}$$

and

$$\mathcal{H}^{N-1}(\partial V) \leqslant c\varepsilon^{-2}\mathcal{H}^{N-1}(K). \tag{3.19}$$

Theorem 3.1 follows by applying the above construction to the case $A = \overline{\Omega} \cap B_r(\bar{x})$ and $D = \partial\Omega \cap B_r(\bar{x})$, where r is small enough for a given smooth Ω. For ε small enough we consider the set V as defined above. By Lemma 3.1 we can claim that (H1) holds with a suitable c_H, while Eqs. (3.18) and (3.19) allow us to get (H2) and (H3) with a suitable $c(\varepsilon, \Omega)$.

4. Boundary excision method and density estimates

The inner estimates of [21] are based on the fact that, under certain circumstances, one can modify a closed set K on a ball $B \subset \Omega$ in such a way to make E decrease. More precisely, given a ball $B \subset \Omega$, with suitably small radius $R \leqslant 1$, centered at a point $\bar{x}$ of the discontinuity set K, the excision method works if *u is Hölder continuous out of a thin neighborhood V of $K \cap B$*, which means that conditions (V1), (V2), (V3) in [21], which we are going to reformulate, hold with a suitably small value of the constant α. In this paper we extend the approach to cover the case $B \not\subset \Omega$.

The new assumptions must be consistent with the former ones, when $\partial\Omega \cap B = \emptyset$. If s is any given positive number, in this section and in the last one $B(s)$ will stand for $B_s(\bar{x})$. We shall set $\sigma = \mathcal{H}^{N-1}(B(R/2) \cap K)$. Since we shall allow the ball B to be not entirely contained in Ω, the lower bound on the density of K will depend also on the regularity properties of $\partial\Omega$. We shall require u to be Hölder continuous out of $V \subset \Omega$ and to be consistent with the boundary datum $\hat{u}$. So, in order to reformulate (V1) we introduce, according to the notation in the previous section, the function u^* defined as follows

$$u^* = \begin{cases} u & \text{on } (B \setminus V) \cap \Omega, \\ \hat{u} & \text{on } B \cap \partial\Omega. \end{cases}$$

We shall assume that the set V enjoys the following properties.

$$u^* \text{ is Hölder continuous with exponent } \tfrac{1}{2} \text{ and norm } c_H, \tag{V1}$$

$$\mathcal{H}^{N-1}(\partial V) \leqslant c_2\sigma, \tag{V2}$$

$$|V| \leqslant \alpha R\sigma. \tag{V3}$$

$$\begin{gathered} \text{There exists } \mu > \tfrac{1}{2} \text{ such that } \hat{u} = u^* \text{ is Hölder continuous} \\ \text{with exponent } \mu \text{ and norm } c_\mu, \text{ on } \partial\Omega \cap B. \end{gathered} \tag{V4}$$

We remark that (V4) is meaningless when $\partial\Omega \cap B = \emptyset$, this is the reason for it does not appear in the inner version studied in [21], while (V1) can be restated as in [21] when $\partial\Omega \cap B = \emptyset$. Furthermore, it is worth to notice that the new assumptions (V1) and (V4) are not merely technical devices employed to deal with the Dirichlet problem. They represent the key regularity conditions underlying the present approach, as we shall show through suitable counterexamples.

In the last section we shall prove the following Excision Theorem, which can be regarded as an extension of the inner version proved in [21, Theorem 4.6].

THEOREM 4.1 (Excision). – *Let c_H, c_2, c_μ, μ be given. Then there exists $\bar{\alpha} > 0$ such that, if K is a closed subset of $\overline{\Omega}$ such that* (V1), (V2), (V3) *and* (V4) *hold with $\alpha \leqslant \bar{\alpha}$, then the trace of K on $\hat{B}(R)$ can be modified in order to obtain a new set K' satisfying $E_{\hat{u}}(K') < E_{\hat{u}}(K) - \frac{1}{2}\sigma$.*

Let now K and $B = B_R(x)$ be given with $x \in K$ and $R \leqslant 1$ and let us suppose that

$$E(K) = E_{\hat{u}}(K) < \inf E + \frac{1}{2}\sigma. \tag{4.20}$$

We assume that $\hat{u}$ is Hölder continuous with exponent μ and norm c_μ on $\partial\Omega \cap B$; in particular, since $R \leqslant 1$, it results to be Hölder continuous with exponent $\frac{1}{2}$. Furthermore, let us assume that $u = u(K)$ satisfies (WS), so we are in a position to apply Theorem 3.1. Under the additional assumption

$$\mathcal{H}^{N-1}(K \cap B) \leqslant c\sigma \tag{4.21}$$

which states that $\mathcal{H}^{N-1}(K \cap B)$ and σ are of the same order of magnitude, (V1), (V2) and (V3) can be immediately deduced from (H1), (H2) and (H3) of Theorem 3.1, by taking $\alpha = c(\varepsilon, \Omega)\max(\nu_b, \nu_i, \nu_0)$. (V4) follows by the assumptions made on $\hat{u}$. We call $\alpha(\varepsilon)$ the value of the threshold $\bar{\alpha}$ (multiplied by $c(\varepsilon, \Omega)^{-1}$) established in Theorem 4.1 corresponding to the constants $c_H(\varepsilon)$, $c_2(\varepsilon)$, c_μ and μ founded by Theorem 3.1, which in turn gives them as function of ε. If $\max(\nu_b, \nu_i, \nu_0) \leqslant \alpha(\varepsilon)$ and therefore (V3) holds with $\alpha \leqslant \alpha(\varepsilon)$, then by the excision theorem we get the existence of a set K' such that $E(K') < E(K) - \frac{1}{2}\sigma$, which, by (4.20), leads to a contradiction, which shows that

$$\max(\nu_b, \nu_i, \nu_0) \geqslant \alpha(\varepsilon). \tag{4.22}$$

This is the same to claim that there exists a ball $B \in \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ whose diameter is greater or equal to $c\alpha(\varepsilon)R$. If such a ball $B \in \mathcal{B}_1$, then we can modify E by adding $\partial \hat{B}''$ to K and putting $u = \text{const.}$ inside B''; by (2.11) this modification gives rise to a gain

$$\begin{aligned} E(K) - E(K') &\geqslant (c_0 + 1)\mathcal{H}^{N-1}(\partial B'') - \mathcal{H}^{N-1}(\partial \hat{B}'') - |\hat{B}''| \\ &\geqslant \mathcal{H}^{N-1}(\partial B'') \geqslant c\big(\alpha(\varepsilon)R\big)^{N-1}. \end{aligned} \tag{4.23}$$

If we also assume that

$$E(K) < \inf E + c_{(4.23)}\big(\alpha(\varepsilon)R\big)^{N-1}, \tag{4.24}$$

we get an evident contradiction. Therefore, we deduce that B must belong to $\mathcal{B}_2 \cup \mathcal{B}_3$ and this is equivalent to say that

$$\max(\nu_b, \nu_i) \geqslant \alpha(\varepsilon). \tag{4.25}$$

Since, if K is a minimum, then (4.20), (4.24) are trivially satisfied, while condition (WS) follows from (1.5), we can claim the following.

THEOREM 4.2 (Bisection-Insulation property). – *Let Ω be an open smooth bounded subset of $\mathbb{R}^N$. If $\hat{u} \in C^{0,\mu}(\partial\Omega)$ with $\mu > \frac{1}{2}$ and K is a minimum for $E_{\hat{u}}$, then the Bisection-Insulation property holds.*

By virtue of the results in Section 2, the following corollary is then trivially implied.

COROLLARY 4.3 (Uniform Concentration property). – *Let Ω be an open smooth bounded subset of $\mathbb{R}^N$. If $\hat{u} \in C^{0,\mu}(\partial\Omega)$ with $\mu > \frac{1}{2}$ and K is a minimum for $E_{\hat{u}}$, then the Uniform Concentration property holds.*

It is worth to notice that, since the concentration property is clearly stronger than the density property, Theorem 1.1 is in turn a corollary of the last one. On the other hand, as far as Theorem 1.1 is concerned, the extra assumption (4.21) is not a restriction since it can be forced by choosing a suitable scale as in [21, Section 7]. Finally, whence Theorem 1.1 is proved, (4.21) follows as a trivial consequence in view of (1.6) so the previous results are established in the case of a minimal K, without any restriction.

5. Semicontinuity properties and existence of a minimum

In [17] some lower semicontinuity properties of E with respect to the Hausdorff distance, have been shown. More precisely, a meaningful decomposition of the discontinuity set was there introduced, namely $K = \tilde{K} \cup K^*$, where $\tilde{K} = K \setminus K^*$ and K^*, identified as the *noise* part of K, is defined as the set of points where K has mean density less than a fixed small threshold β^*, on some scale less or equal to one. By using Theorem 4.1, instead of the inner version of the excision theorem, and by taking advantage of the results stated in [17, Section 8], which allow to force (WS), we can repeat the same arguments of [17] and show that, if $(K_n)_{n\in\mathbb{N}}$ is a minimizing sequence for E, then $\mathcal{H}^{N-1}(K_n^*)\to 0$. Note that the closure of $\mathcal{M}\cup K$ stated in [17, Lemma 3.3] is now replaced by the same property of $\mathcal{M}\cup K\cup H$, where H is as in Theorem 1.2. The assumption that H is a closed negligible set makes the two facts applicable in the same way (see [17, Remark 8.1]).

If K is the Hausdorff limit of $(\tilde{K}_n)_{n\in\mathbb{N}}$, we can find a Vitali covering $\mathcal{B}$ of $K\backslash H$, consisting of balls on which $\hat{u}$ has the smoothness required in the previous section, namely $\hat{u}\in C^{0,\mu}$ with $\mu>\frac{1}{2}$. By neglecting subsets of K with arbitrarily small measure, we can also assume that μ is the same for every ball in $\mathcal{B}$ and $||\hat{u}||_\mu$ is uniformly bounded. Then an application of the results in the previous section, instead of their inner version considered in [17], allow us to find for every $\varepsilon>0$ a threshold $\alpha(\varepsilon)$ such that for every $B\in\mathcal{B}$ we have $\nu_c(B,\tilde{K}_n,\varepsilon)\geqslant\alpha(\varepsilon)$ definitively. We can apply the results of Section 2, Eq. (2.10) and give the following theorem.

