A MORSE LEMMA AT INFINITY FOR YAMABE TYPE PROBLEMS ON DOMAINS

Mohamed BEN AYED *, Hichem CHTIOUI, Mokhless HAMMAMI
Département de mathématiques, faculté des sciences de Sfax, route Sokra 3018, km 3.5, Sfax, Tunisia

Received 5 November 2001, revised 19 March 2002

ABSTRACT. – In this paper we consider the following nonlinear elliptic problem (P): $-\Delta u = u^p$, $u > 0$ in $\Omega$, $u = 0$ on $\partial \Omega$, where $\Omega$ is a bounded and smooth domain in $\mathbb{R}^n$, $n \geq 4$, $p + 1 = 2n/(n - 2)$ is the critical Sobolev exponent. We prove a version of Morse lemmas at infinity for this problem. As application of these lemmas we will give a characterization of the critical points at infinity of the functional corresponding to (P).

MSC: 35J65

Keywords: Morse lemma; Elliptic problems with critical Sobolev exponent

1. Introduction and the main results

We prove in this paper a version of a Morse Lemma at infinity for Yamabe-type problems on domains in $\mathbb{R}^n$, $n \geq 4$. These problems are related to the study of the equation

$$
\begin{cases}
-\Delta u = u^{\frac{2n}{n-2}} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(P)

* Corresponding author.

E-mail addresses: Mohamed.Benayed@fss.rnu.tn (M. Ben Ayed), Mokhless.Hammami@fss.rnu.tn (M. Hammami).
where $\Omega \subset \mathbb{R}^n$, $n \geq 4$, $\Omega$ is a bounded and regular set. This problem has a variational structure. The related functional is
\[
J(u) = \left( \int_{\Omega} |u|^{\frac{2n}{n-2}} \right)^{-\frac{n-2}{2}}
\]
defined on
\[
\Sigma = \left\{ u \in H^1_0(\Omega) \ | \ \int_{\Omega} |\nabla u|^2 = 1 \right\}.
\]
The problem (P) is delicate from a variational viewpoint because the functional $J$ does not satisfy the Palais–Smale condition (P.S. for short). This means that there exist sequences along which $J$ is bounded, its gradient goes to zero and which do not converge. The P.S. condition fails for $J$ on
\[
\Sigma^+ = \{ u \in \Sigma \ s.t. \ u \geq 0 \}.
\]
Its failure has been analyzed throughout the works of [13,14,16]. The analysis carried out in [11] and [15] comes out here virtually without any change. These various studies have led to the characterization of the sequences failing the Palais–Smale condition. In order to describe this characterization, we need to introduce some notations.

Let, for $a \in \Omega$ and $\lambda > 0$ given
\[
d_0(a,\lambda)(x) = c_0 \left( \frac{\lambda}{1 + \lambda^2|x-a|^2} \right)^{\frac{n-2}{2}} \tag{1.1}
\]
c0 is chosen so that $d_0(a,\lambda)$ is the family of solutions of the Yamabe problem on $\mathbb{R}^n$. Let $P$ be the projection from $H^1(\Omega)$ onto $H^1_0(\Omega)$ i.e. $u = Pf$ is the solution of
\[
\Delta u = \Delta f \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega.
\]
Let, for $\varepsilon > 0$, $p \in \mathbb{N}^*$ and $w$ either a solution of (P) or zero
\[
V(p, \varepsilon, w) = \left\{ u \in \Sigma \ s.t. \ \exists (a_1, \ldots, a_p) \in \Omega^p, \ \exists (\lambda_1, \ldots, \lambda_p) \in (0, +\infty)^p, \ \exists (\alpha_0, \alpha_1, \ldots, \alpha_p) \in (0, +\infty)^{p+1} \ s.t. \ \lambda_i d(a_i, \partial \Omega) > \varepsilon^{-1}, \right. \\
\left. \left| u - \alpha_0 w - \sum_{i=1}^{p} \alpha_i P_\delta(a_i, \lambda_i) \right|_{H^1_0} < \varepsilon, \ \left| \frac{\alpha_i}{\alpha_j} - 1 \right| < \varepsilon, \ v_{ij} < \varepsilon \right\} \tag{1.2}
\]
where, for $i \neq j$,
\[
v_{ij}^{-1} = \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{\frac{n-2}{2}}.
\]
The failure of Palais–Smale condition can be described as follows.
PROPOSITION 1.1. – Let \((u_k) \in \Sigma^+\) be a sequence such that \((\partial J(u_k))\) tends to zero and \((J(u_k))\) is bounded. Then, after possibly having extracted a subsequence, there exists \(p \in \mathbb{N}^*\), a sequence \((\varepsilon_k)\), \(\varepsilon_k\) tends to zero, and \(w\) (either a solution of \((P)\) or zero) such that \(u_k \in V(p, \varepsilon_k, w)\).

Thinking of these sequences which do not satisfy the \((P.S.)\) condition as “critical points”, a natural idea is to try to find suitable parameters in order to complete a Morse Lemma at infinity. For manifolds without boundary, this program has been completed in [4] and [9]. We would here extend the proof of the existence of a Morse Lemma at infinity to the case of Dirichlet boundary conditions. We introduce the following parameterization of the set \(V(p, \varepsilon, w)\) (where \(w\) is a solution of \((P)\)). We denote by \(W_u(w)\) the unstable manifold of \(w\) for a decreasing pseudogradient of \(J\). If a function \(u\) belongs to \(V(p, \varepsilon, w)\) then, for \(\varepsilon > 0\) small enough, the minimization problem

\[
\min \left\{ \left| u - \alpha_0(w + h) - \sum_{i=1}^{p} \alpha_i P \delta(a_i, \lambda_i) \right|_{H_0^1} \mid \alpha_i > 0, \ a_i \in \Omega, \ \lambda_i > 0, \ h \in T_w(W_u(w)) \right\}
\]

(1.3)

has an unique solution, up to permutation (see [4–6]).

Therefore, for \(\varepsilon > 0\) sufficiently small, any \(u\) in \(V(p, \varepsilon, w)\) can be uniquely written as

\[
u = \alpha_0(w + h) + \sum_{i=1}^{p} \alpha_i P \delta(a_i, \lambda_i) + v
\]

where \(v\) satisfies the following conditions

\[
(V_0) \begin{cases}
(v, P \delta(a_i, \lambda_i))_{H_0^1(\Omega)} = (v, \frac{\partial}{\partial a_i} P \delta(a_i, \lambda_i))_{H_0^1(\Omega)} = 0, \\
(v, \frac{\partial}{\partial \lambda_i} P \delta(a_i, \lambda_i)) = (v, w)_{H_0^1(\Omega)} = (v, h)_{H_0^1(\Omega)} = 0,
\end{cases}
\]

(1.4)

and the \(\alpha_i\)’s satisfy

\[
\frac{\alpha_i}{\alpha_j} \in [1 - \varepsilon, 1 + \varepsilon] \ \forall i, \ \forall j.
\]

We denote by \(G\) the Green’s function of the Laplacian with Dirichlet boundary condition on \(\Omega\) and by \(H\) its regular part i.e.

\[
G(x, y) = |x - y|^{2-n} - H(x, y) \quad \text{for } (x, y) \in \Omega^2, \\
\Delta_x H = 0 \quad \text{in } \Omega^2, \quad \quad G = 0 \quad \text{on } \partial(\Omega^2).
\]

For \(q \in \mathbb{N}^*\), and \(x = (x_1, \ldots, x_q) \in \Omega^q\), such that \(x_i \neq x_j \ \forall i \neq j\), we denote by \(M(x) = (m_{ij})_{1 \leq i, j \leq q}\) the matrix defined by

\[
m_{ij} = H(x_i, x_i), \quad m_{ij} = -G(x_i, x_j) \quad \text{for } i \neq j,
\]

(1.5)

by \(\rho(x)\) its least eigenvalue and by \(e(x)\) the eigenvector corresponding to \(\rho(x)\) whose norm is 1 and whose components are strictly positive (see [8] and [9]).
In this paper we assume that zero is a regular value of \( \rho \) for each \( q \leq p \) (such assumption is true if, for example, \( \Omega \) is a thin or a large annuli (see [1,2])). Our main results are the following Morse lemmas at infinity.

