Ann. I. H. Poincaré – AN 20, 1 (2003) 107–143

S0294-1449(02)00008-2/FLA

CLUSTERING LAYERS AND BOUNDARY LAYERS IN SPATIALLY INHOMOGENEOUS PHASE TRANSITION PROBLEMS

COUCHES LIMITES DANS DES PROBLÈMES DE TRANSITION DE PHASE SPATIALEMENT INHOMOGENES

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Received 12 February 2001

RÉSUMÉ. – Nous étudions dans cet article l'existance de solutions « multi-couches » du problème de transition de phase suivant :

$$-\varepsilon^2 u_{xx} + W_u(x,u) = 0 \quad \text{in } (0,1),$$
$$u_x(0) = u_x(1) = 0$$

où $\varepsilon > 0$ est un petit paramètre et $W(x,u)$ est un potentiel à double parts « équlibre ». En particulier, nous montrons l'existence de solutions avec couches limites et couches.

0. Introduction

In this paper we study the existence of solutions with multiple transition layers for the following spatially inhomogeneous problem of Allen–Cahn type:

$$-\varepsilon^2 u_{xx} + W_u(x,u) = 0 \quad \text{in } (0,1),$$
$$u_x(0) = u_x(1) = 0. \tag{0.1}$$

Here $\varepsilon > 0$ is a small parameter and $W(x,u)$ is a double-well potential. A typical example of $W(x,u)$ is $\frac{1}{4}h(x)^2(1-u^2)^2$ with $h(x): [0,1] \to (0,\infty)$ and in this case

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1 Authors are partially supported by Waseda University Grant for Special Research Projects.

(0.1) is called (spatially inhomogeneous) Allen–Cahn equation. Such problems and their higher dimensional versions appear in various situations related to phase transitions.

Here we assume

(W1) $W(x,u) \in C^2([0,1] \times \mathbf{R})$,

(W2) there exist 2 functions $\alpha_-(x), \alpha_+(x) \in C^2([0,1])$ such that

$$\alpha_-(x) < 0 < \alpha_+(x) \quad \text{for all } x \in [0,1], \tag{0.2}$$

$$W(x,\alpha_-(x)) = W(x,\alpha_+(x)) = 0 \quad \text{for all } x \in [0,1], \tag{0.3}$$

$$W_u(x,\alpha_-(x)) = W_u(x,\alpha_+(x)) = 0 \quad \text{for all } x \in [0,1], \tag{0.4}$$

$$W_{uu}(x,\alpha_-(x)) > 0, \ W_{uu}(x,\alpha_+(x)) > 0 \quad \text{for all } x \in [0,1], \tag{0.5}$$

$$W(x,s) > 0 \quad \text{for all } s \in \mathbf{R} \setminus \{\alpha_-(x), \alpha_+(x)\}. \tag{0.6}$$

Two functions $\alpha_-(x), \alpha_+(x)$ represent two stable states of the potential $W(x,u)$. It is not difficult to find two stable solutions of (0.1) which are close to $\alpha_-(x)$ or $\alpha_+(x)$ for all $x \in [0,1]$. Besides such stable solutions (0.1) gives rise to solutions with finitely many sharp transition layers joining two stable states $\alpha_-(x)$ and $\alpha_+(x)$ for small $\varepsilon > 0$. The main purpose of this paper is to study location and multiplicity of such transition layers. Especially we are interested in *clustering layers* and *boundary layers*. Here we mean, by clustering layers, multiple transition layers which are not isolated and appear in a small neighborhood of a particular point in $[0,1]$. By boundary layers, we mean transition layers which approach to the boundary 0 or 1 of $[0,1]$ as $\varepsilon \to 0$.

We remark that a typical feature of our problem is the property (0.3) and our potential $W(x,u)$ is "balanced", that is, depth of two wells are equal for all $x \in [0,1]$. For the study of layered solutions for the "unbalanced" case, we refer to Angenent, Maret-Paret and Peletier [3] and Ai and Hastings [1]. We also refer to Kath [13] and Gedeon, Kokubu, Mischaikow and Oka [8] for the study of slowly varying planar Hamiltonian systems from the dynamical point of view. [2] In particular, in recent paper [8], Gedeon, Kokubu, Mischaikow and Oka showed the existence of complicated dynamics, which is described in terms of symbolic sequence of integers, by means of the Conley index theory.

In previous papers [15,16], the first author studies a special class of balanced potentials; her potential has a form:

$$W(x,u) = h(x)^2 F(u)$$

and she assumes conditions related to (W1)–(W2); in particular, for some $\alpha_- < 0 < \alpha_+$, she assumes that $F(\alpha_-) = F(\alpha_+) = 0$, $F(u) > 0$ for $u \in \mathbf{R} \setminus \{\alpha_-, \alpha_+\}$, $F_u(0) = 0$, $F_{uu}(0) < 0$ and

$$\frac{F_u(u)}{u} < F_{uu}(u) \quad \text{for all } u \neq 0. \tag{0.7}$$

We remark that spatially inhomogeneous Allen–Cahn equation satisfies these assumptions.

[2] After completing this work, the Authors has learned about works [13,8] from Professor H. Kokubu. The Authors would like to thank to Professor H. Kokubu for this information.

We observe in [15,16] that transition layers appear only in a neighborhood of critical points of $h(x)$ and boundary points 0 and 1 of [0, 1]:

$$\{x \in (0,1);\ h'(x) = 0\} \cup \{0,\ 1\}.$$

Moreover at most one transition layer can appear in a neighborhood of interior local minimum of $h(x)$.

Under non-degeneracy assumption on $h(x)$:

$$h''(x) \neq 0 \quad \text{if } h'(x) = 0$$

we studied in [16] non-degeneracy and the existence of solutions with interior clustering layers, but without boundary layers. Our proof of the existence in [16] is based on the non-degeneracy of solutions and global bifurcation theory.

In this paper, we continue a study of layered solutions and we investigate the existence of solutions with boundary and interior layers in more general setting (0.1). We do not assume non-degeneracy conditions. Here we use a variational argument and we find layered solutions rather in a constructive way.

Now we state our main results. In our setting (0.1), the following function plays an important role.

$$G(t) = \sqrt{2} \int_{\alpha_-(t)}^{\alpha_+(t)} \sqrt{W(t,\tau)}\, \mathrm{d}\tau : [0,1] \to \mathbf{R}.$$

We can determine location of layers of solutions by $G(t)$. For sufficiently small $\varepsilon > 0$, we can find a solution $u_\varepsilon(x)$ of (0.1) which has prescribed number of zeros near local minima and maxima of $G(t)$; as numbers of zeros of solutions, we can prescribe 0 or 1 at interior local minima of $G(t)$ and any non-negative integers at interior local maxima of $G(t)$. At boundary points 0, 1 of the spatial region [0, 1], we can prescribe any non-negative integer if $G(t)$ takes local maxima there. To state our existence result precisely, we assume

(W3) $G(t)$ has finitely many critical points in $(0,1)$

in addition to (W1)–(W2). We use notation

$$M_+ = \{p \in [0,1];\ G(t) \text{ takes a local maximum in } [0,1] \text{ at } p\},$$
$$M_- = \{p \in (0,1);\ G(t) \text{ takes a local minimum at } p\},$$
$$M = M_+ \cup M_-.$$

We remark that 0 (1 respectively) belongs to M_+ if $G'(0) < 0$ ($G'(1) > 0$ respectively). Our main result is the following theorem:

THEOREM 0.1. – *Assume* (W1)–(W3). *Then for any* $\delta > 0$ *and for any sequence of non-negative integers* $(n_p)_{p \in M}$ *satisfying*

$$n_p \in \{0,1\} \quad \textit{if } p \in M_-,$$

there exists an $\varepsilon_0 = \varepsilon_0((n_p), \delta) > 0$ *such that for any* $0 < \varepsilon \leqslant \varepsilon_0$, (0.1) *has solutions* $u_\varepsilon^+(x)$, $u_\varepsilon^-(x)$ *with the following properties*:

(i) $\pm u_\varepsilon^\pm(0) > 0$,

(ii) $u_\varepsilon^\pm(x)$ *has exactly* n_p *zeros in* $[p-\delta, p+\delta]$ *for all* $p \in M$,

(iii) $u_\varepsilon^\pm(x) \neq 0$ *for all* $x \in [0,1] \setminus \bigcup_{p\in M}[p-\delta, p+\delta]$.

Remark 0.2. – (i) If $W(x,u) = h(x)^2 F(u)$, we have $G(t) = Ch(t)$, where $C = \sqrt{2}\int_{-1}^1 F(s)\,\mathrm{d}s$ is a constant independent of t, and $M_\pm$ are sets of points where $h(x)$ takes local maximum in [0, 1] or local minimum in (0, 1).

(ii) We can generalize (W3) slightly. See Remark 5.1 below.

Solutions of (0.1) can be characterized as critical points of the following functional:

$$I_\varepsilon(u) = \int_0^1 \frac{\varepsilon}{2}|u_x|^2 + \frac{1}{\varepsilon} W\big(x, u(x)\big)\,\mathrm{d}x$$

and solutions $u_\varepsilon^\pm(x)$ in Theorem 0.1 will be obtained as a critical points satisfying

$$I_\varepsilon(u_\varepsilon) \to \sum_{p\in M} n_p G(p) \quad \text{as } \varepsilon \to 0. \tag{0.8}$$

By virtue of (0.3), we can also show that solutions $u_\varepsilon(x)$ of (0.1) with finite energy $\limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) < \infty$ must be very close to $\alpha_-(x)$ or $\alpha_+(x)$ in most of the spatial region [0, 1] for ε small and $u_\varepsilon(x)$ has finitely many zeros. Moreover any zero of $u_\varepsilon(x)$ belongs to a neighborhood of critical points of $G(t)$ or boundaries 0, 1. More precisely, we have

THEOREM 0.3. – *Assume* (W1) *and* (W2). *Then for any* $A > 0$, $\delta > 0$ *there exist* $\varepsilon_0 = \varepsilon_0(A, \delta) > 0$, $n_0 = n_0(A) \in \mathbf{N}$, $C(\delta) > 0$ *with the following properties*: *if* $\varepsilon \in (0, \varepsilon_0]$ *and solutions* $u_\varepsilon(x)$ *of* (0.1) *satisfy*

$$I_\varepsilon(u_\varepsilon) \leqslant A, \tag{0.9}$$

then

(i) $\mathrm{meas}\{x \in [0,1];\ u_\varepsilon(x) \notin [\alpha_-(x) - \delta, \alpha_-(x) + \delta] \cup [\alpha_+(x) - \delta, \alpha_+(x) + \delta]\} \leqslant C(\delta)\varepsilon$,

(ii) $u_\varepsilon(x)$ *has at most* n_0 *zeros in* [0, 1] *and any of zeros belongs to* δ*-neighborhood of critical points of* $G(t)$ *or boundaries* {0, 1}.

Proofs of Theorems 0.1 and 0.3 will be given in the following sections. To find solutions $u_\varepsilon^\pm(x)$ stated in Theorem 0.1, we use a method quite different from [1,3,8, 15,16]; we employ variational arguments and we apply a method originally introduced by Hemple [11] and Chen [4]. More precisely, we set

$$\Delta = \big\{(t_1, t_2, \ldots, t_n);\ 0 < t_1 < t_2 < \cdots < t_n < 1\big\}$$

and define for $(t_1, \ldots, t_n) \in \Delta$

$$f_\varepsilon^\pm(t_1, t_2, \ldots, t_n) = \inf\{I_\varepsilon(u);\ u \in H^1(0,1), \pm(-1)^j u(x) \geqslant 0 \text{ in } [t_j, t_{j+1}] \text{ for } j = 0, 1, 2, \ldots, n\}. \tag{0.10}$$

Here we use convention $t_0 = 0$ and $t_{n+1} = 1$.

We will show that critical points of $f_\varepsilon^\pm(t_1, t_2, \ldots, t_n)$ are corresponding to critical points of $I_\varepsilon(u)$ with transition layers at $t_1, t_2, \ldots, t_n$ under condition $(t_{j+1} - t_j)/\varepsilon \gg 1$ for all j (see Corollary 1.6 below). Conversely, if a solution $u_\varepsilon(x)$ satisfies (0.9) for $\varepsilon > 0$ small, then it is also characterized as a critical point of (0.10).

We study the behavior of $f_\varepsilon^\pm(t_1, \ldots, t_n)$ and its derivatives to find critical points of $f_\varepsilon^\pm(t_1, \ldots, t_n)$. We observe that

$$f_\varepsilon^\pm(t_1, \ldots, t_n) \sim \sum_{j=1}^n G(t_j) - \sum_{j=0}^n \exp\left(-\rho_{\pm,j}(t_j)\frac{t_{j+1} - t_j}{\varepsilon}\right). \tag{0.11}$$

Here $\rho_{\pm,j}(t)$ $(j = 0, 1, \ldots, n)$ are positive continuous functions of t. For the precise meaning of (0.11) we refer to Proposition 1.15 and Remark 1.16 below. Estimates of derivatives of $f_\varepsilon^\pm$ related to (0.11) are important for the proofs of Theorems 0.1, 0.3 and this is a key of our proof. In particular, the second term of (0.11) seems to be related to the interaction phenomena between two layers or between a layer and a boundary.

Finally we would like to give a mention about works [2,5–7,9,10,12,14,17–21] where a similar question for nonlinear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = u^p \quad \text{in } \mathbf{R}^N$$

is studied via variational arguments and our approach in this paper is largely motivated by these works.

1. Minimizing problems in subintervals and variational formulation of (0.1)

1.1. Variational formulation of (0.1)

To give our variational formulation to (0.1), we use the following notation: for $0 \leqslant s < t \leqslant \infty$

$$\begin{aligned}
E_{NN}^\pm(s,t) &= \{u \in H^1(s,t);\ \pm u(x) \geqslant 0 \text{ in } [s,t]\},\\
E_{DN}^\pm(s,t) &= \{u \in E_{NN}^\pm(s,t);\ u(s) = 0\},\\
E_{ND}^\pm(s,t) &= \{u \in E_{NN}^\pm(s,t);\ u(t) = 0\},\\
E_{DD}^\pm(s,t) &= \{u \in E_{NN}^\pm(s,t);\ u(s) = u(t) = 0\}
\end{aligned}$$

and

$$I_{\varepsilon,(s,t)}(u) = \int_s^t \frac{\varepsilon}{2}|u_x|^2 + \frac{1}{\varepsilon}W(x,u)\,\mathrm{d}x \quad \text{for } u \in E_{NN}^\pm(s,t).$$

Here D and N stand for Dirichlet and Neumann boundary conditions. We define

$$m^{\pm}_{DD}(\varepsilon; s,t) = \inf_{u\in E^{\pm}_{DD}(s,t)} I_{\varepsilon,(s,t)}(u),$$
$$m^{\pm}_{ND}(\varepsilon; s,t) = \inf_{u\in E^{\pm}_{ND}(s,t)} I_{\varepsilon,(s,t)}(u),$$
$$m^{\pm}_{DN}(\varepsilon; s,t) = \inf_{u\in E^{\pm}_{DN}(s,t)} I_{\varepsilon,(s,t)}(u).$$

With this notation, our functional $f^{\pm}_{\varepsilon}(t_1,t_2,\dots,t_n)$ given in (0.10) can be written as

$$f^{\pm}_{\varepsilon}(t_1,t_2,\dots,t_n) = m^{\pm}_{ND}(0,t_1) + m^{\mp}_{DD}(t_1,t_2) + \cdots + m^{\pm(-)^{n-1}}_{DD}(t_{n-1},t_n) + m^{\pm(-)^n}_{DN}(t_n,1). \tag{1.1}$$

Here

$$(-)^n = \begin{cases} - & \text{if } n \text{ is odd,} \\ + & \text{if } n \text{ is even.} \end{cases}$$

Thus analysis of minimizing problems $m^{\pm}_{DD}(\varepsilon;s,t)$, $m^{\pm}_{ND}(\varepsilon;s,t)$, $m^{\pm}_{DN}(\varepsilon;s,t)$ is essential in our approach.

To analyze minimizing problems $m^{\pm}_{DD}(\varepsilon;s,t)$ etc., we introduce the following minimizing problem for $s\in[0,1]$ and $\ell\in(0,\infty]$

$$b^{\pm}(s,\ell) = \inf_{u\in E^{\pm}_{DN}(0,\ell)} \int_0^{\ell} \frac{1}{2}|v_y|^2 + W\big(s,v(y)\big)\,dy. \tag{1.2}$$

After scaling $v(y)=u(s+\varepsilon y)$, the above minimizing problem appears as a limit problem of $m^{\pm}_{DN}(\varepsilon;s,t)$ or $m^{\pm}_{DD}(\varepsilon;s,t)$. See Lemma 1.8 below.