THEOREM 5.1. – *Let $(K_n)_{n\in\mathbb{N}}$ be a minimizing sequence for E. Then for all n the decomposition $K_n=\tilde{K}_n\cup K_n^*$ holds with $\mathcal{H}^{N-1}(K_n^*)\to 0$ and, if K is the limit by the Hausdorff distance of a subsequence $\tilde{K}_{i_n}$ then*

$$\mathcal{H}^{N-1}(K)\leqslant\liminf_n\mathcal{H}^{N-1}(\tilde{K}_{i_n}).$$

Thus, as a simple corollary, we deduce the existence of a closed minimum K by the compactness of the set of the closed subsets of $\overline{\Omega}$ and by the semicontinuity of J with respect to the weak convergence (see [17, Section 2] and [18, Lemma 13.6]).

Finally we can also get a boundary version of the Density Theorem for nonminimal sets K [17, Section 5], which does not require the ball to be contained in Ω.

THEOREM 5.2. – *Let Ω be an open smooth bounded subset of $\mathbb{R}^N$ and let $\hat{u}:\partial\Omega\to\mathbb{R}$ be Hölder continuous with exponent $\mu>\frac{1}{2}$ and norm c_μ. For any given positive constant $\alpha<1$, there exist two positive constants β and c^* (which depend on α, on the dimension N, on μ, on c_μ and on the regularity of Ω) such that, for every K closed subset of $\overline{\Omega}$, for every $R\leqslant 1$ and for every $x\in K$, either one of the two following conditions is always satisfied*:

(i) $\mathcal{H}^{N-1}(K\cap B(x,R))\geqslant\beta R^{N-1}$,

(ii) $\mathcal{H}^{N-1}(K\cap B(x,\alpha R))<c^*(E_{\hat{u}}(K)-m_{\hat{u}})$,

where $m_{\hat{u}}=\inf E_{\hat{u}}(K)$.

6. Counterexamples and optimality of the conditions

The counterexamples in this section will be given for the simple functional $J(u,K)=\int_{\Omega\backslash K}|\nabla u|^2$, for which the proofs of the existence theorems become easier. More precisely, the existence theory and the estimates developed in the paper can be referred more in general to quasiminimal sets, see [4] for the definition, while the counterexamples in this section will be given even under a full minimality assumption.

Therefore, we show that, even in this simpler case, the Hölder continuity assumptions on $\hat{u}$ cannot be weakened. Let Ω be a smooth domain contained in $\mathbb{R}^N$.

LEMMA 6.1. – *There exists $c > 0$ such that, if (u, K) is a minimum for E and the trace of u on $\partial\Omega$ is Hölder continuous with exponent $\frac{1}{2}$ and norm bigger than c, then $K \neq \emptyset$.*

Proof. – If $K = \emptyset$ and (u, K) is a minimum, by Morrey Hölder continuity theorem, see [21, Theorem 6.1], u is Hölder continuous with exponent $\frac{1}{2}$ and norm smaller than $c\|u\|^*_{2N}$, which is in turn estimated by (1.4). □

Now we state a weaker version of Counterexample 1.1, which shows that a bound on $\hat{u}$ in the $C^{0,1/2}$ norm is not enough to determine a constant $\beta > 0$ such that (1.7) holds.

COUNTEREXAMPLE 6.1. – *There exists a constant $M > 0$ such that, for every $\beta > 0$ there exists $\hat{u}$ satisfying $\|\hat{u}\|_{\frac{1}{2}} \leqslant M$ and, such that, if (u, K) is a minimum of $E_{\hat{u}}$, then we can find a point of K and a radius $R \leqslant 1$ for which* (1.7) *does not hold.*

Proof. – Let $\bar{x} \in \partial\Omega$. We fix a function $v \in C_0^\infty$ with $\|v\|_{\frac{1}{2}} \geqslant c$, spherically symmetric around $\bar{x}$. For all $\lambda > 0$, we consider the scaled function v_λ, such that $\|v_\lambda\|_{\frac{1}{2}}$ is constant with respect to λ, i.e. $v_\lambda(x) = \lambda^{1/2} v(\lambda^{-1}x)$.

For every λ we consider the minimum problem (P_λ) with boundary value $\hat{u}_\lambda = v_{\lambda_{|\partial\Omega}}$. If K_λ is a solution of (P_λ), then we have

$$\mathcal{H}^{N-1}(K_\lambda) \leqslant \min_{u|_{\partial\Omega\setminus K} = \hat{u}_\lambda} E(u, K) \leqslant E\big(0, \overline{S_\lambda}\big) \leqslant c\,\lambda^{N-1}, \tag{6.26}$$

where $S_\lambda = \{x \in \partial\Omega \mid \hat{u}_\lambda(x) \neq 0\}$. By Lemma 6.1 we have that $K_\lambda \neq \emptyset$. Let $x \in K_\lambda$ and $\rho > 0$. If the density estimate (1.7), for some positive β and x, holds and B_ρ is a ball centered in x, we have that $\mathcal{H}^{N-1}(K_\lambda) \geqslant \beta\rho^{N-1}$, which combined with (6.26) gives $\rho \leqslant c\lambda$. So, when λ is suitably small, (1.7) cannot hold for every $\rho \leqslant 1$. Note that, as $\lambda \to 0$ we have that $\|\hat{u}_\lambda\|_\mu$, for all $\mu > \frac{1}{2}$, goes to infinity. □

Remark 6.1. – Since for all $\lambda > 0$ we have $\hat{u}_\lambda \in C^\infty$, there exists a density constant $\beta_\lambda > 0$. For all $\mu > \frac{1}{2}$, $\|\hat{u}_\lambda\|_\mu$ is not bounded and we have that $\beta_\lambda \to 0$. However, for every $x \notin S_\lambda$, a constant $\beta > 0$ such that (1.7) holds for $\rho \leqslant d(x, S_\lambda)$ can be found thanks to the local character of (V4). Then the above arguments still imply $\rho \leqslant c\lambda$. By the arbitrariness of $x \in K_\lambda$, we deduce that K_λ is contained in a $(c\lambda)$-neighborhood of S_λ.

In the previous example we have kept $\|\hat{u}\|_{\frac{1}{2}}$ bounded, but we have fixed it large enough in order to deduce $K_\lambda \neq \emptyset$ as a consequence of Lemma 6.1. One can still wonder if a small enough bound on $\|\hat{u}_\lambda\|_{\frac{1}{2}}$ could replace (V4). To answer (negatively) such a question, we shall now establish a more technical variant of the above construction which will allow us to take $\|\hat{u}_\lambda\|_{\frac{1}{2}}$ as small as we want. In order to avoid useless details, we shall now work in two dimensions and we take Ω in such a way that its boundary contains a segment.

LEMMA 6.2. – *There exists a Hölder continuous function $\hat{u} : \partial\Omega \to \mathbb{R}$, with exponent $\frac{1}{2}$, which is not the trace of any function in $H^1(\Omega)$. The measure of the support of such a function can be taken arbitrarily small.*

We note that for such a function, the Hölder norm of exponent $\frac{1}{2}$ can be also taken arbitrarily small, by multiplying it for a small constant.

Proof. – We shall take a segment $S \subset \partial\Omega$ on which we shall define a Hölder continuous function f_∞ with exponent $\frac{1}{2}$ such that, for every continuous extension u, on a neighborhood W of S, which is C^1 out of the segment, it results

$$\int_W |\nabla u|^2 = +\infty. \tag{6.27}$$

Such a function will be taken to be zero at the endpoints of S and so, extended by zero on the rest of $\partial\Omega$, will give the desired function $\hat{u}$. The segment S can be taken of arbitrarily small length and it becomes $[0,2]$, under a suitable frame. So the function will be defined on $[0,1]$ with $f_\infty(0)=0$ and extended by reflection on $[0,2]$, i.e. we shall take $f_\infty(t)=f_\infty(2-t)$ for $t\in[1,2]$. This function will be obtained as the uniform limit of the sequence of mappings we are going to define. For every positive integer n let $\mathcal{S}_n$ denote the set of the closed subintervals obtained by dividing $[0,1]$ in 4^n equal parts. Let f_0 be the identity function defined on $[0,1]$, then $||f_0||_{\frac{1}{2}}=1$. Let f_1 be the function which is linear on each interval in $\mathcal{S}_1$ and takes the values $f_1(0)=0$, $f_1(\frac{1}{4})=\frac{1}{2}$, $f_1(\frac{1}{2})=1$, $f_1(\frac{3}{4})=\frac{1}{2}$ and $f_1(1)=1$. Next we pass to a function f_2 obtained by replacing the linear pieces of f_1 on the intervals in $\mathcal{S}_1$, with scaled copies of f_1 in such a way to keep the previous values in the points 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ and 1. So, on $[0,\frac{1}{4}]$ we have $f_2(x)=\frac{1}{2}f_1(4x)$, on $[\frac{1}{4},\frac{1}{2}]$ $f_2(x)=\frac{1}{2}+\frac{1}{2}f_1(4(x-\frac{1}{4}))$ and so on.

Then we iterate the construction by substituting the linear pieces of f_2 on the intervals in $\mathcal{S}_2$ with a suitable scaled copy of f_1 and so on. In such a way we get a sequence of functions $(f_n)_{n\in\mathbb{N}}$ with the following properties:

(a) For every $n\in\mathbb{N}$: $|f_n'|=2^n$ a.e.
(b) For every $n\in\mathbb{N}$: $\|f_n-f_{n+1}\|_{L^\infty}=\frac{1}{2^{n+1}}$.
(c) For every $n\in\mathbb{N}$: on every $I\in\mathcal{S}_n$, f_n is linear and has an oscillation equal to 2^{-n}.
(d) If $n\geqslant m$ and $I\in\mathcal{S}_m$, then $f_n(I)=f_m(I)$.
(e) If $I\in\mathcal{S}_n$, then $|f_{n+1}'|=2^{n+1}$ on the first half of I.

