ASYMPTOTIC BEHAVIOR OF GROUND STATES OF QUASILINEAR ELLIPTIC PROBLEMS WITH TWO VANISHING PARAMETERS

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\textbf{Abstract.} – We study the asymptotic behavior of the radially symmetric ground state solution of a quasilinear elliptic equation involving the $m$-Laplace operator. The case of two vanishing parameters is considered: we show that these two parameters have opposite effects on the asymptotic behavior. Moreover the results highlight a surprising phenomenon: different asymptotic are obtained according to whether $n > m^2$ or $n \leq m^2$, where $n$ is the dimension of the underlying space.

\textbf{Résumé.} – Nous étudions le comportement asymptotique de l’état fondamental à symétrie radiale d’une équation elliptique quasilinéaire contenant le $m$-Laplacien. Le cas de deux paramètres tendant vers 0 est considéré : nous montrons que ces deux paramètres sont en compétition. Les résultats obtenus découvrent un nouveau surprenant phénomène : deux comportements asymptotiques complètement différents sont obtenus suivant une relation entre le paramètre $m$ et la dimension $n$ de l’espace.

\section{Introduction}

Let $\Delta_m u = \text{div}(|\nabla u|^{m-2}\nabla u)$ denote the degenerate $m$-Laplace operator and consider the quasilinear elliptic equation

\[-\Delta_m u = -\delta u^{m-1} + u^{p-1} \quad \text{in } \mathbb{R}^n,\]

\textbf{(P\textsubscript{δ}p)}

where $n > m > 1$, $m < p < m^*$, $\delta > 0$ and

$$m^* = \frac{nm}{n-m}.$$
By the results in [6,10] (see also [1,4] for earlier results in the case \( m = 2 \)) we know that 
\((P_{\delta}^{p})\) admits a ground state for all \( p, \delta \) in the given ranges. Here, by a ground state we mean a \( C^{1}(\mathbb{R}^{n}) \) positive distribution solution of \((P_{\delta}^{p})\), which tends to zero as \(|x| \to \infty\).

Since in this paper we only deal with radial solutions of \((P_{\delta}^{p})\), from now on by a ground state we shall mean precisely a radial ground state. It is known [14,17] moreover that radial ground states of \((P_{\delta}^{p})\) are unique.

Equation \((P_{\delta}^{p})\) is of particular interest because of the choice of the power \( m-1 \) for the lower order term: if \( m = 2 \) (i.e. \( R\Delta K_{m} = R\Delta K \)) this is just the linear case, while for any \( m > 1 \) the lower order term has the same homogeneity as the differential operator \( R\Delta K_{m} \), a fact which allows the use of rescaling methods. Moreover, this case is precisely the borderline between compact support and positive ground states, see [7, Section 1.3].

It is our purpose to study the behavior of (radial) ground states of \((P_{\delta}^{p})\) as \( p \to m^{*} \), \( \delta \to 0 \). As far as we are aware, the asymptotic behavior of solutions of \((P_{\delta}^{p})\) has been studied previously only for the vanishing parameter \( \varepsilon = m^{*} - p \) and only in the case of bounded domains, see [3,8,9,11,15,16] and references therein.

Consider first the case when \( \delta = 0 \). Then \((P_{\delta}^{p})\) becomes

\[-\Delta_{m}u = u^{p-1} \quad \text{in } \mathbb{R}^{n}, \quad (P_{p}^{0})\]

which by [13, Theorem 5] admits no ground states (recall \( p < m^{*} \)). It is of interest therefore to study the behavior of the ground states \( u \) of \((P_{\delta}^{p})\) as \( \delta \to 0 \) and \( p \) is fixed: in Theorem 1 below we prove in this case that \( u \to 0 \) uniformly on \( \mathbb{R}^{n} \) and moreover estimate the rate of convergence. As a side result, the arguments used in the proof of Theorem 1 allow us to show that the corresponding ground states \( u \) converge to a Dirac measure concentrated at the origin, namely, \( u(0) \to \infty \) and \( u(x) \to 0 \) for all \( x \neq 0 \), while also, at the same time, \( u \) converges strongly to 0 in any Lebesgue space \( L^{q}(\mathbb{R}^{n}) \) with \( m - 1 \leq q < m^{*} \). Our study also reveals a striking and unexpected phenomenon: the asymptotic behavior is different in the two cases \( n \leq m^{2} \) and \( n > m^{2} \); for instance, in the case \( m = 2 \) (i.e. \( \Delta_{m} = \Delta \)) there is a difference of behavior between the space dimensions \( n = 3, 4 \) and \( n \geq 5 \). More precisely, if \( n > m^{2} \) we show that \( u(0) \) blows up asymptotically like \( \varepsilon^{-(n-m)/m^{2}} \) while if \( n \leq m^{2} \) it blows up at a stronger rate, essentially \( \varepsilon^{-(m-1)/m} \). This phenomenon is closely related with the \( L^{m} \) summability of functions which achieve the best constant in the Sobolev embedding \( D^{1,m} \subset L^{m^{*}} \), see [18] and (1) below for the explicit form of these functions.

Finally, let both \( p = m^{*} \) and \( \delta = 0 \); then equation \((P_{p}^{\delta})\) reads

\[-\Delta_{m}u = u^{m^{*}-1} \quad \text{in } \mathbb{R}^{n}, \quad (P_{m^{*}}^{0})\]
which admits the one-parameter family of ground states

\[ U_d(x) = d \left[ 1 + D \left( (m-n) m^{-1} x \right) \right]^{-\frac{m-n}{m-n}} \quad (d > 0), \quad (1) \]

where \( D = D_{m,n} = (m-1)/(n-m)n^{1/(m-1)} \) and \( U_d(0) = d \). Since the effects of vanishing \( m^* - p \) and \( \delta \) are in some sense “opposite”, it is reasonable to conjecture that there exists a continuous function \( h \), with \( h(0) = 0 \), such that if \( \delta = h(\varepsilon) \), \( p = m^* - \varepsilon \), then ground states \( u \) of \((P_{\delta}^p)\) converge neither to a Dirac measure nor to 0!

In Theorem 4 below we prove the surprising fact that when \( n > m^2 \) this equilibrium occurs exactly when \( \delta \) and \( \varepsilon \) are linearly related, \( h(\varepsilon) \approx \text{Const} \varepsilon \). Moreover in this case the corresponding ground states \( u \) then converge uniformly to a suitably concentrated ground state of \((P_0^m)\), namely a function of the family (1), with the parameter \( d = U_d(0) \) representing a “measure of concentration” and depending on the limiting value of the ratio \( h(\varepsilon)/\varepsilon \).

Let us heuristically describe the phenomena highlighted by our results. When \( p \to m^* \) with \( \delta \) fixed, the mass of the ground state \( u \) of \((P_{\delta}^p)\) tends to concentrate near the point \( x = 0 \), that is, all other points of the graph are attracted to this point: in order to “let the other points fit near \( x = 0 \)” the maximum level \( u(0) \) is forced to blow up. When \( \delta \to 0 \) with \( p \) fixed, the ground state spreads, since now \( x = 0 \) behaves as a repulsive point, forcing the maximum level to blow down in order “not to break the graph”. When both \( \varepsilon = m^* - p \) and \( \delta \) tend to 0 at the “equilibrium velocity” \( \delta = h(\varepsilon) \), the point \( x = 0 \) is neither attractive nor repulsive: in this case, a further striking fact is that the exponential decay of the solution \( u \) of \((P_\delta^p)\) at infinity reverts to a polynomial decay.

The outline of the paper is as follows. In the next section we state our main results, Theorems 1–5. Then in Section 3 we present background material on radial ground states, including an estimate for the asymptotic decay as \( r \to \infty \) of ground states of \((P_{\delta}^p)\), see Theorem 8. This estimate, along with Theorems 6 and 7 in Section 3, seems to be new and may be useful in other contexts. These results allow us to give a simple proof of Theorem 5 while the proofs of Theorems 1–4 are given in subsequent sections.

2. Main results

The existence and uniqueness of radial ground states for equation \((P_\delta^p)\) is well known \([10,17]\). We state this formally as

**Proposition 1.** – For all \( n > m > 1, m < p < m^* \) and \( \delta > 0 \) equation \((P_\delta^p)\) admits a unique radial ground state \( u = u(r) \), \( r = |x| \). Moreover \( u'(r) < 0 \) for \( r > 0 \).

We start the asymptotic analysis of \((P_\delta^p)\) by maintaining \( p \) fixed and letting \( \delta \to 0 \). An important role will be played by the rescaled problem (\( \delta = 1 \))

\[ -\Delta_m v = -v^{m-1} + v^{p-1} \quad \text{in} \ \mathbb{R}^m. \quad (Q_p) \]

By Proposition 1 there exists a unique (radial) ground state \( v \) of \((Q_p)\), so that the constant

\[ \beta = v(0) \]
is a well-defined function of the parameters \( m, n, p \).