When $\ell<\infty$, (1.2) is corresponding to positive (or negative) solution of

$$\begin{aligned} &-v_{yy} + W_u\big(s,v(y)\big) = 0 \quad \text{for } y\in(0,\ell), \\ &v(0)=0, \\ &v_y(\ell)=0. \end{aligned} \tag{1.3}$$

We remark that under (W1)–(W2), solutions of (1.4) can be extended to 4ℓ-periodic solutions of $-v_{yy}+W_u(s,v(y))=0$ in $\mathbf{R}$. When $\ell=\infty$, (1.2) is corresponding to a solution of

$$\begin{aligned} &-v_{yy} + W_u(s,v(y)) = 0 \quad \text{for } y\in(0,\infty), \\ &v(0)=0, \\ &v(y)\to\alpha_{\pm}(s) \quad \text{as } y\to\infty. \end{aligned} \tag{1.4}$$

We remark that the solution of (1.5) can be extended to a heteroclinic solution joining $\alpha_{\mp}(s)$ and $\alpha_{\pm}(s)$:

$$\begin{aligned} &-v_{yy} + W_u\big(s,v(y)\big) = 0 \quad \text{for } y\in(-\infty,\infty), \\ &v(0)=0, \\ &v(y)\to\alpha_{\mp}(s) \quad \text{as } y\to-\infty, \\ &v(y)\to\alpha_{\pm}(s) \quad \text{as } y\to\infty. \end{aligned} \tag{1.5}$$

Uniqueness of the heteroclinic solution of (1.5) is easily seen and we denote the unique heteroclinic solution by $\omega^{\pm}(s,\infty;y)$.

Here we give some remarks on solutions $v(y)$ of

$$-v_{yy}+W_u\big(s,v(y)\big)=0. \tag{1.6}$$

It is not difficult to show the following

LEMMA 1.1. – *Assume* (W1)–(W2)*. For any fixed* $s\in[0,1]$ *and for any solution* $v(y)$ *of* (1.6)*, we have the following properties*:

(i) $E_v\equiv\frac{1}{2}|v_y(y)|^2-W(s,v(y))$ *is independent of* y*. Moreover we have* $E_v<0$ *for all periodic solutions* $v(y)$ *of* (1.6)*, in particular, for minimizers of* $b(s,\ell)$ $(\ell<\infty)$*,* $E_v=0$ *for heteroclinic solutions joining* $\alpha_+(s)$ *and* $\alpha_-(s)$*, in particular, for minimizer of* $b(s,\infty)$*.* $E_v\leqslant 0$ *for all bounded solutions of* (1.6)*.*

(ii) *If a bounded solution* $v(y)$ *of* (1.6) *satisfies*

$$v(y_n)\to\alpha_+(s)\quad or\quad \alpha_-(s)$$

for some sequence y_n *satisfying* $y_n\to\infty$ *or* $y_n\to-\infty$*, then* $v(y)$ *is a heteroclinic solutions joining* $\alpha_-(s)$ *and* $\alpha_+(s)$*.*

We will use the above properties repeatedly in the following arguments.

Next we give basic properties of $m^{\pm}_{DD}(\varepsilon;s,t)$, $m^{\pm}_{ND}(\varepsilon;s,t)$, $m^{\pm}_{DN}(\varepsilon;s,t)$, $b^{\pm}(s,\ell)$. Proofs of the following proposition will be given later in Section 6.

PROPOSITION 1.2. – *There exist* $\varepsilon_0>0$ *and* $\ell_0>0$ *such that*

(i) *for* $\varepsilon\in(0,\varepsilon_0]$*,* $(t-s)/\varepsilon\geqslant\ell_0$ *the minimizing problems* $m^{\pm}_{DD}(\varepsilon;s,t)$*,* $m^{\pm}_{ND}(\varepsilon;s,t)$*,* $m^{\pm}_{DN}(\varepsilon;s,t)$ *have unique minimizers* $u(x)$*. The minimizers satisfy*

$$-\varepsilon^2u_{xx}+W_u(x,u)=0\quad in\ (s,t),$$
$$\pm u(x)>0\quad in\ (s,t)$$

and

$$\begin{cases} u(s)=u(t)=0 & for\ m^{\pm}_{DD}(\varepsilon;s,t),\\ u_x(s)=u(t)=0 & for\ m^{\pm}_{ND}(\varepsilon;s,t),\\ u(s)=u_x(t)=0 & for\ m^{\pm}_{DN}(\varepsilon;s,t),\end{cases}$$

(ii) *for* $\ell\in[\ell_0,\infty]$*, the minimizing problem* (1.2) *has a unique minimizer* $\omega(y)$ *and it satisfies* (1.4) *or* (1.5)*.*

In what follows, we denote the unique minimizers by

$$u^{\pm}_{DD}(\varepsilon,s,t;x),\qquad u^{\pm}_{ND}(\varepsilon,s,t;x),\qquad u^{\pm}_{DN}(\varepsilon,s,t;x),\qquad \omega^{\pm}(s,\ell;y)$$

for $\varepsilon\in(0,\varepsilon_0]$, $(t-s)/\varepsilon\geqslant\ell_0$, $\ell\geqslant\ell_0$.

By the uniqueness of the minimizers, we can see that the minimizers depend on the parameters ε, s, t, ℓ continuously in C^1-sense. In particular, setting

$$\mathcal{D}=\big\{(\varepsilon,s,t);\ \varepsilon\in(0,\varepsilon_0],\ 0\leqslant s<t\leqslant 1,\ (t-s)/\varepsilon\geqslant\ell_0\big\},$$

we have

LEMMA 1.3. – *Functions* $u^{\pm}_{DD,x}(\varepsilon, s, t; s)$, $u^{\pm}_{DD,x}(\varepsilon, s, t; t)$, $u^{\pm}_{ND,x}(\varepsilon, s, t; t)$, $u^{\pm}_{DN,x}(\varepsilon, s, t; s) : \mathcal{D} \to \mathbf{R}$ *are functions of class* C^1.

We remark here that the minimizer $u(x)$ is uniquely determined by its initial data $u(s)$ and $u_x(s)$. For $\omega^{\pm}(s, \ell; y)$, we can see that $\omega^{\pm}(s, \ell; y)$ is continuous also at $\ell = \infty$. In particular,

LEMMA 1.4. – *For any* $L > 0$ *and* $\delta > 0$, *there exists* $\eta = \eta(L, \delta) \geqslant \ell_0$ *such that for* $\ell \geqslant \eta$

$$\left\|\omega^{\pm}(s, \ell; y) - \omega^{\pm}(s, \infty; y)\right\|_{C^2([0,L])} < \delta.$$

The following lemma is essentially due to Hemple [11] and it is easily derived from C^1-dependence of $u^{\pm}_{DD}$, $u^{\pm}_{ND}$, $u^{\pm}_{DN}$ on s, t.

LEMMA 1.5. – (i) $m^{\pm}_{DD}$, $m^{\pm}_{ND}$, $m^{\pm}_{DN} : \mathcal{D} \to \mathbf{R}$ *are differentiable and*

$$\frac{\partial}{\partial s} m^{\pm}_{DD}(\varepsilon; s, t) = \frac{\varepsilon}{2}\left|u^{\pm}_{DD,x}(\varepsilon, s, t; s)\right|^2 - \frac{1}{\varepsilon} W(s, 0),$$
$$\frac{\partial}{\partial t} m^{\pm}_{DD}(\varepsilon; s, t) = -\frac{\varepsilon}{2}\left|u^{\pm}_{DD,x}(\varepsilon, s, t; t)\right|^2 + \frac{1}{\varepsilon} W(t, 0),$$
$$\frac{\partial}{\partial t} m^{\pm}_{ND}(\varepsilon; s, t) = -\frac{\varepsilon}{2}\left|u^{\pm}_{ND,x}(\varepsilon, s, t; t)\right|^2 + \frac{1}{\varepsilon} W(t, 0),$$
$$\frac{\partial}{\partial s} m^{\pm}_{DN}(\varepsilon; s, t) = \frac{\varepsilon}{2}\left|u^{\pm}_{DN,x}(\varepsilon, s, t; s)\right|^2 - \frac{1}{\varepsilon} W(s, 0).$$

(ii) $b^{\pm}(s, \ell) : [0, 1] \times [\ell_0, \infty) \to \mathbf{R}$ *is differentiable and*

$$\frac{\partial}{\partial \ell} b^{\pm}(s, \ell) = -\frac{1}{2}\left|\omega_y(s, \ell; \ell)\right|^2 + W\big(s, \omega(s, \ell; \ell)\big) = W\big(s, \omega(s, \ell; \ell)\big).$$

In the setting of Lemma 1.5(ii), we can see $-\frac{1}{2}|\omega_y(y)|^2 + W(s, \omega(y))$ is independent of y. Thus we also have

$$\frac{\partial}{\partial \ell} b^{\pm}(s, \ell) = -\frac{1}{2}\left|\omega_y(s, \ell; 0)\right|^2 + W(s, 0).$$

Here we omit proofs of Lemmas 1.3–1.5.

As to the corollary to the above lemma, we can give a variational formulation of (0.1).

COROLLARY 1.6. – *The function* $f^{\pm}_{\varepsilon}(t_1, t_2, \ldots, t_n)$ *defined in* (1.1) *is differentiable in* $(0, \varepsilon_0] \times \{(t_1, t_2, \ldots, t_n);\ t_1/\varepsilon, (t_2 - t_1)/\varepsilon, \ldots, (t_n - t_{n-1})/\varepsilon, (1 - t_n)/\varepsilon \geqslant \ell_0\}$. *Moreover* $(t_1, t_2, \ldots, t_n)$ *with* $t_1/\varepsilon, (t_2 - t_1)/\varepsilon, \ldots, (t_n - t_{n-1})/\varepsilon, (1 - t_n)/\varepsilon \geqslant \ell_0$ *is a critical point of* $f^{\pm}_{\varepsilon}(t_1, t_2, \ldots, t_n)$ *if and only if*

$$u_\varepsilon(x) = \begin{cases} u^{\pm}_{ND}(\varepsilon, 0, t_1; x) & \text{for } x \in [0, t_1], \\ u^{\pm(-)^j}_{DD}(\varepsilon, t_j, t_{j+1}; x) & \text{for } x \in [t_j, t_{j+1}]\ (j = 1, 2, \ldots, t_n), \\ u^{\pm(-)^n}_{DN}(\varepsilon, t_n, 1; x) & \text{for } x \in [t_n, 1] \end{cases}$$

is a solution of (0.1).

Proof. – Recall that

$$f_\varepsilon^\pm(t_1, t_2, \dots, t_n) = m_{ND}^\pm(\varepsilon; 0, t_1) + m_{DD}^\mp(\varepsilon; t_1, t_2) + \cdots$$
$$+ m_{DD}^{\pm(-)^{n-1}}(\varepsilon; t_{n-1}, t_n) + m_{DN}^{\pm(-)^n}(\varepsilon; t_n, 1)$$

and $u_{ND}^\pm(\varepsilon, 0, t_1; x)$, $u_{DD}^\mp(\varepsilon; t_1, t_2; x), \dots$ are the corresponding minimizers. It is clear that $u_\varepsilon(x)$ satisfies (0.1) except n points $t_1, t_2, \dots, t_n$ and it solves (0.1) if and only if

$$\frac{\partial}{\partial t_1} u_{ND}^\pm(\varepsilon, 0, t_1; t_1) = \frac{\partial}{\partial t_1} u_{DD}^\mp(\varepsilon, t_1, t_2; t_1),$$
$$\frac{\partial}{\partial t_j} u_{ND}^{\pm(-)^{j-1}}(\varepsilon, t_{j-1}, t_j; t_j) = \frac{\partial}{\partial t_j} u_{DD}^{\pm(-)^j}(\varepsilon, t_j, t_{j+1}; t_j) \quad \text{for } j = 2, 3, \dots, n-1,$$
$$\frac{\partial}{\partial t_n} u_{ND}^{\pm(-)^{n-1}}(\varepsilon, t_{n-1}, t_n; t_n) = \frac{\partial}{\partial t_n} u_{DD}^{\pm(-)^n}(\varepsilon, t_n, 1; t_n).$$

We can easily see that these are equivalent to $\nabla f_\varepsilon^\pm(t_1, t_2, \dots, t_n) = 0$. □

Remark 1.7. – If $W(x, u)$ has a form:

$$W(x, u) = h(x)^2 F(u)$$

and $F(u)$ satisfies (0.7) in addition to (W1)–(W2), Hemple [11] showed the uniqueness of minimizer $m_{DD}^\pm(\varepsilon; s, t)$ without assumption of smallness of ε and largeness of $(t-s)/\varepsilon$. His proof of uniqueness works also for $m_{ND}^\pm(\varepsilon; s, t)$, $m_{DN}^\pm(\varepsilon; s, t)$ after minor modification. Thus under the assumption (0.7), all solutions of (0.1) can be characterized as critical points of $f_\varepsilon^\pm(t_1, \dots, t_n)$.

1.2. Properties of $m_{DD}^\pm(\varepsilon; s, t)$, $m_{ND}^\pm(\varepsilon; s, t)$, $m_{ND}^\pm(\varepsilon; s, t)$

From now on, we try to find critical points of $\nabla f_\varepsilon^\pm(t_1, t_2, \dots, t_n)$. We remark that re-scaled function $v(y) = u_{DD}^\pm(\varepsilon, s, t; s + \varepsilon y)$, etc. satisfies

$$-v_{yy} + W_u\big(s + \varepsilon y, v(y)\big) = 0 \quad \text{in } \left(0, \frac{t-s}{\varepsilon}\right).$$

We use frequently the following properties of minimizers.

LEMMA 1.8. – *For any $L > 0$ and $\delta > 0$ there exists $\varepsilon_1 = \varepsilon_1(L, \delta) > 0$ independent of s, t such that for $\varepsilon \in (0, \varepsilon_1]$ and $(t-s)/\varepsilon \geqslant \ell_0$*

$$\left\| u_{DD}^\pm(\varepsilon, s, t; s + \varepsilon y) - \omega^\pm\left(s, \frac{t-s}{2\varepsilon}; y\right) \right\|_{C^2([0, \tilde{L}/2])} < \delta,$$
$$\left\| u_{DD}^\pm(\varepsilon, s, t; t - \varepsilon y) - \omega^\pm\left(t, \frac{t-s}{2\varepsilon}; y\right) \right\|_{C^2([0, \tilde{L}/2])} < \delta,$$
$$\left\| u_{ND}^\pm(\varepsilon, s, t; t - \varepsilon y) - \omega^\pm\left(t, \frac{t-s}{\varepsilon}; y\right) \right\|_{C^2([0, \tilde{L}])} < \delta,$$

$$\left\| u^{\pm}_{DN}(\varepsilon, s, t; s+\varepsilon y) - \omega^{\pm}\left(s, \frac{t-s}{\varepsilon}; y\right) \right\|_{C^2([0,\tilde{L}])} < \delta,$$

where $\tilde{L} = \min\{L, \frac{t-s}{\varepsilon}\} > 0$.

From Lemma 1.4, we have

COROLLARY 1.9. – *For any* $L > 0$ *and* $\delta > 0$ *there exist* $\varepsilon_1 = \varepsilon_1(L, \delta) > 0$ *and* $\eta = \eta(L, \delta) \geqslant \ell_0$ *such that for* $\varepsilon \in (0, \varepsilon_1]$ *and* $(t-s)/\varepsilon \geqslant \eta$

$$\begin{aligned}
&\left\| u^{\pm}_{DD}(\varepsilon, s, t; s+\varepsilon y) - \omega^{\pm}(s, \infty; y) \right\|_{C^2([0,L])} < \delta, \\
&\left\| u^{\pm}_{DD}(\varepsilon, s, t; t-\varepsilon y) - \omega^{\pm}(t, \infty; y) \right\|_{C^2([0,L])} < \delta, \\
&\left\| u^{\pm}_{ND}(\varepsilon, s, t; t-\varepsilon y) - \omega^{\pm}(t, \infty; y) \right\|_{C^2([0,L])} < \delta, \\
&\left\| u^{\pm}_{DN}(\varepsilon, s, t; s+\varepsilon y) - \omega^{\pm}(s, \infty; y) \right\|_{C^2([0,L])} < \delta.
\end{aligned}$$

LEMMA 1.10. – *For any* $\delta > 0$ *there exist* $L_1 = L_1(\delta) > 0$ *and* $\varepsilon_1 = \varepsilon_1(\delta) > 0$ *such that for* $\varepsilon \in (0, \varepsilon_1]$ *and* $(t-s)/\varepsilon \geqslant \ell_0$

$$\int_{s+\varepsilon\tilde{L}_1/2}^{t-\varepsilon\tilde{L}_1/2} \frac{\varepsilon}{2}\left|u^{\pm}_{DD,x}(\varepsilon, s, t; x)\right|^2 + \frac{1}{\varepsilon} W(x, u^{\pm}_{DD})\,\mathrm{d}x < \delta,$$

$$\int_{s}^{t-\varepsilon\tilde{L}_1} \frac{\varepsilon}{2}\left|u^{\pm}_{ND,x}(\varepsilon, s, t; x)\right|^2 + \frac{1}{\varepsilon} W(x, u^{\pm}_{ND})\,\mathrm{d}x < \delta,$$

$$\int_{s+\varepsilon\tilde{L}_1}^{t} \frac{\varepsilon}{2}\left|u^{\pm}_{DN,x}(\varepsilon, s, t; x)\right|^2 + \frac{1}{\varepsilon} W(x, u^{\pm}_{DN})\,\mathrm{d}x < \delta,$$

where $\tilde{L}_1 = \min\{L_1, (t-s)/\varepsilon\} > 0$.