**Theorem 1.2.** For \( \varepsilon > 0 \) sufficiently small given, there exists a change of variables, such that for any \( u = \sum_{i=1}^{p} \alpha_i P \delta(a_i, \lambda_i) + v \) belongs to \( V(p, \varepsilon) \) where \( V \) belongs to a neighborhood of zero in a fixed Hilbert space so that

\[
J \left( \sum_{i=1}^{p} \alpha_i P \delta(a_i, \lambda_i) + v \right) = J \left( \sum_{i=1}^{p} \alpha_i P \delta(a_i', \lambda_i') \right) + |V|^2.
\]

Furthermore, if each \( a_i \) belongs to a neighborhood of \( x_i \) such that \( \rho(X) > 0 \) and \( \rho'(X) = 0 \), where \( X = (x_1, \ldots, x_p) \), then we can find another change of variables \( (a_i, \lambda_i) \rightarrow (a_i', \lambda_i') \) such that, for \( \eta \) a fixed small constant

\[
J \left( \sum_{i=1}^{p} \alpha_i P \delta(a_i, \lambda_i) + \alpha_0 (w + h) + v \right) = \Psi(\alpha, a', \lambda') := \frac{(S_n)^{2/n} \sum_{i=1}^{p} \alpha_i^2}{\rho S_n} \frac{1}{(1 + \left( c_1' \rho S_n - \eta \right) \rho(a')} \sum_{i=1}^{p} \frac{1}{\lambda_i n - 2} \right) \]

where \( \alpha = (\alpha_1, \ldots, \alpha_p) \), \( a' = (a'_1, \ldots, a'_p) \), \( \lambda' = (\lambda'_1, \ldots, \lambda'_p) \) and \( c_1' \) is a positive constant and where \( S_n = \int_{\mathbb{R}^n} \delta^{2/(n-2)} \).

**Theorem 1.3.** For \( \varepsilon > 0 \) sufficiently small given, there exists a change of variables, such that for any \( u = \sum_{i=1}^{p} \alpha_i P \delta(a_i, \lambda_i) + \alpha_0 (w + h) + v \) belongs to \( V(p, \varepsilon, w) \), \( (a_i, \lambda_i, h, v) \rightarrow (a'_i, \lambda'_i, H, V) \) where each of \( H \) and \( V \) belongs to a neighborhood of zero in a fixed Hilbert space so that

\[
J \left( \sum_{i=1}^{p} \alpha_i P \delta(a_i, \lambda_i) + \alpha_0 (w + h) + v \right) = J \left( \sum_{i=1}^{p} \alpha_i P \delta(a'_i, \lambda'_i) + \alpha_0 w \right) + |V|^2 - |H|^2.
\]

The proof of these theorems is of course quite difficult and extremely technical. In principle, it relies on the construction of a suitable pseudogradient \( Z \) at infinity, as in [4,5,9], which in turn relies on very delicate expansion of \( J \) and \( \partial J \) near infinity.

This construction is even more difficult when there is a boundary, because the distance to the boundary appears (in the denominator) in the expansion (see Section 2). We need to complete much more delicate and careful expansion.

However, these Theorems are useful. They should be useful for the study of the existence of multiple solutions to (P). At this point, we will illustrate their usefulness through the following three results.

**Proposition 1.4.** Assume that zero is a regular value of \( \rho \). There are no critical points at infinity and no critical points in a neighborhood of \( V(p, \varepsilon, w) \) for \( w \neq 0 \) i.e. we can define a pseudogradient which satisfies the (P.S.) condition on decreasing flow lines and has no asymptotes in this neighborhood.

**Proposition 1.5.** Assume that zero is a regular value of \( \rho \). Then we have:
For ε small enough, J does not have any critical point in V(p, ε).

The only critical points at infinity of J correspond to \( \sum_{i=1}^{p} \delta_{(x_i, \infty)} \) where \( p \in N^* \) and the \( x_i \)'s satisfy

\[
\rho(x_1, \ldots, x_p) > 0 \quad \text{and} \quad \rho'(x_1, \ldots, x_p) = 0.
\]

There is \( p_0 \in N^* \) such that, above \( (p_0 S_n)^{2/n} \), J does not have any critical point at infinity.

We also give a new proof, based on these theorems, of the formula for the difference \( \partial P_{\delta i} \).

The remainder of the present paper is organized as follow. Section 2 will be devoted to the expansion of J and its gradient. In Section 3 we will study the \( \nu \)-part of \( u \). In Section 4 we will construct a suitable pseudogradient and then we will prove Theorem 1.2, Proposition 1.5 and Proposition 1.6. The proof of Theorem 1.3 and Proposition 2.1 are given in Section 5. Lastly, the proofs require some technical estimates which may be found in appendices.

2. The expansion of the functional and its gradient

In this section, we will give the expansion of \( J(\sum_{i=1}^{p} \alpha_i P_{\delta i} + v) \), \( (\partial J(\sum_{i=1}^{p} \alpha_i P_{\delta i}), \frac{1}{\lambda_{i}} \frac{\partial P_{\delta i}}{\partial \alpha_i}) \), \( (\partial J(\sum_{i=1}^{p} \alpha_i P_{\delta i}), \frac{1}{\lambda_{i}} \frac{\partial P_{\delta i}}{\partial \alpha_i}) \).

Proposition 2.1. For \( p \in N^*, \varepsilon > 0 \), small and \( u = \sum_{i=1}^{p} \alpha_i P_{\delta i} + v \in V(p, \varepsilon) \), we have the following expansion

\[
J(u) = \frac{\left(\sum_{i=1}^{p} \alpha_i^2\right) S_n^{2/n} \left(\sum_{i=1}^{p} \alpha_i^{2n/(n-2)}\right) \left(\sum_{i=1}^{p} \alpha_i^{2n/(n-2)}\right)}{S_n^{2/n} \left(\sum_{i=1}^{p} \alpha_i^{2n/(n-2)}\right) \left(\sum_{i=1}^{p} \alpha_i^{2n/(n-2)}\right)} \left[ 1 + \frac{c'_1}{S_n} \sum_{i=1}^{p} \frac{H(a_i, a_i)}{\lambda_{i}^{n-2}} - \frac{\alpha_i^2}{\sum \alpha_j^2} \right] - \frac{c'_1}{S_n} \sum_{i \neq j} \left( e_{ij} - \frac{2 a_i (a_i + 1)}{a_j (a_j - 1)} \right) \left( \sum_{i=1}^{p} \alpha_i^{2n/(n-2)} \right) \left( \sum_{j \neq i} \alpha_j^{2n/(n-2)} \right) + \frac{1}{\sum \alpha_j^2 S_n} Q(v, v) - (f, v) + O(\sum_{k=1}^{p} \log(\lambda_k d_k)\left(\frac{\alpha_k}{\lambda_k d_k}\right)^n) + \sum_{i \neq j} \frac{e_{ij}^*}{\lambda_{ij}} \log(\varepsilon_{ij}^{-1}) + O(|v|^{\min(2n, 3)})
\]
where $c_1'$ is a positive constant,

$$Q(v, v) = |v|_{H_0^1}^2 - \frac{n + 2}{n - 2} \sum \frac{\alpha_k^2}{\alpha_k^{2n/(n-2)}} \int \left( \sum \alpha_i P\delta_i \right)^{\frac{2n}{n-2}} v^2$$

and

$$(f, v) = 2 \left( \sum \alpha_j^{2n} S_n \right)^{-1} \int \left( \sum \alpha_i P\delta_i \right)^{\frac{2n}{n-2}} v.$$ 

Proof. –

$$J(u) = \left| \sum \alpha_i P\delta_i + v \right|^2 \left( \int \left( \sum \alpha_i P\delta_i + v \right)^{2n/(n-2)} (\frac{2n}{n-2})^{n/2} \right)^{n/(n-2)} = \frac{N}{D^{(n-2)/n}},$$

$$N = \left| \sum \alpha_i P\delta_i + v \right|^2 = \sum \alpha_i^2 |P\delta_i|^2 + |v|^2 + \sum_{i \neq j} \alpha_i \alpha_j (P\delta_i, P\delta_j).$$

From Lemmas A.1 and A.3 (in Appendix A), we obtain

$$N = \sum \alpha_i^2 \left( S_n - c_1' \frac{H(a_i, a_i)}{\lambda_i^{n-2}} \right) + \sum \alpha_i \alpha_j c_1' \left( \epsilon_{ij} - \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right)$$

$$+ |v|^2 + R_2,$$  \hspace{1cm} (2.1)

where $R_2$ satisfies

$$R_2 = O \left( \sum_k \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \sum_{i \neq j} e_{ij} \frac{1}{n} \log(e_{ij}^{-1}) \right).$$  \hspace{1cm} (2.2)

We are left for $D$. Observe that, for $q = 2n/(n - 2)$, we have

$$D = \int \left( \sum \alpha_i P\delta_i \right)^{\frac{2n}{n-2}} q \int \left( \sum \alpha_i P\delta_i \right)^{q-1} v$$

$$+ \frac{q(q - 1)}{2} \int \left( \sum \alpha_i P\delta_i \right)^{q-2} v^2 + O(|v|^{\inf(3, q)}).$$

We have also

$$\int \left( \sum_{i=1}^p \alpha_i P\delta_i \right)^{\frac{2n}{n-2}} q \sum_{i=1}^p \alpha_i^{q-1} \alpha_i \int P\delta_i^{q-1} P\delta_j$$

$$+ O \left( \sum_{i \neq j} \int \sup(\alpha_i P\delta_i, \alpha_j P\delta_j) \frac{4}{n-2} \inf(\alpha_i P\delta_i, \alpha_j P\delta_j)^2 \right).$$