Now, we claim that the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $C^{0,1/2}$. Indeed, given $x,y\in[0,1]$, there exists $\bar{n}$ such that $1/4^{\bar{n}+1}\leqslant d(x,y)\leqslant 1/4^{\bar{n}}$. So x and y belong at most to two contiguous subintervals in $\mathcal{S}_{\bar{n}}$. So by (c), if $n=\bar{n}$:

$$\left|f_n(x)-f_n(y)\right|\leqslant\frac{1}{2^{\bar{n}-1}}=\sqrt{\frac{1}{4^{\bar{n}-1}}}\leqslant 4\sqrt{d(x,y)}. \tag{6.28}$$

The same estimate also holds for $n\geqslant\bar{n}$ by (d). If $n\leqslant\bar{n}$, by (a) we have

$$\frac{|f_n(x)-f_n(y)|}{\sqrt{d(x,y)}}\leqslant\frac{|f_n(x)-f_n(y)|}{d(x,y)}\sqrt{d(x,y)}\leqslant\sup|f_n'|\sqrt{d(x,y)}\leqslant\frac{1}{2^{\bar{n}-n}}\leqslant 1.$$

So for every $n\in\mathbb{N}$: $\|f_n\|_{\frac{1}{2}}\leqslant 4$. By (b) $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the uniform norm. Therefore $(f_n)_{n\in\mathbb{N}}$ converges to a function f_∞ in the uniform metric. The function

f_∞ is Hölder continuous too, since it is the limit of functions which are bounded in $C^{0,1/2}$.

For a fixed $n \in \mathbb{N}$ let $I \in \mathcal{S}_n$ and let x and y be two points on the first half of I, s.t. $|x-y| = \frac{1}{4^{n+1}}$. The increment of f_{n+1} can be easily evaluated by (e) and this is clearly left invariant if we increase the exponent. So we have in the limit

$$\left|f_\infty(x) - f_\infty(y)\right| = \frac{1}{2^{n+1}}. \tag{6.29}$$

Now, let u be any continuous extension of f_∞ such that it is C^1 in a neighborhood of S. We consider an equilateral triangle whose sides have length $\frac{1}{4^{n+1}}$, and one of them, denoted by L, is contained in the first half of a fixed $I \in \mathcal{S}_n$. Let P be the union of the remaining two sides of the triangle, then, by (6.29), the increment of u on L is $\frac{1}{2^{n+1}}$, so

$$\frac{1}{2^{n+1}} \leqslant \int_P |\nabla u| \leqslant \left(\int_P |\nabla u|^2\right)^{1/2} |P|^{1/2} \leqslant \left(\int_P |\nabla u|^2\right)^{1/2} \sqrt{2}\frac{1}{2^{n+1}},$$

therefore

$$\frac{1}{2} \leqslant \int_P |\nabla u|^2.$$

By allowing all the position of L on the first half of I, we can take the triangle contained in a rectangle R_I, which has a side equal to I and the other one of length $\frac{\sqrt{3}}{2\cdot 4^{n+1}}$. By integrating the previous inequality we have

$$2\int_{R_I} |\nabla u|^2 \geqslant \frac{1}{4^{n+1}}\frac{1}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4^{n+2}}. \tag{6.30}$$

Let now R be a rectangle with a side equal to S and the other one of length $\frac{\sqrt{3}}{2\cdot 4^{n+1}}$. Since R contains 4^n rectangles R_I for every $I \in \mathcal{S}_n$, we deduce from (6.30),

$$\int_R |\nabla u|^2 \geqslant \frac{\sqrt{3}}{32}.$$

This shows that $|\nabla u|^2$ cannot be integrable because of the absolute continuity of the Lebesgue integral and because the Lebesgue measure of R tends to zero as $n \to \infty$. $\square$

Proof of Counterexample 1.1. – Let $\hat{u}$ be the function introduced in the proof of the above lemma as an extension of the function f_∞ and, let $\hat{u}_n$ be the analogous extension of the functions f_n. For every $n \in \mathbb{N}$, let u_n be the solution of the Dirichlet problem

$$\begin{cases} -\Delta u_n = 0 & \text{in } \Omega, \\ u_n = \hat{u}_n & \text{on } \partial\Omega. \end{cases}$$

We can observe that $(u_n)_{n\in\mathbb{N}}$ is not bounded in $H^1(\Omega)$; if it were bounded it would contain a weakly converging subsequence to a function $u \in H^1(\Omega)$. Such a function would be an extention of f_∞, which contradicts Lemma 6.2. For every n let $E_n = E_{\hat{u}}$ for $\hat{u} = \hat{u}_n$ and K_n be a closed minimum of the functional E_n; it exists, because $\hat{u}_n$ is a Lipschitz mapping. We claim that $K_n \neq \emptyset$, otherwise:

$$E_n(\emptyset) = J(u_n) \to +\infty$$

while

$$E_n(K_n) = \min E_n \leqslant E_n(S) = \mathcal{H}^{N-1}(S).$$

So $K_n \neq \emptyset$ definitively and

$$\mathcal{H}^{N-1}(K_n) \leqslant E_n(K_n) \leqslant E_n(S) = \mathcal{H}^{N-1}(S). \tag{6.31}$$

Finally, for a fixed radius $\rho > 0$ and for every $\beta > 0$ the density estimate cannot hold since S can be taken of arbitrarily small measure. □

To the aim of proving Counterexample 1.2, we shall now work in arbitrary dimension N and, for the sake of simplicity, we shall assume that $\partial\Omega$ has a flat part S. Let $\varphi : \mathbb{R} \to [0,1]$ a C^∞ function, such that for every $x \leqslant 0$ $\varphi(x) = 0$ and for every $x \geqslant 1$ $\varphi(x) = 1$. We take φ such that $\|\varphi\|_{\frac{1}{2}}$ is big enough to apply Lemma 6.1. Let $\bar{x}$ be a point in the inner part of S on $\partial\Omega$. Let $r \geqslant \varepsilon > 0$ be such that $B_{2r}(\bar{x}) \cap \partial\Omega$ is contained in S. We consider the function $\omega : \partial\Omega \to [0,1]$ defined as $\omega(x) = \sqrt{\varepsilon}\varphi(\varepsilon^{-1}(r + \varepsilon - d(x, \bar{x})))$. We note that $\|\omega\|_{\frac{1}{2}}$ does not depend on ε and r.

LEMMA 6.3. – *Let*

$$c_\omega = \inf E_{\omega},$$

then

$$c_\omega \leqslant c\sqrt{\frac{\varepsilon}{r}}r^{N-1}, \tag{6.32}$$

where c depends on the dimension N.

Proof. – We shall estimate the functional in an admissible pair (v, H). We take H equal to the trace on $\partial\Omega$ of the annulus $B_{r+\sqrt{\varepsilon r}}(\bar{x}) \setminus B_r(\bar{x})$, so

$$\mathcal{H}^{N-1}(H) = cr^{N-2}\sqrt{\varepsilon r}. \tag{6.33}$$

Then we take $v : \Omega \to \mathbb{R}$ such that $v(x) = \sqrt{\varepsilon}\varphi((r\varepsilon)^{-1/2}(r + \sqrt{\varepsilon r} - d(x, \bar{x})))$ and we note that:

$$\int_\Omega |\nabla v|^2 \leqslant c\, r^{N-1} r^{-1} \sqrt{\varepsilon r}$$

which, with (6.33), implies (6.32). □

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ be two decreasing real positive sequences such that

(i) $\varepsilon_n r_n^{-1} \to 0$;

(ii) $r_{n+1} + \sqrt{\varepsilon_{n+1} r_{n+1}} < \frac{1}{2} r_n$.

For every natural n we shall take $\varepsilon_n r_n^{-1}$ small enough. For every $n \in \mathbb{N}$, we denote by $\omega_n : \partial\Omega \to [0,1]$ the function corresponding to ω for $\varepsilon = \varepsilon_n$ and $r = r_n$. Then for every $n \in \mathbb{N}$ we consider the minimum problem with boundary datum $\hat{u}_n = \sum_{j=0}^{n} \omega_j$, we denote by E_n the functional $E_{\hat{u}_n}$ and we set $m_n = \inf_K E_n(K)$. For every $n \in \mathbb{N}$ we denote (v_n, H_n) the pair corresponding to (v, H) of the proof of Lemma 6.32, for $\varepsilon = \varepsilon_n$ and $r = r_n$. Let $i \in \mathbb{N}$. We denote by B_i the ball centered at $\bar{x}$ with radius $r_i + \sqrt{\varepsilon_i r_i}$.

We shall obtain the function $\hat{u}$ in Counterexample 1.2 as the limit of $\hat{u}_n$. To the aim of proving the desired properties, we need to establish several lemmas. We begin by showing some estimates which imply that the minimum level m_i converge.

LEMMA 6.4. – *Let K be a closed subset of $\overline{\Omega}$ such that*

$$E_i(K) - m_i \leqslant N b_N r_{i+1}^{N-1}, \tag{6.34}$$

then

$$E_{i+1}(K \cup H_{i+1}) \leqslant E_i(K) + c\sqrt[4]{\frac{\varepsilon_{i+1}}{r_{i+1}}}\, r_{i+1}^{N-1}. \tag{6.35}$$

Proof. – Let $u = u(K, \hat{u}_i)$. We note that

$$\int_{B_{i+1}} |\nabla u|^2 \leqslant 2\mathcal{H}^{N-1}(\partial B_{r_{i+1}}). \tag{6.36}$$