LEMMA 1.11. – *There are constants* $a_1, a_2, a_3 > 0$ *such that for* $y \in [0, (t-s)/\varepsilon]$

$$\begin{aligned}
&\left|u^{\pm}_{DD}(\varepsilon, s, t; s+\varepsilon y) - \alpha_{\pm}(s+\varepsilon y)\right| + \left|\frac{\mathrm{d}}{\mathrm{d}y}\left(u^{\pm}_{DD}(\varepsilon, s, t; s+\varepsilon y) - \alpha_{\pm}(s+\varepsilon y)\right)\right| \\
&\quad \leqslant a_1\varepsilon^2 + a_2\exp(-a_3 y) + a_2\exp\left(-a_3\left(\frac{t-s}{\varepsilon} - y\right)\right), \\
&\left|u^{\pm}_{ND}(\varepsilon, s, t; s+\varepsilon y) - \alpha_{\pm}(s+\varepsilon y)\right| + \left|\frac{\mathrm{d}}{\mathrm{d}y}\left(u^{\pm}_{ND}(\varepsilon, s, t; s+\varepsilon y) - \alpha_{\pm}(s+\varepsilon y)\right)\right| \\
&\quad \leqslant a_1\varepsilon^2 + a_2\exp\left(-a_3\left(\frac{t-s}{\varepsilon} - y\right)\right), \\
&\left|u^{\pm}_{DN}(\varepsilon, s, t; s+\varepsilon y) - \alpha_{\pm}(s+\varepsilon y)\right| + \left|\frac{\mathrm{d}}{\mathrm{d}y}\left(u^{\pm}_{DN}(\varepsilon, s, t; s+\varepsilon y) - \alpha_{\pm}(s+\varepsilon y)\right)\right| \\
&\quad \leqslant a_1\varepsilon^2 + a_2\exp(-a_3 y).
\end{aligned}$$

Proofs of Lemmas 1.8–1.11 will be given later in Section 6. As a corollary to Lemma 1.11, we have

COROLLARY 1.12. – *There is a constant* $a_4 > 0$ *such that for* $(t-s)/\varepsilon \geqslant 2a_4|\log\varepsilon|$

$$\left|u^{\pm}_{DD}(\varepsilon,s,t;s+\varepsilon y)-\alpha_{\pm}(s+\varepsilon y)\right|+\left|\frac{\mathrm{d}}{\mathrm{d}y}\left(u^{\pm}_{DD}(\varepsilon,s,t;s+\varepsilon y)-\alpha_{\pm}(s+\varepsilon y)\right)\right|\leqslant 2a_1\varepsilon^2$$

for $y\in\left[a_4|\log\varepsilon|,(t-s)/\varepsilon-a_4|\log\varepsilon|\right]$,

$$\left|u^{\pm}_{ND}(\varepsilon,s,t;s+\varepsilon y)-\alpha_{\pm}(s+\varepsilon y)\right|+\left|\frac{\mathrm{d}}{\mathrm{d}y}\left(u^{\pm}_{ND}(\varepsilon,s,t;s+\varepsilon y)-\alpha_{\pm}(s+\varepsilon y)\right)\right|\leqslant 2a_1\varepsilon^2$$

for $y\in\left[0,(t-s)/\varepsilon-a_4|\log\varepsilon|\right]$,

$$\left|u^{\pm}_{DN}(\varepsilon,s,t;s+\varepsilon y)-\alpha_{\pm}(s+\varepsilon y)\right|+\left|\frac{\mathrm{d}}{\mathrm{d}y}\left(u^{\pm}_{DN}(\varepsilon,s,t;s+\varepsilon y)-\alpha_{\pm}(s+\varepsilon y)\right)\right|\leqslant 2a_1\varepsilon^2$$

for $y\in\left[a_4|\log\varepsilon|,(t-s)/\varepsilon\right]$.

1.3. Behavior of $f^{\pm}_{\varepsilon}(t_1,\dots,t_n)$

In this section we give an explanation and a proof of (0.11). We do not need estimates given in this section for the proofs of Theorems 0.1 and 0.3 directly. The readers can skip this section and proceed to Section 2.

First we compare $m^{\pm}_{DD}(\varepsilon;s,t)$, $m^{\pm}_{ND}(\varepsilon;s,t)$, $m^{\pm}_{DN}(\varepsilon;s,t)$ and $b^{\pm}(s,\frac{t-s}{\varepsilon})$, $b^{\pm}(t,\frac{t-s}{\varepsilon})$. Here we use results in previous subsections.

LEMMA 1.13. – *For any* $\delta>0$ *there exists* $\varepsilon_1>0$ *such that for any* $\varepsilon\in(0,\varepsilon_1]$ *and* $(t-s)/\varepsilon\geqslant\ell_0$

$$\left|m^{\pm}_{DD}(\varepsilon;s,t)-b^{\pm}\left(s,\frac{t-s}{2\varepsilon}\right)-b^{\pm}\left(t,\frac{t-s}{2\varepsilon}\right)\right|\leqslant\delta,$$
$$\left|m^{\pm}_{ND}(\varepsilon;s,t)-b^{\pm}\left(t,\frac{t-s}{\varepsilon}\right)\right|\leqslant\delta,$$
$$\left|m^{\pm}_{DN}(\varepsilon;s,t)-b^{\pm}\left(s,\frac{t-s}{\varepsilon}\right)\right|\leqslant\delta.$$

Proof. – We only prove the first inequality. By Lemma 1.10, we can find $L>0$ such that

$$\int_{s+\varepsilon\tilde L}^{t-\varepsilon\tilde L}\frac{\varepsilon}{2}\left|u^{\pm}_{DD,x}(\varepsilon,s,t;x)\right|^2+\frac{1}{\varepsilon}W(x,u^{\pm}_{DD})\,\mathrm{d}x<\frac{\delta}{3}\quad\text{for sufficiently small }\varepsilon>0,\qquad(1.7)$$

where

$$\tilde L=\min\left\{L,\frac{t-s}{2\varepsilon}\right\}.$$

On the other hand

$$\int_{s}^{s+\varepsilon\tilde L}\frac{\varepsilon}{2}\left|u^{\pm}_{DD,x}(\varepsilon,s,t;x)\right|^2+\frac{1}{\varepsilon}W(x,u^{\pm}_{DD})\,\mathrm{d}x$$

$$= \int_0^{\tilde{L}} \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}y}u_{DD}^{\pm}(\varepsilon, s, t; s+\varepsilon y)\right|^2 + W(s+\varepsilon y, u_{DD}^{\pm})\,\mathrm{d}y \tag{1.8}$$

and by Corollary 1.9 we have for sufficiently small ε

$$\left|\int_0^{\tilde{L}} \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}y}u_{DD}^{\pm}(\varepsilon, s, t; s+\varepsilon y)\right|^2 + W(s+\varepsilon y, u_{DD}^{\pm})\,\mathrm{d}y \right.$$
$$\left. - \int_0^{\tilde{L}} \frac{1}{2}\left|\omega_y^{\pm}\left(s, \frac{t-s}{2\varepsilon}; y\right)\right|^2 + W\left(s, \omega^{\pm}\left(s, \frac{t-s}{2\varepsilon}; y\right)\right)\mathrm{d}y\right| < \frac{\delta}{6}. \tag{1.9}$$

We can also see

$$\left|\int_0^{\tilde{L}} \frac{1}{2}\left|\omega_y^{\pm}\left(s, \frac{t-s}{2\varepsilon}; y\right)\right|^2 + W\left(s, \omega^{\pm}\left(s, \frac{t-s}{2\varepsilon}; y\right)\right)\mathrm{d}y - b^{\pm}\left(s, \frac{t-s}{2\varepsilon}\right)\right| < \frac{\delta}{6} \tag{1.10}$$

for large $L \geqslant 1$ independent of s, t satisfying $(t-s)/\varepsilon \geqslant \ell_0$. It follows from (1.8)–(1.10) that

$$\left|\int_s^{s+\varepsilon\tilde{L}} \frac{\varepsilon}{2}\left|u_{DD,x}^{\pm}(\varepsilon, s, t; x)\right|^2 + \frac{1}{\varepsilon}W(x, u_{DD}^{\pm})\,\mathrm{d}x - b^{\pm}\left(s, \frac{t-s}{2\varepsilon}\right)\right| < \frac{\delta}{3}. \tag{1.11}$$

Similarly we have

$$\left|\int_{t-\varepsilon\tilde{L}}^{t} \frac{\varepsilon}{2}\left|u_{DD,x}^{\pm}(\varepsilon, s, t; x)\right|^2 + \frac{1}{\varepsilon}W(x, u_{DD}^{\pm})\,\mathrm{d}x - b^{\pm}\left(t, \frac{t-s}{2\varepsilon}\right)\right| < \frac{\delta}{3}. \tag{1.12}$$

Thus by (1.7), (1.11), (1.12) we have

$$\left|m_{DD}^{\pm}(\varepsilon; s, t) - b^{\pm}\left(s, \frac{t-s}{2\varepsilon}\right) - b^{\pm}\left(t, \frac{t-s}{2\varepsilon}\right)\right| \leqslant \delta. \qquad \square$$

Next we give an estimate for $b^{\pm}(s, \ell)$.

LEMMA 1.14. – *It holds that*

$$b^{\pm}(s,\ell) \sim b^{\pm}(s,\infty) - \exp\left(-2\sqrt{W_{uu}(s,\alpha_{\pm}(s))}\,\ell\right)$$

for large ℓ. More precisely for any $\nu > 0$ there exists $\tilde{\ell}(\nu) \geqslant \ell_0$ such that

$$-\exp\left(-\left(2\sqrt{W_{uu}(s,\alpha_{\pm}(s))} - \nu\right)\ell\right) \leqslant b^{\pm}(s,\ell) - b^{\pm}(s,\infty)$$
$$\leqslant -\exp\left(-\left(2\sqrt{W_{uu}(s,\alpha_{\pm}(s))} + \nu\right)\ell\right) \tag{1.13}$$

for $\ell \geqslant \tilde{\ell}(\nu)$.

Proof. – We deal with '+' case. '−' case can be dealt in a similar way. By (ii) of Lemma 1.5,

$$\frac{\partial}{\partial \ell} b^+(s,\ell) = W\big(s, \omega^+(s,\ell;\ell)\big).$$

We remark that $y \mapsto -\frac{1}{2}|\omega_y(s,\ell;y)|^2 + W(s,\omega^+(s,\ell;y))$ is independent of y and we have

$$\omega_y^+(s,\ell;y) = \sqrt{2\big(W(s,\omega^+(s,\ell;y)) - W(s,\omega^+(s,\ell;\ell))\big)}$$

for all $y \in [0,\ell]$. Thus

$$\ell = \int_0^\ell dy = \int_0^{\omega(s,\ell;\ell)} \frac{1}{\sqrt{2(W(s,\zeta) - W(s,\omega^+(s,\ell;\ell)))}}\, d\zeta. \tag{1.14}$$

We remark that

$$\omega^+(s,\ell;\ell) \to \alpha_+(s) - 0 \quad \text{as } \ell \to \infty.$$

To analyze the behavior of $\omega^+(s,\ell;\ell)$ as $\ell \to \infty$ precisely, we fix small $h_0 > 0$ and consider behavior of

$$\int_{\alpha_+(s)-h_0}^{\alpha_+(s)-h} \frac{1}{\sqrt{2(W(s,\zeta) - W(s,\alpha_+(s)-h))}}\, d\zeta \tag{1.15}$$

as $h \to +0$. We remark that we can find constant $C_0(h_0)$ depending only on h_0 such that

$$\int_0^{\alpha_+(s)-h_0} \frac{1}{\sqrt{2(W(s,\zeta) - W(s,\alpha_+(s)-h))}}\, d\zeta \leqslant C_0(h_0). \tag{1.16}$$

Introducing $\widetilde{W}(\tau) = W(s,\alpha_+(s)-\tau)$, we can rewrite (1.15) as

$$\int_h^{h_0} \frac{1}{\sqrt{2(\widetilde{W}(\tau) - \widetilde{W}(h))}}\, d\tau.$$

Using Cauchy's mean value theorem, we can find $\theta_1 \in (h,h_0)$ and $\theta_2 \in (0,\theta_1) \subset (0,h_0)$ such that

$$\frac{\widetilde{W}(\tau) - \widetilde{W}(h)}{\tau^2 - h^2} = \frac{\widetilde{W}_u(\theta_1)}{2\theta_1} = \frac{1}{2}\widetilde{W}_{uu}(\theta_2).$$

Thus, setting $C_\pm(\nu) = \sqrt{W_{uu}(s,\alpha_+(s)) \pm \frac{1}{2}\nu}$, for any $\nu > 0$ we can choose small $h_0 > 0$ such that

$$\frac{1}{2}C_-(\nu)^2\big(\tau^2 - h^2\big) \leqslant \widetilde{W}(\tau) - \widetilde{W}(h) \leqslant \frac{1}{2}C_+(\nu)^2\big(\tau^2 - h^2\big) \quad \text{for } 0 < h < \tau < h_0.$$

Thus we have

$$\frac{1}{C_+(\nu)}\int_h^{h_0}\frac{1}{\sqrt{\tau^2-h^2}}\,\mathrm{d}\tau \leqslant \int_h^{h_0}\frac{1}{\sqrt{2(\widetilde{W}(\tau)-\widetilde{W}(h))}}\,\mathrm{d}\tau \leqslant \frac{1}{C_-(\nu)}\int_h^{h_0}\frac{1}{\sqrt{\tau^2-h^2}}\,\mathrm{d}\tau.$$

Since

$$\int_h^{h_0}\frac{1}{\sqrt{\tau^2-h^2}}\,\mathrm{d}\tau = \log\left(\frac{h_0}{h}+\sqrt{\left(\frac{h_0}{h}\right)^2-1}\right) \in \left[\log\left(\frac{2h_0}{h}-1\right), \log\left(\frac{2h_0}{h}\right)\right],$$

we have

$$\frac{1}{C_+(\nu)}\log\left(\frac{2h_0}{h}-1\right) \leqslant \int_h^{h_0}\frac{1}{\sqrt{2(\widetilde{W}(\tau)-\widetilde{W}(h))}}\,\mathrm{d}\tau \leqslant \frac{1}{C_-(\nu)}\log\left(\frac{2h_0}{h}\right). \tag{1.17}$$

Setting $h=\alpha_+(s)-\omega^+(s,\ell;\ell)$, we have from (1.14), (1.16) that

$$\ell - C_0(h_0) \leqslant \int_h^{h_0}\frac{1}{\sqrt{2(\widetilde{W}(\tau)-\widetilde{W}(h))}}\,\mathrm{d}\tau \leqslant \ell.$$

Thus by (1.17)

$$\exp\bigl(C_-(\nu)\bigl(\ell - C_0(h_0)\bigr)\bigr) \leqslant \frac{2h_0}{h} \leqslant \exp\bigl(C_+(\nu)\ell\bigr)+1.$$

That is, for any $\nu>0$ we can find

$$\exp\bigl(-C_+(2\nu)\ell\bigr) \leqslant \alpha_+(s)-\omega^+(s,\ell;\ell) \leqslant \exp\bigl(-C_-(2\nu)\ell\bigr) \quad \text{for } l \gg 1.$$

Using

$$\frac{\partial}{\partial \ell}b^+(s,\ell) = W\bigl(s,\omega^+(s,\ell;\ell)\bigr) \sim \frac{1}{2}W_{uu}\bigl(s,\alpha_+(s)\bigr)\bigl(\alpha_+(s)-\omega(s,\ell;\ell)\bigr)^2$$

and

$$b^+(s,\ell)-b^+(s,\infty) = -\int_\ell^\infty \frac{\partial}{\partial \ell}b^+(s,\ell)\,\mathrm{d}\tau,$$

we can get the desired result (1.13). □

Combining Lemmas 1.13 and 1.14, we have

PROPOSITION 1.15. – *For any* $\delta > 0$ *and* $\nu > 0$ *there exist* $\varepsilon_1 = \varepsilon_1(\delta) > 0$ *and* $\ell_1 = \ell_1(\nu) \geqslant \ell_0$ *such that for any* $\varepsilon \in (0, \varepsilon_1]$ *and for any* $(t_1, \ldots, t_n)$ *satisfying* $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = 1$ *and* $(t_{j+1} - t_j)/\varepsilon \geqslant \ell_1$ *it holds that*

$$
\begin{aligned}
&-\exp\left(-\left(\rho_\pm(t_1) - \nu\right)\frac{t_1 - t_0}{\varepsilon}\right) - \sum_{j=1}^{n-1}\exp\left(-\left(\rho_{\pm(-)^j}(t_j) - \nu\right)\frac{t_{j+1} - t_j}{2\varepsilon}\right) \\
&\quad - \exp\left(-\left(\rho_{\pm(-)^n}(t_n) - \nu\right)\frac{t_{n+1} - t_n}{\varepsilon}\right) - \delta \\
&\quad \leqslant f_\varepsilon^\pm(t_1, \ldots, t_n) - \sum_{j=1}^{n} G(t_j) \\
&\quad \leqslant -\exp\left(-\left(\rho_\pm(t_1) + \nu\right)\frac{t_1 - t_0}{\varepsilon}\right) - \sum_{j=1}^{n-1}\exp\left(-\left(\rho_{\pm(-)^j}(t_j) + \nu\right)\frac{t_{j+1} - t_j}{2\varepsilon}\right) \\
&\quad - \exp\left(-\left(\rho(t_n)_{\pm(-)^n} + \nu\right)\frac{t_{n+1} - t_n}{\varepsilon}\right) + \delta,
\end{aligned}
$$

where $\rho_\pm(t) = 2\sqrt{W_{uu}(t, \alpha_\pm(t))}$.