Using Lemmas A.2, A.3 and A.4, and the fact that $4/(n - 2) \leq 2$, we get

$$D = \sum \alpha_i^q \left( S_n - c_1' \frac{H(a_i, a_i)}{\lambda_i^{n-2}} \right) + q \sum_{i \neq j} \alpha_i^{q-1} \alpha_j \left( \epsilon_{ij} - \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right)$$

$$+ q \int \left( \sum \alpha_i P\delta_i \right)^{q-1} v + \frac{q(q - 1)}{2} \int \left( \sum \alpha_i P\delta_i \right)^{q-2} v^2$$

$$+ R_2 + O(|v|^{\inf(3, q)}).$$  \hspace{1cm} (2.3)
Combining (2.1) and (2.3), the proof of Proposition 2.1 follows. □

**Proposition 2.2.** Let \( n \geq 4 \). For \( u = \sum_{i=1}^{p} \alpha_i P \delta_i \in V(p, \varepsilon) \), we have the following expansion

\[
\left( \partial J(u), \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \right) = 2J(u)c'_1 \left[ -\frac{n-2}{2}\alpha_i \frac{H(a_i, a_i)}{\lambda_i^{n-2}} (1 + o(1)) - \sum_{j \neq i} \alpha_j \left( \frac{\partial \delta_{ij}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right) (1 + o(1)) + R_2 \right],
\]

where \( R_2 \) is defined in (2.2).

**Proof.** We have

\[
\partial J(u) = 2J(u) \left[ u + J(u) \frac{n}{n-2} \Delta^{-1} \alpha_j^{\frac{n-2}{n}} \right].
\]  

Thus

\[
\left( \partial J(u), \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \right) = 2J(u) \left[ \sum \alpha_j \left( P \delta_j, \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \right) - J(u) \frac{n}{n-2} \int \left( \sum \alpha_j P \delta_j \right)^{\frac{n-2}{n}} \frac{\partial P \delta_i}{\partial \lambda_i} \right].
\]  

Observe that, since \( n \geq 4 \)

\[
\left( \sum \alpha_j P \delta_j \right)^{\frac{n-2}{n}} = \sum (\alpha_j P \delta_j)^{\frac{n-2}{n}} + \frac{n+2}{n-2} \sum_{i \neq j} (\alpha_i P \delta_i)^{\frac{n-2}{n}} \alpha_j P \delta_j \\
+ O \left( \sum_{i \neq j} (\alpha_i P \delta_i)^{\frac{n-2}{n}} (\alpha_j P \delta_j)^{\frac{n-2}{n}} \right) \\
+ O \left( \sum_{k, j \neq i} (\alpha_j P \delta_j)^{\frac{n-2}{n}} (\alpha_k P \delta_k) \right).
\]  

Combining (2.6), (2.5), Lemmas A.5–A.9 and the following facts: \(|\lambda_i \partial \delta_i / \partial \lambda_i| \leq c \delta_i, P \delta_k \leq \delta_k \) and \( J(u)^{n/(n-2)} \alpha_j^{4/(n-2)} = 1 + o(1), \forall j = 1, \ldots, p \), Proposition 2.5 follows. □

**Proposition 2.3.** Let \( n \geq 4 \). For \( u = \sum_{i=1}^{p} \alpha_i P \delta_i \) belonging to \( V(p, \varepsilon) \), we have the following expansion

\[
\left( \partial J(u), \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \right) = J(u)c'_1 \left[ \frac{\alpha_i}{\lambda_i^{n-1}} \frac{\partial H(a_i, a_i)}{\partial a_i} (1 + o(1)) + 2 \sum_{j \neq i} \alpha_j \left( \frac{1}{\lambda_i} \frac{\partial \delta_{ij}}{\partial a_i} - \frac{\partial H(a_i, a_j)}{\partial a_j} \frac{1}{\lambda_i (\lambda_i \lambda_j)^{(n-2)/2}} \right) \times \left( 1 - J(u)^{\frac{n-2}{n}} (\alpha_j^{\frac{4}{n-2}} + \alpha_i^{\frac{4}{n-2}}) \right) \right] + R_3,
\]

where \( R_3 \) satisfies
\[ R_3 = O\left( \sum_i \frac{\lambda_i d_i}{(\lambda_i d_i)^n} + \sum_{k \neq j} \frac{a_j}{\lambda_j} \log(\lambda_j) + \sum_j \lambda_j |a_j| + \frac{\lambda_j^{n+1}}{|a_j|} \right). \]  

(2.7)

As in the proof of Proposition 2.2, we get (2.5) but with \( \lambda_i \partial P \delta_i / \partial \lambda_i \) changed by \( \lambda_i^{-1} \partial P \delta_i / \partial a_i \). Thus, using Lemmas A.10–A.14, the proposition follows.

**Proposition 2.4.** – Let \( n \geq 4 \), for \( u = \sum_{i=1}^p \alpha_i P \delta_i + \alpha_0 (w + h) \in V(p, \varepsilon, w) \), we have

\[ (-\partial J(u), h) \geq c|h|^2 + O\left( \sum_i \lambda_i^{2-n} \right). \]

**Proof.** – Let \( u = \sum \alpha_i P \delta_i + \alpha_0 (w + h) \), using (2.4) and the fact that

\[ \left( \sum \alpha_i P \delta_i + \alpha_0 (w + h) \right)^{\frac{n+2}{n+4}} = a_0^{\frac{n+2}{n+4}} \left( w^{\frac{n+2}{n+4}} + \frac{n+2}{n-2} w^{\frac{1}{n-2}} \right) + O(|h|^{\frac{n+2}{n+4}}) \]

\[ + O\left( |h|^{\inf\left( \frac{n+2}{n+4}, \frac{1}{n-2} \right)} + \sum (w^{\frac{1}{n-2}} + |h|^{\frac{1}{n-2}}) P \delta_i + P^{\frac{n+2}{n+4}} \right) \]

we need to estimate

\[ \int_\Omega \delta_i |h| \leq |h|_{L^\infty} \int_\Omega \delta_i = O\left( \lambda_i^{-\frac{2n}{n-2}} |h|_{L^\infty} \right) = o(|h|_{L^\infty}^2) + O\left( \frac{1}{\lambda_i^{2-n}} \right), \]

(2.9)

\[ \int_\Omega \delta_i |h| \leq |h|_{L^\infty} \int_\Omega \delta_i = O\left( |h|_{L^\infty}^2 \right) = o(|h|_{L^\infty}^2) + O\left( \frac{1}{\lambda_i^{2-n}} \right), \]

(2.10)

where \( \Omega \subset B = B(a_i, R) \).

The function \( h \) belongs to \( T_{\omega}(W_\varepsilon(w)) \) which has a dimension equal to the index of \( w \). Thus

\[ |h|_{L^\infty} = O\left( |h|_{H^1_0}^2 \right); \quad \int w^{\frac{1}{n-2}} h = (w, h) = 0. \]  

(2.11)

Therefore

\[ (\partial J(u), h) = 2J(u) \left[ a_0 |h|_{H^1_0}^2 - \frac{n+2}{n-2} J(u) w^{\frac{1}{n-2}} a_0^{\frac{n+2}{n-2}} \int w^{\frac{1}{n-2}} h^2 \right. \]

\[ \left. + o(|h|_{H^1_0}^2) + O\left( \sum \frac{1}{\lambda_i^{2-n}} \right) \right]. \]

Since \( u = \sum \alpha_i P \delta_i + \alpha_0 (w + h) \in V(p, \varepsilon, w) \), we have

\[ J(u) w^{\frac{1}{n-2}} a_0^{\frac{n+2}{n-2}} = 1 + o(1), \quad |h|^2 - \frac{n+2}{n-2} \int w^{\frac{1}{n-2}} h^2 \leq -\beta_0 |h|^2. \]

Thus the result follows. \( \Box \)

**Proposition 2.5.** – Let \( n \geq 4 \), for \( u = \sum_{i=1}^p \alpha_i P \delta_i + \alpha_0 (w + h) \in V(p, \varepsilon, w) \) we have

\[ \left( \partial J(u), \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \right) = 2J(u) \left[ \frac{n-2}{2} a_0 \frac{w(a_i)}{\lambda_i^{(n-2)/2}} (1 + o(1)) - \frac{n-2}{2} \frac{H(a_i, a_i)}{\lambda_i^{n-2}} (1 + o(1)) \right]. \]
\[ -\sum_{j \neq i} \alpha_j \left( \lambda_i \frac{\partial \epsilon_{ij}}{\partial \lambda_i} + \frac{n - 2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right) (1 + o(1)) + o \left( |h|^2 + \sum \frac{1}{\lambda_i^{n/2}} + \sum \varepsilon_{kr}^{n/2} \right) + R_2 = O \left( \frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}} \right). \]

where \( R_2 \) is defined in (2.2).