**Theorem 1.** For all \( \delta > 0 \), let \( u \) be the unique ground state of \((P_{\delta}^p)\) with \( m < p < m^* \). Then \( u(0) = \delta^{1/(p-m)} \beta \), while for fixed \( p \) and \( x \neq 0 \) there holds

\[
\frac{u(x)}{u(0)} = 1 - \frac{m-1}{m} \left( \frac{\beta^{p-m} - 1}{n} \delta \right)^{\frac{1}{m-1}} |x|^{\frac{m}{m-1}} + \alpha \left( \delta^{\frac{1}{m-1}} |x|^{\frac{m}{m-1}} \right) \quad \text{as} \ \delta \to 0. \tag{3}
\]

Also, putting \( \ell = n(p-m)/m \), there exists \( \alpha_{m,n,p} > 0 \) independent of \( \delta \) such that

\[
\int_{\mathbb{R}^n} u \ell = \alpha_{m,n,p} \quad \forall \delta > 0.
\]

From Theorem 1 we can also obtain a result which, while slightly beyond the scope of the paper, is nevertheless worth noting. It states that the unique solution of \((P_{\delta}^p)\) for fixed \( p < m^* \) tends to a Dirac measure as \( \delta \to \infty \), see Theorem 9 in Section 4.

We now maintain \( \delta > 0 \) fixed and let \( p \to m^* \). In order to state our main asymptotic result for this case, it is convenient to introduce the beta function \( B(\cdot, \cdot) \) defined by

\[
B(a, b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}} \, dt, \quad a, b > 0.
\]

Then we put

\[
\beta_{m,n} = \left( n \left( \frac{m}{n-m} \right)^2 \frac{B(n(m-1), n-m^2)}{B(n, n-m)} \right)^{(n-m)/m^2} \quad \text{for } n > m^2,
\]

and

\[
\gamma_{m,n} = \omega_n \frac{m - 1}{m} \left( n \left( \frac{m}{n-m} \right)^{m-1} \right)^{n/m} B \left( \frac{n(m-1)}{m}, \frac{n}{m} \right) \quad (\omega_n = \text{measure } S^{n-1}).
\]

We also put \( C_{m,n} = D^{-(m-1)(n-m)/m} \), where \( D = D_{m,n} \) is given in Eq. (1).

These coefficients allow us to describe the exact behavior of ground states when \( n > m^2 \); in particular note that \( \beta_{m,n} \to \infty \) as \( m \uparrow \sqrt{n} \).

**Theorem 2.** For all \( m < p < m^* \), let \( u \) be the unique ground state for equation \((P_{\delta}^p)\) with fixed \( \delta > 0 \). Then, writing \( \varepsilon = m^* - p \), we have

\[
\lim_{\varepsilon \to 0} \left[ \frac{\varepsilon}{\delta} \right]^{(n-m)/m^2} u(0) = \begin{cases} 
\beta_{m,n} & \text{if } n > m^2, \\
\infty & \text{if } n \leq m^2.
\end{cases} \tag{4}
\]

Moreover for all \( x \neq 0 \)

\[
\lim_{\varepsilon \to 0} \{ u(0) u^{m-1}(x) \} \leq C_{m,n} |x|^{-(n-m)} \tag{5}
\]
uniformly outside of any neighborhood of the origin, while also
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} u^q = 0 \quad \forall q \in [m-1, m^*), \quad \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} u^{m^*} = \gamma_{m,n}.
\] (6)

Theorem 2 gives a complete description of the asymptotic behavior of \( u \) when \( n > m^2 \); it leaves open the exact behavior when \( n \leq m^2 \). This latter question is considered in more detail in Section 5.2. The results given there, while not as precise as in the case \( n > m^2 \), nevertheless provide significant insight into the behavior of \( u(0) \) as \( \varepsilon \to 0 \) beyond that described in the second case of (4). In particular from Lemmas 7 and 8 we have the following additional asymptotic results as \( \varepsilon \to 0 \).

Let \( \delta = 1 \). If \( n = m^2 \), then
\[
\left( \frac{\varepsilon}{|\log \varepsilon|} \right)^{(m-1)/m} u(0) \approx 1,
\]
while if \( m < n < m^2 \), then for appropriate positive constants we have
\[
\text{Const.} |\log \varepsilon|^{(n-m^2)/m^2} \leq \varepsilon^{(m-1)/m} u(0) \leq \text{Const.} |\log \varepsilon|^{(n-m)/m^2}.
\]

The picture below describes this striking phenomenon; let
\[
\mu = \inf \{ \gamma > 0; \lim_{\varepsilon \to 0} [u(0)e^{\gamma}] = 0 \},
\]
then, \( \mu = (m-1)/m \) when \( n \leq m^2 \) and \( \mu = (n-m)/m^2 \) when \( n > m^2 \). The figure represents the map \( \mu = \mu(n) \) in the case \( m = 2 \).
Condition (6) shows that, as \( \varepsilon \to 0 \), not only does \( u \) approach a Dirac measure \((u(0)) \to \infty \) and \( u(|x|) \to 0 \) for \(|x| \neq 0 \), but also that the \( L^m \) norm of \( u \) approaches a non-zero finite limit. It is a remarkable fact, also, that the limit relation (6) is independent of the value of \( \delta \). It is worthwhile to note as well that by (6) and interpolating, the \( L^q \) norm of \( u \) becomes \( \infty \) if \( q > m^* \).

Remark. – The constants in Theorem 2 in the important case \( m = 2 \) are given by

\[
\beta_{2,n} = \left( \frac{4n}{(n-2)^2} \frac{B\left(\frac{n}{2}, \frac{n-2}{2}\right)}{B\left(\frac{n}{4}, \frac{n}{2}\right)} \right)^{(n-2)/4}, \quad \gamma_{2,n} = \frac{\omega_n}{2} [n(n-2)]^{n/2} B\left(\frac{n}{2}, \frac{n}{2}\right),
\]

and \( C_{2,n} = [n(n-2)]^{(n-2)/2} \).

The results of Theorem 2 can be supplemented with the following asymptotic estimates for the gradient \( \nabla u \) of a ground state.

Theorem 3. – For all \( m < p < m^* \), let \( u \) be the unique ground state for equation \((P_\delta^p)\) with fixed \( \delta > 0 \). Then for all \( x \neq 0 \) we have

\[
\lim_{\varepsilon \to 0} \{ u(0)|\nabla u(x)|^{m-1} \} \leq \left( \frac{n-m}{m-1} \right)^{m-1} C_{m,n} |x|^{1-n}
\]

and

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |\nabla u|^q = 0 \quad \forall q \in \left( \frac{m}{n-1}, m \right), \quad \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |\nabla u|^m = \gamma_{m,n}.
\]

Finally, we may accurately describe the behavior of the ground states of \((P_\delta^p)\) when \( \varepsilon = m^* - p \) and \( \delta \) approach zero simultaneously.

Theorem 4. – For \( \delta > 0 \) and \( m < p < m^* \), let \( u \) be the unique ground state of \((P_\delta^p)\). Then for all \( d > 0 \) there exists a positive continuous function \( \tau(\varepsilon) = \tau(\varepsilon, d) \) such that

(i) \( \tau(\varepsilon) \to (d/\beta_{m,n})^m (n-m) \) as \( \varepsilon \to 0 \) (when \( n > m^2 \)), and \( \tau(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) (when \( n \leq m^2 \)).

(ii) If \( \delta = \varepsilon \tau(\varepsilon), \ p = m^* - \varepsilon, \) then \( u(0) = d \). Moreover

\[
u \to U_d \quad \text{as} \quad \varepsilon = m^* - p \to 0
\]

uniformly on \( \mathbb{R}^n \), where \( U_d \) is the function defined in (1).

If \( \varepsilon, \delta \to 0 \) without respecting the equilibrium behavior \( \delta \approx \text{Const} \varepsilon \) (in the case \( n > m^2 \)), the central height \( u(0) \) of the ground state may either converge to zero or diverge to infinity. We note finally that as soon as the asymptotic behavior of \( u(0) \) as \( p \to m^* \) is more accurately determined in the case \( n \leq m^2 \) of (4), one also gets a more precise statement of (i): of course, the equilibrium behavior will no longer be \( \delta \approx \text{Const} \varepsilon \).

To conclude the section, we supply two global estimates for \( u(0) \), supplementing the asymptotic conditions (3) and (4).
Table 1

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\[ \beta(m, n, p) \]

Table 1

*Table 1*

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**THEOREM 5.** Let \( u \) be a ground state of \( (P_p^\delta) \). Then

\[
u(0) > \left( \frac{mp}{mn - p(n-m)\delta} \right)^{1/(p-m)},
\]

and, provided that \( p < n/(n-1) \),

\[
u(0) < \left( \frac{p n - m(n-1)}{m n - p(n-1)\delta} \right)^{1/(p-m)}.
\]

The proof of this result is given in the next section. By setting \( \delta = 1 \) in Theorem 5 we obtain related estimates for the parameter \( \beta = \nu(0) \) in Theorem 1. Also from Theorem 2 we have the following asymptotic formula for \( \beta \), with \( \varepsilon = m^* - p \to 0 \),

\[
\beta = \beta_{m,n} \varepsilon^{-1/m^2} (1 + o(1)) \text{ if } n > m^2;
\]

see also Lemmas 5–8 in Sections 5.