Indeed if (6.36) does not hold, we can consider the admissible pair (u', K') such that $K' = K \cup (\partial B_{r_{i+1}} \cap \overline{\Omega})$, $u' = \text{const}$ on $B_{r_{i+1}} \cap \Omega$ and $u' = u$ on $\Omega \setminus B_{r_{i+1}}$. We can observe that, since, as $\varepsilon_i r_i^{-1} \to 0$, $\mathcal{H}^{N-1}(\partial B_i \cap \Omega)$ is equal to $\frac{N}{2} b_N r_i^{N-1}$, modulo an infinitesimum of higher order, so

$$\mathcal{H}^{N-1}(\partial B_i \cap \Omega) < N b_N r_i^{N-1}. \tag{6.37}$$

Therefore, by (6.34) and (6.37) we have:

$$E_i(u', K') < E_i(K) + N b_N r_{i+1}^{N-1} - \int_{B_{i+1}} |\nabla u|^2 < E_i(K) - N b_N r_{i+1}^{N-1} \leqslant m_i,$$

which leads to a contradiction and proves (6.36). By (6.36), with similar estimates to those used in the proof of Lemma 6.32, since $(u + v_{i+1}, K \cup H_{i+1})$ is an admissible pair for E_{i+1}, we have:

$$\begin{aligned} E_{i+1}(K \cup H_{i+1}) &\leqslant E_{i+1}(u + v_{i+1}, K \cup H_{i+1}) \\ &\leqslant E_i(K) + \int_\Omega |\nabla v_{i+1}|^2 + \mathcal{H}^{N-1}(H_{i+1}) + 2\sqrt{\int_{B_{i+1}} |\nabla u|^2}\sqrt{\int_\Omega |\nabla v_{i+1}|^2} \\ &\leqslant E_i(K) + c\sqrt{\frac{\varepsilon_{i+1}}{r_{i+1}}}\, r_{i+1}^{N-1} + c\sqrt[4]{\frac{\varepsilon_{i+1}}{r_{i+1}}}\, r_{i+1}^{N-1}. \qquad \square \end{aligned}$$

Now we shall give some simple variants of Lemma 6.4. Let $\hat{u}_\infty = \sum_{i=0}^{\infty} \omega_i$. We denote by E_∞ the functional $E_{\hat{u}_\infty}$ and $m_\infty = \inf_K E_\infty(K)$. Let $H_\infty^i = \bigcup_{j=i}^{\infty} H_j \cup \{\bar{x}\}$.

LEMMA 6.5. – *If K is a closed subset of $\overline{\Omega}$ which satisfies* (6.34), *then*

$$E_\infty\big(K \cup H_\infty^{i+1}\big) \leqslant E_i(K) + c\sqrt[4]{\frac{\varepsilon_{i+1}}{r_{i+1}}} r_{i+1}^{N-1}. \tag{6.38}$$

Proof. – Let $u = u(K, \hat{u}_i)$. We note that the supports H_i of ∇v_i are disjoint, so by Pitagora's theorem

$$\int_\Omega \left| \sum_{j=i}^{\infty} \nabla v_j \right|^2 = \sum_{j=i}^{\infty} \int_\Omega |\nabla v_j|^2 \leqslant c \sum_{j=i}^{\infty} \sqrt{\frac{\varepsilon_j}{r_j}} r_j^{N-1} = c\sqrt{\frac{\varepsilon_i}{r_i}} r_i^{N-1}. \tag{6.39}$$

Finally, by the same computation as in Lemma 6.4 and by (6.39), we have

$$E_\infty\big(K \cup H_\infty^{i+1}\big) \leqslant E_\infty\left(u + \sum_{j=i+1}^{\infty} v_j, K \cup H_\infty^{i+1}\right) \leqslant E_i(u, K) + c\sqrt[4]{\frac{\varepsilon_{i+1}}{r_{i+1}}} r_{i+1}^{N-1}. \qquad \square$$

LEMMA 6.6. – *If K is a closed subset of $\overline{\Omega}$ such that*:

$$E_{i+1}(K) - m_{i+1} \leqslant N b_N r_{i+1}^{N-1}, \tag{6.40}$$

then

$$E_i(K \cup H_{i+1}) \leqslant E_{i+1}(K) + c\sqrt[4]{\frac{\varepsilon_{i+1}}{r_{i+1}}} r_{i+1}^{N-1}. \tag{6.41}$$

Proof. – The proof of this lemma is analogous to the proof of Lemma 6.4, it is only sufficient to estimate the functional E_i in the admissible pair $(u - v_{i+1}, K \cup H_{i+1})$, where $u = u(K, \hat{u}_{i+1})$. $\square$

In the same way of Lemma 6.5 we have, by lemma 6.6, the following lemma:

LEMMA 6.7. – *If K is a closed subset of $\overline{\Omega}$ which satisfies*

$$E_\infty(K) - m_\infty \leqslant N b_N r_{i+1}^{N-1}, \tag{6.42}$$

then

$$E_i\big(K \cup H_\infty^{i+1}\big) \leqslant E_\infty(K) + c\sqrt[4]{\frac{\varepsilon_{i+1}}{r_{i+1}}} r_{i+1}^{N-1}. \tag{6.43}$$

By Lemmas 6.4 and 6.6 we deduce:

COROLLARY 6.1. –

$$|m_{i+1} - m_i| \leqslant c\sqrt[4]{\frac{\varepsilon_{i+1}}{r_{i+1}}} r_{i+1}^{N-1}. \tag{6.44}$$

By Lemmas 6.5 and 6.7:

COROLLARY 6.2. –

$$|m_i - m_\infty| \leqslant c\sqrt[4]{\frac{\varepsilon_{i+1}}{r_{i+1}}}r_{i+1}^{N-1}. \tag{6.45}$$

For every $i \in \mathbb{N}$ we consider the following condition

$$E_i(K) - m_i \leqslant \sqrt[5]{\frac{\varepsilon_{i+1}}{r_{i+1}}}r_{i+1}^{N-1}. \tag{n_i}$$

LEMMA 6.8. – *If K satisfies (n_i), then $K \cup H_i$ verifies (n_{i-1}).*

Proof. – By Lemma 6.6 and Corollary 6.1, we have

$$E_{i-1}(K \cup H_i) \leqslant E_i(K) + c\sqrt[4]{\frac{\varepsilon_i}{r_i}}r_i^{N-1} \leqslant m_i + c\sqrt[4]{\frac{\varepsilon_i}{r_i}}r_i^{N-1} \leqslant m_{i-1} + c\sqrt[4]{\frac{\varepsilon_i}{r_i}}r_i^{N-1}.$$

By taking all the terms ε_i/r_i small enough we deduce (n_{i-1}). □

In the following we shall apply Theorem 5.2, since we shall make use of the density results for nonminimal sets in order to estimate inductively the density of K on $B_{\frac{r_i}{2}}$, if K satisfies (n_i).

LEMMA 6.9. – *For every $i \in \mathbb{N}$, if K satisfies (n_i), we have*:

$$\mathcal{H}^{N-1}(K \cap B_{\frac{r_i}{2}}) \leqslant c\sqrt[5]{\frac{\varepsilon_{i+1}}{r_{i+1}}}r_{i+1}^{N-1}. \tag{6.46}$$

Proof. – If $i > 0$, we have from Lemma 6.8 that $(K \cup H_i)$ verifies (n_{i-1}). So, by induction we can assume the thesis for $i - 1$. Then, by Lemma 6.8, we have:

$$\mathcal{H}^{N-1}(K \cap B_{r_i}) \leqslant \mathcal{H}^{N-1}\big((K \cup H_i) \cap B_{2r_i}\big) \leqslant c\sqrt[5]{\frac{\varepsilon_i}{r_i}}r_i^{N-1} < \beta(2r_i)^{N-1},$$

where β is the density constant of Theorem 5.2. If $i = 0$, the same conclusion follows from Lemma 6.3 which implies

$$\begin{aligned}\mathcal{H}^{N-1}(K \cap B_{r_0}) &\leqslant \mathcal{H}^{N-1}(K) \leqslant E_0(K) \leqslant m_0 + \sqrt[5]{\frac{\varepsilon_0}{r_0}}r_0^{N-1}\\ &= c_{\omega_0} + \sqrt[5]{\frac{\varepsilon_0}{r_0}}r_0^{N-1} \leqslant c\sqrt{\frac{\varepsilon_0}{r_0}}r_0^{N-1} + \sqrt[5]{\frac{\varepsilon_0}{r_0}}r_0^{N-1}.\end{aligned}$$

This implies, by Theorem 5.2 and (n_i),

$$\mathcal{H}^{N-1}(K \cap B_{r_i}) < c\big(E_i(K) - m_i\big) < c\sqrt[5]{\frac{\varepsilon_{i+1}}{r_{i+1}}}r_{i+1}^{N-1}. \qquad \square$$

The next lemma is a simple consequence of Lemma 6.7 and Corollary 6.2.

LEMMA 6.10. – *Let K be a minimum of E_∞, then for every $i \in \mathbb{N}$ the set $K \cup H_\infty^{i+1}$ satisfies (n_i).*

The following lemma allows us to take $\hat{u} = \hat{u}_\infty$ in Counterexample 1.2.

LEMMA 6.11. – *The function $\hat{u}_\infty$ is Hölder continuous with exponent $\frac{1}{2}$.*

Proof. – Let $x,\ y \in \partial\Omega$ be given. Let $i(x) = \sup\{i \in \mathbb{N} \mid x \in B_i\}$ and, analogously, $i(y) = \sup\{i \in \mathbb{N} \mid y \in B_i\}$. If $i(x) = i(y)$ we have two cases:

(1) $i(x) = i(y) = k$, then

$$\big|\hat{u}_\infty(x) - \hat{u}_\infty(y)\big| = \big|\omega_k(x) - \omega_k(y)\big| \leqslant \|\varphi\|_{\frac{1}{2}}\ \sqrt{d(x,y)};$$

(2) $i(x) = i(y) = \infty$, then $x = y = \bar{x}$ so $\hat{u}_\infty(x) = \hat{u}_\infty(y)$.