Remark 1.16. – Proposition 1.15 means that (0.11) holds with

$$
\begin{aligned}
\rho_{\pm,0}(t) &= 2\sqrt{W_{uu}(t, \alpha_\pm(t))}, \\
\rho_{\pm,j}(t) &= \sqrt{W_{uu}(t, \alpha_{\pm(-)^j}(t))} \quad \text{for } j = 1, 2, \ldots, n-1, \\
\rho_{\pm,n}(t) &= 2\sqrt{W_{uu}(t, \alpha_{\pm(-)^n}(t))}.
\end{aligned}
$$

Proof. – We remark that $G(s) = \int_{-\infty}^{\infty} \frac{1}{2}|\omega_y^+(s, \infty; y)|^2 + W(s, \omega_y^+(s, \infty; y))\, \mathrm{d}y$ and it holds that $G(s) = b^+(s, \infty) + b^-(s, \infty)$. Since

$$
f_\varepsilon^\pm(t_1, \ldots, t_n) = m_{ND}^\pm(\varepsilon; 0, t_1) + m_{DD}^\mp(\varepsilon; t_1, t_2) + \cdots + m_{DN}^{\pm(-)^n}(\varepsilon; t_n, 1),
$$

Proposition 1.15 follows from Lemmas 1.13, 1.14 and the continuity of $W_{uu}(t, \alpha_\pm(t))$ easily. □

Thus we have (0.11). Heuristically we can derive our existence result from (0.11). Here we explain the case $n = 1$. Since

$$
f_\varepsilon^\pm(t_1) \sim G(t_1) \quad \text{for } \frac{t_1}{\varepsilon}, \frac{1 - t_1}{\varepsilon} \gg 1,
$$

we can find critical points in a neighborhood of strict interior minima and maxima.

As to boundary layers, we assume that 0 is a strict local maximum of $G(t)$. Choose $\eta \in (0, 1)$ so that $G(t) < G(0)$ in $(0, \eta]$. We have

$$
f_\varepsilon^\pm(t_1) \sim G(t_1) - \exp\left(-\frac{\rho_\pm(0) t_1}{\varepsilon}\right) \quad \text{near } t_1 \sim 0.
$$

More precisely by Proposition 1.15, for any $\delta > 0$ and $\nu > 0$ there exists $\varepsilon_1(\delta) > 0$ and $\ell_1(\nu) \geqslant \ell_0$ such that for $\varepsilon \in (0, \varepsilon_1]$ and $\ell \geqslant \ell_1(\nu)$, it holds that

$$f_\varepsilon^\pm(\varepsilon\ell) \leqslant G(\varepsilon\ell) - \exp\bigl(-(\rho_\pm(\varepsilon\ell)+\nu)\ell\bigr) + \delta,$$
$$f_\varepsilon^\pm(2\varepsilon\ell) \geqslant G(2\varepsilon\ell) - \exp\bigl(-2(\rho_\pm(2\varepsilon\ell)-\nu)\ell\bigr) - \delta.$$

We choose ν, ℓ, δ in the following way: first we choose small ν so that

$$2\bigl(\rho_\pm(0)-\nu\bigr) > \rho_\pm(0)+\nu.$$

Next we choose large $\ell \geqslant \ell_1(\nu)$ so that

$$G(0) - \exp\bigl(-2(\rho_\pm(0)-\nu)\ell\bigr) > G(\eta).$$

Finally we choose $\delta > 0$ so that

$$-\exp\bigl(-(\rho_\pm(0)+\nu)\ell\bigr) + \delta < -\exp\bigl(-2(\rho_\pm(0)-\nu)\ell\bigr) - \delta,$$
$$G(0) - \exp\bigl(-2(\rho_\pm(0)-\nu)\ell\bigr) - \delta > G(\eta).$$

Letting $\varepsilon \to 0$, we have

$$\limsup_{\varepsilon\to 0} f_\varepsilon^\pm(\varepsilon\ell) \leqslant G(0) - \exp\bigl(-(\rho_\pm(0)+\nu)\ell\bigr) + \delta,$$
$$\liminf_{\varepsilon\to 0} f_\varepsilon^\pm(2\varepsilon\ell) \geqslant G(0) - \exp\bigl(-2(\rho_\pm(0)-\nu)\ell\bigr) - \delta,$$
$$\lim_{\varepsilon\to 0} f_\varepsilon^\pm(\eta) = G(\eta).$$

Thus we have

$$f_\varepsilon^\pm(2\varepsilon\ell) > \max\bigl\{f_\varepsilon^\pm(\varepsilon\ell),\, f_\varepsilon^\pm(\eta)\bigr\} \quad \text{for sufficiently small } \varepsilon > 0.$$

Therefore $f_\varepsilon^\pm(t_1)$ has a critical point in $(\varepsilon\ell, \eta)$. Since we can choose $\eta > 0$ arbitrarily, we can see that $f_\varepsilon^\pm(t_1)$ has a critical point in a neighborhood of 0.

To find critical points of $f_\varepsilon^\pm(t_1,\dots,t_n)$ for $n \geqslant 1$, we need to use minimax methods or degree arguments and we need estimates of $\partial_{t_j} f_\varepsilon^\pm(t_1,\dots,t_n)$ which will be developed in the following sections.

2. A constraint for $f_\varepsilon^\pm(t_1,\dots,t_n)$

From now on, we try to find a critical point of $f_\varepsilon^\pm(t_1,\dots,t_n)$ with a profile given in Theorem 0.1. We choose a small $h > 0$ such that

$$h < \frac{1}{6}\min\bigl\{|p-p'|;\ p, p' \in M \cup \{0,1\},\ p \neq p'\bigr\}.$$

Then $p \in M_+$ ($p \in M_-$ respectively) implies

$$G'(t) > 0 \ (< 0 \text{ respectively}) \quad \text{for } t \in [p-2h, p),$$
$$G'(t) < 0 \ (> 0 \text{ respectively}) \quad \text{for } t \in (p, p+2h].$$

We write $M = \{p_1, p_2, \dots, p_N\}$ ($p_1 < p_2 < \cdots < p_N$). For a sequence $n_j = n_{p_j}$ ($j = 1,\dots,N$) of non-negative integers satisfying the assumption of Theorem 0.1, i.e.,

$$n_j \in \{0,1\} \quad \text{if } p_j \in M_-$$

we try to find a critical point $u_\varepsilon(x)$ of $I_\varepsilon(x)$ which has exactly n_j zeros in a neighborhood of p_j for each j. To do so, we arrange $t_1, t_2, \ldots, t_n$ ($n = \sum_{i=1}^N n_{p_i}$) into N groups. We write

$$(t_1, \ldots, t_n) = (t_{11}, \ldots, t_{1n_1}, t_{21}, \ldots, t_{2n_2}, \ldots, t_{N1}, \ldots, t_{Nn_N})$$

and we assume the ith group $(t_{i1}, \ldots, t_{in_i})$ lie in $[p_i - 2h, p_i + 2h]$. (Some of these groups may be empty.)

If $p_i \notin \{0, 1\}$, we set

$$\Delta_\varepsilon^i = \big\{(\tau_1, \ldots, \tau_{n_i});\ p_i - 2h \leqslant \tau_1 < \tau_2 < \cdots < \tau_{n_i} \leqslant p_i + 2h,\\ (\tau_{j+1} - \tau_j)/\varepsilon \geqslant \ell_0 \text{ for } j = 1, 2, \ldots, n_i - 1\big\}. \tag{2.1}$$

If $p_i = 0$, we have $i = 1$ and set

$$\Delta_\varepsilon^1 = \big\{(\tau_1, \ldots, \tau_{n_1});\ \varepsilon\ell_0 \leqslant \tau_1 < \tau_2 < \cdots < \tau_{n_1} \leqslant 2h,\\ (\tau_{j+1} - \tau_j)/\varepsilon \geqslant \ell_0 \text{ for } j = 1, 2, \ldots, n_1 - 1\big\}. \tag{2.2}$$

If $p_i = 1$, we have $i = N$ and set

$$\Delta_\varepsilon^N = \big\{(\tau_1, \ldots, \tau_{n_i});\ 1 - 2h \leqslant \tau_1 < \tau_2 < \cdots < \tau_{n_N} \leqslant 1 - \varepsilon\ell_0,\\ (\tau_{j+1} - \tau_j)/\varepsilon \geqslant \ell_0 \text{ for } j = 1, 2, \ldots, n_N - 1\big\}. \tag{2.3}$$

For sufficiently small $\varepsilon > 0$ we try to find a critical point of

$$f_\varepsilon^\pm : \Delta_\varepsilon^1 \times \cdots \times \Delta_\varepsilon^N \to \mathbf{R}. \tag{2.4}$$

We will show the existence of a critical point of (2.4) by means of Brouwer degree; that is, we will show

$$\deg\big(\nabla f_\varepsilon^\pm, \Delta_\varepsilon^1 \times \cdots \times \Delta_\varepsilon^N, 0\big) \neq 0. \tag{2.5}$$

Estimates of $\nabla f_\varepsilon^\pm$ on $\partial(\Delta_\varepsilon^1 \times \cdots \times \Delta_\varepsilon^N)$ are important in the proof of (2.5).

We choose $i \in \{1, 2, \ldots, N\}$ and we deal with estimates of

$$(\partial_{t_{i1}} f_\varepsilon^\pm, \ldots, \partial_{t_{in_i}} f_\varepsilon^\pm) : \Delta_\varepsilon^i \to \mathbf{R}^{n_i}$$

for fixed $(t_{11}, \ldots, t_{i-1,n_{i-1}}, t_{i+1,1}, \ldots, t_{Nn_N}) \in \Delta_\varepsilon^1 \times \cdots \times \Delta_\varepsilon^{i-1} \times \Delta_\varepsilon^{i+1} \times \cdots \times \Delta_\varepsilon^N$. By the definition of $f_\varepsilon^\pm$, it is clear that $(\partial_{t_{i1}} f_\varepsilon^\pm, \ldots, \partial_{t_{in_i}} f_\varepsilon^\pm)$ depends only on $t_{i-1,n_{i-1}}, t_{i1}, \ldots, t_{in_i}$, $t_{i+1,1}$.

For the sake of simplicity of notation, we write

$$\begin{aligned}
&p = p_i, \qquad n = n_i,\\
&\tau_0 = t_{i-1,n_{i-1}},\ \tau_1 = t_{i1},\ \tau_2 = t_{i2},\ \ldots,\ \tau_n = t_{in},\ \tau_{n+1} = t_{i+1,1},\\
&\nabla_\tau = (\partial_{\tau_1}, \ldots, \partial_{\tau_n}),\\
&\Delta_\varepsilon = \big\{(\tau_1, \ldots, \tau_n);\ p - 2h \leqslant \tau_1 < \tau_2 < \cdots < \tau_n \leqslant p + 2h,\\
&\qquad (\tau_{j+1} - \tau_j)/\varepsilon \geqslant \ell_0 \text{ for } j = 0, 1, \ldots, n\big\}
\end{aligned} \tag{2.6}$$

and compute $\deg(\nabla_\tau g_\varepsilon, \Delta_\varepsilon, 0)$, where

$$g_\varepsilon(\tau_1, \ldots, \tau_n) = m_{DD}^+(\varepsilon; \tau_0, \tau_1) + m_{DD}^-(\varepsilon; \tau_1, \tau_2) + \cdots + m_{DD}^{(-)^n}(\varepsilon; \tau_n, \tau_{n+1}). \tag{2.7}$$

If $i = 1$ or $i = N$, we regard

$$\begin{cases} \tau_0 = 0 & \text{if } i = 1, \\ \tau_{n+1} = 1 & \text{if } i = N \end{cases}$$

and we replace the first term of (2.7) by m^+_{ND} if $\tau_0 = 0$ and the last term by $m^{(-)^n}_{DN}$ if $\tau_{n+1} = 1$. We remark that we may assume $\varepsilon\ell_0 < h$ and any set Δ^i_ε in (2.1)–(2.3) can be written in form (2.6) in a unified way.

We will estimate $\nabla_\tau g_\varepsilon$ on the boundary $\partial\Delta_\varepsilon$ of Δ_ε to show $\deg(\nabla_\tau g_\varepsilon, \Delta_\varepsilon, 0) \neq 0$. We remark that if $p \notin \{0, 1\}$,

$$\partial\Delta_\varepsilon = \big\{(\tau_1, \ldots, \tau_n);\ \tau_1 = p - 2h\big\} \cup \big\{(\tau_1, \ldots, \tau_n);\ \tau_n = p + 2h\big\}$$
$$\cup \big\{(\tau_1, \ldots, \tau_n);\ (\tau_{j+1} - \tau_j)/\varepsilon = \ell_0 \text{ for some } j \in \{1, 2, \ldots, n-1\}\big\}.$$

If $p = 0$, then

$$\partial\Delta_\varepsilon = \big\{(\tau_1, \ldots, \tau_n);\ \tau_n = p + 2h\big\}$$
$$\cup \big\{(\tau_1, \ldots, \tau_n);\ (\tau_{j+1} - \tau_j)/\varepsilon = \ell_0 \text{ for some } j \in \{0, 1, 2, \ldots, n-1\}\big\}.$$

If $p = 1$, then

$$\partial\Delta_\varepsilon = \big\{(\tau_1, \ldots, \tau_n);\ \tau_1 = p - 2h\big\}$$
$$\cup \big\{(\tau_1, \ldots, \tau_n);\ (\tau_{j+1} - \tau_j)/\varepsilon = \ell_0 \text{ for some } j \in \{1, 2, \ldots, n\}\big\}.$$

We remark that $j = 0$ or $j = n$ takes a place if $p = 0$ or $p = 1$.

In what follows, we show two types of estimates of $\nabla_\tau g_\varepsilon$ on $\partial\Delta_\varepsilon$. The first type of estimates deal with the case $\tau_1 = p - 2h$ or $\tau_n = p + 2h$ and it reflects the influence of the function $G(t)$, i.e., the first term of (0.11). The second type of estimates deal with the case $(\tau_{j+1} - \tau_j)/\varepsilon = \ell_0$ and it reflects the interaction between two layers or between a layer and boundary 0, 1, i.e., the second term of (0.11).