**Remark.** The condition \( p < n/(n-1) \) implies \( p < m/(m-1) \), since \( n > m \); therefore, the upper bound in (10) is obtained only for values \( m < 2 \) (because \( p > m \) and values \( p \) "far" from the critical exponent \( m^* \), that is \( m^* - p > n^2(m-1)/(n-m)(n-1) \)). However, in the restricted range of values \( p < n/(n-1) \), inequality (10) gives useful information about \( \nu(0) = \beta \); we quote here some numerical computations (Table 1).

3. Preliminary results about ground states

In this section we consider the ground state problem for the general equation

\[
-\Delta_m u = f(u) \text{ in } \mathbb{R}^n,
\]

where the function \( f \) is assumed only to be continuous on \([0, \infty)\) and to obey the condition

\[
f(0) = 0, \quad f(u) < 0 \quad \text{for } u \text{ near } 0.
\]
A radial ground state \( u = u(r), \ r = |x|, \) of (11) is in fact a \( C^1 \) solution of the ordinary differential equation

\[
\left( |u'|^{m-2} u' \right)' + \frac{n-1}{r} |u'|^{m-2} u' + f(u) = 0, \quad r > 0, \\
u(0) = \alpha > 0, \quad u'(0) = 0
\]

(13)

for some initial value \( \alpha > 0. \) For our purposes the dimension \( n \) may in fact be considered as any real number greater than \( m. \)

Put

\[
F(u) = \int_0^u f(s) \, ds
\]

(14)

and introduce the energy function

\[
E = E(r) = \frac{m-1}{m} |u'(r)|^m + F(u(r)).
\]

(15)

The following properties of ground states are well-known [7].

**Proposition 2.** – A radial ground state \( u = u(r) \) of (13) has the properties

\[
\frac{|u'(r)|^{m-1}}{r} \to \frac{f(\alpha)}{n} \text{ as } r \to 0,
\]

\[
r^{n-1} |u'(r)|^{m-1} \to \text{Finite limit as } r \to \infty,
\]

\[
F(\alpha) = (n-1) \int_0^\infty \frac{|u'(r)|^m}{r} \, dr
\]

and

\[
E(r) > 0 \quad \forall \ r \geq 0, \quad E(r) \to 0 \quad \text{as } r \to \infty.
\]

In the next result we recall a Pohozaev-type identity [12].

**Proposition 3.** – Let \( u = u(r) \) be a radial ground state of (13), and put

\[
Q(r) = nm F(u) - (n-m)uf(u).
\]

(16)

Then the functions \( r^{n-1} Q(r) \) and \( r^{n-1} F(u(r)) \) are in \( L^1(0, \infty) \), and moreover

\[
\int_0^\infty Q(r) r^{n-1} \, dr = 0.
\]

(17)

\[2\] Formula (17) is given in [12] for the case \( m = 2, \) see (3.7) and put \( a = (n-2)/2; \) the case for general \( m \) moreover is implicit in Section 4, Case (V) of [12].
Remark. – In other terms, the result of Proposition 3 says that the functions 
$$Q(|x|)$$ and 
$$F(u(|x|))$$ are in $$L^1(\mathbb{R}^n)$$ and that 
$$\int_{\mathbb{R}^n} Q(|x|) \, dx = 0.$$ 

For completeness we give a proof of Proposition 3. By direct calculation, using (13), 
we find that 
$$P(r) = \int_{0}^{r} Q(t) t^{n-1} \, dt, \quad r > 0,$$ 
where 
$$P(r) = (n-m) r^{n-1} u(r) u'(r) |u'(r)|^{m-2} + m r^n E(r).$$ 

Since 
$$E = \frac{m-1}{m} |u'|^m + F(u(r)) > 0$$ 
and because $$f(s) < 0$$ for $$s$$ near 0, we get 

$$|F(u(r))|, \quad E(r) \leq \frac{m-1}{m} |u'(r)|^m$$ 
for all sufficiently large $$r$$. Using Proposition 2 then gives 
$$r^{n-1} |u'|^{m-1} \leq \text{Const.}$$ and 
$$r^n |F(u(r))|, \quad r^n E(r) \leq \text{Const.} r^{-(n-m)/(m-1)}$$ (18) 
for sufficiently large $$r$$. Hence $$P(r) \to 0$$ as $$r \to \infty$$, which yields 
$$\lim_{r \to \infty} \int_{0}^{r} Q(t) t^{n-1} \, dt = 0.$$ 

But from (18) we get 
$$r^{n-1} |F(u(r))| \in L^1(0, \infty),$$ 
while also $$uf(u) < 0$$ for all sufficiently large $$r$$. Thus the previous equation together with the definition of $$Q(r)$$ shows in fact that 
$$r^{n-1} Q(r)$$ is in $$L^1(0, \infty)$$ and that (17) holds. This completes the proof. 

Proposition 3 has the following important consequence.

THEOREM 6. – Suppose there exists $$\gamma > 0$$ such that 
$$nmF(s) - (n-m)sf(s) < 0 \quad \text{for} \quad 0 < s < \gamma.$$ (19) 

Then $$\alpha > \gamma$$. 

Proof. – Suppose for contradiction that $$\alpha \leq \gamma$$. Then since $$u' < 0$$ for $$r > 0$$, it follows that $$u(r) < \gamma$$ for all $$r > 0$$. In turn, by the hypothesis (19) we have 
$$Q(r) = nmF(u) - (n-m)uf(u) < 0 \quad \text{for} \quad r > 0,$$ 
which contradicts Proposition 3. 

An upper bound for $$u(0)$$ can also be obtained in some circumstances, as in the following.

THEOREM 7. – Suppose $$f'(s) \geq 0$$ whenever $$f(s) > 0$$ and that there exists $$\mu > 0$$ such that 
$$nF(s) - (n-1)sf(s) \geq 0 \quad \text{for} \quad s \geq \mu.$$ (20) 

Then $$\alpha < \mu$$. 

Proof. We assert that the function $r \mapsto \Phi(r) = r^{-1}|u'(r)|^{m-1}$ is decreasing on $(0, \infty)$. By direct calculation, using (13),

$$r\Phi'(r) = f(u) - n\Phi(r).$$

If $f(u) \leq 0$ then $\Phi' < 0$. On the other hand, for all $r$ such that $f(u) > 0$, we have $(f(u) - n\Phi(r))' = f'(u)u' - n\Phi'(r) \leq -n\Phi'(r)$, by hypothesis. Consequently

$$(r\Phi')' \leq -n\Phi'.$$

By integration this gives $r^{n+1}\Phi'(r) \leq r_1^{n+1}\Phi'(r_1)$ on any interval $(r_1, r)$ where $f(u) > 0$. The assertion now follows by an easy argument, once one notes notes that $r^{n+1}\Phi'(r) \to 0$ as $r \to 0$.

Now by Proposition 2 and the assertion, we have

$$F(\alpha) = (n-1) \int_0^\infty \frac{|u'(r)|^m}{r} \, dr = (n-1) \int_0^\infty \Phi(r)|u'(r)| \, dr$$

$$< (n-1)\Phi(0) \int_0^\infty |u'(r)| \, dr = (n-1)\alpha\Phi(0).$$

Since by Proposition 2 we also have $\Phi(0) = f(\alpha)/n$, this gives $nF(\alpha) - (n-1)\alpha f(\alpha) < 0$. The conclusion now follows from the main hypothesis (20).

Using Theorems 6 and 7 it is now easy to obtain the

Proof of Theorem 5. – Equation $(P_\delta^p)$ can be written in the form (11), or (13), with

$$f(s) = -\delta s^{m-1} + s^{p-1}, \quad Q(r) = -\delta^{m} + \frac{m - p(n-m)}{p} u^p.$$

Hence for this case we can take

$$\gamma = \left( \frac{mp}{mn - p(n-m)} \delta \right)^{1/(p-m)}$$

in (19), giving the first conclusion of Theorem 5 as a consequence of Theorem 6. Moreover

$$nF(s) - (n-1)sf(s) = -\frac{n-m(n-1)}{m} \delta s^m + \frac{n - p(n-1)}{p} s^p.$$

Thus we can take

$$\mu = \left( \frac{p n - m(n-1)}{mn - p(n-1)} \delta \right)^{1/(p-m)}$$

in (20), giving the second conclusion as a consequence of Theorem 7.
We conclude the section by showing that radial ground states \( u = u(r) \) of \((P_{\delta}^\varepsilon)\) have
exponential decay as \( r \) approaches infinity. This is well-known in the case \( m = 2 \), see [4, 
Theorem 1(iv)]: we give here a different proof in the general case \( m > 1 \).