If $i(x) \neq i(y)$, since we may always assume $k = i(x) < i(y)$, then we have $x \in B_k \setminus B_{k+1}$ and $y \in B_{k+1}$. Now, if $d(x,y) \leqslant \frac{1}{2}r_{k+1}$, then,

$$\big|\hat{u}_\infty(x) - \hat{u}_\infty(y)\big| = \big|\omega_{k+1}(x) - \omega_{k+1}(y)\big| \leqslant \|\varphi\|_{\frac{1}{2}}\ \sqrt{d(x,y)};$$

otherwise, for $h = k$ or $h = k+1$ we have $d(x,y) \geqslant \frac{1}{2}r_h$ and then by (ii),

$$\frac{|\hat{u}_\infty(x) - \hat{u}_\infty(y)|}{\sqrt{d(x,y)}} \leqslant \sum_{i=h}^{\infty}\sqrt{\varepsilon_i}\left(\frac{1}{2}r_h\right)^{-1/2} = c. \qquad \square$$

Proof of Counterexample 1.2. – Let K be a closed minimum of E_∞. For every i, a local application of Lemma 6.1, made possible by our choice of φ, shows that $B_{r_i}(\bar{x}) \cap K \neq \emptyset$, so $\bar{x} \in \overline{K} = K$. For every $\rho > 0$, we take $i \in \mathbb{N}$ such that $\frac{r_{i+1}}{2} \leqslant \rho \leqslant \frac{r_i}{2}$. Then by Lemma 6.9 and by Lemma 6.10 we find

$$\frac{\mathcal{H}^{N-1}(B_\rho(\bar{x}) \cap K)}{\rho^{N-1}} \leqslant \frac{\mathcal{H}^{N-1}(B_{\frac{r_i}{2}}(\bar{x}) \cap (K \cup H_\infty^{i+1}))}{(r_{i+1}/2)^{N-1}} \leqslant c2^{N-1}\sqrt[5]{\frac{\varepsilon_{i+1}}{r_{i+1}}}.$$

By (i) we have that the density of K in $\bar{x}$ is zero. $\square$

In the last counterexample we are going to show that the condition $\mu > \frac{1}{2}$ is also necessary for Theorem 1.2. We shall denote by c^* the same constant c appearing in Lemma 6.1.

Proof of Counterexample 1.3. – Let $\hat{u}_0 : B \to [0, 2c^*]$, with $B = B_1(0)$, be a $C_0^\infty(B)$ function, such that $\hat{u}_0(0) = 2c^*$. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in $\partial\Omega$ and let $(\lambda_n)_{n\in\mathbb{N}}$

be a decreasing sequence such that $\lambda_0 = 1$ and for every $i > 0$ $\lambda_{i+1} = \frac{\lambda_i}{c_\lambda (i+1)^2}$, where c_λ is a suitable constant.

For every $k \in \mathbb{N}$ we define $\hat{u}_k(x) = \sum_{n=k}^{\infty} \hat{u}_{\lambda_n}(x)$, where $\hat{u}_{\lambda_n}(x) = \lambda_n^{1/2} \hat{u}_0\ (\lambda^{-1}(x - x_n) + x_n)$. Note that for all n, there exist $y_n \in \partial\Omega$ such that $d(x_n, y_n) = \lambda_n$ and

$$\left|\hat{u}_{\lambda_n}(x_n) - \hat{u}_{\lambda_n}(y_n)\right| = 2c^* \sqrt{d(x_n, y_n)}. \tag{6.47}$$

Let now $x, y \in \partial\Omega$. For every $m \in \mathbb{N}$ such that $d(x, y) \leqslant \lambda_m$ we have:

$$\begin{aligned}
\sum_{i=0}^{m-1} \left|\hat{u}_{\lambda_i}(x) - \hat{u}_{\lambda_i}(y)\right| &\leqslant \sum_{i=0}^{m-1} \sup |\nabla \hat{u}_{\lambda_i}| d(x, y) \\
&\leqslant \sum_{i=0}^{m-1} \sup |\nabla \hat{u}_0| (\lambda_i)^{-1/2} \sqrt{\lambda_m}\ \sqrt{d(x, y)} \\
&\leqslant \sup |\nabla \hat{u}_0| \left(m \frac{\sqrt{\lambda_m}}{\sqrt{\lambda_{m-1}}} \right) \sqrt{d(x, y)} \\
&\leqslant \frac{1}{\sqrt{c_\lambda}} \sup |\nabla \hat{u}_0| \sqrt{d(x, y)},
\end{aligned} \tag{6.48}$$

and for every $m \in \mathbb{N}$ such that $d(x, y) \geqslant \lambda_m$

$$\begin{aligned}
\sum_{i=m+1}^{\infty} \left|\hat{u}_{\lambda_i}(x) - \hat{u}_{\lambda_i}(y)\right| &\leqslant \sum_{i=m+1}^{\infty} osc\, \hat{u}_{\lambda_i} \\
&\leqslant \sum_{i=m+1}^{\infty} \sqrt{\lambda_i}\, osc\, \hat{u}_0 \leqslant osc\, \hat{u}_0 \sum_{i=m+1}^{\infty} \frac{\sqrt{\lambda_i}}{\sqrt{\lambda_m}} \sqrt{d(x, y)} \\
&\leqslant \frac{\sqrt{c_\lambda}}{\sqrt{c_\lambda} - 1}\, osc\, \hat{u}_0 \sqrt{d(x, y)}.
\end{aligned} \tag{6.49}$$

By summing (6.48) and (6.49) we have that for every $x, y \in \partial\Omega$ and for every $n, m \in \mathbb{N}$, $n \leqslant m$ such that $d(x, y) = \lambda_m$,

$$\left|(\hat{u}_n - \hat{u}_{\lambda_m})(x) - (\hat{u}_n - \hat{u}_{\lambda_m})(y)\right| \leqslant \left(\frac{\sup |\nabla \hat{u}_0|}{\sqrt{c_\lambda}} + \frac{\sqrt{c_\lambda}\, osc\, \hat{u}_0}{\sqrt{c_\lambda} - 1} \right) \sqrt{d(x, y)}. \tag{6.50}$$

We choose c^* such that

$$\left(\frac{1}{\sqrt{c_\lambda}} \sup |\nabla \hat{u}_0| + \frac{\sqrt{c_\lambda}}{\sqrt{c_\lambda} - 1}\, osc\, \hat{u}_0 \right) \leqslant c^*.$$

We claim that for every $k \in \mathbb{N}$ $\hat{u}_k$ is Hölder continuous, as one easily sees from (6.48) and (6.49).

Moreover, by (6.47) and (6.50), for every $n, k \in \mathbb{N}$, $k \leqslant n$ we have

$$\begin{aligned}
\left|\hat{u}_k(x_n) - \hat{u}_k(y_n)\right| &\geqslant \left|\hat{u}_{\lambda_n}(x_n) - \hat{u}_{\lambda_n}(y_n)\right| - \left|(\hat{u}_k - \hat{u}_{\lambda_n})(x_n) - (\hat{u}_k - \hat{u}_{\lambda_n})(y_n)\right| \\
&\geqslant 2c^* \sqrt{d(x_n, y_n)} - c^* \sqrt{d(x_n, y_n)} \\
&= c^* \sqrt{d(x_n, y_n)}.
\end{aligned} \tag{6.51}$$

For every $k \in \mathbb{N}$, we consider in Ω the minimum problem with boundary datum $\hat{u}_k$. If K is a closed minimum of E, by applying locally Lemma 6.1, since $\hat{u}_k$ satisfies (6.51), we have that for every $n \in \mathbb{N}$, $n \geqslant k$:

$$K \cap B_{\lambda_n}(x_n) \neq \emptyset.$$

Thus, since $(x_n)_{n\in\mathbb{N}}$ is dense in $\partial\Omega$, we have that $\partial\Omega \subset K$. On the other side

$$\begin{aligned}\inf(E_{\hat{u}_k}) \leqslant E_{\hat{u}_k}(\emptyset) \leqslant \|\hat{u}_k\|^2_{H^1} &= \left\| \sum_{n=k}^{\infty} \hat{u}_{\lambda_n} \right\|^2_{H^1} \\ &\leqslant \left(\sum_{n=k}^{\infty} \|\hat{u}_{\lambda_n}\|_{H^1} \right)^2 = \|\hat{u}_0\|^2_{H^1} \left(\sum_{n=k}^{\infty} \lambda_n^{\frac{N-1}{2}} \right)^2. \end{aligned} \tag{6.52}$$

Consequently, by (6.52), for k large, $\inf(E_{\hat{u}_k})$ is smaller than $\mathcal{H}^{N-1}(\partial\Omega)$, a contradiction. □

7. Hölder continuity properties of solutions to (EL)

To prove Theorem 4.1, we need some preliminary results about the Hölder continuity of the solutions to the differential equation in (EL), on $\hat{B}$ with Hölder continuous boundary data. We introduce the notation $S^e = \partial\Omega \cap B$ and $S^i = \partial B \cap \Omega$ (thus $\partial\hat{B} = S^i \cup S^e$), $d_i(x) = d(x, S^i)$, $d_e(x) = d(x, S^e)$ and $d = \min(d_e, d_i)$.

We make the following *regularity assumption* on $\partial\Omega$. For every $\bar{x} \in \partial\Omega$, we have $\Omega \subset C_{\bar{x}}$, where, roughly speaking, $C_{\bar{x}}$ is a cone with vertex $\bar{x}$ and amplitude $\pi + 2\delta$. More precisely, this means that, *after setting* $\bar{x} = 0$ *and under a suitable choice of the first axis, for every* $x \in \Omega$, *it results*

$$x_1 \geqslant -\frac{\delta}{2}|x|. \tag{R_δ}$$

The constant $\delta > 0$ will be assumed sufficiently small depending on μ. We remark that, in the case Ω is convex the above regularity property is trivially satisfied for every δ.

Let v be a real function defined on $\hat{B}$, such that

(i) v is Hölder continuous with exponent $\mu > \frac{1}{2}$ on S^e and norm c_μ,

(ii) v is Hölder continuous with exponent $\frac{1}{2}$ on $\partial\hat{B}$ and norm c_H,

(iii) $\forall x \in \hat{B}$: $|\Delta v(x)| \leqslant R^{-3/2}$.

The aim of this section consists in proving the following estimate concerning a function v satisfying (i), (ii) and (iii).

THEOREM 7.1. – *If* (R_δ) *holds with* $\delta < \frac{1-\mu}{N-1}$ *then, for every* $\delta(N-1) < \beta < \frac{1}{2}$,

$$\left|\nabla v(x)\right| \leqslant c\big(d_i(x)^{-1/2} + d_e(x)^{\mu-1} + d_i(x)^{\beta-1/2} d_e(x)^{-\beta}\big). \tag{7.53}$$

For the proof of the theorem, we shall take advantage of the two following lemmas which estimate the oscillation of v near a point of the boundary.