3. Estimates of $\nabla_\tau g_\varepsilon$

3.1. Estimates of derivatives of $m^\pm_{DD}(\varepsilon; s, t)$, $m^\pm_{ND}(\varepsilon; s, t)$, $m^\pm_{DN}(\varepsilon; s, t)$ for relatively large $(t - s)/\varepsilon$

The aim of this subsection is to show the following estimates:

PROPOSITION 3.1. – *For any $\delta > 0$ there exists $\varepsilon_2 > 0$ such that if $\varepsilon \in (0, \varepsilon_2]$ and $(t - s)/\varepsilon \geqslant 3a_4|\log\varepsilon|$ ($a_4 > 0$ is given in Corollary 1.12), then*

$$\left|\frac{\partial}{\partial s} m^\pm_{DD}(\varepsilon; s, t) - G^{\pm\prime}(s)\right| \leqslant \delta, \tag{3.1}$$

$$\left|\frac{\partial}{\partial t} m^\pm_{DD}(\varepsilon; s, t) - G^{\pm\prime}(t)\right| \leqslant \delta, \tag{3.2}$$

$$\left|\frac{\partial}{\partial t} m^\pm_{ND}(\varepsilon; s, t) - G^{\pm\prime}(t)\right| \leqslant \delta, \tag{3.3}$$

$$\left|\frac{\partial}{\partial s} m^\pm_{DN}(\varepsilon; s, t) - G^{\pm\prime}(s)\right| \leqslant \delta. \tag{3.4}$$

Here $G^{\pm}(t)$ is defined by

$$G^{+}(t)=\sqrt{2}\int_{0}^{\alpha_{+}(t)}\sqrt{W(t,\tau)}\,\mathrm{d}\tau,\qquad G^{-}(t)=\sqrt{2}\int_{\alpha_{-}(t)}^{0}\sqrt{W(t,\tau)}\,\mathrm{d}\tau.$$

Before giving a proof of Proposition 3.1, we remark that

$$\begin{aligned}G^{+}(t)&=\sqrt{2}\int_{0}^{\alpha_{+}(t)}\sqrt{W(t,\tau)}\,\mathrm{d}\tau\\&=\sqrt{2}\int_{0}^{\infty}\sqrt{W(t,\omega^{+}(t,\infty;y))}\omega_{y}^{+}(t,\infty;y)\,\mathrm{d}y\\&=\int_{0}^{\infty}\frac{1}{2}\left|\omega_{y}^{+}(t,\infty;y)\right|^{2}+W\bigl(t,\omega^{+}(t,\infty;y)\bigr)\,\mathrm{d}y.\end{aligned}$$

Thus we have

$$\begin{aligned}G^{+\prime}(t)&=\int_{0}^{\infty}\bigl(-\omega_{yy}^{+}+W_{u}(t,\omega^{+})\bigr)\omega_{t}^{+}+W_{x}\bigl(t,\omega^{+}(t,\infty;y)\bigr)\,\mathrm{d}y\\&=\int_{0}^{\infty}W_{x}\bigl(t,\omega^{+}(t,\infty;y)\bigr)\,\mathrm{d}y.\end{aligned}\tag{3.5}$$

In a similar way, we have

$$G^{-\prime}(t)=\int_{-\infty}^{0}W_{x}\bigl(t,\omega^{+}(t,\infty;y)\bigr)\,\mathrm{d}y.\tag{3.6}$$

Proof. – We give a proof of (3.1). (3.2)–(3.4) can be proved in a similar way. We fix $t\in(0,1]$ and we show (3.1) for '+' sign. Here we write $u(x)=u_{DD}^{+}(\varepsilon,s,t;x)$. Let $\varphi(\tau):[0,\infty)\to\mathbf{R}$ be a function of class C^{1} such that

$$\begin{aligned}&\varphi(\tau)=1\quad\text{for }\tau\in[0,1],\\&\varphi(\tau)=0\quad\text{for }\tau\in[2,\infty),\\&\varphi'(\tau)\leqslant 0\quad\text{for }\tau\in[0,\infty)\end{aligned}$$

and we set

$$\mu_{\varepsilon}=a_{4}|\log\varepsilon|,$$

where $a_{4}>0$ is a constant appeared in Corollary 1.12.

We suppose $(t-s)/\varepsilon\geqslant 3\mu_{\varepsilon}$ and it follows from Lemma 1.5 that

$$\frac{\partial}{\partial s}m_{DD}^{+}(\varepsilon;s,t)=\frac{\varepsilon}{2}\left|u_{x}(s)\right|^{2}-\frac{1}{\varepsilon}W(s,0)$$

$$
\begin{aligned}
&= \frac{\varepsilon}{2}\left|u_x(s)\right|^2 - \frac{1}{\varepsilon}W\big(s,u(s)\big)\\
&= \int_s^t \frac{\mathrm{d}}{\mathrm{d}x}\left\{\varphi\left(\frac{x-s}{\varepsilon\mu_\varepsilon}\right)\left(-\frac{\varepsilon}{2}\left|u_x(x)\right|^2 + \frac{1}{\varepsilon}W\big(x,u(x)\big)\right)\right\}\mathrm{d}x\\
&= \int_s^t \frac{1}{\varepsilon\mu_\varepsilon}\varphi'\left(\frac{x-s}{\varepsilon\mu_\varepsilon}\right)\left(-\frac{\varepsilon}{2}\left|u_x(x)\right|^2 + \frac{1}{\varepsilon}W\big(x,u(x)\big)\right)\mathrm{d}x\\
&\qquad + \int_s^t \varphi\left(\frac{x-s}{\varepsilon\mu_\varepsilon}\right)\frac{1}{\varepsilon}W_x\big(x,u(x)\big)\,\mathrm{d}x.
\end{aligned}
$$

Changing variable $x = s+\varepsilon y$ and introducing $v(y) = u(s+\varepsilon y) = u^+_{DD}(\varepsilon, s, t; s+\varepsilon y)$, we have

$$
\begin{aligned}
\frac{\partial}{\partial s}m^+_{DD}(\varepsilon; s, t) &= \int_{\mu_\varepsilon}^{2\mu_\varepsilon} \frac{1}{\varepsilon\mu_\varepsilon}\varphi'\left(\frac{y}{\mu_\varepsilon}\right)\left(-\frac{1}{2}|v_y|^2 + W\big(s+\varepsilon y, v(y)\big)\right)\mathrm{d}y\\
&\quad + \int_0^{2\mu_\varepsilon} \varphi\left(\frac{y}{\mu_\varepsilon}\right)W_x\big(s+\varepsilon y, v(y)\big)\,\mathrm{d}y\\
&= (\mathrm{I}) + (\mathrm{II}).
\end{aligned}
\tag{3.7}
$$

By Corollary 1.12 and (W3), we have

$$
\left|-\frac{1}{2}|v_y|^2 + W\big(s+\varepsilon y, v(y)\big)\right| \leqslant C\varepsilon^2 \quad \text{for } y \in [\mu_\varepsilon, 2\mu_\varepsilon],
$$

where $C > 0$ is independent of $\varepsilon > 0$. Thus

$$
|(\mathrm{I})| \leqslant \int_{\mu_\varepsilon}^{2\mu_\varepsilon} -\frac{1}{\varepsilon\mu_\varepsilon}\varphi'\left(\frac{y}{\mu_\varepsilon}\right)C\varepsilon^2\,\mathrm{d}y = \frac{1}{2}\big(\varphi(1)-\varphi(2)\big)C\varepsilon = \frac{C}{2}\varepsilon. \tag{3.8}
$$

Using Corollary 1.9 and Lemma 1.10, we have

$$
\begin{aligned}
(\mathrm{II}) &= \int_0^{2\mu_\varepsilon} \varphi\left(\frac{y}{\mu_\varepsilon}\right)W_x\big(s+\varepsilon y, v(y)\big)\,\mathrm{d}y\\
&\to \int_0^{\infty} W_x\big(s, \omega^+(s,\infty; y)\big)\,\mathrm{d}y = G^{+\prime}(s)
\end{aligned}
\tag{3.9}
$$

as $\varepsilon \to 0$ uniformly in $s \in [0,1]$. Here we used (3.5).

Combining (3.7)–(3.9), we get (3.1). □

We also need the following estimate.

PROPOSITION 3.2. – *For any* $\delta > 0$ *there exist* $\varepsilon_2 > 0$ *and* $\ell_2 \geqslant \ell_0$ *such that for* $\varepsilon \in (0, \varepsilon_2]$ *and* t, s *satisfying* $(t-s)/\varepsilon \geqslant \ell_2$,

$$\left|\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial t}\right)m^{\pm}_{DD}(\varepsilon; s, t) - \left(G^{\pm\prime}(s) + G^{\pm\prime}(t)\right)\right| \leqslant \delta. \tag{3.10}$$

Proof. – We deal with just '+' case. Here we write $u(x) = u^{+}_{DD}(\varepsilon, s, t; x)$ and $v(y) = u(s+\varepsilon y) = u^{+}_{DD}(\varepsilon, s, t; s+\varepsilon y)$. As in the proof of Proposition 3.1, we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial t}\right)m^{+}_{DD}(\varepsilon; s, t)\\
&\quad = -\left(-\frac{\varepsilon}{2}\left|u_x(s)\right|^2 + \frac{1}{\varepsilon}W(s,0)\right) + \left(-\frac{\varepsilon}{2}\left|u_x(t)\right|^2 + \frac{1}{\varepsilon}W(t,0)\right)\\
&\quad = \int_s^t \frac{\mathrm{d}}{\mathrm{d}x}\left\{-\frac{\varepsilon}{2}|u_x|^2 + \frac{1}{\varepsilon}W\big(x, u(x)\big)\right\}\mathrm{d}x\\
&\quad = \int_s^t \frac{1}{\varepsilon}W_x\big(x, u(x)\big)\,\mathrm{d}x = \int_0^{(t-s)/\varepsilon} W_x\big(s+\varepsilon y, v(y)\big)\,\mathrm{d}y.
\end{aligned}$$

Thus by Corollary 1.9 and Lemma 1.10, for sufficiently small $\varepsilon > 0$ and sufficiently large $(t-s)/\varepsilon$, we have

$$\begin{aligned}
\Bigg| &\int_0^{(t-s)/\varepsilon} W_x\big(s+\varepsilon y, v(y)\big)\,\mathrm{d}y - \int_0^{\infty} W_x\big(s, \omega^{+}(s,\infty; y)\big)\,\mathrm{d}y\\
&- \int_0^{\infty} W_x\big(t, \omega^{+}(t,\infty; y)\big)\,\mathrm{d}y\Bigg| \leqslant \delta.
\end{aligned}$$

By (3.5), we have (3.10). □

3.2. Estimates of derivatives of $m^{\pm}_{DD}(\varepsilon; s, t)$, $m^{\pm}_{ND}(\varepsilon; s, t)$, $m^{\pm}_{DN}(\varepsilon; s, t)$ for relatively small $(t-s)/\varepsilon$

Next we deal with estimates of $\frac{\partial}{\partial s}m^{\pm}_{DD}(\varepsilon; s, t)$, $\frac{\partial}{\partial t}m^{\pm}_{DD}(\varepsilon; s, t)$, $\frac{\partial}{\partial t}m^{\pm}_{ND}(\varepsilon; s, t)$, $\frac{\partial}{\partial s}m^{\pm}_{DN}(\varepsilon; s, t)$ for relatively small $(t-s)/\varepsilon$.

PROPOSITION 3.3. – (i) *For any* $\ell \geqslant \ell_0$ *there exists* $\rho(\ell) > 0$ *and* $\varepsilon_3(\ell) > 0$ *such that for* $(t-s)/\varepsilon \in [\ell_0, \ell]$ *and* $\varepsilon \in (0, \varepsilon_3(\ell)]$, *it holds that*

$$\begin{aligned}
\varepsilon\frac{\partial}{\partial s}m^{\pm}_{DD}(\varepsilon; s, t) &\leqslant -\rho(\ell),\\
\varepsilon\frac{\partial}{\partial t}m^{\pm}_{DD}(\varepsilon; s, t) &\geqslant \rho(\ell),\\
\varepsilon\frac{\partial}{\partial t}m^{\pm}_{ND}(\varepsilon; s, t) &\geqslant \rho(\ell),
\end{aligned}$$

$$\varepsilon\frac{\partial}{\partial s}m^{\pm}_{DN}(\varepsilon;s,t)\leqslant -\rho(\ell).$$

(ii) *For any* $\delta>0$ *there exists* $\ell(\delta)\geqslant\ell_0$ *and* $\varepsilon_4>0$ *such that for* $(t-s)/\varepsilon\geqslant\ell(\delta)$ *and* $\varepsilon\in(0,\varepsilon_4]$, *it holds that*

$$\varepsilon\left|\frac{\partial}{\partial s}m^{\pm}_{DD}(\varepsilon;s,t)\right|,\ \varepsilon\left|\frac{\partial}{\partial t}m^{\pm}_{DD}(\varepsilon;s,t)\right|,\ \varepsilon\left|\frac{\partial}{\partial t}m^{\pm}_{ND}(\varepsilon;s,t)\right|,\ \varepsilon\left|\frac{\partial}{\partial s}m^{\pm}_{DN}(\varepsilon;s,t)\right|\leqslant\delta.$$

Proof. – We prove just for $\frac{\partial}{\partial s}m^{+}_{DD}(\varepsilon;s,t)$.

(i) We argue indirectly. If the conclusion of (i) does not hold, there exist sequences s_j, t_j, ε_j such that

$$\begin{aligned}&(t_j-s_j)/\varepsilon_j\in[\ell_0,\ell],\\&\varepsilon_j\to 0,\\&\liminf_{j\to\infty}\varepsilon_j\frac{\partial}{\partial s}m^{+}_{DD}(\varepsilon_j;s_j,t_j)\geqslant 0.\end{aligned}\tag{3.11}$$

We may assume $s_j\to\tilde s$, $t_j\to\tilde s$ and $(t_j-s_j)/\varepsilon_j\to\tilde\ell\in[\ell_0,\ell]$. Let $u_j(x)$ be a minimizer corresponding to $m^{+}_{DD}(\varepsilon_j;s_j,t_j)$ and set $v_j(y)=u_j(s_j+\varepsilon_j y)$. Then by Lemma 1.8, we have

$$\left\|v_j(y)-\omega^+(\tilde s,\tilde\ell/2;y)\right\|_{C^2}\to 0\quad\text{as } j\to\infty.$$

Since $\omega^+(\tilde s,\tilde\ell/2;y)$ can be extended to a $2\tilde\ell$-periodic solution of

$$-v_{yy}+W_u\big(\tilde s,v(y)\big)=0,$$

we have

$$\frac{1}{2}\left|\omega^+_y(\tilde s,\tilde\ell/2;0)\right|^2-W_u(\tilde s,0)<0.$$

On the other hand by Lemma 1.5, we have

$$\varepsilon_j\frac{\partial}{\partial s}m^{+}_{DD}(\varepsilon_j;s_j,t_j)=\frac{1}{2}\left|v_{j,y}(0)\right|^2-W(s_j,0)\to\frac{1}{2}\left|\omega^+_y(\tilde s,\tilde\ell/2;0)\right|^2-W(\tilde s,0)<0$$

as $j\to\infty$. This is a contradiction to (3.11).

(i) Since $\omega^+(s,\infty;y)$ is a heteroclinic solution of (1.5), it satisfies

$$\frac{1}{2}\left|\omega^+_y(s,\infty;0)\right|^2-W(s,0)=0.$$

Using this property, we can deduce the second statement of Proposition 3.3 from Lemma 1.5 and Corollary 1.9. □

3.3. Estimates of $\nabla_\tau g_\varepsilon$ when $\tau_1=p-2h$ or $\tau_n=p+2h$ holds

From now on we give estimates $\nabla_\tau g_\varepsilon$ on $\partial\Delta_\varepsilon$. First we deal with the case $\tau_1=p-2h$ or $\tau_n=p+2h$. Mainly we consider the case $p\in M_+\setminus\{0\}$ and $\tau_1=p-2h$.