**Theorem 8.** Suppose that there exist constants \( \delta, \lambda, \rho > 0 \) such that \( f \) satisfies
the inequality
\[
-\delta s^{m-1} \leq f(s) \leq -\lambda s^{m-1} \quad \text{for } 0 < s < \rho. \tag{21}
\]
Then there exist constants \( \mu_0, \mu_1, \mu_2, \nu > 0 \) (depending on \( m, n, \delta, \lambda \)) such that, for \( r \) suitably large,
\[
u u(r) > (\lambda/(m-1))^{1/m} \forall r \geq R. \tag{23}
\]
Integrating this inequality on the interval \([R, r]\) yields the first part of the result, with
\[
\mu_0 = \rho e^{\nu R}, \quad \nu = (\lambda/(m-1))^{1/m}. \tag{24}
\]

For the other estimates, we rewrite (13) in the form
\[
(r^{n-1}|u'(r)|^{m-1})' = r^{n-1} f(u(r)). \tag{25}
\]
Since \( f(u) < 0 \) for \( u \) near 0, it follows that \( r^{n-1}|u'(r)|^{m-1} \) is ultimately decreasing,
clearly to a non-negative limit as \( r \to \infty \) (this is the first result of Proposition 2). By the
exponential decay proved above, the limit must be \( 0 \). Therefore we can integrate (25) on
\([r, \infty)\) for \( r \geq R \) to obtain, with the help of (21),
\[
r^{n-1}|u'(r)|^{m-1} = - \int_r^\infty t^{n-1} f(u(t)) \, dt < \delta \int_r^\infty t^{n-1} u^{m-1}(t) \, dt
\]
\[
\leq \delta \mu_0^{m-1} \int_r^\infty t^{n-1} e^{-(m-1)\nu t} \, dt.
\]
With \( n - 1 \) integrations by parts, this proves that
\[
|u'(r)| \leq \mu_1 e^{-\nu r} \quad \forall r \geq R.
\]

Finally, we write (13) as
\[
(m - 1)|u'(r)|^{m-2}u''(r) = \frac{n - 1}{r}|u'(r)|^{m-1} - f(u).
\]

From the right hand inequality of (21) we get \( f(u) \leq 0 \) for \( r \geq R \), which shows that \( u''(r) > 0 \) for all \( r \geq R \). Further, from the left hand inequality,
\[
u''(r) < \frac{n - 1}{(m - 1)R}|u'(r)| + \frac{\delta}{m-1} \frac{u^{m-1}(r)}{|u'(r)|^{m-2}}.
\]

Hence by (23) and by the exponential decay of \( u \) and \( u' \), this yields
\[
0 < u''(r) < \frac{n - 1}{(m - 1)R}|u'(r)| + \frac{\delta}{m-1} \left( \frac{m-1}{\lambda} \right)^{(m-2)/m} u(r) \leq \mu_2 e^{-\nu r} \quad \forall r \geq R.
\]

The proof of Theorem 8 is now complete. 

Remarks. – The first estimate of (22) requires only the right hand inequality of (21) for its validity.
It almost goes without saying that the function \( f(u) = -\delta u^{m-1} + u^{p-1} \) satisfies (21) for suitable \( \lambda, \rho \).

4. Proof of Theorem 1

Let \( u = u(r) \) be a ground state of \( (P^\delta_p) \). Define \( v = v(r) \) by means of the rescaling
\[
v(r) = \delta^{-1/(p-m)}u \left( \frac{r}{\delta^{1/m}} \right),
\]

so that \( v \) is the unique ground state of the rescaled equation \( (Q_p) \). By definition (2) and by (26) one has \( u(0) = \delta^{1/(p-m)} \beta \).

Next, from \( (Q_p) \) we find, as in (25),
\[
|v'(r)|^{m-1} = \frac{1}{r^{n-1}} \int_0^r s^{n-1} \left\{ -v^{m-1}(s) + v^{p-1}(s) \right\} ds
= \frac{1}{r^{n-1}} \int_0^r s^{n-1} \left\{ -\beta^{m-1} + \beta^{p-1} + o(1) \right\} ds
= \frac{r}{n} \left\{ -\beta^{m-1} + \beta^{p-1} + o(1) \right\}
\]
as } r \to 0. \text{ Taking the } 1/(m-1) \text{ root and integrating from } 0 \text{ to } r \text{ then gives}

\[ v(r) = \beta - \frac{m-1}{m} \left( \frac{\beta^{p-1} - \beta^{m-1}}{n} \right)^{1/(m-1)} r^{m/(m-1)} + o\left(r^{m/(m-1)}\right) \quad \text{as } r \to 0. \tag{27} \]

This, together with (26), yields (3).

The final part of theorem is an almost obvious consequence of (26) and the change of
variables } s = \delta^{1/m} r; \text{ in particular } \alpha_{m,n,p} = \int_{R^n} v_{\ell}.

\[ \square \]

When } \delta \to \infty \text{ we can obtain a partial companion result to (3) in Theorem 1.

**Theorem 9.** – For fixed } x \neq 0 \text{ we have

\[ u(x) = o\left(e^{-\nu\delta^{1/m}|x|}\right) \]

as } \delta \to \infty, \text{ where } v \text{ is any (positive) number less than } 1/(m - 1)^{1/m}.

**Proof.** – We apply Theorem 7 for ground states of } (Q_p). \text{ Here } f(s) = -s^{m-1} + s^{p-1}, \text{ so that one can take } \lambda \text{ to be any number less than } 1 \text{ in (21), provided that } \rho \text{ is chosen appropriately near } 0. \text{ Thus by Theorem 8 we have}

\[ v(r) \lesssim \mu_0 e^{-v r} \]

for all sufficiently large } r, \text{ where, see (24), } v \text{ is any number less than } 1/(m - 1)^{1/m}. \text{ Hence, by (26),}

\[ u(x) = \delta^{1/(p-m)} v\left(\delta^{1/m}|x|\right) \lesssim \mu_0 \delta^{1/(p-m)} e^{-\nu\delta^{1/m}|x|} \]

for all fixed } x \neq 0 \text{ and sufficiently large } \delta. \text{ Finally, taking } \tilde{v} = v - \theta, \text{ with } \theta \text{ small, we get}

\[ u(x) \lesssim \mu_0 \delta^{1/(p-m)} e^{-\theta\delta^{1/m}|x|} e^{-\tilde{\nu}\delta^{1/m}|x|} = o\left(e^{-\tilde{\nu}\delta^{1/m}|x|}\right) \]

as } \delta \to \infty. \text{ The conclusion now follows at once, since clearly by appropriate choice of } v \text{ and } \theta \text{ we can assume that } \tilde{v} \text{ is any number less than } 1/(m - 1)^{1/m}. \quad \square \]

**5. Proof of Theorem 2**

The argument is delicate, covering a number of pages. For the proof of (4) we need to
distinguish the two cases } n > m^2 \text{ and } m < n \leq m^2; \text{ this is done in Sections 5.1 and 5.2
below. The proof of (5) and (6) is given in Section 5.3. \text{ We shall prove (4) first, for the case } \delta = 1, \text{ and then obtain the general estimate by means of the rescaling (26).}

Thus we assume that } u = u(r) \text{ satisfies (13) with } f(s) = -s^{m-1} + s^{p-1}, \text{ namely

\[ \left(|u'|^{m-2} u'\right)' + \frac{n-1}{r} |u'|^{m-2} u' - u^{m-1} + u^{p-1} = 0 \tag{28} \]
with \( u(0) = \alpha \). From the estimate (9) in Theorem 5 we have always \( \alpha > 1 \) (since \( p > m \)) and, more precisely,

\[
\alpha > \left( \frac{mp}{\varepsilon(n-m)} \right)^{1/(p-m)}
\]

where

\[ p = m^* - \varepsilon. \]

Hence

\[
\alpha > \left( \frac{m^2}{n-m} \right) \frac{1}{1/(p-m)},
\]

which gives the important condition

\[
\omega \equiv \varepsilon \alpha^{p-m} \geq K \quad \forall \varepsilon \in (0, m^* - m), \tag{29}
\]

where \( K = m^2/(n-m) \).

We make a second rescaling

\[
w(r) = \frac{1}{\alpha} u(\alpha^{-(p-m)/m}r), \tag{30}
\]

so that if \( u = u(r) \) solves (28), then \( w = w(r) \) satisfies

\[
\begin{cases}
(\varepsilon n w')' + \frac{n-1}{r} w^{m-2} w' - \eta w^{m-1} + w^{p-1} = 0, \\
w(0) = 1, \quad w'(0) = 0, \tag{31}
\end{cases}
\]

where \( \eta = \alpha^{-(p-m)} \). Note that \( \eta < 1 \) since \( \alpha > 1 \), and also, by (29), \( \eta \to 0 \) as \( \varepsilon \to 0 \).

Now define the modified nonlinearity

\[
f_\eta(s) = -\eta s^{m-1} + s^{p-1}
\]

and the corresponding functions (see (14) and (16))

\[
F_\eta(s) = -\frac{\eta}{m} s^m + \frac{1}{p} s^p, \quad Q_\eta(r) = -m \eta w^m(r) + \frac{\varepsilon(n-m)}{p} w^p(r). \tag{32}
\]

Also, for \( r \geq 0 \) let us define the function

\[
z(r) = (1 + (1 - \eta)^{1/(m-1)} D m/(m-1))^{-(n-m)/m} \tag{33}
\]

where the constant \( D = D_{m,n} \) is given in (1).