LEMMA 7.1. – *Let $y \in \partial \hat{B}$, then*

$$\forall s > 0: \quad \operatorname*{osc}_{\hat{B}_R \cap B(y,s)} v \leqslant c s^{1/2}. \tag{7.54}$$

Proof. – Let $y \in \partial \hat{B}$. We can assume, without any restriction, $y = 0$ and $v(0) = 0$. Moreover, we can take a coordinate system such that (R_δ) holds. We are going to introduce a function w, defined on $\hat{B}$, such that

$$\text{(a)} \quad |v| \leqslant w \quad \text{on } \partial \hat{B}, \qquad \text{(b)} \quad \Delta w \leqslant -|\Delta v| \quad \text{on } \hat{B}, \tag{7.55}$$

and so

$$-\Delta(w - |v|) \geqslant 0.$$

Then, by the maximum principle we can say that $w - |v| \geqslant 0$ on the whole of $\hat{B}$. Thus, the oscillation of v on $B(s)$ will be controlled by the oscillation of w on $B(s)$, then

$$\operatorname*{osc}_{\hat{B}_R \cap B(y,s)} v \leqslant 2 \sup_{\hat{B}_R \cap B(y,s)} w. \tag{7.56}$$

We shall show that a function w as above is given, for a suitable value of the constant $\bar{c}_0 \geqslant 0$, by setting

$$w(x) = c_H \big(|x|^{1/2} + \bar{c}_0 (x_1 + \delta |x|)^{1/2} \big).$$

Such an equality, combined with (7.56), will produce the thesis. In order to conclude the proof, we just have to show that w satisfies (7.55). Condition (7.55(a)) is easily verified since $v \in C^{0,1/2}(\partial \hat{B})$, with norm c_H, $x_1 + \delta |x| \geqslant 0$ and $\bar{c}_0 \geqslant 0$.

For (7.55(b)) we need to estimate Δw. Since we shall need similar computations also in the next lemma, we shall work more in general with the function

$$w_0(x) = \bar{c} \big(|x|^\tau + \bar{c}_0 (x_1 + \delta |x|)^\tau \big),$$

with $\tau < 1 - \delta(N-1)$, which will give the required information for $\tau = \frac{1}{2}$. By an easy computation,

$$\begin{aligned} \bar{c}^{-1} \Delta w_0(x) &= \tau(\tau + N - 2)|x|^{\tau-2} \\ &\quad + \bar{c}_0 \tau (x_1 + \delta|x|)^{\tau-2} \left[(\tau - 1)\left(1 + 2\delta \frac{x_1}{|x|} + \delta^2\right) + \delta(N-1)\left(\frac{x_1}{|x|} + \delta\right)\right]. \end{aligned}$$

Now, since $(\frac{x_1}{|x|} + \delta) \leqslant (1 + 2\delta \frac{x_1}{|x|} + \delta^2)$, $(x_1 + \delta|x|)^{\tau-2} \geqslant (1+\delta)^{\tau-2}|x|^{\tau-2}$, $\delta < \frac{1-\tau}{N-1}$, i.e. $(\tau - 1 + \delta(N-1)) \leqslant 0$, and $2x_1 < \delta|x|$, we get

$$\Delta w_0(x) \leqslant \bar{c}\tau \big(N + \tau - 2 + \bar{c}_0 (1+\delta)^{\tau-2} \big(\tau - 1 + \delta(N-1)\big)\big)|x|^{\tau-2}. \tag{7.57}$$

Therefore, by taking in (7.57)

$$\bar{c}_0 = \frac{(2 - N - \tau) - 4(\bar{c}\tau)^{-1}}{(1+\delta)^{\tau-2}(\tau - 1 + \delta(N-1))} > 0,$$

we obtain

$$\Delta w_0 \leqslant -4|x|^{\tau-2} \leqslant -R^{\tau-2},$$

since $|x| \leqslant 2R$. For $\bar{c} = c_H$ and $\tau = \frac{1}{2}$ we estimate Δw and so we obtain (7.55(b)). □

In the case of the points of the exterior boundary S^e we can also produce the following estimate.

LEMMA 7.2. – *Let $y \in S^e$, then $\forall s > 0$*

$$\underset{\hat{B}_R \cap B(y,s)}{osc} v \leqslant c\bigl(c_\mu s^\mu + c_H \bigl(d_i(y)\bigr)^{\beta-\frac{1}{2}} s^{1-\beta}\bigr), \tag{7.58}$$

where $c = c(\beta, \mu, \delta)$.

Proof. – The proof involves the same arguments employed to prove the previous lemma, so, after setting $\bar{d}_i = d_i(y)$, we perform the same change of variables as before in order to have (R_δ) for every $x \in \hat{B}$. The only variant relies on the choice of the function $w(x)$ which, in the present case, is given by the sum $w_1 + w_2$, where

$$w_1(x) = c_\mu\bigl(|x|^\mu + \bar{c}_1(x_1 + \delta|x|)^\mu\bigr),$$
$$w_2(x) = c_H(\bar{d}_i)^{\beta-1/2}\bigl(|x|^{1-\beta} + \bar{c}_2(x_1 + \delta|x|)^{1-\beta}\bigr).$$

Now, w_1 and w_2 are positive functions, moreover $|v| \leqslant w_1$ on S^e, from (i) and $|v| \leqslant w_2$ on S^i, from (ii). So, by combining the last two inequalities, we get (7.55(a)). We remark that either w_1 and w_2 are particular cases of w_0, obtained for $\tau = \mu$ and $\tau = 1 - \beta$. Thus, by taking

$$\bar{c}_1 = \frac{2 - N - \tau}{(1+\delta)^{\tau-2}(\tau - 1 + \delta(N-1))} > 0,$$

for $\tau = \mu$, we obtain $\Delta w_1 \leqslant 0$, while by taking $\bar{c}_2$ as we have taken c_0 in the previous proof, we get, for $\tau = 1 - \beta$

$$\Delta w_2 \leqslant -4(\bar{d}_i)^{\beta-\frac{1}{2}}|x|^{\tau-2} \leqslant -R^{-\frac{3}{2}},$$

since both $\bar{d}_i$ and $|x|$ are less or equal to $2R$. By combining the last two inequalities with (iii), (7.55b) is proved. □

Proof of Theorem 7.1. – Let φ be a spherically symmetric mollifier with a support contained in the unitary ball. We consider the rescaled functions φ_λ, normalized in L^1, defined as the functions which send x in $\lambda^{-N}\varphi(\lambda^{-1}x)$, for $\lambda > 0$.

Firstly we recall that by a standard calculation, if ψ is any symmetric mollifier, then

$$v = v * \psi - \Delta v * n_\psi, \tag{7.59}$$

where, by introducing the real function $\bar{\psi}$ such that, for every $x \in \mathbb{R}^N$ $\bar{\psi}(|x|) = \psi(x)$, we have set

$$n_\psi(x) = \int_{|x|}^{\infty} \frac{1}{N b_N (N-2)} \left(\frac{1}{|x|^{N-2}} - \frac{1}{\rho^{N-2}}\right) \bar{\psi}(\rho)\, \mathrm{d}\rho.$$

We can estimate ∇n_ψ as follows,

$$\left|\nabla n_\psi(x)\right| = \left|\int_{|x|}^{\infty} \frac{1}{Nb_N|x|^{N-1}}\bar{\psi}(\rho)\,\mathrm{d}\rho\right| \leqslant \frac{1}{Nb_N|x|^{N-1}}. \tag{7.60}$$

By differentiating (7.59) we have

$$\nabla v = v * \nabla\psi - \Delta v * \nabla n_\psi. \tag{7.61}$$

Let $x \in \hat{B}_R$ be given. We take $\lambda = d(x)$, then for $\psi = \varphi_\lambda$, since $\operatorname{supp}\varphi_\lambda \subset B_\lambda(0)$, from (7.60) we have $\|\nabla n_{\varphi_\lambda}\|_{L^1} \leqslant \lambda$ and by (iii) we get

$$\left|\Delta v * \nabla n_{\varphi_\lambda}(x)\right| \leqslant \|\nabla n_{\varphi_\lambda}\|_{L^1}\|\Delta v\|_{L^\infty} \leqslant CR^{-3/2}d \leqslant CR^{-1/2} \leqslant c{d_i}^{-1/2}. \tag{7.62}$$

In order to estimate the first term in (7.61) we distinguish two cases:

(a) If $2d_e(x) \leqslant d_i(x)$ we have $\lambda = d_e(x)$ and we can fix $y \in S^e$ such that $d(x,y) = d_e(x) = \lambda$. By the triangular inequality $\frac{1}{2}d_i(x) \leqslant d_i(y)$. By Lemma 7.2 we have the estimate

$$\begin{aligned}\left|v * \nabla\varphi_\lambda(x)\right| &\leqslant \frac{1}{2}\|\nabla\varphi_\lambda\|_{L^1} \operatorname*{osc}_{\hat{B}_R\cap B_{2\lambda}(y)} v \\ &\leqslant \frac{1}{2}\lambda^{-1}\|\nabla\varphi\|_{L^1} c\left(c_\mu(2\lambda)^\mu + c_H\left(d_i(y)\right)^{\beta-\frac{1}{2}}(2\lambda)^{1-\beta}\right) \\ &\leqslant c\left(c_\mu\lambda^{\mu-1} + c_H\left(d_i(x)\right)^{\beta-\frac{1}{2}}\lambda^{-\beta}\right).\end{aligned}$$

Since $\lambda = d_e(x)$, we get

$$\left|v * \nabla\varphi_\lambda(x)\right| \leqslant c\left(c_\mu\left(d_e(x)\right)^{\mu-1} + c_H\left(d_i(x)\right)^{\beta-\frac{1}{2}}\left(d_e(x)\right)^{-\beta}\right). \tag{7.63}$$

(b) If $2d_e(x) > d_i(x)$ we have $\lambda \geqslant \frac{1}{2}d_i(x)$ and fix $y \in S^i$ such that $d(x,y) = d_i(x) \leqslant 2\lambda$. Thus, in the same way as the previous case, by Lemma 7.1, we get

$$\begin{aligned}\left|v * \nabla\varphi_\lambda(x)\right| &\leqslant \frac{1}{2}\|\nabla\varphi_\lambda\|_{L^1} \operatorname*{osc}_{\hat{B}_R\cap B_{3\lambda}(y)} v \\ &\leqslant c\frac{1}{2}\lambda^{-1}\|\nabla\varphi\|_{L^1}(3\lambda)^{1/2} \leqslant cc_H\left(d_i(x)\right)^{-1/2}.\end{aligned} \tag{7.64}$$

By combining (7.62), (7.63) and (7.64), we can deduce (7.53) from (7.61). □

A simpler variant of Theorem 7.1, involving only Lemma 7.1, allows to show that

$$|\nabla v(x)| \leqslant cc_H\left(d(x)\right)^{-1/2}, \tag{7.65}$$

from which, in the case of a regular boundary, we deduce $v \in C^{0,1/2}(\overline{\hat{B}_R})$ with a constant of the same order of c_H.