**Proof.** We have

\[
\int \left( \sum \alpha_j P_{ij} \right) \frac{\partial P_{ij}}{\partial \lambda_i} + \alpha_0 (w + h) \frac{\partial P_{ij}}{\partial \lambda_i} = \sum_{j=1}^{P} \int P \delta_i \frac{\partial P_{ij}}{\partial \lambda_i} + \alpha_0 \int w \frac{\partial P_{ij}}{\partial \lambda_i} + \frac{n + 2}{n - 2} \int \left( \int P \delta_i \frac{\partial P_{ij}}{\partial \lambda_i} \right) \left( \sum \alpha_j P_{ij} \right) + o(w) \]

\[
+ \int \sum_{j \neq \lambda_i} \delta_j \frac{\partial P_{ij}}{\partial \lambda_i} + \frac{n + 2}{n - 2} \int \left( \int P \delta_i \frac{\partial P_{ij}}{\partial \lambda_i} \right) \left( \sum \alpha_j P_{ij} \right) + o(w) \]

Observe that

\[
\int_{B(a_i, d_i)} \frac{\partial P_{ij}}{\partial \lambda_i} = O \left( \frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}} \right),
\]

\[
\int_{B^c(a_i, d_i)} \frac{\partial P_{ij}}{\partial \lambda_i} \leq \frac{c}{\lambda_i^{(n-2)/2}} \frac{1}{(\lambda_i d_i)^2} = O \left( \frac{1}{\lambda_i^{n/2}} \right) + O \left( \frac{1}{(\lambda_i d_i)^2} \right).
\]

\[
\int_{B^c(a_i, d_i)} \delta_i^{2} = (n \geq 5) \left( \frac{1}{\lambda_i^{n/2}} + O \left( \frac{1}{(\lambda_i d_i)^2} \right) \right) + (n = 4) \alpha \left( \frac{d_i}{\lambda_i} + \frac{1}{(\lambda_i d_i)^2} \right).
\]

We also have, for \( n \geq 5 \), using Holder’s inequality

\[
\int \delta_i \delta_j \leq \left( \varepsilon_{ij}^{n/2} \left( \log \epsilon_{ij} \right)^{n/2} \right) \left( \log(\lambda_i d_i) \frac{\lambda_j}{\lambda_j^{n/2} \lambda_i^{n/2}} \right) \left( \log(\lambda_j) \frac{\lambda_i}{\lambda_i^{n/2} \lambda_j^{n/2}} \right)^{\frac{4}{n}} = o \left( \epsilon_{ij}^{n/2} + \lambda_i^{n/2} + \lambda_j^{n/2} \right)
\]

(2.12)

(since \( (n - 4)/(n - 1) + 4/n > 1 \)). For \( n = 4 \),

\[
\int \delta_i \delta_j \leq \frac{1}{\lambda_i} \left( \frac{d_i}{\lambda_i} + \frac{1}{(\lambda_i d_i)^2} \right) \frac{1}{\lambda_j} \left( \frac{d_i}{\lambda_i} + \frac{1}{(\lambda_i d_i)^2} + \frac{1}{\lambda_j} \right).
\]
Observe also, using Holder’s inequality, we obtain
\[
\int_{w \leq \delta_j} \delta_j^{n-1} w \delta_i \leq c \int \delta_j^{n-1} \delta_i \leq \left( \epsilon_{ij}^{n-1} (\log \epsilon_{ij}^{-1})^{n+1} \right)^{\frac{n-1}{n}} \left( \frac{\log \lambda_i \lambda_j}{\lambda_j^{1/2}} \right)^{\frac{1}{2}} = o(\epsilon_{ij}^{n-1} + \lambda_i + \lambda_j^{-\frac{n}{2}})
\]
(since \((n-2)/(n-1) + 2/n > 1\) for \(n \geq 4\)). Using the Lemmas A.5–A.9, A.15, A.16, and the fact that \(J(u)^{n/4} \alpha_i = 1 + o(1)\) for each \(i = 0, \ldots, p\), the result follows. □

**Proposition 2.6.** For \(n \geq 4\) and \(u = \sum_{i=1}^{p} \alpha_i P \delta_i + \alpha_0 (w + h)\) belonging to \(V(p, \varepsilon, w)\), we have
\[
\left( \frac{\partial J(u)}{\partial \alpha_i} \right) = J(u) c_i' \left[ \left( \frac{\alpha_i}{\lambda_i^{n/2}} \frac{\partial H(a_i, a_i)}{a_i} - \frac{\alpha_0}{\lambda_i^{n/2}} Dw(a_i) \right)(1 + o(1))
\]
\[
+ o \left( \sum_{j \neq i} \frac{1}{\lambda_j^{n/2}} \right) + 2 \sum_{j \neq i} \alpha_j \left( \frac{1}{\lambda_j} \frac{\partial \epsilon_{ij} \epsilon_{ij}^{-1}}{\partial a_i} - \frac{\partial H(a_i, a_i)}{\partial \alpha_i} \frac{\partial \epsilon_{ij} \epsilon_{ij}^{-1}}{\partial \alpha_i} \right)
\]
\[
\times \left( 1 - J(u)^{\frac{n+2}{n}} (\alpha_j^{1/n} + \alpha_i^{1/n}) \right) + R_3 + o(|h|^2)
\]
where \(R_3\) is defined in Proposition 2.3.

**Proof.**
\[
\left( \frac{\partial J(u)}{\partial \alpha_i} \right) = 2J(u) \left[ \sum_{j=1}^{p} \alpha_j \left( P \delta_j, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \right) + \alpha_0 \left( w + h, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \right)
\]
\[
- J(u)^{\frac{n+2}{n}} \left( \int \left( \sum_{j=1}^{p} \alpha_j P \delta_j + \alpha_0 w \right) \frac{\partial P \delta_i}{\partial a_i} \right)
\]
\[
+ (if \ n \geq 5)O \left( \sum_{j \neq i} \frac{1}{\lambda_j^{n/2}} \right) + o(|h|^2)
\]
\[
+ (if \ n = 4)O \left( \frac{1}{\lambda_i^{1/2}} + \frac{1}{(\lambda_i, d_i)^{3/2}} + \sum_{j \neq i} \epsilon_{ij}^{3/2} \right)
\]

We consider now the term
\[
\int \left( \sum_{j=1}^{p} \alpha_j P \delta_j + \alpha_0 w \right) \frac{\partial P \delta_i}{\partial a_i}
\]
\[
= \sum_{j=1}^{p} \alpha_j \int P \delta_j^{\frac{n+2}{n-1}} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} + \alpha_0 \int w \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i}
\]
\[
+ \frac{n+2}{n-2} \alpha_i \int P \delta_i^{\frac{n+2}{n-1}} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \left( \sum_{j \neq i} \alpha_j P \delta_j + \alpha_0 w \right)
\]
\[
+ O \left( \sum_{j \neq i} \epsilon_{ij}^{\frac{n+2}{n-1}} \log (\epsilon_{ij}^{-1}) \right) + \int w \delta_j^{\frac{n-2}{n}} \left( \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \right) + \int w \delta_j^{\frac{n-2}{n}} \delta_i
\]
Therefore

\[
\int \frac{w^{\frac{n}{2}}}{\delta} \frac{1}{\lambda} \frac{\partial P}{\partial a} \leq c \int \frac{1}{\delta} \frac{\partial P}{\partial a} + c \int \frac{1}{\delta} \frac{\partial \theta}{\partial a}
\]

\[
\leq c \int |x-a|^{\frac{n+2}{2}} + c \left( \frac{1}{\delta} \frac{\partial \theta}{\partial a} \right)_{\Omega} \left( \int \delta^{\frac{n+2}{2}} \right).
\]

For \( n \geq 6 \), we have \( 2n/(n+2) \geq n/(n-2) \), thus

\[
\int \delta^{\frac{n+2}{2}} = (\text{if } n \geq 6) O \left( \frac{\log(\lambda)}{\lambda^{n/2}} \right) \quad \text{and} \quad \text{(if } n = 4, 5) o \left( \frac{1}{\lambda} \right).
\]

Therefore

\[
\int \frac{w^{\frac{n}{2}}}{\delta} \frac{1}{\lambda} \frac{\partial P}{\partial a} = o \left( \frac{1}{\lambda^{n/2}} + \frac{1}{(\lambda d)^{n-1}} \right).
\]

\[
\int \frac{w^{\frac{n}{2}-1}}{\delta} \frac{1}{\lambda} \frac{\partial P}{\partial a} \leq \int \frac{w^{\frac{n}{2}-1}}{\delta} \frac{1}{\lambda} \frac{\partial \delta}{\partial a} + \int \frac{w^{\frac{n}{2}-1}}{\delta} \frac{1}{\lambda} \frac{\partial \theta}{\partial a}
\]

\[
\leq \int w^{\frac{n}{2}} |x-a|^{\frac{n}{2}} + \frac{c}{(\lambda d)^{n/2} d^{(n-2)/2}} \log(\lambda)
\]

\[
= o \left( \frac{1}{\lambda^{n/2}} + \frac{1}{(\lambda d)^{n-1}} \right).
\]