Suppose that $p \in M_+$ and $(\tau_1, \dots, \tau_n) \in \Delta_\varepsilon$ satisfies $\tau_1 = p - 2h$. Then for sufficiently small $\varepsilon > 0$ we can find $j \in \{1, 2, \dots, n\}$ such that

$$p - 2h = \tau_1 < \tau_2 < \cdots < \tau_j \leqslant p - h, \tag{3.12}$$

$$(\tau_{j+1} - \tau_j)/\varepsilon \geqslant 3a_4|\log \varepsilon|. \tag{3.13}$$

We choose $\delta > 0$ sufficiently small so that

$$\min_{p-2h \leqslant t \leqslant p-h} G'(t) > \delta.$$

Applying Propositions 3.1 and 3.2, we choose $\varepsilon_2 > 0$ so that (3.1)–(3.4) holds for $\varepsilon \in (0, \varepsilon_2]$. Then we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial \tau_1} + \cdots + \frac{\partial}{\partial \tau_j}\right) g_\varepsilon(\tau_1, \dots, \tau_n) \\
&\quad = \frac{\partial}{\partial \tau_1} m_{DD}^{\pm}(\varepsilon; \tau_0, \tau_1) + \left(\frac{\partial}{\partial \tau_1} + \frac{\partial}{\partial \tau_2}\right) m_{DD}^{\mp}(\varepsilon; \tau_1, \tau_2) + \cdots \\
&\qquad + \left(\frac{\partial}{\partial \tau_{j-1}} + \frac{\partial}{\partial \tau_j}\right) m_{DD}^{\pm(-)^{j-1}}(\varepsilon; \tau_{j-1}, \tau_j) + \frac{\partial}{\partial \tau_j} m_{DD}^{\pm(-)^j}(\varepsilon; \tau_j, \tau_{j+1}) \\
&\quad \geqslant \left(G^{\pm\prime}(\tau_1) - \delta\right) + \left(G^{\mp\prime}(\tau_1) + G^{\mp\prime}(\tau_2) - \delta\right) + \cdots \\
&\qquad + \left(G^{\pm(-)^{j-1}\prime}(\tau_{j-1}) + G^{\pm(-)^{j-1}\prime}(\tau_j) - \delta\right) + \left(G^{\pm(-)^j\prime}(\tau_j) - \delta\right) \\
&\quad = G'(\tau_1) + \cdots + G'(\tau_j) - j\delta \\
&\quad > 0.
\end{aligned}$$

We can argue in a similar way and we have

PROPOSITION 3.4. – (i) *Suppose that* $p \in M_+ \setminus \{0\}$ ($p \in M_- \setminus \{0\}$ *respectively*) *and* $\tau_1 = p - 2h$. *Then there exist* $j \in \{1, 2, \dots, n\}$ *such that* (3.12) *and* (3.13) *hold. For such a* j, *we have*

$$\left(\frac{\partial}{\partial \tau_1} + \cdots + \frac{\partial}{\partial \tau_j}\right) g_\varepsilon(\tau_1, \dots, \tau_n) > 0 \quad (< 0 \textit{ respectively}). \tag{3.14}$$

(ii) *A similar result holds for the case* $p \in M_+ \setminus \{1\}$ ($p \in M_- \setminus \{1\}$ *respectively*) *and* $\tau_n = p + 2h$. *More precisely, there exists* $j \in \{1, 2, \dots, n\}$ *such that*

$$p + h \leqslant \tau_j < \tau_{j+1} < \cdots < \tau_n = p + 2h, \tag{3.15}$$

$$(\tau_j - \tau_{j-1})/\varepsilon \geqslant 3a_4|\log \varepsilon| \tag{3.16}$$

and for such a j *we have*

$$\left(\frac{\partial}{\partial \tau_j} + \cdots + \frac{\partial}{\partial \tau_n}\right) g_\varepsilon(\tau_1, \dots, \tau_n) < 0 \quad (> 0 \textit{ respectively}). \tag{3.17}$$

3.4. Estimates of $\nabla_\tau g_\varepsilon$ when $(\tau_{j+1}-\tau_j)/\varepsilon=\ell_0$ for some $j\in\{0,1,\dots,n\}$

Next we deal with the case $(\tau_{j+1}-\tau_j)/\varepsilon=\ell_0$. Here we use Proposition 3.3. We choose $\ell_1,\ell_2,\dots,\ell_n$ in the following way. First we apply Proposition 3.3(i) to choose

$$\rho_0=\rho(\ell_0).$$

Next we apply (ii) of Proposition 3.3 for $\delta=\rho_0/2>0$ and let

$$\ell_1=\ell(\rho_0/2).$$

We continue this process and set

$$\rho_1=\rho(\ell_1),\ \ell_2=\ell(\rho_1/2),\ \rho_2=\rho(\ell_2),\ \ell_3=\ell(\rho_2/2),\ \dots,\ \ell_n=\ell(\rho_{n-1}/2),\ \rho_n=\rho(\ell_n).$$

By the definition, we have

$$\rho_0>\rho_1>\cdots>\rho_n,$$
$$\ell_0<\ell_1<\cdots<\ell_n.$$

As a consequence of Proposition 3.3, we have the following

PROPOSITION 3.5. – *Suppose that* $(\tau_1,\dots,\tau_n)\in\Delta_\varepsilon$ *satisfies*

$$(\tau_{i+1}-\tau_i)/\varepsilon=\ell_0\quad\text{for some } i\in\{0,1,\dots,n\}. \tag{3.18}$$

Then we can find $j\in\{1,2,\dots,n\}$ *and* $k\in\{1,2,\dots,n\}$ *such that*

$$\begin{cases}(\tau_j-\tau_{j-1})/\varepsilon\in[\ell_0,\ell_k],\\(\tau_{j+1}-\tau_j)/\varepsilon\in[\ell_{k+1},\infty)\end{cases} \tag{3.19}$$

or

$$\begin{cases}(\tau_j-\tau_{j-1})/\varepsilon\in[\ell_{k+1},\infty),\\(\tau_{j+1}-\tau_j)/\varepsilon\in[\ell_0,\ell_k].\end{cases} \tag{3.20}$$

For such j,k, *we have*

$$\left(\frac{\partial}{\partial\tau_j}\right)g_\varepsilon(\tau_1,\dots,\tau_n)\ \begin{cases}>0 & \textit{if } (3.19) \textit{ holds},\\ <0 & \textit{if } (3.20) \textit{ holds}.\end{cases} \tag{3.21}$$

Proof. – First we show (3.19) or (3.20) holds. If (3.19) does not hold, we have for any j,k

$$(\tau_j-\tau_{j-1})/\varepsilon\in[\ell_0,\ell_k]\quad\Rightarrow\quad(\tau_{j+1}-\tau_j)/\varepsilon\in[\ell_0,\ell_{k+1}]. \tag{3.22}$$

Since (3.18) holds, we apply (3.22) repeatedly and we get

$$(\tau_{i+1}-\tau_i)/\varepsilon\in[\ell_0,\ell_1],$$
$$(\tau_{i+2}-\tau_{i+1})/\varepsilon\in[\ell_0,\ell_2],$$
$$\vdots$$
$$(\tau_{n+1}-\tau_n)/\varepsilon\in[\ell_0,\ell_{n-i+1}].$$

If (3.20) does not hold, we have

$$(\tau_{j+1}-\tau_j)/\varepsilon \in [\ell_0,\ell_k] \quad \Rightarrow \quad (\tau_j-\tau_{j-1})/\varepsilon \in [\ell_0,\ell_{k+1}]$$

and

$$\begin{aligned}(\tau_{i+1}-\tau_i)/\varepsilon &\in [\ell_0,\ell_1],\\ (\tau_i-\tau_{i-1})/\varepsilon &\in [\ell_0,\ell_2],\\ &\vdots\\ (\tau_1-\tau_0)/\varepsilon &\in [\ell_0,\ell_{i+1}].\end{aligned}$$

Thus, if both of (3.19) and (3.20) do not hold, we have

$$(\tau_{j+1}-\tau_j)/\varepsilon \leqslant \ell_n \quad \text{for all } j \in \{0,1,\dots,n\}.$$

In particular, we have

$$\tau_{n+1}-\tau_0 \leqslant n\ell_n\varepsilon. \tag{3.23}$$

Since $\tau_{n+1}-\tau_0 \geqslant h$, we can see (3.23) is impossible for small $\varepsilon>0$. Therefore (3.19) or (3.20) holds for suitable j and k.

Next we assume that (3.19) holds for some j, k and prove (3.21). The case, where (3.20) holds, can be treated in a similar way. By Proposition 3.3, we have for small $\varepsilon>0$

$$\varepsilon\frac{\partial}{\partial\tau_j}m_{DD}^{\pm}(\varepsilon;\tau_{j-1},\tau_j) \geqslant \rho(\ell_k)=\rho_k, \tag{3.24}$$

$$\varepsilon\left|\frac{\partial}{\partial\tau_j}m_{DD}^{\pm}(\varepsilon;\tau_j,\tau_{j+1})\right| \leqslant \frac{\rho_k}{2}. \tag{3.25}$$

Thus we have

$$\begin{aligned}\frac{\partial}{\partial\tau_j}g_\varepsilon(\tau_1,\dots,\tau_n) &= \frac{\partial}{\partial\tau_j}m_{DD}^{(-)^{j-1}}(\varepsilon;\tau_{j-1},\tau_j)+\frac{\partial}{\partial\tau_j}m_{DD}^{(-)^{j}}(\varepsilon;\tau_j,\tau_{j+1})\\ &\geqslant \frac{1}{2\varepsilon}\rho_k>0.\end{aligned} \tag{3.26}$$

Here we remark that we need to modify our proof slightly if $\tau_0=0$ or $\tau_{n+1}=1$. If $\tau_0=0$ and $j=1$, we replace $m_{DD}^{\pm}(\varepsilon;\tau_0,\tau_1)$ in (3.24)–(3.26) by $m_{ND}^{\pm}(\varepsilon;\tau_0,\tau_1)$. If $\tau_{n+1}=1$ and $j=n$, we replace $m_{DD}^{\pm}(\varepsilon;\tau_n,\tau_{n+1})$ by $m_{DN}^{\pm}(\varepsilon;\tau_n,\tau_{n+1})$. □

4. Brouwer degree of $\nabla_\tau g_\varepsilon$

By the estimates developed in the previous section, we have

$$\nabla_\tau g_\varepsilon \neq 0 \quad \text{on } \partial\Delta_\varepsilon$$

for sufficiently small $\varepsilon>0$ and $\deg(\nabla_\tau g_\varepsilon,\Delta_\varepsilon,0)$ is well-defined. In this section we show that $\deg(\nabla_\tau g_\varepsilon,\Delta_\varepsilon,0)=\pm1$. We consider 3 cases:

Case 1: $p \in M_-$ and $n = 1$.
Case 2: $p \in M_+ \setminus \{0, 1\}$ and $n \geqslant 1$.
Case 3: $p \in M_+ \cap \{0, 1\}$ and $n \geqslant 1$.

In each case we set

Case 1: $\Phi_\varepsilon(\tau_1) = \frac{1}{2}(\tau_1 - p)^2$,
Case 2: $\Phi_\varepsilon(\tau_1, \ldots, \tau_n) = -\frac{1}{2}(\tau_1 - p)^2 - \frac{1}{2}(\tau_n - p)^2 - \sum_{j=1}^{n-1} \exp(-\frac{\tau_{j+1}-\tau_j}{\varepsilon})$,
Case 3: If $p = 0$, we set $\Phi_\varepsilon(\tau_1, \ldots, \tau_n) = -\frac{1}{2}\tau_n^2 - \sum_{j=0}^{n-1} \exp(-\frac{\tau_{j+1}-\tau_j}{\varepsilon})$, where $\tau_0 = 0$.
If $p = 1$, we set $\Phi_\varepsilon(\tau_1, \ldots, \tau_n) = -\frac{1}{2}(1 - \tau_1)^2 - \sum_{j=1}^{n} \exp(-\frac{\tau_{j+1}-\tau_j}{\varepsilon})$, where $\tau_{n+1} = 1$.

In each case we will see that $\nabla_\tau g_\varepsilon$ and $\nabla_\tau \Phi_\varepsilon$ is homotopic in Δ_ε, that is,

$$(1-\theta)\nabla_\tau g_\varepsilon(\tau_1, \ldots, \tau_n) + \theta \nabla_\tau \Phi_\varepsilon(\tau_1, \ldots, \tau_n) \neq 0 \quad \text{for all } \theta \in [0, 1] \text{ and } (\tau_1, \ldots, \tau_n) \in \partial \Delta_n, \tag{4.1}$$

and

$$\deg(\nabla_\tau g_\varepsilon, \Delta_\varepsilon, 0) = \deg(\nabla_\tau \Phi, \Delta_\varepsilon, 0) = \begin{cases} 1 & \text{in Case 1,} \\ (-1)^n & \text{in Cases 2, 3.} \end{cases} \tag{4.2}$$

In the following subsections we show (4.1) and (4.2) for each case.

4.1. Case 1: $p \in M_-$ and $n = 1$

In this case we have $p \notin \{0, 1\}$, $n = 1$ and $\Delta_\varepsilon = [p - 2h, p + 2h]$.

Proof of (4.1) in Case 1. – We remark that $\tau_0 < p - 4h < p - 2h \leqslant \tau_1 \leqslant p + 2h < p + 4h < \tau_2$ and $(\tau_1 - \tau_0)/\varepsilon, (\tau_2 - \tau_1)/\varepsilon \geqslant 2h/\varepsilon \geqslant 3a_4|\log \varepsilon|$ for sufficiently small $\varepsilon > 0$. Applying Proposition 3.4 with $\tau_1 = p \pm 2h$, $j = 1$, we have

$$\frac{\partial}{\partial \tau_1} g_\varepsilon(p - 2h) < 0, \qquad \frac{\partial}{\partial \tau_1} g_\varepsilon(p + 2h) > 0$$

for sufficiently small $\varepsilon > 0$. Since we have $\frac{\partial}{\partial \tau_1}\Phi_\varepsilon(p \pm 2h) = \pm 2h$, we have (4.1). □

Proof of (4.2) in Case 1. – By the homotopy invariance of Brouwer degree and (4.1), we have $\deg(\nabla_\tau g_\varepsilon, [p - 2h, p + 2h], 0) = \deg(\nabla_\tau \Phi_\varepsilon, [p - 2h, p + 2h], 0)$. It is clear that $\deg(\nabla_\tau \Phi_\varepsilon, [p - 2h, p + 2h], 0) = 1$. □

4.2. Case 2: $p \in M_+ \setminus \{0, 1\}$ and $n \geqslant 1$

Proof of (4.1) in Case 2. – It suffices to show that $\Phi_\varepsilon(\tau_1, \ldots, \tau_n)$ has similar properties to $g_\varepsilon(\tau_1, \ldots, \tau_n)$. More precisely,

(i) (3.12)–(3.13) implies $(\frac{\partial}{\partial \tau_1} + \cdots + \frac{\partial}{\partial \tau_j})\Phi_\varepsilon(\tau_1, \ldots, \tau_n) > 0$,
(ii) (3.15)–(3.16) implies $(\frac{\partial}{\partial \tau_j} + \cdots + \frac{\partial}{\partial \tau_n})\Phi_\varepsilon(\tau_1, \ldots, \tau_n) < 0$,
(iii) (3.19) or (3.20) imply (3.21) for $\Phi_\varepsilon(\tau_1, \ldots, \tau_n)$.

Suppose that (3.12) and (3.13) hold. Straightforward computation gives us

$$\left(\frac{\partial}{\partial \tau_1} + \cdots + \frac{\partial}{\partial \tau_j}\right)\Phi_\varepsilon(\tau_1, \ldots, \tau_n) = -(\tau_1 - p) - \frac{1}{\varepsilon}\exp\left(-\frac{\tau_{j+1} - \tau_j}{\varepsilon}\right) \geqslant 2h - \varepsilon^{3a_4-1}.$$

We can assume $3a_4 - 1 > 1$ and we have for sufficiently small $\varepsilon > 0$

$$\left(\frac{\partial}{\partial\tau_1} + \cdots + \frac{\partial}{\partial\tau_j}\right)\Phi_\varepsilon(\tau_1, \ldots, \tau_n) > 0.$$

Thus we get (i). We can also show (ii) in a similar way.

Next we show that (3.19) implies $\frac{\partial}{\partial\tau_j}\Phi_\varepsilon(\tau_1, \ldots, \tau_n) > 0$. In fact we have under (3.19) that

$$\begin{aligned}\frac{\partial}{\partial\tau_j}\Phi_\varepsilon(\tau_1, \ldots, \tau_n) &= -\frac{1}{\varepsilon}\left(\exp\left(-\frac{\tau_{j+1} - \tau_j}{\varepsilon}\right) - \exp\left(-\frac{\tau_j - \tau_{j-1}}{\varepsilon}\right)\right)\\ &\geqslant -\frac{1}{\varepsilon}\bigl(\exp(-\ell_{k+1}) - \exp(-\ell_k)\bigr) > 0.\end{aligned}$$

In a similar way, we can see that (3.20) implies $\frac{\partial}{\partial\tau_j}\Phi_\varepsilon(\tau_1, \ldots, \tau_n) < 0$. □

Proof of (4.2) in Case 2. – By the homotopy invariance of Brouwer degree and (4.1), we have $\deg(\nabla_\tau g_\varepsilon, \Delta_\varepsilon, 0) = \deg(\nabla_\tau \Phi_\varepsilon, \Delta_\varepsilon, 0)$. We can also see that Φ_ε has unique critical point in Δ_ε and it is corresponding to a strict local maximum. Thus

$$\deg(\nabla_\tau \Phi_\varepsilon, \Delta_\varepsilon, 0) = (-1)^n. \qquad \square$$

4.3. Case 3: $p \in M_+ \cap \{0, 1\}$ and $n \geqslant 1$

Here we assume $0 \in M_+$ and we deal with the case $p = 0$ and $n \geqslant 1$. The case $p = 1$ can be treated in a similar way.