We can now prove the following comparison result, closely related to Lemma 2.1 of [11].

\[^3\] The idea of a uniform upper bound for a scaled function \( w(r) \) first appears (for the case \( m = 2 \)) in [2].
LEMMA 1. – We have

\[ w(r) < z(r) \quad \forall r > 0. \]  (34)

Proof. – We make use of the function \( H \) introduced in Lemma 2.1 in [11]: here however it will be applied without a previous Emden–Fowler inversion. Thus set

\[ H(r) = (m-1)r^n |w'(r)|^m - (n-m)r^{n-1}w(r) |w'(r)|^{m-1} + \frac{n-m}{n}r^n w(r) f_\eta(w(r)). \]

Then by using the fact that \( w \) solves (31) we obtain

\[ H'(r) = r^n \left( m^2 \eta w^{m-1}(r) - (n-m)w^{p-1}(r) \right) w'(r). \]

Let \( R \) be the unique value of \( r \) where \( w(R) = \left( \frac{m^2}{n-m} \right)^{1/(p-m)} \in (0, 1) \); see (29) and recall from Proposition 2 that \( w' < 0 \) and \( w < 1 \) for \( r > 0 \). Hence it is easy to see that \( H \) is strictly increasing on \([0, R]\) and strictly decreasing on \([R, \infty)\). Moreover, \( H(0) = 0 \) and \( \lim_{r \to \infty} H(r) = 0 \) by Theorem 8. Consequently

\[ H(r) > 0 \quad \forall r > 0. \]  (35)

Consider the function

\[ \Psi(r) = \frac{|w'(r)|^{m-1} \Phi(r)}{r w^{m(n-1)/(n-m)}(r)} = \frac{\Phi(r)}{w^{m(n-1)/(n-m)}(r)}, \]

where \( \Phi(r) = |w'(r)|^{m-1} / r \) (see the proof of Theorem 7). By using (31) again we find that

\[ \Psi'(r) = \frac{n}{n-m} \frac{1}{r^{n+1} w^{m(n-1)/(n-m)}(r)} H(r). \]

From (35) it follows that \( \Psi \) is strictly increasing on \([0, \infty)\). Therefore, by Proposition 2 we have

\[ \Psi(r) > \lim_{t \to 0} \Psi(t) = \frac{f_\eta(1)}{n} = \frac{1-\eta}{n}, \]

hence

\[ \frac{|w'(r)|}{w^{n/(n-m)}(r)} > \left( \frac{1-\eta}{n} \right)^{1/(m-1)} r^{1/(m-1)} \frac{|z'(r)|}{z^{n/(n-m)}(r)} \quad \forall r > 0. \]

The conclusion (34) follows upon integration, and the proof is complete. \( \Box \)

For later use we observe that the function \( z = z(r) \) defined in (33) satisfies the equation

\[ \left( |z'|^{m-2} z' \right)' + \frac{n-1}{r} |z'|^{m-2} z' + (1-\eta) z^{m-1} = 0 \]  (36)
(the easiest way to check this is to note from (1) that 
\[ z = d^{-1} U_d \] 
for 
\[ d = (1 - \eta)^{(n-m)/m^2}, \]
so that 
\[ z \]
then satisfies 
\[ (P_{m^*}) \]
with the extra coefficient 
\[ (1 - \eta) \]
inserted on the right side).

Now let
\[ C_1 = C_1(\varepsilon) = \left( \frac{n-m}{m^2 \varepsilon} \right)^{\varepsilon/(p-m)}. \]  
(37)

Then by differential calculus (recalling that 
\[ p = m^* - \varepsilon \]
and 
\[ \eta = \alpha - (p - m) \]) we find without difficulty that
\[ f_\eta(s) \leq C_1 \alpha \varepsilon s^{m^* - 1} \quad \forall s > 0 \] 
and 
\[ \lim_{\varepsilon \to 0} C_1 = 1. \]  
(38)

This allows us to obtain the following partial converse of Lemma 1.

**LEMMA 2.** – There exists a positive function 
\[ C_2 = C_2(\varepsilon) \] 
such that 
\[ \lim_{\varepsilon \to 0} C_2 = 1 \] 
and 
\[ w(r) > C_2 \alpha \varepsilon^{(m-1)} z(r) - (C_2 \alpha \varepsilon^{(m-1)} - 1) \quad \forall r > 0. \]  
(39)

Moreover 
\[ C_2 \alpha \varepsilon^{(m-1)} > 1. \]

**Proof.** – Eq. (31) may be rewritten as
\[ |r^{n-1} w'(r)|^{m-1} = r^{n-1} f_\eta(w). \]  
(40)

Integrating on \([0, r]\), and taking into account (38) and Lemma 1, yields
\[ r^{n-1} |w'(r)|^{m-1} = \int_0^r t^{n-1} f_\eta(w(t)) \, dt < C_1 \alpha \varepsilon^{(m-1)} z^{m^* - 1}(t) \, dt \]
\[ = \frac{C_1}{1 - \eta} \alpha \varepsilon^{(m-1)} z(r)^{m^* - 1}, \]
the last equality being obtained by a similar integration of (36) on \([0, r]\). Therefore,
\[ |w'(r)| < C_2 \alpha \varepsilon^{(m-1)} |z'(r)| \quad \forall r > 0, \]  
(41)

where
\[ C_2 = \left( \frac{C_1}{1 - \eta} \right)^{1/(m-1)}. \]

Integrating (41) on \([0, r]\) then gives (39).

Finally, from (38) one sees that 
\[ C_2 \to 1 \] 
as \( \varepsilon \to 0 \), while by (34) and (39) we infer that
\[ (C_2 \alpha \varepsilon^{(m-1)} - 1)(z(r) - 1) < 0 \quad \forall r > 0, \]
that is, 
\[ C_2 \alpha \varepsilon^{(m-1)} - 1 > 0 \] 
since 
\[ z(r) < 1 \] 
for \( r > 0 \) by (34) and the fact that \( \eta < 1 \). This completes the proof. \( \Box \)

The following technical lemmas will be crucial in the sequel. To simplify their presentation, we shall think of the functions 
\[ w = w(r) \] 
and 
\[ z = z(r), \] 
given in (30)
and (33), to be defined over the space $\mathbb{R}^n$ instead of on $r \geq 0$; that is, $w = w(|x|)$ and $z = z(|x|)$. In particular, $w$ then satisfies the partial differential equation

$$- \Delta_m w = f_\eta(w) = -\eta w^{m-1} + w^{p-1}, \quad \eta = \alpha^{(p-m)}. \quad (42)$$

We observe also that $w(|x|)$ decays exponentially as $|x| \to \infty$, so that the integrals below are well defined.

**Lemma 3.** We have

$$c_1 \omega \int_{\mathbb{R}^n} w^p \leq \int_{\mathbb{R}^n} w^m \leq c_2 \omega \int_{\mathbb{R}^n} w^p,$$

where $\omega = \varepsilon \alpha^{p-m}$, $p = m^* - \varepsilon$, and

$$c_1 = \frac{1}{n} \left( \frac{n-m}{m} \right)^2, \quad c_2 = \frac{n-m}{m^2}.$$

**Proof.** By Proposition 3 applied to the ground state $w$ of (31) we get, with the help of the second part of (32),

$$- m \eta \int_{\mathbb{R}^n} w^m + \frac{\varepsilon (n-m)}{p} \int_{\mathbb{R}^n} w^p = 0, \quad \text{that is}, \quad \int_{\mathbb{R}^n} w^m = \frac{m - m}{mp} \omega \int_{\mathbb{R}^n} w^p.$$ 

But $p \in (m, m^*)$, so the conclusion follows at once. □

**Lemma 4.** We have

$$\int_{\mathbb{R}^n} w^m \geq (C \alpha^\varepsilon)^{-(n-m)/m}, \quad \int_{\mathbb{R}^n} |\nabla w|^m \geq (C \alpha^\varepsilon)^{-(n-m)/m},$$

where $C$ is a Sobolev constant for the embedding of $D^{1, m}(\mathbb{R}^n)$ into $L^{m^*}(\mathbb{R}^n)$.

**Proof.** If we multiply (42) by $w$ and integrate by parts, we obtain

$$\int_{\mathbb{R}^n} |\nabla w|^m = -n \int_{\mathbb{R}^n} w^m + \int_{\mathbb{R}^n} w^p < \int_{\mathbb{R}^n} w^p. \quad (43)$$

Using (38) and the fact that $C_1 \leq 1$ by (37), Eq. (42) can also be written in the form $- \Delta_m w = f_\eta(w) \leq \alpha^\varepsilon w^{m-1}$. Thus, as before,

$$\int_{\mathbb{R}^n} |\nabla w|^m \leq \alpha^\varepsilon \int_{\mathbb{R}^n} w^m \leq C \alpha^\varepsilon \left( \int_{\mathbb{R}^n} |\nabla w|^m \right)^{m^*/m}, \quad (44)$$

by the Sobolev inequality. Solving this relation for $\int_{\mathbb{R}^n} |\nabla w|^m$ gives the second inequality of the lemma; the first is then obtained from (43). This completes the proof. □
5.1. The case $n > m^2$

By (33) we see that $z(|x|) \approx |x|^{(n-m)/(m-1)}$ as $|x| \to \infty$, so $z \in L^m(\mathbb{R}^n)$ if and only if $n > m^2$. This allows us to derive

**Lemma 5.** Let $n > m^2$. Then there exists $A > 0$ (depending only on $m, n$) such that

$$\alpha \leq \left( \frac{A}{\varepsilon} \right)^{(n-m)/m} \quad \text{for all } \varepsilon \in \left( 0, \frac{m-1}{n-n-m} \right).$$

**Proof.** Define $\hat{z}(|x|)$ to be the function given by (33) with the parameter $\eta$ fixed at the value

$$\hat{\eta} = \frac{(m-1)(n-m)}{n^2 - m(m-1)}.$$

Using (9) with $\delta = 1$, an easy calculation shows that for $\varepsilon$ in the range stated in the lemma we have $\eta = \alpha^{-(p-m)} \in (0, \hat{\eta})$. Hence, for the given range of $\varepsilon$, we infer from (34) that

$$\int_{\mathbb{R}^m} w^m \leq \int_{\mathbb{R}^m} z^m \leq \int_{\mathbb{R}^m} \hat{z}^m \equiv \hat{c}$$

(recall $n > m^2$, and observe specifically that $\hat{c} = \hat{c}(m, n)$).