By the same way,

\[
\int \frac{w^{\frac{n}{2}}}{\delta} \frac{1}{\lambda} \frac{\partial P}{\partial a} \leq \int |x-a| \frac{\partial \delta}{\partial a} + \frac{1}{\lambda} \frac{\partial \theta}{\partial a} \left( \frac{\log(\lambda)}{\lambda^{n/2}} \right)
\]

\[
= o \left( \frac{1}{\lambda^{n/2}} + \frac{1}{(\lambda d)^{n-1}} \right).
\]

For \( n = 4 \),
\[ \int w^2 P \frac{1}{\lambda} \frac{\partial P \delta}{\partial a} = \int w^2 \frac{1}{\lambda} \frac{\partial \delta}{\partial a} + O \left( \int \frac{1}{\lambda} \frac{\partial \theta}{\partial a} \right) = o \left( \frac{1}{\lambda^2} + \frac{1}{(\lambda d)^3} \right) \]

and

\[ \int w^2 P \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} = \int w^2 \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} + O \left( \int \frac{1}{\lambda_i} \frac{\partial \theta}{\partial a_i} \right) = o \left( \frac{3}{2} + \frac{1}{\lambda_i} + \frac{1}{(\lambda_i d_i)^3} \right). \]

Using Lemmas A.10–A.14, A.17, A.18, the result follows. □

3. The \( v \)-part of \( u \)

In this section we deal with the \( v \)-part of \( u \), in order to show that it is negligible to the concentration phenomenon.

Lemma 3.1. There is a \( C^1 \)-map which to each \( (\alpha, a, \lambda) \) such that \( \sum_{i=1}^p \alpha_i P_\delta(\alpha_i, \lambda_i) \) belongs to \( V(p, \varepsilon) \) associates \( \bar{v} = \bar{v}(\alpha, a, \lambda) \) satisfying

\[ J \left( \sum \alpha_i P \delta_i + \bar{v} \right) = \min \left\{ J \left( \sum \alpha_i P \delta_i + v \right), \ v \text{ satisfies } (V_0) \right\} \]

and we have the following estimate

\[ |\bar{v}| \leq \begin{cases} \sum \frac{\log(\lambda_i d_i)}{\lambda_i d_i^{1/n}} + \sum_{j \neq i} \frac{\varepsilon_{ij}^{1/n}}{\lambda_i d_i^{1/n}} (\log(\varepsilon_{ij}^{-1}))^{2/n} & \text{if } n \geq 6, \\ \sum_i \frac{\log(\lambda_i d_i)}{\lambda_i d_i^{1/n}} + \sum_{j \neq i} \frac{\varepsilon_{ij}^{1/n}}{\lambda_i d_i^{1/n}} (\log(\varepsilon_{ij}^{-1}))^{2/n} & \text{if } n \leq 5. \end{cases} \]

Proof. Since \( \alpha_i/\alpha_j = 1 + o(1) \) then the quadratic form \( Q \) defined in Proposition 2.1 is close to

\[ |v|^2_{H^1_0} = \frac{n+2}{n-2} \sum \int P \delta_i^{n-2} v \]

and therefore it is a quadratic positive form on \( v \) (see [3]). Since \( \bar{v} \) minimize \( J \) in the \( v \)-space, it is easy to check the following claim (see [3])

\[ \exists \alpha > 0 \text{ s.t. } \alpha |\bar{v}|_{H^1_0}^2 \leq |(f, \bar{v})| \leq |f| |\bar{v}|_{H^1_0}. \]

Thus, it is sufficient to estimate \( |f| \) where \( f \) is defined in Proposition 2.1. We have

\[ (f, v) = 2 \left( \sum \alpha_j^q S_n \right)^{-1} \sum_{i=1}^p \alpha_i^{q-1} \int P \delta_i^{q-1} v + O \left( \sum_{j \neq i} \int P \delta_j^{q-2} P \delta_j |v| \right). \]

Observe that
The proof is similar to the proof of Lemma 3.1. So we will omit it.
4. Construction of a pseudogradient

In this section we construct a pseudogradient $Z$ near infinity as in the Proposition A.2 of [4]. The new fact here comes from the boundary. We need a new technical point showing:

(1) how the expansion behave due to the boundary. We will show that terms of the type $O((\lambda_i d_i)^{2-n})$ are added.

(2) we can construct $Z$ so that, on decreasing flow-lines, the minimal distance to the boundary only increases if it is small enough.

This new property requires a careful study of the behavior of the Green’s function and its regular part near the boundary.

We begin by giving the following main result.

**THEOREM 4.1.** – There exists a pseudogradient $Z$ so that the following holds. There is a constant $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i P^i \delta_i$ in $V(p, \varepsilon)$ so that:

(i) $(-\partial J(u), Z) \geq c \left( \sum_{i} \frac{1}{(\lambda_i d_i)^{n-1}} + \sum_{k \neq r} \frac{c_{kr}}{e^{kr}} \right)$.

(ii) $(-\partial J(u + \tilde{v}), Z + \frac{\partial \tilde{v}}{\partial (\alpha_r, a_i, \lambda_i)}(Z)) \geq c \left( \sum_{i} \frac{1}{(\lambda_i d_i)^{n-1}} + \sum_{k \neq r} \frac{c_{kr}}{e^{kr}} \right)$.

(iii) $|Z|$ is bounded.

(iv) $Z$ satisfies the Palais–Smale condition away from the critical points at infinity.

(v) The minimal distance to the boundary only increases if it is small enough.

(vi) The $\lambda_i$’s are bounded away from the case where $\rho(X) > 0$ and $\rho'(X) = o(1)$ where $X = (a_1, \ldots, a_p)$.

We will prove Theorem 4.1 at the end of the section. We now prove Theorem 1.2 and Proposition 1.5.

**Proof of Theorem 1.2.** – Using Theorem 4.1, the proof is similar to some argument in Appendix 2 [4] (see also [9]). □

**Proof of Proposition 1.5.** – On the one hand, from the normal form of $J$ and claims (ii) and (vi) of Theorem 4.1, we immediately derive (i) and (ii) of Proposition 1.5. On the other hand, since $\Omega$ is bounded, we easily deduce claim (iii) of Proposition 1.5. □

Next we will prove some technical results needed for the proof of Theorem 4.1.

For $u = \sum_{i=1}^{p} \alpha_i P^i \delta_i \in V(p, \varepsilon)$, we introduce the following condition: for $i \in \{1, \ldots, p\}$

\[
\frac{1}{2^{p+1}} \sum_{i \neq k} \varepsilon_{ki} \leq \sum_{j=1}^{p} \frac{H(\alpha_i, \alpha_j)}{(\lambda_i \lambda_j)^{(n-2)/2}}. \tag{4.1}
\]

We divide the set $\{1, \ldots, p\}$ into $T_1 \cup T_2$ where

$T_1 = \{i \text{ s.t. } i \text{ satisfies (4.1)}\}$,

$T_2 = \{i \text{ s.t. } i \text{ does not satisfy (4.1)}\}$.  

In $T_2$, we order the $\lambda_i$’s: $\lambda_{i_1} \leq \lambda_{i_2} \leq \cdots \leq \lambda_{i_s}$.

**Lemma 4.2.** There exists a vector-field $X_1$ defined on the variables $\lambda_j$’s, for $j \in T_2$, as follows

\[ X_1 = -\sum_{k=1}^{s} 2^k a_{ik} \frac{\partial P_{ik}}{\partial \lambda_{ik}}. \]

This vector-field satisfies

\[ (-\partial J(u), X_1) \geq c \sum_{i \in T_2} \left( \sum_{j \neq i} \epsilon_{ij} + \frac{1}{(\lambda_i d_i)^{n-2}} \right) + R_2 \]

where $R_2$ is defined in (2.2).