We shall apply the above estimates to cases in which the condition $R \leqslant 1$ holds. Therefore (iii) will be satisfied whenever $|\Delta v(x)| \leqslant 1$ and this is the case for the solutions to (EL). Moreover, when $R \leqslant 1$ it happens that $d_i(x) \leqslant 2$, so, by increasing the value of the constant in (7.53), we can put it in the homogeneous form

$$|\nabla v| \leqslant c\big(d_i{}^{-\frac{1}{2}} + d_i{}^{\frac{1}{2}-\mu} d_e{}^{\mu-1} + d_i{}^{\beta-\frac{1}{2}} d_e{}^{-\beta}\big)$$

which, by taking $\beta = 1-\mu > \delta(N-1)$, becomes

$$|\nabla v| \leqslant c \sum_{\tau} d_i{}^{-\tau} d_e{}^{\tau-\frac{1}{2}}, \tag{7.66}$$

where τ takes the values $\frac{1}{2}$ and $\mu - \frac{1}{2}$. Then

$$|\nabla v|^2 \leqslant c \sum_{\tau} d_i{}^{-\tau} d_e{}^{\tau-1}, \tag{7.67}$$

where τ takes the values 1, μ and $2\mu - 1$.

8. Proof of the excision theorem

In this section we shall prove Theorem 4.1, so we are in the situation described in the beginning of Section 4; we shall take $\bar{x} = 0$. Let us begin with some comments about conditions (V1–V4) listed therein. In [21] it is shown that conditions (V2) and (V3) are equivalent to ask the existence of a finite decomposition of V, consisting in a family of piecewise regular open sets $(V_j)_j$ enjoying the properties:

$$\sum_j \mathcal{H}^{N-1}(\partial V_j) \leqslant c_2\sigma, \tag{V2$'$}$$

$$\sup_j(\operatorname{diam} V_j) \leqslant \alpha R. \tag{V3$'$}$$

Moreover, the arguments in [21] allow us to fix a finite covering $\mathcal{B}$ of V, made by balls B_i, with radius $r_i \leqslant \alpha R$. In [21, Section 2] it is shown that, thanks to the properties of V, we can take B_i with small radius and such that the sum of the $(N-1)$-dimensional measures of their boundaries is of the order of the measure of ∂V.

We recall the notation used in [21] and we introduce some new one due to the possible presence of $\partial\Omega$ in the ball. Let K be a given closed subset of $\overline{\Omega}$, $u = u(K)$. Let V be a neighborhood of the set $\overline{B(R)} \cap K$ and $(V_j)_j$ a decomposition of V such that (V1), (V2$'$), (V3$'$) and (V4) hold. In the following computations we shall not assume that K is a minimum, nevertheless we shall ask that (1.4) holds with $c = 2$ for every subset A of $\overline{\Omega}$, such that

$$\big(K \cap B(R/2)\big) \subset A \subset B(R). \tag{8.68}$$

Thus the results of this section will depend by the last hypothesis, therefore in order to employ them in the proof of Theorem 4.1, we shall provide to force such a condition.

A simple consequence of (1.4) (note that (8.68) holds for $A = V$), (V2), and $R \leqslant 1$ is

$$\int_V |\nabla u|^2 \leqslant \mathcal{H}^{N-1}(\partial V) + |V| \leqslant (c_2 + \alpha)\sigma. \tag{8.69}$$

By employing Hölder Inequality, (V3) and (8.69) we deduce:

$$\sqrt{R} \int_V |\nabla u| \leqslant \sqrt{\alpha(c_2 + \alpha)} R\sigma. \tag{8.70}$$

Let us denote by $\tilde{u}$ a Hölder continuous extension [14, 2.10.44], with norm c_1 and with exponent $\frac{1}{2}$, of u^* to $\hat{B}(R)$ and by $\bar{u}$ a truncation of u by two constants such that $\bar{u} = u^*$, where the last one is defined, and

$$\operatorname*{osc}_{\hat{B}(R)} \bar{u} \leqslant c_H \sqrt{2R}. \tag{8.71}$$

Note that (see [15]) $\bar{u} \in H^1(\hat{B}(R) \setminus K)$ and that

$$\nabla u \nabla \bar{u} \leqslant |\nabla u|^2. \tag{8.72}$$

For every positive $s < R$ we denote by $S^i(s)$ and $S^e(s)$ the terms S^i and S^e introduced in Section 7 for $\hat{B} = \hat{B}(s)$. Moreover we set

$$V(s) = V \cap B(s)$$

and we also set, for every exponent j,

$$V_j(s) = V_j \cap B(s).$$

Let v_s be the solution of the Dirichlet problem

$$\begin{cases} -\Delta v_s + v_s = g & \text{in } \hat{B}(s), \\ v_s = \tilde{u} & \text{on } \partial \hat{B}(s). \end{cases} \tag{8.73}$$

We proved in Section 7 that v_s is Hölder continuous with exponent $\frac{1}{2}$ and norm cc_H (see (7.65) and the following considerations) and that (7.66) and (7.67) hold for $v = v_s$.

We introduce the notation $d_{i,s}(x) = d(x, S^i(s))$, $d_{e,s}(x) = d(x, S^e(s))$. Since the center of the ball $B(s)$ is $\bar{x} = 0$, then $d_{i,s}(x) \geqslant s - |x|$, if $|x| \leqslant s$. We put $d_\Omega(x) = d(x, \partial\Omega) \leqslant d_{e,s}(x)$ so, under mild regularity assumptions on $\partial\Omega$, for every $0 < \tau < 1$ with $c_\tau = c(\tau, \Omega)$, we obtain

$$\begin{aligned} \int_V d_\Omega(x)^{\tau-1}\,\mathrm{d}x &\leqslant \sum_i \int_{B_i} c_\tau d_\Omega(x)^{\tau-1}\,\mathrm{d}x \\ &\leqslant \sum_i c_\tau r_i^{N+\tau-1} \leqslant c_\tau \big(\sup_i r_i\big)^\tau \sum_i r_i^{N-1} \\ &\leqslant c_\tau (\alpha R)^\tau \sigma. \end{aligned} \tag{8.74}$$

Now, for all $s \in [R/2, R]$ we define an "excised set" K_s and an "excised function" u_s. In this perspective we consider the sets

$$T_0(s) = S^i(s) \cap V,$$

$$T_1(s) = S^i(s - \sqrt{\alpha}R) \cap V,$$

$$T_2(s) = \partial V \cap \bigl(B(s) \setminus B(s - \sqrt{\alpha}R)\bigr).$$

Then we define the "cut set"

$$T(s) = T_0(s) \cup T_1(s) \cup T_2(s),$$

the "cushion set"

$$U(s) = V(s) \setminus B\bigl(s - \sqrt{\alpha}R\bigr)$$

and the "excised set"

$$K_s = \bigl(K \setminus \overline{B}(s)\bigr) \cup T(s).$$

Finally we can set

$$u_s = \begin{cases} u & \text{on } \Omega \setminus B(s), \\ v_s & \text{on } B(s) \setminus U(s), \\ 0 & \text{on } U(s). \end{cases}$$

We notice that u_s belongs to the set of admissible functions for E, since it satisfies the boundary condition. In general $u_s \neq u(K_s)$.

The crucial step in the proof of the main excision theorem relies on comparing $\int_{R/2}^{R} (J(u_s) - J(u))\,\mathrm{d}s$ with $R\sigma$. So we shall essentially have to show that some terms, which will be employed to the aim of estimating that integral, are much smaller than $R\sigma$, when α is small. As in [21], we prefer to start by showing in a systematical way some of such inequalities and, finally, we shall combine them in order to prove the theorem. So, let $s \in [R/2, R]$ be given and let us begin by taking $x_j \in \partial V_j(s)$ such that

$$\bigl|(\tilde{u} - v_s)(x_j)\bigr| = \min_{x \in \partial V_j(s)} \bigl|(\tilde{u} - v_s)(x)\bigr|$$

and put $c_j = (\tilde{u} - v_s)(x_j)$. Then, for all $x \in \partial V_j(s)$

$$\begin{aligned} \bigl|(\tilde{u} - v_s)(x) - c_j\bigr| &\leqslant \min\Bigl(\operatorname*{osc}_{\partial V_j(s)} (\tilde{u} - v_s), 2\bigl|(\tilde{u} - v_s)(x)\bigr|\Bigr) \\ &\leqslant \Bigl(\operatorname*{osc}_{\partial V_j(s)} (\tilde{u} - v_s)\Bigr)^{\lambda} \bigl(2\bigl|(\tilde{u} - v_s)(x)\bigr|\bigr)^{1-\lambda}, \end{aligned} \tag{8.75}$$

for every $0 \leqslant \lambda \leqslant 1$. Since $\tilde{u} - v_s$ is Hölder continuous, then for every $x \in \partial V_j(s)$ we have

$$d_{e,s}(x)^{-1/2} \leqslant d_{\Omega}(x)^{-1/2} \leqslant c\bigl|(\tilde{u} - v_s)(x)\bigr|^{-1}, \tag{8.76}$$

so by (7.66) and (8.76) we get, for some positive constant c,

$$\bigl|\nabla v_s(x)\bigr| \leqslant c \sum_{\tau} |\tilde{u} - v_s|^{(\tau-1)} d_{i,s}(x)^{-\tau/2}, \tag{8.77}$$

where τ varies in the set $\{1, 2\mu - 1\}$ (note the change of variable). For every τ, by (8.75) with $\lambda = \tau$, (8.77) and by the divergence theorem, we can establish the following estimate

$$\begin{aligned}\int\limits_{\partial V_j(s)} (\tilde u - v_s)\frac{\partial v_s}{\partial n} &\leqslant \int\limits_{\partial V_j(s)} |\tilde u - v_s - c_j||\nabla v_s| + \int\limits_{V_j(s)} c_j|\Delta v_s| \\ &\leqslant \int\limits_{\partial V_j(s)} |\tilde u - v_s - c_j| c\sum_\tau |\tilde u - v_s|^{(\tau-1)} d_{i,s}(x)^{-\tau/2} + |V_j| \\ &\leqslant c\sum_\tau \int\limits_{\partial V_j(s)} \bigl(\operatorname*{osc}_{\partial V_j}(\tilde u - v_s)\bigr)^\tau d_{i,s}(x)^{-\tau/2} + |V_j|. \end{aligned} \tag{8.78}$$