On the other hand, by Lemmas 3 and 4,

$$\int_{\mathbb{R}^m} w^m \geq c_1 \omega \int_{\mathbb{R}^p} w^p \geq c_1 \left( C \alpha^p \right)^{-(n-m)/m} \omega.$$

Combining the two previous lines, and remembering that $\omega = \varepsilon \alpha^{p-m}$, $p = m^* - \varepsilon$, we obtain

$$\alpha^{m^* - m - \varepsilon} \leq \frac{A}{\varepsilon},$$

where $A \equiv (\hat{c}/c_1)C^{(n-m)/m}$ depends only on $m, n$. Finally, using the given restriction

$$0 < \varepsilon \leq \frac{m-1}{n-n-m} \frac{m^2}{m-m}$$

(note $m^* - m = m^2/(n-m)$), one derives from (46) that

$$\alpha^{m/(n-m)} \leq \frac{A}{\varepsilon};$$

(45) now follows immediately, and the proof is complete. \hfill \Box

Together with the inequality $\alpha > 1$, Lemma 5 implies the important conclusion

$$\alpha^\varepsilon \to 1 \quad \text{as } \varepsilon \to 0.$$
LEMMA 6. – Let $n > m^2$. Then there exists $K' > 0$ (depending only on $m, n$) such that
\[ \omega = \varepsilon \alpha^{p-m} \leq K' \text{ for all } \varepsilon \in \left(0, \frac{m-1}{n \cdot m^2} \right). \]

Proof. – We have
\[ \alpha^{p-m} = \alpha^{m^2-m-\varepsilon} \cdot \alpha^{\varepsilon \frac{m}{m^2}} \leq A \cdot \left( \frac{A}{\varepsilon} \right)^{\varepsilon \left( \frac{m}{m^2} \right)^2}, \]
by (45) and (46). Hence
\[ \omega = \varepsilon \alpha^{p-m} \leq A \cdot \left( \frac{A}{\varepsilon} \right)^{\varepsilon \left( \frac{m}{m^2} \right)^2}. \]
It remains to show that the right side is bounded, but this follows directly from the fact that $(1/s)^s$ is bounded ($\preceq \epsilon^{1/2}$) on $(0, \infty)$. The proof is complete. \(\square\)

Remark. – A short calculation, taking into account restriction (47), shows that in fact we can choose $K' = A^{m(n-m+1)/n} \omega^{(n-m)/em^2}$.

We can now complete the proof of (4). Here it is convenient to revert to the original understanding that $w = w(r)$ and $z = z(r)$. We first rewrite the results of Lemmas 1, 2 as
\[ 0 < z - w < C_3 - 1 \text{ for all } r > 0, \]
where $C_3 = C_3(\varepsilon) = C_2 \alpha^{\varepsilon/(m-1)} \to 1$ as $\varepsilon \to 0$; of course also $C_3 > 1$ by Lemma 2.

From Proposition 3 applied to equation (31) we obtain
\[ \int_0^\infty Q_\eta(r)r^{n-1} \, dr = 0, \]
where $Q_\eta(r)$ is defined by (32); see the same argument in Lemma 3.
Now by (29) and Lemma 6 we know that $\varepsilon/\eta = \omega \in [K, K']$. Then, since $w \leq 1$, it follows from (32) that
\[ |Q_\eta(r)| \leq \text{Const } m \eta w^m \leq \text{Const } m^\eta \hat{z}^m, \]
see the proof of Lemma 5. Recalling that $\hat{z}^m \in L^1(\mathbb{R}^n)$, we can therefore apply the Lebesgue dominated convergence theorem to (50) when $\varepsilon \to 0$. Clearly $\omega$ converges to some limit $\omega_0 \in [K, K']$, up to a subsequence (in fact we will determine a unique possible value for $\omega_0$, which shows that $\omega \to \omega_0$ on the continuum $\varepsilon > 0$). Moreover by (49) and the fact that $\eta \to 0$ as $\varepsilon \to 0$, we have
\[ z(r) \to z_0(r) \equiv \left(1 + D_{\rho m/(m-1)}\right)^{-(n-m)/m}. \]
pointwise for all $r \geq 0$. Consequently there results
\[ \int_0^\infty z_0^m(r)r^{n-1} dr = \omega_0 \frac{(n - m)^2}{nm^2} \int_0^\infty z_0^m(r)r^{n-1} dr. \]

Both $z_0^m r^{n-1}$ and $z_0^m r^{n-1}$ are in $L^1(0, \infty)$ since $n > m^2$.

By means of the change of variables $s = Dr^{m/(m-1)}$ one obtains
\[ \int_0^\infty z_0^m(r)r^{n-1} dr = \frac{m-1}{m} D^{-\frac{m-1}{m}} B \left( \frac{n(m-1)}{m}, \frac{n-m^2}{m} \right) \] (51)
and
\[ \int_0^\infty z_0^m(r)r^{n-1} dr = \frac{m-1}{m} D^{-\frac{m-1}{m}} B \left( \frac{n(m-1)}{m}, \frac{n}{m} \right). \] (52)

Hence,
\[ \omega_0 = n \left( \frac{m}{n-m} \right)^2 B \left( \frac{n(m-1)}{m}, \frac{n-m^2}{m} \right). \]

We can now prove the asymptotic relation (4). Indeed,
\[ \varepsilon^{(n-m)/m^2} \alpha = (\omega \alpha^\varepsilon)^{(n-m)/m^2} \alpha \rightarrow \omega_0^{(n-m)/m^2} \beta_{m,n} \]
as $\varepsilon \to 0$ (recall $\alpha^\varepsilon \to 1$), which is just (4) for the case $\delta = 1$. Since for general $\delta$ one has $u(0) = \delta^{1/(p-m)} \alpha \approx \delta^{(n-m)/m^2} \alpha$, relation (4) is proved (case $n > m^2$).

5.2. The case $n \leq m^2$

Here $z \notin L^m(\mathbb{R}^n)$ and the crucial Lemma 6 does not hold; nevertheless, we can prove the following result.

**Lemma 7.** Assume that $n \leq m^2$. Then there exists $K' = K'(m, n) > 0$ such that
\[ \varepsilon^{m/(m-1)} \leq K' \log \varepsilon^{(n-m)/m(m-1)}. \]

**Proof.** We argue as in the proof of Lemmas 5 and 6, with several major changes. Let $\ell$ be an exponent greater than $n(m-1)/(n-m)$ to be determined later. Then, from (34) we have
\[ \int_{\mathbb{R}^n} w^{\ell} \leq \int_{\mathbb{R}^n} z^{\ell} \leq \int_{\mathbb{R}^n} \hat{d} < \infty \] (53)
since $\hat{d} \in L^\ell(\mathbb{R}^n)$; here $\hat{d}$ of course depends on $\ell$. On the other hand, by Lemmas 3 and 4 we find
\[ \int_{\mathbb{R}^n} w^{\ell} \geq c_1 \omega \int_{\mathbb{R}^n} w^{\ell} \geq c_1 \omega (C \alpha^\varepsilon)^{-(n-m)/m}. \] (54)
Next, integrating (40) over \((0, \infty)\) and taking into account the exponential decay of \(w\) and \(w'\), as well as (34), we get

\[
\int_{\mathbb{R}^n} w^{m-1} = \alpha^{p-m} \int_{\mathbb{R}^n} w^{p-1} \leq \alpha^{p-m} \int_{\mathbb{R}^n} \hat{z}^{p-1} = \hat{d}_1 \alpha^{p-m},
\]

(55)

where we have used the fact that \(\hat{z} \in L^{p-1}(\mathbb{R}^n)\) (for \(\varepsilon < m/(n-m)\)).