**Proof.** Using Proposition 2.2, we derive

\[ (-\partial J(u), X_1) = 2J(u) c_1 \sum_{k=1}^{s} \left[ -\sum_{j \neq i} 2^k a_j a_{ik} \frac{\partial \epsilon_{ij}}{\partial \lambda_{ik}} (1 + o(1)) \right. \]

\[ - \frac{n-2}{2} \sum_{j=1}^{p} 2^k a_j a_{ik} \frac{H(a_j, a_{ik})}{(\lambda_i \lambda_j)^{(n-3)/2}} (1 + o(1)) + R_2 \right]. \tag{4.2} \]

Observe that

\[ -\lambda_i \frac{\partial \epsilon_{ij}}{\partial \lambda_i} = \frac{n-2}{2} \epsilon_{ij} \left( 1 - 2 \frac{\lambda_j}{\lambda_i} \frac{\epsilon_{ij}}{e_{ij}} \right). \tag{4.3} \]

Thus, if $\lambda_i \geq \lambda_j$

\[ -2\lambda_i \frac{\partial \epsilon_{ij}}{\partial \lambda_i} - \lambda_j \frac{\partial \epsilon_{ij}}{\partial \lambda_j} \geq -\lambda_i \frac{\partial \epsilon_{ij}}{\partial \lambda_i} \geq \frac{n-2}{4} \epsilon_{ij} \tag{4.4} \]

and for $j \in T_1$ and $i \neq j$,

\[ \frac{\lambda_j}{\lambda_i} \frac{\epsilon_{ij}}{e_{ij}} = o(1). \tag{4.5} \]

Indeed, if $\frac{1}{2} d_j \leq d_i \leq 2d_j$, we use the fact that $j$ satisfies (4.1) and $H(a_i, a_k) \leq (d_i d_k)^{-(n-3)/2}$ (see Lemma B.2) and in the other case we use the inequality $|a_i - a_j| \geq \frac{1}{2} \max(d_i, d_j)$. Thus

\[ (-\partial J(u), X_1) \geq \frac{n-2}{4} c \sum_{i \in T_2} \left( \sum_{j \neq i} \epsilon_{ij} - 2^p \sum_{j=1}^{p} \frac{H(a_j, a_j)}{(\lambda_i \lambda_j)^{(n-3)/2}} + R_2 \right). \tag{4.6} \]

Since $i \in T_2$ and $H(a_i, a_i) \geq (d_i d_i)^{2-n}$ (see Lemma B.2), the Lemma follows. \(\square\)

We order all the $\lambda_i d_i$: $\lambda_1 d_1 \leq \lambda_2 d_2 \leq \cdots \leq \lambda_p d_p$. Let us define

\[ I_1 = \{1\} \cup \{i \text{ s.t. } \forall k \leq i, \ c_1 \lambda_k d_k \leq \lambda_{k-1} d_{k-1} \leq \lambda_k d_k \} \]

where $c_1$ is a constant chosen small enough.
COROLLARY 4.3. – If $T_2 \cap I_1 \neq \emptyset$ and $I_1 \neq \{1, - , p\}$, the vector-field $X_1$ satisfies the condition (i) of Theorem 4.1.  

Proof. – The lower-bound of the estimate of Lemma 4.2 is limited to the indices of $T_2$, thus if $T_2 \cap I_1 \neq \emptyset$, we can make the term $(\lambda_1 d_1)^{- (n - 2)}$ appears in this lower-bound and therefore we can also make all the $(\lambda_i d_i)^{- (n - 2)}$ appear in this formula. For $i \in T_1$ (i.e. $i$ satisfies (4.1)), we have

$$\sum_{j=1}^{p} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \leq \frac{p}{(\lambda_1 d_1)^{n-2}}.$$  

(4.7)

Therefore, from $(\lambda_1 d_1)^{- (n - 2)}$, we can make the term $\sum_{i \in T_1} \varepsilon_{ij}$ appear in this lower bound. Hence, the proof of the corollary. □

If $I_1 = \{1, - , p\}$, we change the vector-field $X_1$, so that along its flow-lines, $I_1$ is defined continuously even though there could be a switch in the relative ordering of the $\lambda_id_i$’s. To this effect, let $i_1 = \min\{i \text{ s.t. } i \in T_2\}$, we define

$$F_{i_1} = \{ j \notin T_2 \text{ s.t. } \lambda_i \lambda d_i \leq 2c_1 \lambda_j d_j \}.$$  

LEMMA 4.4. – If $T_2 \cap I_1 \neq \emptyset$ and $I_1 = \{1, \ldots , p\}$, the following vector-field

$$X_2 = \sum_{i \in T_2 \cup F_{i_1}} \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i}$$

satisfies the condition (i) of Theorem 4.1.  

Proof. – Using Proposition 2.2, we obtain (4.2) but with indices in $T_2 \cup F_{i_1}$. Since $I_1 = \{1, \ldots , p\}$, we have for each $i \neq k$

$$\varepsilon_{ik} = \left( \frac{1}{\lambda_i \lambda_k |a_i - a_k|^2} \right) \left( 1 + O \left( \frac{1}{\lambda_i^2 |a_i - a_k|^2} + \frac{1}{\lambda_k^2 |a_i - a_k|^2} \right) \right)$$  

$$= \left( \frac{1}{\lambda_i \lambda_k |a_i - a_k|^2} \right) + O \left( \frac{1}{(\lambda_1 d_1)^{n-2}} + \varepsilon_{ik}^{\frac{n-2}{2}} \right).$$  

(4.8)

Indeed if $d_i \leq \frac{1}{2} d_k$ or $d_i \geq 2d_k$, we have $|a_i - a_k| \geq \frac{1}{2} \max(d_i, d_k)$ and the result follows. In the other case i.e. $\frac{1}{2} d_k \leq d_i \leq 2d_k$, using that $i, k \in I_1$, thus $\lambda_i/\lambda_k$ and $\lambda_k/\lambda_i$ are bounded. Therefore $(\lambda_r |a_i - a_k|)^{-2} = O(\varepsilon_{ik}^{2/(n-2)})$, for $r = i, k$. Thus, for $i \in F_{i_1}$, using (4.8) and (4.3)

$$\sum_{j \neq i} \left( -\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \frac{n-2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right)$$  

$$= n - 2 \sum_{j \neq i} \frac{G(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} + O \left( \frac{1}{(\lambda_1 d_1)^n} + \sum_{j \neq i} \varepsilon_{ij}^{\frac{n-2}{2}} \right).$$  

(4.9)

Furthermore, if $i \in F_{i_1}$

$$\frac{H(a_i, a_i)}{\lambda_i^{n-2}} = O \left( \frac{1}{(\lambda_i d_1)^{n-2}} \right) = O \left( \frac{(2c_1)^{n-2}}{(\lambda_i d_1)^{n-2}} \right) = o \left( \frac{1}{(\lambda_i d_1)^{n-2}} \right).$$
For \( i \in T_2 \), using (4.3), (4.8) and the fact that \( i \) does not satisfy (4.1) then

\[
\sum_{j \neq i} -\lambda_i \frac{\partial \epsilon_{ij}}{\partial \lambda_i} - \sum_{j=1}^{p} \frac{n-2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \geq \frac{c}{\sum_{j \neq i} \frac{1}{(\lambda_i d_i)^{n-2}}}.
\]

Therefore, using the fact that the Green’s function is positive, thus we derive

\[
(-\partial J(u), X_2) \geq c \sum_{i \in T_2} \left( \sum_{j \neq i} \epsilon_{ij} + \frac{1}{(\lambda_i d_i)^{n-2}} \right) + R_2.
\]

Using the fact that \( T_2 \cap I_1 \neq \emptyset \), the result follows as in the Proof of Corollary 4.3. 

Now, let us define, for \( c_2 \) a fixed small constant

\[
L = \{ j \in T_1 \text{ s.t. } \exists i \in T_1 \text{ s.t. } c_2 \max(d_i, d_j) \geq |a_i - a_j| \}.
\]

(4.10)

For \( i \in L \), we denote \( i_0 \) the index such that

\[
c_2 \max(d_i, d_{i_0}) \geq |a_i - a_{i_0}|.
\]

(4.11)

**Lemma 4.5.** – If there exist two indices \( i \) and \( i_0 \) belonging to \( T_1 \) and satisfying (4.11), then, we can assume that \( \lambda_i \geq \lambda_{i_0} \), there exists a vector-field \( X_3 \), defined on the variable \( \lambda_i \) as follows

\[
X_3 = -a_i \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i}.
\]

This vector-field satisfies

\[
(-\partial J(u), X_3) \geq c \left( \epsilon_{i_{i_0}} + \frac{1}{(\lambda_i d_i)^{n-2}} \right) + R_2 + O \left( \sum_{k \in T_2} \epsilon_{ik} \right).
\]

**Proof.** – Using Proposition 2.2, we derive (4.2) (for \( s = 1 \)). The terms with \( j \in T_2 \) can be seen like \( O(\epsilon_{ij}) \). We are interesting now with the indices \( j \in T_1 \). Observe that, for \( i, k \in T_1 \), we have (4.8). Indeed, if \( 2d_i \leq d_k \) or \( d_i \geq 2d_k \), we have \( 2|a_i - a_k| \geq \max(d_i, d_k) \) and the result follows. In the other case i.e. \( \frac{1}{2}d_k \leq d_i \leq 2d_k \), using that \( i, k \in T_1 \), we have, as in (4.5)

\[
\epsilon_{ik} = O\left( \frac{\lambda_i}{\lambda_k} \frac{\lambda_k}{\lambda_i} \right).
\]