Proofs of (4.1) and (4.2) in Case 3. – In this case we have

$$\begin{aligned}\Delta_\varepsilon = \bigl\{&(\tau_1, \ldots, \tau_n);\ \varepsilon\ell_0 \leqslant \tau_1 < \tau_2 < \cdots < \tau_n \leqslant 2h,\\ &(\tau_{j+1} - \tau_j)/\varepsilon \geqslant \ell_0 \text{ for all } j = 0, 1, \ldots, n-1\bigr\},\\ \partial\Delta_\varepsilon = \bigl\{&(\tau_1, \ldots, \tau_n) \in \Delta_\varepsilon;\ (\tau_{j+1} - \tau_j)/\varepsilon = \ell_0 \text{ for some } j = 0, 1, \ldots, n-1\bigr\}\\ &\cup \bigl\{(\tau_1, \ldots, \tau_n) \in \Delta_\varepsilon;\ \tau_n = 2h\bigr\}.\end{aligned}$$

Thus it suffices to show

(i) (3.15)–(3.16) implies $(\frac{\partial}{\partial\tau_j} + \cdots + \frac{\partial}{\partial\tau_n})\Phi_\varepsilon(\tau_1, \ldots, \tau_n) < 0$,

(ii) (3.19) or (3.20) imply (3.21) for $\Phi_\varepsilon(\tau_1, \ldots, \tau_n)$.

We can show the above properties essentially as in Case 2 and we omit the proof here. □

5. Proofs of Theorems 0.1 and 0.3

Now we can prove our Theorem 0.1.

Proof of Theorem 0.1. – It suffices to show that

$$\deg(\nabla f_\varepsilon^\pm, \Delta_\varepsilon, 0) = \pm 1.$$

For each $p \in M$ we define $\Phi_\varepsilon^{(p)}(\tau_1, \ldots, \tau_n)$ as in Section 4 and set $\Psi_\varepsilon : \Delta_\varepsilon^1 \times \cdots \times \Delta_\varepsilon^N \to \mathbf{R}^{n_1+\cdots+n_N}$ by

$$\Psi_\varepsilon(t_{11}, \ldots, t_{1n_1}, \ldots, t_{N1}, \ldots, t_{Nn_N}) = \sum_{i=1}^N \Phi_\varepsilon^{(p_i)}(t_{i1}, \ldots, t_{in_i}).$$

Then we can see that $\nabla f_\varepsilon^\pm$ is homotopic to $\nabla \Psi_\varepsilon$ and

$$\begin{aligned} \deg\big(\nabla f_\varepsilon^\pm, \Delta_\varepsilon^1 \times \cdots \times \Delta_\varepsilon^N, 0\big) &= \deg\big(\nabla \Psi_\varepsilon, \Delta_\varepsilon^1 \times \cdots \times \Delta_\varepsilon^N, 0\big) \\ &= \prod_{i=1}^N \deg\big(\nabla \Phi_\varepsilon^{(p_i)}, \Delta_\varepsilon^i, 0\big) \\ &= \pm 1. \end{aligned}$$

Thus $f_\varepsilon^\pm$ has a critical point in $\Delta_\varepsilon^1 \times \cdots \times \Delta_\varepsilon^N$ and it has a desired profile stated in Theorem 0.1. □

Remark 5.1. – From the proof of Theorem 0.1, it is clear that we can get the existence result in more general setting. For example, instead of (W3), we assume

(W3′) $G(t)$ has critical sets $(I_i)_{i\in\Lambda}$ such that
- (i) for each i, I_i is an subinterval of $(0, 1)$. (I_i may be one point.) We write $I_i = [a_i, b_i]$.
- (ii) $I_i \cap I_j = \emptyset$ for $i \neq j$.
- (iii) $G'(t) = 0$ for all $t \in \bigcup I_i$.
- (iv) For each i there exists $\delta_i > 0$ such that

$$\begin{cases} G'(t) > 0 & \text{in } [a_i - \delta_i, a_i), \\ G'(t) < 0 & \text{in } (b_i, b_i + \delta_i], \end{cases} \tag{5.1}$$

or

$$\begin{cases} G'(t) < 0 & \text{in } [a_i - \delta_i, a_i), \\ G'(t) > 0 & \text{in } (b_i, b_i + \delta_i]. \end{cases} \tag{5.2}$$

We classify I_i's into 2 groups:

$$M_+ = \{I_i;\ (5.1) \text{ holds}\}, \qquad M_- = \{I_i;\ (5.2) \text{ holds}\}.$$

Then we can show the following result: *Assume* (W1), (W2) and (W3′). *Then for any given sequence* $(n_i)_{i\in\Lambda}$ *of non-negative integers satisfying*

(i) $n_i = 0$ except for finitely many i,
(ii) $n_i \in \{0, 1\}$ if $I_i \in M_-$,

there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ our problem (0.1) has a solution $u(x)$ with exactly n_i-layers in a neighborhood of I_i. We can also deal with boundary layers.

Finally we give a proof to our Theorem 0.3.

Proof of (i) of Theorem 0.3. – Let $A > 0$ and $\delta > 0$ are given numbers and let $u_\varepsilon(x)$ be a critical point of $I_\varepsilon(u)$ satisfying

$$I_\varepsilon(u_\varepsilon) \leqslant A. \tag{5.3}$$

Setting

$$\nu(\delta)=\min\bigl\{W(x,u);\ x\in[0,1],\ u\notin[\alpha_-(x)-\delta,\alpha_-(x)+\delta]\cup[\alpha_+(x)-\delta,\alpha_+(x)+\delta]\bigr\}>0,$$

we have

$$\nu(\delta)\,\mathrm{meas}\bigl\{x\in[0,1];\ u_\varepsilon(x)\notin[\alpha_-(x)-\delta,\alpha_-(x)+\delta]\cup[\alpha_+(x)-\delta,\alpha_+(x)+\delta]\bigr\}\leqslant\int_0^1 W\bigl(x,u(x)\bigr)\,\mathrm{d}x\leqslant\varepsilon I_\varepsilon(u_\varepsilon)\leqslant\varepsilon A.$$

Thus choosing $C(\delta)=A/\nu(\delta)$, we have (i) of Theorem 0.3. □

Proof of (ii) of Theorem 0.3. – Proof of (ii) of Theorem 0.3 consists of several steps. Let $A>0$ be a given number and let $u_\varepsilon(x)$ be a critical point of $I_\varepsilon(u)$ satisfying (5.3).

We fix small $\rho_0>0$ satisfying

$$\inf_{x\in[0,1],\,|\xi|\leqslant\rho_0} W_{uu}\bigl(x,\alpha_\pm(x)+\xi\bigr)>0.$$

We set

$$L_\varepsilon=\bigl\{y\in[0,1/\varepsilon];\ u_\varepsilon(\varepsilon y)\notin[\alpha_-(\varepsilon y)-\rho_0,\alpha_-(\varepsilon y)+\rho_0]\cup[\alpha_+(\varepsilon y)-\rho_0,\alpha_+(\varepsilon y)+\rho_0]\bigr\}.$$

By (i) of Theorem 0.3, we have

$$\mathrm{meas}\,L_\varepsilon\leqslant C(\rho_0)\ \text{independent of}\ \varepsilon. \tag{5.4}$$

Step 1: *Let $s_\varepsilon\in L_\varepsilon$ be a sequence such that $\tilde s_0=\lim_{\varepsilon\to0}\tilde s_\varepsilon$ exists, where $\tilde s_\varepsilon=\varepsilon s_\varepsilon$. Set $v_\varepsilon(y)=u_\varepsilon(\tilde s_\varepsilon+\varepsilon y)$. Then $v_\varepsilon(y)$ converges in C^2_{loc} to a heteroclinic solution of*

$$-w_{yy}+W_u\bigl(\tilde s_0,w(y)\bigr)=0 \tag{5.5}$$

joining $\alpha_-(\tilde s_0)$ and $\alpha_+(\tilde s_0)$.

In fact, $v_\varepsilon(y)$ converges in C^2_{loc} to a solution $w(y)$ of (5.5). By (i) of Theorem 0.3, for any $\delta>0$ there exists $y_{\varepsilon,\delta}\in[\tilde s_\varepsilon-2C(\delta),\tilde s_\varepsilon+2C(\delta)]$ such that $v_\varepsilon(y_{\varepsilon,\delta})\in[\alpha_-(\tilde s_\varepsilon+\varepsilon y_{\varepsilon,\delta})-\delta,\alpha_-(\tilde s_\varepsilon+\varepsilon y_{\varepsilon,\delta})+\delta]\cup[\alpha_+(\tilde s_\varepsilon+\varepsilon y_{\varepsilon,\delta})-\delta,\alpha_+(\tilde s_\varepsilon+\varepsilon y_{\varepsilon,\delta})+\delta]$. Thus

$$w\bigl([\tilde s_0-2C(\delta),\tilde s_0+2C(\delta)]\bigr)\cap\bigl([\alpha_-(\tilde s_0)-\delta,\alpha_-(\tilde s_0)+\delta]\cup[\alpha_+(\tilde s_0)-\delta,\alpha_+(\tilde s_0)+\delta]\bigr)\neq\emptyset.$$

Therefore there exists a sequence $y_n\to\pm\infty$ such that $w(y_n)\to\alpha_+(\tilde s_0)$ or $w(y_n)\to\alpha_-(\tilde s_0)$ By (ii) of Lemma 1.1, $w(y)$ must be a heteroclinic solution joining $\alpha_-(\tilde s_0)$ and $\alpha_+(\tilde s_0)$.

Step 2: *Let s_ε and t_ε be consecutive zeros of $u_\varepsilon(x)$. Then $\frac{t_\varepsilon-s_\varepsilon}{\varepsilon}\to\infty$ as $\varepsilon\to0$.*

This is a direct consequence of Step 1. In a similar way, we have

Step 3: *Let s_ε be any zero of $u_\varepsilon(x)$. Then $\frac{s_\varepsilon}{\varepsilon} \to \infty$, $\frac{1-s_\varepsilon}{\varepsilon} \to \infty$ as $\varepsilon \to 0$.*

Now we prove

Step 4: The number of zeros of $u_\varepsilon(x)$ is bounded by a constant $n_0(A)$ depending only on A.

In fact, let $\{t_j^{(\varepsilon)};\ j = 1, 2, \ldots\}$ be a set of zeros of $u_\varepsilon(x)$. By Steps 1–3, we can see

$$\limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \geqslant \limsup_{\varepsilon\to 0} \sum_j G\bigl(t_j^{(\varepsilon)}\bigr).$$

Thus by (5.3), the number of zeros of $u_\varepsilon(x)$ is bounded by $n_0 = A/\min_{x\in[0,1]} G(x)$ for sufficiently small ε.

Step 5: *Let $0 < t_1^{(\varepsilon)} < t_2^{(\varepsilon)} < \cdots < t_n^{(\varepsilon)} < 1$ be set of zeros of $u_\varepsilon(x)$. Suppose $u_\varepsilon(0) > 0$ ($u_\varepsilon(0) < 0$ respectively). Then for sufficiently small $\varepsilon > 0$, $u_\varepsilon|_{[0,t_1]}(x)$, $u_\varepsilon|_{[t_1,t_2]}(x), \ldots, u_\varepsilon|_{[t_n,1]}(x)$ are minimizers of $m_{ND}^+(\varepsilon; 0, t_1)$, $m_{DD}^-(\varepsilon; t_1, t_2), \ldots, m_{DN}^{(-)^n}(\varepsilon; t_n, 1)$ ($m_{ND}^-(\varepsilon; 0, t_1)$, $m_{DD}^+(\varepsilon; t_1, t_2), \ldots, m_{DN}^{(-)^{n+1}}(\varepsilon; t_n, 1)$ respectively).*

We prove just for $u_\varepsilon|_{[t_j,t_{j+1}]}(x)$ ($j = 1, 2, \ldots, n-1$). By Steps 1–3 and (5.4), we can see $u_\varepsilon|_{[t_j,t_{j+1}]}(x)$ satisfies

$$\bigl|u_\varepsilon(t_j + \varepsilon y) - \alpha_\pm(t_j + \varepsilon y)\bigr| \leqslant \rho_0 \quad \text{for } y \in \left[C(\rho_0), \frac{t_{j+1} - t_j}{\varepsilon} - C(\rho_0)\right]. \tag{5.6}$$

Thus applying Lemma 6.4 in Section 6, we can get uniqueness of solutions in (t_j, t_{j+1}) satisfying (5.6). The minimizer of $m_{DD}^\pm(\varepsilon; t_j, t_{j+1})$ is also a solution satisfying (5.6). Therefore $u_\varepsilon|_{[t_j,t_{j+1}]}(x)$ is a minimizer of $m_{DD}^\pm(\varepsilon; t_j, t_{j+1})$.

Step 6: *Conclusion.*

For sufficiently small $\varepsilon > 0$, $u_\varepsilon(x)$ is characterized as a critical point of $f_\varepsilon^\pm(t_1, \ldots, t_n)$. Thus by the arguments in Section 3, we can get the desired result. □

6. Properties of $m_{DD}^\pm(\varepsilon; s, t)$, $m_{ND}^\pm(\varepsilon; s, t)$, $m_{DN}^\pm(\varepsilon; s, t)$

Here we give proofs to Proposition 1.2 and Lemmas 1.8–1.11.

First we deal with Proposition 1.2. We mainly consider $m_{DD}^+(\varepsilon; s, t)$. Setting $\ell = (t-s)/\varepsilon$ and $v(y) = u(s + \varepsilon y)$, it suffices to prove uniqueness of minimizer of the following minimizing problem:

$$c(\varepsilon, s, \ell) = \inf_{v\in E_{DD}^+(s,\ell)} J(\varepsilon, s, \ell; v)$$

for sufficiently small ε and large ℓ. Here

$$J(\varepsilon, s, \ell; v) = \int_0^\ell \frac{1}{2}|v_y|^2 + W\bigl(s + \varepsilon y, v(y)\bigr)\,dy.$$

We start with the following lemma.

LEMMA 6.1. – *For sufficiently small* ε *and large* ℓ, $c(\varepsilon; s, \ell)$ *has a minimizer* $v(y)$ *which satisfies*

$$v(y) > 0 \quad \textit{in } (0, \ell).$$

Proof. – It is clear that $c(\varepsilon, s, \ell)$ is achieved and the corresponding minimizer $v(y)$ satisfies

$$v(y) \geqslant 0 \quad \text{in } [0, \ell].$$

Under (W1)–(W2) we can easily see that $v(y) \not\equiv 0$ for large ℓ.

Suppose that $v(y) = 0$ in $[0, \delta]$ $(\delta > 0)$ and $v(y) > 0$ in (δ, m) $(m \leqslant \ell)$. Then we have

$$-v_{yy} + W_u\big(s + \varepsilon y, v(y)\big) = 0 \quad \text{in } (\delta, m).$$

For $h \in (0, \delta)$, we set

$$v_h(y) = \begin{cases} 0 & \text{in } [0, \delta - h), \\ \frac{v(\delta+h)}{2h}(y - (\delta - h)) & \text{in } [\delta - h, \delta + h), \\ v(y) & \text{in } [\delta + h, \ell]. \end{cases}$$

Then we can see

$$\frac{\mathrm{d}}{\mathrm{d}h}\bigg|_{h=0} J(\varepsilon, s, \ell; v_h) = -\frac{1}{4}\big|v_y(\delta)\big|^2 < 0.$$

Thus $J(\varepsilon, s, \ell; v_h) < J(\varepsilon, s, \ell; v)$ for small $h > 0$. This is a contradiction and we have $\delta = 0$. In a similar way, we can see that $v(y) > 0$ in $(0, \ell)$. □

As a fundamental property of $c(\varepsilon, s, \ell)$ we have

LEMMA 6.2. – *There exists a constant* $A_1 > 0$ *independent of* $\varepsilon \in (0, 1]$, $s \in [0, 1]$, $\ell \geqslant 2$ *such that*

$$c(\varepsilon, s, \ell) \leqslant A_1 \quad \textit{for all } \varepsilon, s, \ell.$$

Proof. – We set

$$v(y) = \begin{cases} s\alpha_+(s + \varepsilon) & \text{for } y \in [0, 1], \\ \alpha_+(s + \varepsilon y) & \text{for } y \in [1, \ell - 1], \\ (\ell - s)\alpha_+(s + \varepsilon(\ell - 1)) & \text{for } y \in [\ell - 1, \ell]. \end{cases}$$

Then we have $\int_{[0,1]\cup[\ell-1,\ell]} \frac{1}{2}|v_y|^2 + W(s + \varepsilon y, v(y))\,\mathrm{d}y \leqslant C$, where $C > 0$ is independent of ε, s, ℓ. Thus we have

$$c(\varepsilon, s, \ell) \leqslant C + \int_1^{\ell-1} \frac{1}{2}|v_y|^2 + W\big(s + \varepsilon y, v(y)\big)\,\mathrm{d}y$$

$$= C + \int_1^{\ell-1} \frac{1}{2}\varepsilon^2\big|\alpha_+'(s + \varepsilon y)\big|^2\,\mathrm{d}y$$

$$= C + \frac{\varepsilon}{2} \int_{s+\varepsilon}^{s+\varepsilon(\ell-1)} \left|\alpha_+'(x)\right|^2 \mathrm{d}x$$
$$\leqslant C + \frac{\varepsilon}{2} \int_0^1 \left|\alpha_+'(x)\right|^2 \mathrm{d}x.$$

Thus we get Lemma 6.2. □

Next we show the minimizer $v(y)$ stays near $\alpha_+(s+\varepsilon y)$ except neighborhoods of boundaries of $[0,\ell]$.