Fubini Theorem and (V2′) allow us to deduce

$$\begin{aligned}&\int\limits_{R/2}^{R} \sum_j \Biggl(\int\limits_{\partial V_j(s)} (\tilde u - v_s)\frac{\partial v_s}{\partial n}\Biggr) \mathrm{d}s \\ &\quad\leqslant c\sum_\tau\sum_j\Biggl(\int\limits_{\partial V_j} \bigl(\operatorname*{osc}_{\partial V_j}(\tilde u - v_s)\bigr)^\tau \int\limits_{|x|}^{R} \frac{\mathrm{d}s}{(s-|x|)^{\tau/2}} + |V_j|\Biggr) \\ &\quad\leqslant c\sum_\tau\sum_j \int\limits_{\partial V_j} (\alpha R)^{\tau/2} R^{1-\frac{\tau}{2}} + |V| \\ &\quad\leqslant c\sum_\tau\sum_j R\alpha^{\tau/2}\mathcal{H}^{N-1}(\partial V_j) + |V| \ll R\sigma. \end{aligned} \tag{8.79}$$

Moreover, for every $0 < \tau < 1$, by Hölder Inequality and (8.74), we get

$$\begin{aligned}&\int\limits_V |\nabla u|\Biggl(\int\limits_{|x|}^{R} (s-|x|)^{-\tau/2} d_\Omega(x)^{\frac{\tau-1}{2}}\,\mathrm{d}s\Biggr)\mathrm{d}x \\ &\quad\leqslant R^{1-\frac{\tau}{2}} c_\tau \Biggl(\int\limits_V |\nabla u|^2\Biggr)^{1/2}\Biggl(\int\limits_V d_\Omega(x)^{\tau-1}\Biggr)^{1/2} \\ &\quad\leqslant R^{1-\frac{\tau}{2}} c_\tau \sqrt{\sigma}\sqrt{(\alpha R)^\tau\sigma} \ll R\sigma. \end{aligned} \tag{8.80}$$

Finally, by Fubini Theorem, (7.66), (8.76) and (8.80), we obtain the estimate

$$\begin{aligned}\int\limits_{R/2}^{R}\Biggl(\int\limits_{V(s)} |\nabla u|\,|\nabla v_s|\Biggr)\mathrm{d}s &\leqslant \int\limits_V\Biggl(\int\limits_{|x|}^{R} |\nabla u|\,|\nabla v_s|\,\mathrm{d}s\Biggr)\mathrm{d}x \\ &\leqslant \sum_\tau \int\limits_V |\nabla u|\Biggl(\int\limits_{|x|}^{R} (s-|x|)^{-\frac{\tau}{2}} d_\Omega(x)^{\frac{\tau-1}{2}}\,\mathrm{d}s\Biggr)\mathrm{d}x \\ &\ll R\sigma. \end{aligned} \tag{8.81}$$

Now we are in a position to prove the excision theorem. Let $\delta(s) = J(u_s) - J(u)$ be the increment of the integral part of the functional under the excision operation, we shall estimate $\int_{R/2}^{R} \delta(s)\,\mathrm{d}s \ll R\sigma$. To this aim, we shall employ the computation in the beginning of the proof of the [21, Lemma 4.1] to estimate $\delta(s)$. Indeed, since both u and v_s satisfy the elliptic equations (EL) and (8.73), $u = \tilde{u}$ on $\hat{B}(s) \setminus V$ and $\tilde{u} = v_s$ on $\partial\hat{B}(s)$, then (see [21] for more information) we have that

$$J_{\hat{B}(s)\setminus V}(v_s) - J_{\hat{B}(s)\setminus V}(u) = \int\limits_{\partial V(s)} \frac{\partial(v_s + u)}{\partial n}(\tilde{u} - v_s).$$

Therefore we can split the term $J(u_s) - J(u)$ as follows:

$$\begin{aligned} J(u_s) - J(u) &= J_{\hat{B}(s)\setminus V}(v_s) - J_{\hat{B}(s)\setminus V}(u) + J_{V(s)}(u_s) - J_{V(s)}(u) \\ &= \delta_1(s) + \delta_2(s) + \delta_3(s) + \delta_4(s) + \delta_5(s), \end{aligned}$$

where

$$\begin{aligned} &\delta_1(s) = \int\limits_{\partial V(s)} \frac{\partial v_s}{\partial n}(\tilde{u} - v_s), \qquad \delta_2(s) = \int\limits_{\partial V(s)} \frac{\partial u}{\partial n}\bar{u} - J_{V(s)}(u), \\ &\delta_3(s) = \int\limits_{T_1(s)} \frac{\partial u}{\partial n}(\tilde{u} - \bar{u}), \qquad \delta_4(s) = -\int\limits_{\partial V(s)} \frac{\partial u}{\partial n} v_s, \\ &\delta_5(s) = J_{V(s)}(u_s). \end{aligned}$$

We remark that in the above calculations we have used the condition $u = \bar{u}$ where $u = \tilde{u}$.

LEMMA 8.1. – *If V is thin enough and* (1.4) *holds for $A = V$ and $A = \hat{B}$, then we have*

$$\int\limits_{R/2}^{R} \delta(s)\,\mathrm{d}s \ll R\sigma. \tag{8.82}$$

Proof. – For $i = 1$ the estimate is (8.79). For $i = 2$ and $i = 3$, the computations of [21] remain unchanged. Let us evaluate the addendum for $i = 4$ by using the divergence theorem.

$$\delta_4(s) = \int\limits_{V(s)} (-\Delta u\ v_s - \nabla u \nabla v_s) \leqslant |V| + \int\limits_{V(s)} |\nabla u|\,|\nabla v_s| \tag{8.83}$$

and the desired inequality follows from (8.81). Finally for $i = 5$, since $u_s = 0$ on the cushion set $U(s)$, by (7.67) and (8.74), we get, for τ taking the same values of (7.67),

$$\begin{aligned} \delta_5(s) &= \int\limits_{V(s)\setminus U(s)} |\nabla v_s|^2 + \int\limits_{V(s)} |v_s - g|^2 \\ &\leqslant \sum_{\tau} \int\limits_{V} d_{i,s}(x)^{-\tau} d_{e,s}(x)^{\tau-1} + |V| \\ &\leqslant \sum_{\tau} \int\limits_{V} \left(\sqrt{\alpha} R\right)^{-\tau} d_{\Omega}(x)^{\tau-1} + |V| \end{aligned}$$

$$\leqslant \sum_{\tau} \bigl(\sqrt{\alpha} R\bigr)^{-\tau} (\alpha R)^{\tau} \sigma + |V|, \tag{8.84}$$

so we can conclude

$$\int_{R/2}^{R} \delta_5(s)\, \mathrm{d}s \ll R\sigma. \qquad \square \tag{8.85}$$

Proof of Theorem 4.1. – Let us distinguish two cases. If there exists a subset A such that (8.68) holds and (1.4) does not hold, then we proceed as follows. Firstly, by continuity we can assume $A \subset B(s)$ for $s < R$, then we can take as K' the set $(K \setminus A) \cup \partial A$. Let u' be the function defined as follows

$$u' = \begin{cases} u & \text{on } \Omega \setminus A, \\ 0 & \text{on } A, \end{cases}$$

and note that the negation of (1.4) implies

$$E_A(K) > 2\bigl(\mathcal{H}^{N-1}(\partial A) + |A|\bigr). \tag{8.86}$$

Then we have

$$\begin{aligned} E(K') &\leqslant E(u', K') \\ &\leqslant E(K) + \mathcal{H}^{N-1}(\partial A) + \int_A |g - u|^2 - E_A(K) \\ &\leqslant E(K) - \frac{1}{2} E_A(K) \leqslant E(K) - \frac{1}{2}\sigma, \end{aligned} \tag{8.87}$$

since $\sigma \leqslant \mathcal{H}^{N-1}(K \cap A) \leqslant E_A(K)$. So, in this case the theorem is proved. Otherwise, if (1.4) holds for every A satisfying (8.68), we have all the assumptions needed in this section and so we can apply Lemma 8.1. Let us also remark that

$$\int_{R/2}^{R} \mathcal{H}^{N-1}\bigl(T(s)\bigr)\, \mathrm{d}s \leqslant 2|V| + \sqrt{\alpha} c_2 R\sigma \ll R\sigma,$$

therefore we can find $s \in [R/2, R]$ such that

$$\delta(s) + \mathcal{H}^{N-1}\bigl(T(s)\bigr) \leqslant \frac{\sigma}{2}.$$

Now let us take $K' = K_s$ for such a value of s, then

$$\begin{aligned} E(K') = E(K_s) &\leqslant E(u_s, K_s) = J(u_s) + \mathcal{H}^{N-1}(K_s) \\ &\leqslant J(u) + \delta(s) + \mathcal{H}^{N-1}(K) + \mathcal{H}^{N-1}\bigl(T(s)\bigr) - \mathcal{H}^{N-1}\bigl(K \cap B(s)\bigr) \\ &\leqslant J(u) + \delta(s) + \mathcal{H}^{N-1}(K) + \mathcal{H}^{N-1}\bigl(T(s)\bigr) - \sigma \\ &\leqslant E(K) - \frac{\sigma}{2}. \qquad \square \end{aligned} \tag{8.88}$$

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