Therefore

\[
\frac{1}{\lambda_i^2 |a_i - a_k|^2} = \frac{1}{\lambda_k} \frac{1}{\lambda_i |a_i - a_k|^2} \leq \frac{\lambda_k}{\lambda_i} \frac{1}{\epsilon_{ik}^{n-2}} \leq \frac{\lambda_i}{\lambda_k} \frac{1}{\lambda_i d_i} \frac{1}{\lambda_1 d_1} = O\left( \frac{1}{(\lambda_1 d_1)^2} \right)
\]

and we use the same argument for \( (\lambda_i |a_i - a_k|)^{n-2} \). Thus we obtain (4.9) with the indices \( j \in T_1 \). Using the fact that the Green’s function is positive and that \( H(a_i, a_i) \leq d_i^{2-n} \) (see Lemma B.2), thus

\[
\left( \partial J(u), \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \right) \geq c \left( \frac{1}{(\lambda_i d_i)^{n-2}} + \frac{G(a_i, a_{i_0})}{(\lambda_i \lambda_{i_0})^{(n-2)/2}} \right) + O \left( R_2 + \sum_{k \in T_2} \epsilon_{ik} \right).
\]
where \(i\) and \(i_0\) are the indices satisfying (4.11). Using (4.11) and \(\lambda_i \geq \lambda_{i_0}\) thus

\[
\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{H(a_i, a_{i_0})}{(\lambda_i \lambda_{i_0})^{(n-2)/2}} \leq \left( \frac{c_2^2}{\lambda_i \lambda_{i_0} |a_i - a_{i_0}|^2} \right)^{\frac{n-2}{n-1}}.
\]  

(4.13)

Using (4.8) and (4.12), the lemma follows. \(\square\)

**Corollary 4.6.** – If \(T_2 \cap I_1 = \emptyset\) and there are two indices \(i\) and \(i_0\) belonging to \(I_1\) such that (4.11) is satisfied with \(i\) and \(i_0\). Thus, for \(m_1\) a fixed large constant, the vector-field \(X_3 + m_1 X_1\) satisfies the condition (i) of Theorem 4.1.

**Proof.** – Since \(i\) belongs to \(I_1\) and the term \((\lambda_i d_i)^{(2-n)}\) appears in the lower-bound of the estimate of Lemma 4.5, as in the proof of Corollary 4.3, we can make all the \((\lambda_k d_k)^{(2-n)}\) and \(\sum_{k \in T_1} \varepsilon_{kr}\) appear in this lower-bound. Thus, using Lemma 4.2 and Lemma 4.5, for \(m_1\) a fixed large constant, the corollary follows. \(\square\)

For \(d_0\) a fixed small constant, we introduce the set

\[I'_1 = \{ i \in I_1 \text{ s.t. } d_i < d_0 \}.\]

**Lemma 4.7.** – If \(T_2 \cap I_1 = \emptyset\) and, for each \(i\) and \(j\) belonging to \(I_1\), (4.11) is not satisfied, if we have also \(I'_1 \neq \emptyset\), then there exists a vector-field \(X_4\), defined on the variables \(a_i\)'s, for \(i \in I'_1\), as follows

\[
X_4 = \sum_{i \in I'_1} a_i \frac{\partial \delta_i}{\partial a_i} \left( -\frac{n_i}{\lambda_{j_0}} \right)
\]

where \(\lambda_{j_0} = \max\{\lambda_i, i \in I'_1\}\) and \(n_i\) is the outward normal to \(\partial \Omega_d_i = \{ x \in \Omega \text{ s.t. } d(x, \partial \Omega) = d_i \}\) at \(a_i\). This vector-field satisfies

\[
(-\partial J(u), X_4) \geq c \sum_{i=1}^p \frac{1}{(\lambda_i d_i)^{n-1}} + O \left( \sum_{i \in I'_1, j \in T_2 \cup L_i} \varepsilon_{ij} \right) + R_3
\]

where \(R_3\) is defined in Proposition 2.3 and

\[L_i = \{ j \in L \text{ s.t. } i \text{ and } j \text{ satisfy (4.11)} \}.
\]

**Proof.** – Using the Proposition 2.3, we derive

\[
(-\partial J(u), X_4) = \frac{c_1}{\lambda_{j_0}} J(u) \sum_{i \in I'_1} \left[ \alpha_i^2 \frac{\partial H(a_i, a_{i_0})}{\partial n_i} (1 + o(1)) + O \left( \sum_{j \in T_2 \cup L_i} \lambda_i \varepsilon_{ij} \right) \right]
\]

\[
+ 2 \sum_{j \in T_1 \setminus L_i} \alpha_i \alpha_j \left( \frac{\partial \varepsilon_{ij}}{\partial n_i} - \frac{\partial H(a_i, a_j)}{\partial n_i} \frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} \right)
\]

\[
\times \left( 1 - J(u)^{n-2} (\alpha_i^{n-2} + \alpha_j^{n-2}) \right) + \lambda_i R_3
\]

(4.14)
where $R_3$ is the remainder term in Proposition 2.3. For $i \in I'_1$ and $j \in T_1 \setminus (I_1 \cup L_i)$, using Lemma B.2

\[
\frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} \frac{\partial H}{\partial n_i}(a_i, a_j) \leq \frac{c H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}d_i} \leq \frac{c}{d_i (\lambda_i d_i)(\lambda_j d_j)(n-2)/2} = o \left( \frac{1}{d_i (\lambda_i d_i)^{n-2}} \right).
\]

For $i$ and $j$ in $I'_1$, if $d_i/d_j$, $d_j/d_i$, and $|a_i - a_j|/d_i$ are bounded, we use the estimate of $(\partial H/\partial n_i)(a_i, a_j)$ provided in the Appendix B and proved in the appendix of [8], we derive $(\partial H/\partial n_i)(a_i, a_j) > 0$. In the other case, we have

\[
\frac{\partial H}{\partial n_i}(a_i, a_j) \leq H(a_i, a_j) \leq \frac{1}{d_i \max(d_i, d_j, |a_i - a_j|)^{n-2}} = o \left( \frac{1}{(d_i d_j)^{(n-1)/2}} \right)
\]

and therefore

\[
(\lambda_i \lambda_j)^{-2(n-2)/2} \frac{\partial H}{\partial n_i}(a_i, a_j) = o \left( \frac{1}{d_i (\lambda_i d_i)^{n-2}} + \frac{1}{d_j (\lambda_j d_j)^{n-2}} \right).
\]

(4.15) We also have for $i \in I'_1$ (see Lemma B.2)

\[
\frac{\partial H(a_i, a_i)}{\partial n_i} = \frac{n-2}{2} \frac{1}{d_i^{n-1}} (1 + o(1)).
\]

For $i$ and $j$ belong to $I'_1$, $n_i - n_j = O(|a_i - a_j|)$ therefore

\[
\frac{\partial \varepsilon_{ij}}{\partial a_i} n_i + \frac{\partial \varepsilon_{ij}}{\partial a_j} n_j = \frac{n-2}{2} \lambda_i \lambda_j (a_i - a_j) \varepsilon_{ij}^{\omega_2} (n_j - n_i) = O(\varepsilon_{ij}).
\]

For each $i \in I'_1$ and $j \in T_1 \setminus (I_1 \cup L_i)$ we have $c_2 \max(d_i, d_j) \leq |a_i - a_j|$, therefore

\[
\left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| = (n-2) \lambda_i \lambda_j |a_i - a_j| \varepsilon_{ij}^{\omega_2} \leq \frac{c}{(\lambda_i \lambda_j)^{n-2/2}d_i} \frac{1}{|a_i - a_j|^{n-1}}
\]

\[
= O \left( \frac{1}{c_2^{n-1}(\lambda_i \lambda_j)^{(n-2)/2}d_i} \right) = O \left( \frac{c_1}{c_2^{n-1}(\lambda_i d_i)^{n-2}d_i} \right)
\]

\[
= o \left( \frac{1}{d_i (\lambda_i d_i)^{n-2}} \right)
\]

(4.16) if we choose $c_1$ and $c_2$ so that $c_1^{(n-2)/2} = o(c_2^{n-1})$. For $i \in I'_1$, $j \in I_1 \setminus I'_1$, we claim

\[
\frac{\partial H(a_i, a_j)}{\partial n_i} \frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} = \frac{1}{\partial a_i} n_i
\]

\[
= - \frac{\partial G(a_i, a_j)}{\partial n_i} \frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} + o \left( \frac{\lambda_i}{(\lambda_i d_i)^{n+1}} \right).
\]

(4.17) Indeed, since $T_2 \cap I_1 = \emptyset$, then $i$ and $j$ belong to $T_1$. Using (4.8) and the fact that (4.11) is not satisfied, thus

\[
\frac{\partial \varepsilon_{ij}}{\partial a_i} = (n-2)(a_j - a_i) \lambda_i \lambda_j \varepsilon_{ij}^{\omega_2} = \frac{(n-2)(a_j - a_i)}{(\lambda_i \lambda_j)^{(n-2)/2}|a_i - a_j|^n} \left( 1 + O \left( \frac{1}{(\lambda_i d_i)^2} \right) \right).
\]
Therefore
\[ \frac{\partial \varepsilon_{ij}}{\partial d_i} = \frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} \frac{\partial}{\partial a_i} \left( \frac{1}{|a_i - a_j|^{n-2}} \right) + O \left( \frac{\lambda_i}{\varepsilon_i^{n-1} (\lambda_i d_i)^{n+1}} \right) \]
and our claim follows. Thus
\[ (-\partial J(u), X_4) \geq c \sum_{i \in I_1, j \in I_i} \left[ \frac{1}{(\lambda_i d_i)^{n-2} d_i} - \sum_{j \in I_1 \setminus I'_1} \frac{1}{(\lambda_i \lambda_j)^{n-2}/2} \frac{\partial G(a_i, a_j)}{\partial n_i} \right. \\
\left. + O \left( \sum_{j \in T_2} \varepsilon_{ij} \right) + O \left( \sum_{j \in T_2} \lambda_i \varepsilon_{ij} \right) + \lambda_i R_3 + O \left( \lambda_i \sum_{j \in T_1} \varepsilon_{ij} \right) \right]. \]