LEMMA 6.3. – *For any* $\rho > 0$ *there exists* $L_2(\rho) > 0$ *such that for any minimizer* $v(y)$ *of* $c(\varepsilon, s, \ell)$ *it holds that*

$$\left|v(y) - \alpha_+(s+\varepsilon y)\right| \leqslant \rho \quad \textit{for all } y \in \left[L_2(\rho), \ell - L_2(\rho)\right]. \tag{6.1}$$

Proof. – First we show that for any $\eta > 0$ there exists $A_2(\eta) > 0$ such that

$$\operatorname{meas}\left\{y \in (0,\ell);\ \left|v(y) - \alpha_+(s+\varepsilon y)\right| \geqslant \eta\right\} \leqslant A_2(\eta). \tag{6.2}$$

In fact, setting

$$\nu_\eta = \inf_{t\in[0,1],\ \xi\in[0,\alpha_+(t)-\eta]\cup[\alpha_+(t)+\eta,\infty)} W\left(t, \alpha_+(t)+\xi\right) > 0,$$

we have from Lemma 6.2 that

$$\nu_\eta \operatorname{meas}\left\{y \in (0,\ell);\ |v(y) - \alpha_+(s+\varepsilon y)| \geqslant \eta\right\} \leqslant J(\varepsilon, s, \ell; v) = c(\varepsilon, s, \ell) \leqslant A_1.$$

Thus we have (6.2) for $A_2(\eta) = A_1/\nu_\eta$. Next we show the following property holds for any minimizer $v(y)$ of $c(\varepsilon, s, t)$:

For any $\delta > 0$ *there exist* $d(\delta) > 0$ *and* $\varepsilon(\delta) > 0$ *such that for* $\varepsilon \in (0, \varepsilon(\delta)]$ *and* y_1, $y_2 \in (0,\ell)$ *satisfying*

$$\left|v(y_i) - \alpha_+(s+\varepsilon y_i)\right| \leqslant d(\delta) \quad \textit{for } i = 1, 2, \tag{6.3}$$

it holds that

$$\int_{y_1}^{y_2} \frac{1}{2}|v_y|^2 + W\left(s+\varepsilon y, v(y)\right) \mathrm{d}y \leqslant \delta. \tag{6.4}$$

To prove this fact, we consider two cases: $|y_2 - y_1| \geqslant 3$ and $|y_2 - y_1| < 3$. When $|y_2 - y_1| \geqslant 3$, we consider a function $w(y)$ defined by

$$w(y) = \begin{cases} (y-y_1)\alpha_+(s+\varepsilon y_1) + (1-y+y_1)v(y_1) & \text{for } y \in [y_1, y_1+1], \\ \alpha_+(s+\varepsilon y) & \text{for } y \in [y_1+1, y_2-1], \\ (y_2-y)\alpha_+(s+\varepsilon y_2) + (1+y-y_2)v(y_2) & \text{for } y \in [y_2-1, y_2]. \end{cases}$$

When $|y_2 - y_1| < 3$, we consider

$$w(y) = \frac{y - y_1}{y_2 - y_1} v(y_1) + \frac{y_2 - y}{y_2 - y_1} v(y_2).$$

In both cases, we have

$$\int_{y_1}^{y_2} \frac{1}{2}|w_y|^2 + W\big(s + \varepsilon y, w(y)\big)\,\mathrm{d}y \to 0$$
$$\text{as } \varepsilon,\ \big|v(y_1) - \alpha_+(s + \varepsilon y_1)\big|,\ \big|v(y_2) - \alpha_+(s + \varepsilon y_2)\big| \to 0.$$

Since $v(y)$ is a minimizer, we have

$$\int_{y_1}^{y_2} \frac{1}{2}|v_y|^2 + W\big(s + \varepsilon y, v(y)\big)\,\mathrm{d}y \leqslant \int_{y_1}^{y_2} \frac{1}{2}|w_y|^2 + W\big(s + \varepsilon y, w(y)\big)\,\mathrm{d}y$$

and we can find desired $\varepsilon(\delta)$, $d(\delta)$.

Suppose $|v(y_0) - \alpha_+(s + \varepsilon y_0)| > \rho$ and our goal is to show that y_0 lies in a neighborhood of 0 or ℓ. We assume $y_0 \in [1, \ell - 1]$ and we remark that there exists a constant $c(\rho) > 0$ depending only on ρ such that

$$\int_{y_0-1}^{y_0+1} \frac{1}{2}|v_y|^2 + W\big(s + \varepsilon y, v(y)\big)\,\mathrm{d}y \geqslant c(\rho). \tag{6.5}$$

For $\delta = c(\rho)/2 > 0$ we choose $d(\delta)$, $\varepsilon(\delta) > 0$ so that (6.3) implies (6.4). Applying (6.2) with $\eta = d(\delta) > 0$, we can find $y_1 \in [0, A_2(\eta)]$, $y_2 \in [\ell - A_2(\eta), \ell]$ such that (6.3) holds. Thus we have

$$\int_{y_1}^{y_2} \frac{1}{2}|v_y|^2 + W\big(\alpha_+(s + \varepsilon y), v(y)\big)\,\mathrm{d}y \leqslant \delta = c(\rho)/2. \tag{6.6}$$

Comparing (6.5) with (6.6), we have $y_0 \in [0, \ell] \setminus [A_2(\eta) + 1, \ell - A_2(\eta) - 1]$. Thus we get (6.1) for $L_2(\rho) = A_2(\eta) + 1$. □

For the proof we choose $\rho_0 > 0$ such that

$$\delta_0 = \inf_{t \in [0,1],\, |\xi| \leqslant \rho_0} W_{uu}\big(t, \alpha_+(t) + \xi\big) > 0. \tag{6.7}$$

Next lemma is an essential part of the proof of Proposition 1.2. It is also used in the proof of Theorem 0.3.

LEMMA 6.4. – *Let $\rho_0 > 0$ be a number satisfying* (6.7). *For any $L_0 > 0$ there exist $\bar{\ell}_0 > 0$ and $\bar{\varepsilon}_0 > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_0]$, $\ell \geqslant \bar{\ell}_0$ satisfying $s + \varepsilon\ell \leqslant 1$, positive* (*or negative*) *solution $v(y)$ of*

$$-v_{yy} + W_u\big(s + \varepsilon y, v(y)\big) = 0 \quad \textit{in } (0, \ell), \tag{6.8}$$

$$v(0) = v(\ell) = 0 \tag{6.9}$$

with a property:

$$\big|v(y) - \alpha_+(s + \varepsilon y)\big| \leqslant \rho_0 \quad \textit{in } [L_0, \ell - L_0] \tag{6.10}$$

is unique.

Proof. – We argue indirectly and suppose that there exist sequences $(\varepsilon_n)_{n=1}^{\infty}$, $(s_n)_{n=1}^{\infty}$, $(\ell_n)_{n=1}^{\infty}$ such that

$$\varepsilon_n \to 0, \qquad \ell_n \to \infty$$

and the corresponding problem (6.8)–(6.9) has two distinct solutions $v_n^1(y)$ and $v_n^2(y)$ which satisfy the property (6.10). We may assume

$$s_n \to s_0, \quad s_n + \varepsilon_n \ell_n \to t_0 \in (0, 1].$$

It is not difficult to see that $v_n^1(y)$, $v_n^2(y)$ converge in C^2_{loc} to some solution $w(y)$ of

$$-w_{yy} + W_u\big(s_0, w(y)\big) = 0. \tag{6.11}$$

Solutions of (6.11) can be classified into 3 classes; unbounded solutions, periodic solutions and heteroclinic solutions. Since $v_n^1(y)$, $v_n^2(y)$ has no zeros in $[0, \ell_n]$ and $\ell_n \to \infty$, the limit function $w(y)$ must be a heteroclinic solution which is asymptotic to $\alpha_+(s_0)$ by the property (6.10). Thus we have

$$v_n^1(y),\ v_n^2(y) \to \omega^+(s_0, \infty; y) \quad \text{uniformly in } [0, L] \text{ as } n \to \infty$$

for any $L > 0$. Similarly we have for any $L > 0$

$$v_n^1(\ell_n - y),\ v_n^2(\ell_n - y) \to \omega^+(t_0, \infty; y) \quad \text{uniformly in } [0, L] \text{ as } n \to \infty.$$

Since both of $v_n^1(y)$ and $v_n^2(y)$ satisfy (6.8) in $[0, \ell_n]$, we can find that $h_n(y) = (v_n^1(y) - v_n^2(y))/\|v_n^1(y) - v_n^2(y)\|_{H^1}$ satisfies

$$\begin{aligned} &-h_{nyy} + W_{uu}\big(s_n + \varepsilon_n y, \theta_n v_n^1 + (1 - \theta_n) v_n^2\big) h_n = 0, \\ &h_n(0) = h_n(\ell_n) = 0, \\ &\|h_n\|_{H^1} = 1 \end{aligned} \tag{6.12}$$

for a suitable $\theta_n = \theta_n(y) \in (0, 1)$.

Multiplying $h_n(y)$ to (6.12) and integrating over $[0, \ell_n]$, we can find

$$\int_0^{\ell_n} |h_{ny}|^2 + W_{uu}\big(s_n + \varepsilon_n y, \theta_n v_n^1 + (1 - \theta_n) v_n^2\big) h_n^2 \, \mathrm{d}y = 0. \tag{6.13}$$

Thus by (6.7) and (6.13)

$$\int_0^{\ell_n} |h_{ny}|^2 + \delta_0 \int_{L_2(\rho_0)}^{\ell_n - L_2(\rho_0)} |h_n|^2 \,\mathrm{d}y$$
$$\leqslant \int_{[0,L_2(\rho_0)]\cup[\ell_n-L_2(\rho_0),\ell_n} W_{uu}\big(s_n + \varepsilon_n y, \theta_n v_n^1 + (1-\theta_n)v_n^2\big)h_n^2 \,\mathrm{d}y. \tag{6.14}$$

Therefore we have

$$\|h_n\|_{L^2(0,L_2(\rho_0))} \not\to 0 \quad \text{or} \quad \|h_n\|_{L^2(\ell_n - L_2(\rho_0),\ell_n)} \not\to 0 \quad \text{as } n \to \infty.$$

In fact, if not, (6.14) implies $\|h_n\|_{H^1} \to 0$ and it is a contradiction to $\|h_n\|_{H^1} = 1$.

Here we assume $\|h_n\|_{L^2(0,L_2(\rho_0))} \not\to 0$. The case $\|h_n\|_{L^2(\ell_n - L_2(\rho_0),\ell_n)} \not\to 0$ can be treated in a similar way. By (6.14), we may assume that $h_n \rightharpoonup h \neq 0$ weakly in H^1. By (6.14), we have

$$\begin{aligned}
&-h_{yy} + W_{uu}\big(s, \omega^+(s,\infty;y)\big)h = 0 \quad \text{in } [0,\infty),\\
&h(0) = 0,\\
&h \in H^1(0,\infty) \quad \text{and} \quad h \not\equiv 0.
\end{aligned}$$

On the other hand $\omega_y^+(y) = \omega_y^+(s,\infty;y)$ also satisfies

$$\begin{aligned}
&-(\omega_y^+)_{yy} + W_{uu}\big(s, \omega^+(s,\infty;y)\big)\omega_y^+ = 0,\\
&\omega_y^+(0) \neq 0.
\end{aligned}$$

Thus $h(y)$ and $\omega_y^+(y)$ are linearly independent solutions of

$$-\zeta_{yy} + W_{uu}\big(s, \omega^+(s,\infty;y)\big)\zeta = 0 \quad \text{in } [0,\infty). \tag{6.15}$$

Since $W_{uu}(s, \omega^+(s,\infty;y)) \to W_{uu}(s, \alpha_+(s)) > 0$ as $y \to \infty$, (6.15) have a unbounded solution $\zeta(y)$. However $\zeta(y)$ must be a linear combination of $h(y)$ and $\omega_y^+(y)$. We remark that both of $h(y)$ and $\omega_y^+(y)$ are bounded in $[0,\infty)$. It is a contradiction and the solution of (6.8)–(6.10) is unique for large ℓ and small ε. □

Now we can complete the proof of Proposition 1.2.

Proof of Proposition 1.2. – Let $v(y)$ be a minimizer for $c(\varepsilon, s, \ell)$. By Lemma 6.3, $v(y)$ satisfies the assumption of Lemma 6.4. Thus by Lemma 6.4, we get uniqueness of minimizer of $m_{DD}^+(\varepsilon; s, t)$ for sufficiently small ε and large $(t-s)/\varepsilon$. Uniqueness for minimizers of $m_{DD}^-(\varepsilon; s, t)$, $m_{ND}^\pm(\varepsilon; s, t)$, $m_{DN}^\pm(\varepsilon; s, t)$, $b^\pm(s, \ell)$ can be proved essentially in same way. □

As to the corollaries to uniqueness result for minimizers, we can prove Lemmas 1.8, 1.10 and 1.11.

Proof of Lemma 1.8. – We give a proof just for $u_{DD}^+(\varepsilon, s, t; s + \varepsilon y)$. If the conclusion of Lemma 1.8 does not hold, we can find $\delta > 0$ and sequences ε_n, s_n, t_n such that $\varepsilon_n \to 0$ and

$$\frac{t_n - s_n}{\varepsilon_n} \geqslant \ell_0,$$
$$\left\| u_{DD}^+(\varepsilon_n, s_n, t_n; s_n + \varepsilon_n y) - \omega^+\left(s_n, \frac{t_n - s_n}{2\varepsilon_n}; y\right) \right\|_{C^2(0,\tilde{L}/2)} \geqslant \delta \qquad (6.16)$$

for all $n \in \mathbf{N}$. We may also assume $s_n \to s_\infty$ and $\ell_\infty = \lim_{n\to\infty}(t_n - s_n)/\varepsilon_n \in [\ell_0, \infty]$ exists.

When $\ell_\infty = \infty$, we can easily see that $u_{DD}^+(\varepsilon_n, s_n, t_n; s_n + \varepsilon_n y) \to \omega^+(s_\infty, \infty; y)$ in C^2_{loc} and this is a contradiction to (6.16).

When $\ell_\infty \in [\ell_0, \infty)$, we can also see that $u_{DD}^+(\varepsilon_n, s_n, t_n; s_n + \varepsilon_n y) \to w(y)$ in $C^2([0, \ell_\infty])$, where $w(y)$ is a unique solution of

$$-w_{yy} + W_u\big(s_\infty, w(y)\big) = 0 \quad \text{in } (0, \ell_\infty),$$
$$w(y) > 0 \quad \text{in } (0, \ell_\infty),$$
$$w(0) = w(\ell_\infty) = 0.$$

We have $w(y) = \omega^+(0, \ell_\infty/2; y)$ in $[0, \ell_\infty/2]$ and this is also contradiction to (6.16). □

Proof of Lemma 1.10. – The result of Lemma 1.10 is essentially obtained in the proof of Lemma 6.3. □

Proof of Lemma 1.11. – We choose $\rho_0 > 0$ so that (6.7) holds. Writing $w(y) = v(y) - \alpha_+(s + \varepsilon y)$ and applying Lemma 6.3, we have

$$-w_{yy} + W_{uu}\big(s + \varepsilon y, w(y) + \alpha_+(s + \varepsilon y)\big) = \varepsilon^2 \alpha_+''(s + \varepsilon y),$$
$$|w(y)| \leqslant \rho_0 \quad \text{in } \big[L_2(\rho_0), \ell - L_2(\rho_0)\big].$$

Thus applying maximal principle to $w(y)$ in $[L_2(\rho_0), \ell - L_2(\rho_0)]$, we have the result of Lemma 1.11. □

Acknowledgements

The authors would like to thank Professor M. del Pino and Professor P. Felmer for their valuable advice and Professor H. Kokubu for informing papers [13,8].

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