By Hölder interpolation,

\[
\int_{\mathbb{R}^n} w^m \leq \left( \int_{\mathbb{R}^n} w^{m-1} \right)^{1-\vartheta} \left( \int_{\mathbb{R}^n} w^{\ell} \right)^{\vartheta},
\]

(56)

where \(\vartheta = 1/(\ell - m + 1) \in (0, 1)\) since \(n \leq m^2\). A short calculation shows moreover that

\[
\hat{d} = O \left( \frac{n-m}{m-1} \ell - n \right)^{-1} \text{ as } \ell \to \frac{n(m-1)}{n-m}.
\]

(57)

Now we choose \(\ell \) near to but slightly larger than \(n(m-1)/(n-m)\), namely

\[
\ell = \frac{m-1}{1 - |\log \varepsilon|^{-1}} \left( \frac{n}{n-m} - \frac{1}{|\log \varepsilon|} \right),
\]

with \(\varepsilon\) so small that \(|\log \varepsilon| > 1\). Then

\[
\vartheta = (\ell - m + 1)^{-1} = \frac{n-m}{m(m-1)} (1 - |\log \varepsilon|^{-1}), \quad \text{and}
\]

\[
\left( \frac{n-m}{m-1} \ell - n \right)^{-1} = \frac{|\log \varepsilon| - 1}{m}.
\]

Inserting (53), (54), (55), (57) into (56) now gives, after a little calculation,

\[
\varepsilon \alpha^{(p-m)\vartheta - c(n-m)/m} \leq A_1 |\log \varepsilon|^\vartheta
\]

where \(A_1 = A_1(m,n)\); hence in turn,

\[
\varepsilon \alpha^{m/(m-1) - \rho/(m-1)} \leq A_1 |\log \varepsilon|^{(n-m)/m(m-1)}
\]

with \(\rho = m|\log \varepsilon|^{-1} + \varepsilon(n-m)\).

For suitably small \(\varepsilon\), say \(\varepsilon \leq \varepsilon_0\), one then obtains (compare Lemma 5)

\[
\alpha \leq \left( \frac{A_1}{\varepsilon} \right)^{2(m-1)/m}.
\]

(58)

As before this implies that \(\alpha^\varepsilon\) and \(\alpha^{1/|\log \varepsilon|}\) are bounded, that is, \(\alpha^\rho\) is bounded, from which the lemma follows at once, subject of course to the previous restrictions given for \(\varepsilon\). □
From (58) it follows that $\alpha \varepsilon \to 1$ as $\varepsilon \to 0$, just as in the case $n > m^2$. In turn (49) holds exactly as before, with $C_3 \to 1$ as $\varepsilon \to 0$.

For the next conclusion, we shall need a sharper form for the behavior of $C_3$. First, it is not difficult to verify that the function $C_1 = C_1(\varepsilon)$ defined in (37) satisfies

$$C_1 \leq 1 + c \varepsilon |\log \varepsilon|$$

for some constant $c > 0$; we understand here and in what follows that $c$ denotes a generic positive constant, depending only on $m$ and $n$. Moreover, by (29) we have $\eta < c \varepsilon$, so the function $C_2 = C_2(\varepsilon)$ defined in (41) also satisfies

$$C_2 \leq 1 + c \varepsilon |\log \varepsilon|.$$  

Finally

$$C_3 = C_2 \alpha \varepsilon^{(m-1)} \leq 1 + c \varepsilon |\log \varepsilon|$$

(59)

for sufficiently small $\varepsilon$.

Next, let $R > 0$ denote the unique value of $r$ where $z(R) = \nu \varepsilon |\log \varepsilon|$, where $\nu > 0$ is a constant to be determined later; note in particular that $R \to \infty$ as $\varepsilon \to 0$. Now, arguing from (39) and the fact that

$$1 < C_3 < 1 + c \varepsilon |\log \varepsilon|,$$

we infer

$$w(r) > C_3 z(r) - (C_3 - 1) \frac{z(r)}{z(R)} > \left(1 - \frac{C_3 - 1}{\nu \varepsilon |\log \varepsilon|}\right) z(r)$$

$$\geq \left(1 - \frac{c}{\nu}\right) z(r) \quad \forall r \in [0, R].$$

In turn, fixing $\nu$ sufficiently large,

$$w(r) \geq \frac{1}{2} z(r) \quad \forall r \in [0, R].$$

(60)

We can now prove a companion result to (29); in particular, it shows that Lemma 6 does not hold when $n \leq m^2$.

**Lemma 8.** – There exists $K_1 = K_1(m, n) > 0$ such that for $\varepsilon$ sufficiently small

$$\varepsilon \alpha^{m/(m-1)} \geq K_1 |\log \varepsilon|^{(n^2 - m^2)/(m-1)} \quad \text{when } m < n < m^2$$

and

$$\varepsilon \alpha^{m/(m-1)} \geq K_1 |\log \varepsilon| \quad \text{when } n = m^2.$$
Proof. – Assume first that \( n < m^2 \). Then for \( \varepsilon \) sufficiently small there holds
\[
\hat{d}_1 \geq \int_{\mathbb{R}^n} z^p \geq \int_{\mathbb{R}^n} u^p \quad \text{by (34)}
\]
\[
\geq \frac{c}{\omega} \int_{\mathbb{R}^n} u^m \quad \text{by Lemma 3}
\]
\[
\geq \frac{c}{\omega} \int_{|x|<R} z^m \quad \text{by (60)}
\]
\[
\geq \frac{c}{\omega} \int_1^R \frac{t^{n-1}}{t^{m(n-m)/(m-1)}} \, dt \quad \text{by (33)}
\]
\[
= \frac{c}{\omega} \left( R^{(m^2-n)/(m-1)} - 1 \right)
\]
\[
\geq \frac{c}{\omega} (\varepsilon |\log \varepsilon|)^{(m^2-n)/(n-m)},
\]
where the last inequality is obtained by solving \( z(R) = \nu \varepsilon |\log \varepsilon| \) (\( \varepsilon \) small). Rearranging with the help of the relation \( \omega = \varepsilon \alpha^{p-m} \leq \varepsilon \alpha^{m^2/(n-m)} \) now yields the first statement of the lemma.

If \( n = m^2 \), the same arguments lead to
\[
\hat{d}_1 \geq \frac{c}{\omega} \int_1^R \frac{dt}{t} = \frac{c}{\omega} \log R \geq \frac{c}{\omega} |\log \varepsilon|,
\]
from which the second statement follows at once. \( \square \)

Lemma 8 shows at once that (4) also holds in the case \( m < n \leq m^2 \), that is whenever \( n > m \).

Remark. – As already mentioned in the introduction, more precision in the asymptotic behavior of \( u(0) \) is needed in the case \( n \leq m^2 \). We conjecture that also in this case there exists a continuous increasing function \( g_{m,n} \) defined on \([0, \infty)\) such that \( g_{m,n}(0) = 0 \) and \( \lim_{\varepsilon \to 0} [g_{m,n}(\varepsilon)u(0)] = 1 \).

5.3. Dirac limits

Here we shall complete the demonstration of Theorem 2 by proving conditions (5) and (6). It will be convenient here and in the sequel not to make the initial assumption \( \delta = 1 \), though we continue to write \( u(0) = \alpha \).

From Section 5.1 we recall the basic estimate (49); with the help of (59) this can be rewritten in the form
\[
0 < z - w < c \varepsilon |\log \varepsilon|.
\]
Here we wish to scale back to the original function \( u \), this being accomplished by means of (26) and (30). More specifically, in (30) it is necessary to replace \( u \) and \( \alpha \) respectively
by $v$ and $\beta$ ($\beta$ as in (2)) because of the initial assumption in Section 5 that $\delta = 1$. The required rescaling is therefore given by

$$w(r) = \frac{1}{\delta^{1/(p-m)} \beta} u \left( \frac{r}{\delta^{1/m} \beta(p-m)/m} \right) = \frac{1}{\alpha} u \left( \frac{r}{\alpha(p-m)/m} \right)$$

(62)

where from Theorem 1 we have $\delta^{1/(p-m)} \beta = \alpha$. After a little calculation, (61) then leads to the basic formula

$$0 < z_\alpha - u \leq c \alpha \varepsilon |\log \varepsilon|,$$

(63)

where

$$z_\alpha = z_\alpha(x) = \alpha z(\alpha(p-m)/m|x|)$$

$$= \alpha / [1 + (1 - \eta)^{(m-1)\alpha(p-m)/(m-1)} D x^{m/(m-1)}]^{(n-m)/m}$$

(64)

and (33) is used at the last step.

Observe from the left hand inequality of (63) that (recall $\eta \to 0$ as $\varepsilon \to 0$)

$$\alpha^{1/(m-1)} u(x) < \alpha^{1/(m-1)} z_\alpha(x) \to D - n - m/m |x| - n/m - 1$$

as $\varepsilon \to 0$,

which immediately yields (5).

To prove (6), let $X = X_R$ denote the Lebesgue space $L^m$ over the domain $|x| < R$, and similarly let $X' = X'_R$ be the space $L^m$ over the domain $|x| \geq R$. By Minkowski’s inequality and (63),

$$\|u\|_X - \|z_\alpha\|_X \leq \|u - z_\alpha\|_X \leq c \alpha \varepsilon |\log \varepsilon| 1\|_X.$$

(65)

In particular, let us make the new choice

$$R = \alpha^{-m/(n-m)+\mu},$$

where $\mu > 0$ is a positive constant to be determined later. Then with the obvious change of variables $s = \alpha^{(p-m)/m} r$, we find

$$\|z_\alpha\|_{m}^m = \alpha^{n/m} \int_0^{\alpha^{-m/m+\mu}} ds \frac{s^{n-1} ds}{[1 + (1-\eta)^{(m-1)\alpha(p-m)/(m-1)} D s^{m/(m-1)}]^n} \gamma_{m,n}$$

(66)

as $\varepsilon \to 0$, see (52) and (48) (which as shown in Section 5.2 is valid for all $n > m$). By the same calculation

$$\|z_\alpha\|_{m'}^m \to 0$$

(67)

as $\varepsilon \to 0$, since the integration is now over the interval $(\alpha^{-m/m+\mu}, \infty)$ and the integral is convergent.

Next, one calculates that

$$\|1\|_X = \frac{\omega_n}{n} R^{n/m} = \frac{\omega_n}{n} \alpha^{-1+\mu(n-m)/m}$$
in view of the definition of $R$. We can now determine the limit as $\varepsilon \to 0$ of the quantity
\[
\alpha \varepsilon |\log \varepsilon| \|1\|_X = (\omega_n/n)\alpha^{(n-m)/m} \varepsilon |\log \varepsilon|.
\]
From Lemmas 6 and 7 it is evident that, whatever the case considered, there exists $\lambda > 0$ (depending only on $m, n$) such that $\alpha < c \varepsilon^{-\lambda}$, provided $\varepsilon$ is small. (One can check that $\lambda = (n-1)/m$ in fact suffices.) Hence
\[
\alpha^{(n-m)/m} \varepsilon |\log \varepsilon| \leq c \varepsilon^{1-\lambda} \alpha^{(n-m)/m} |\log \varepsilon|,
\]
which tends to 0 as $\varepsilon \to 0$ if $\mu$ is chosen small enough. It now follows at once from (65) and (66) that $\|u\|_X^\mu \to \gamma_{m,n}$ as $\varepsilon \to 0$.

We observe finally from the left hand inequality of (63) that
\[
\|u\|_{X^\mu}^\mu < \|z\|_{X^\mu}^\mu \to 0
\]
by (67). Hence
\[
\|u\|_{m^\mu}^\mu = \|u\|_{X^\mu}^\mu + \|u\|_{X'}^\mu \to \gamma_{m,n},
\]
proving the second part of (6).

To obtain the first part, note that integration of $(P_\delta \rho)$ over $\mathbb{R}^n$ and use of Theorem 8 yields
\[
\delta \int_{\mathbb{R}^n} u^{m-1} = \int_{\mathbb{R}^n} u^{p-1}.
\]
But from the left inequality of (63) together with a calculation as in (66), we have
\[
\int_{\mathbb{R}^n} u^{p-1} \leq \int_{\mathbb{R}^n} z^{p-1} = \omega_n \alpha^{-(n-1)/m} \int_0^\infty s^{n-1} ds \frac{1}{1 + (n-1)/m} D_s^{m/n}(1-(n-1)/(n-m)(p-1)/m).
\]
Since the integral is uniformly bounded for any $\varepsilon$ less than $m/(n-m)$, we then get
\[
\int_{\mathbb{R}^n} u^{p-1} \to 0 \quad \text{as } \varepsilon \to 0.
\]
With the help of (68) (and a trivial interpolation) this completes the proof of (6), and therefore of Theorem 2.

6. Proof of Theorem 3

First we prove (8). Multiplying the equation $(P_\delta \rho)$ by $u$ and integrating over $\mathbb{R}^n$ gives
\[
\int_{\mathbb{R}^n} |\nabla u|^m = -\delta \int_{\mathbb{R}^n} u^m + \int_{\mathbb{R}^n} u^p.
\]
We now let \( \varepsilon \to 0 \). The first term on the right approaches 0 by (6).

To treat the second term on the right side of (69), we slightly modify the space \( X \) from its meaning in the previous subsection, so that now it represents the Lebesgue space \( L^p \) over the domain \([|x| < R]\), and similarly for the space \( X' \). Then as in (66) there holds

\[
\| z_\alpha \|_X^p = \omega_n \alpha^{\epsilon(n-m)/m} \int_0^{s_n-1} \left[ 1 + (1 - \eta)^{1/(m-1)} \right]^{(n-m)/m} ds,
\]

the integral being convergent when \( \varepsilon < m/n \).

To evaluate the limit of the right side, note first that on the interval \( 0 < s < \alpha \mu \) there holds (for small \( \varepsilon \))

\[
1 < \left[ 1 + (1 - \eta)^{1/(m-1)} \right]^{(n-m)/m} \varepsilon^{(n-m)/m},
\]

so that by (48), uniformly for \( s \in (0, \alpha \mu) \),

\[
\left[ 1 + (1 - \eta)^{1/(m-1)} \right]^{(n-m)/m} \varepsilon^{(n-m)/m} \to 1.
\]

Hence as in (66), one obtains \( \| z_\alpha \|_X^p \to \gamma_{m,n} \) as \( \varepsilon \to 0 \). Also as before, \( \| z_\alpha \|_{X'}^p \to 0 \), so that finally, again arguing as in the previous subsection,

\[
\| u \|_X^p = \| u \|_X^p + \| u \|_{X'}^p \to \gamma_{m,n},
\]

that is, \( \int_{\mathbb{R}^n} u^p \to \gamma_{m,n} \). The second statement in (8) follows at once from (69). In order to prove the first statement in (8), note that by (62) we have

\[
\int_{\mathbb{R}^n} |\nabla u|^q \leq c \alpha^{p/q} \int_{\mathbb{R}^n} |w'(\alpha^{(n-m)/m})|^q r^{n-1} dr \quad \forall q \geq 1;
\]

note also that \( z \in D^{1,q}(\mathbb{R}^n) \) for all \( q > n(m-1)/(n-1) \) and that \( \| \nabla z \|_q \) remains bounded as \( \varepsilon \to 0 \): therefore, by (41) and an obvious change of variables, we obtain

\[
\int_{\mathbb{R}^n} |\nabla u|^q \leq c \alpha^{p(q-n)/m+n} \int_0^{\infty} |z'(r)|^q r^{n-1} dr \leq c \alpha^{p(q-n)/m+n} \to 0 \quad \forall q \in \left( m, \frac{n(m-1)}{n-1} \right)
\]

which completes the proof of (8).

It remains to prove (7). By evaluating \( z'(r) \) and by using (41) and (59) we obtain

\[
|w'(r)| \leq (1 + c\varepsilon |\log \varepsilon|) \frac{n-m}{m-1} (1 - \eta)^{1/(m-1)} \times D \frac{r^{1/(m-1)}}{(1 + (1 - \eta)^{1/(m-1)} D r^{m/(m-1)})^{n/m}}.
\]

Moreover, according to the “double rescaling” (62) we have

\[
|w'(r)| \leq \frac{1}{\alpha^{p/m}} |u'(r/\alpha^{(p-m)/m})|.
\]
Inserting this in (70), using an obvious change of variables and then letting \( \varepsilon \to 0 \), yields
\[
\lim_{\varepsilon \to 0} \left\{ \alpha^{1/(m-1)} |u'(r)| \right\} \leq \left( \frac{n-m}{m-1} \right)^{n/m} n^{(n-m)/m(m-1)} r^{(1-n)/(m-1)},
\]
which immediately gives (7) since \( \alpha = u(0) \).

### 7. Proof of Theorem 4

We define
\[
\tau(\varepsilon) = \tau(\varepsilon, d) = \frac{1}{\varepsilon} \left( \frac{d}{\beta} \right)^{p-m},
\]
where \( \beta \) is given by (2); here \( \beta \) is a (well-defined) continuous function of \( \varepsilon \) and of course also of \( m, n \). By Theorem 1, when \( \delta = \varepsilon \tau(\varepsilon) \) we have
\[
u(0) = \delta^{1/(p-m)} \beta = d,
\]
proving (ii). Also by Theorem 2 we know that when \( n > m^2 \) (case \( \delta = 1 \))
\[
e^{(n-m)/m^2} \beta \to \beta_{m,n} \quad \text{as} \quad \varepsilon \to 0,
\]
so that
\[
\tau(\varepsilon) = \left( \frac{d}{\varepsilon^{(n-m)/m^2} \beta} \right)^{p-m} \to \left( \frac{d}{\beta_{m,n}} \right)^{m^2/(n-m)} \quad \text{as} \quad \varepsilon \to 0;
\]
similarly, when \( n \leq m^2 \), by Theorem 2 we infer that \( \tau(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Statement (i) is so proved.

To prove the final statement of the theorem, we first use (63), together with the fact that in the present case \( \alpha = u(0) = d \), to infer the fundamental relation
\[
|u - zd| \leq c \varepsilon |\log \varepsilon|.
\]
(71)
But by (64), and since \( \eta \to 0 \) as \( \varepsilon \to 0 \), it now follows that
\[
zd(x) \to d \left[ 1 + D \left( d^{\frac{m}{m^2}} |x| \right)^{\frac{m}{m-1}} \right]^{-\frac{n}{m}} U_d(x)
\]
uniformly for \( x \) in \( \mathbb{R}^n \); see (1) in the introduction. Together with (71) this completes the proof of (ii).

An easy consequence of the above argument is the following companion result for Theorem 4.

**Corollary.** Let \( n > m^2 \). In place of the condition \( \delta = \varepsilon \tau(\varepsilon) \), suppose that \( \delta = a \varepsilon \), where \( a \) is a positive constant. Then \( u \to U_d \) uniformly on \( \mathbb{R}^n \) as \( \varepsilon = p - m \to 0 \), where \( d = a^{(n-m)/m^2} \beta_{m,n} \).
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REFERENCES