Observe that, for \( i \in I'_1 \) and \( j \in I_1 \setminus I'_1 \) we have \( d_i \leq d_j \) then, using Lemma B.3, \(-\partial G(a_i, a_j)/\partial n_i > 0\). For \( i, j \in I'_1 \), using the fact that (4.11) does not satisfied, thus \( \varepsilon_{ij} = O((\lambda_i d_i)^{(2-n)} + (\lambda_j d_j)^{(2-n)}) \) and therefore, since \( d_i \) and \( d_j \) are small, we have \( \varepsilon_{ij} = o((\lambda_i d_i)^{2-n} d_i^{-1} + (\lambda_j d_j)^{2-n} d_j^{-1}) \). Using the fact that \( j_0 \in I'_1 \) and that \( \lambda_{j_0} d_{j_0} \) and \( \lambda_i d_i \) are of the same order, we can make all the \( (\lambda_i d_i)^{(1-n)} \), for \( i \in T_1 \), appear in the lower bound. The result follows. \( \square \)

**LEMMA 4.8.** There exists a vector-field \( X_5 \) satisfying
\[ (-\partial J(u), X_5) \geq c \sum_{i \in I'_1, j \in I_i} \varepsilon_{ij} + o \left( \sum_{i \in I'_1, j \in T_2} \varepsilon_{ij} \right) + R_2. \]

**Proof.** For \( i \in I'_1 \) and \( j \in I_i \), using Lemma 4.5, we can find a vector-field \( X_3(i, j) \) depending on the indices \( i \) and \( j \) and satisfying the estimate of Lemma 4.5. The vector-field \( X_5 \) will be defined as follows
\[ X_5 = \sum_{i \in I'_1, j \in I_i} X_3(i, j). \]

The result follows. \( \square \)

**COROLLARY 4.9.** If \( T_2 \cap I_1 = \emptyset \) and, for each \( i \) and \( j \) belonging to \( I_1 \), (4.11) is not satisfied, if we have also \( I'_1 \neq \emptyset \), then, for \( m_1 \) and \( m_2 \), two fixed large constants, the vector-field \( X_4 + m_1 X_1 + m_2 X_5 \) satisfies the condition (i) of Theorem 4.1.

**Proof.** For \( m_1 \) and \( m_2 \) two fixed large constant, we derive by Lemma 4.2, Lemma 4.7 and Lemma 4.8
\[ (-\partial J(u), X_4 + m_1 X_1 + m_2 X_5) \geq c \left( \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n-1}} + \sum_{j \in T_2} \varepsilon_{ij} \right) + R_2 + R_3. \]

As in the proof of Corollary 4.3, we can make \( \sum_{j \in T_1} \varepsilon_{ij}^{(2-n)/(n-2)} \) appears in the lower-bound of (4.18). Therefore the corollary follows. \( \square \)

**LEMMA 4.10.** If \( T_2 \cap I_1 = \emptyset \) and, for each \( i \) and \( j \) belonging to \( I_1 \), (4.11) is not satisfied, if we have also \( I'_1 = \emptyset \), then there exists a vector-field \( X_6 \) depending on the
values of \( \rho \) and \( \rho' \). This vector-field satisfies

\[
(-\partial J(u), X_b) \geq c \sum_{i \in I_1} \left( \frac{1}{(\lambda_i d_i)^{n-1}} + O \left( \sum_{j \in T_2 \setminus E_i} \epsilon_{ij} \right) \right) + R_2 + R_3.
\]

**Proof.** – Since \( I_1' = \emptyset \), thus, for each \( i \in I_1 \) we have \( d_i \geq d_0 \). We denote by \( M = M(a_i, i \in I_1) \) the matrix defined in (1.5), by \( \rho \) its least eigenvalue and by \( e \) the eigenvector associated to \( \rho \). Let \( \eta > 0 \) be such that for any \( x \) belongs to a neighborhood \( C(e, \eta) \) of \( e \), where

\[
C(e, \eta) \subset \left\{ y \in (R_+^{*})^{\text{card} I_1} \text{ s.t. } \left| y \right| - e < \eta \right\}
\]

we have

\[
th_x M x - \rho |x|^2 \leq \frac{1}{2} |\rho| |x|^2 \quad \text{and} \quad \frac{\partial M}{\partial a_i} x = \left( \frac{\partial \rho}{\partial a_i} + o(1) \right) |x|^2
\]

(4.19)

and for each \( x \in C(e, \eta)^c \) we have

\[
th_x M x - \rho |x|^2 > c |x|^2.
\]

Let \( \Lambda = (\lambda_{j_1}^{-(n-2)/2}, \ldots, \lambda_{j_s}^{-(n-2)/2}) \) (\( \{j_1, \ldots, j_s\} = I_1 \)). If \( \Lambda \) belongs to the set \( C(e, \eta)^c \), then we move the vector \( \Lambda \) to \( C(e, \eta) \), as in [9], along

\[
\Lambda(t) = |\Lambda| \left( \frac{(1 - t) \Lambda + t |\Lambda| e}{(1 - t) \Lambda + t |\Lambda| e} \right).
\]

Using Proposition 2.2, we derive

\[
(-\partial J(u), X_b) = c \left[ -\frac{d}{dt} \left( \frac{\Lambda(t) M \Lambda(t)}{\lambda_i^{n-2}} \right) + O \left( \sum_{i,j \in I_1} \epsilon_{ij} + \sum_{i \in I_1} \frac{1}{\lambda_i^{n-2}} \right) \right. + R_2 + O \left( \sum_{i \in I_1, j \in T_2 \setminus I_1} \epsilon_{ij} \right].
\]

(4.20)

As in [9], we have

\[
\frac{d}{dt} \left( \frac{\Lambda(t) M \Lambda(t)}{\lambda_i^{n-2}} \right) < -c |\Lambda|^2 = -c \sum_{i \in I_1} \frac{1}{\lambda_i^{n-2}}.
\]

For \( i \in I_1, j \in T_1 \setminus I_1 \), using (4.1), we obtain

\[
\epsilon_{ij} = O \left( \frac{1}{(\lambda_i d_i)(\lambda_j d_j)} \right) = O \left( \frac{c_1^{(n-2)/2}}{d_0^{n-2} \lambda_1^{n-2}} \right) = o \left( \frac{c_1^{(n-2)/2}}{d_0^{n-2} \lambda_1^{n-2}} \right)
\]

(4.21)

if we assume that \( c_1 = o(d_0^{n-2}) \). Therefore, the lemma follows in this case. In the case where \( \Lambda \in C(e, \eta) \), the construction of the vector-field depends on the value of \( \rho \) and
|\rho'|. Since zero is a regular value of \rho then there exists a constant \rho_0 > 0 such that either |\rho| > \rho_0 or |\rho'| > \rho_0.

If \rho < -\rho_0 we decrease all the \lambda_i’s, for \(i \in I_1\). If we assume that \(c_1 \frac{n-2}{2} = o(\rho_0 d_0^{n-2})\), using Proposition 2.2, (4.3), (4.5) and (4.21), the result follows as in the above case.

If |\rho'| > \rho_0 and \rho > -\rho_0, then we move the points \(a_i\)’s along \(\lambda_j \partial a_i = -\partial \rho / \partial a_i\) for each \(i \in I_1\) where \(\lambda_{j_0} = \max[\lambda_i, i \in I_1]\). Using Proposition 2.3, we derive

\[
(\partial J(u), X_6) = \frac{1}{\lambda_{j_0}} \sum_{i \in I_1} \left( -\partial J(u), \frac{\partial P \delta_i}{\partial a_i} \right) \left( -\frac{\partial \rho}{\partial a_i} \right)
\]

\[
= \frac{1}{\lambda_{j_0}} \sum_{i \in I_1} \left[ c \frac{\partial \rho}{\partial a_i} \left( \Lambda \frac{\partial M}{\partial a_i} \Lambda \right) + O \left( \sum_{j \in \Lambda_{5 \cap I_1 \cup L_i}} \lambda_i \varepsilon_{ij} \right) \right.
\]

\[
+ \left. \sum_{j \in \Lambda_{5 \cap I_1 \cup L_i}} \frac{1}{\Lambda_{5 \cap I_1 \cup L_i}} \frac{\partial \varepsilon_{ij}}{\partial a_i} \right] + R_3 \right],
\]

where \(R_3\) is defined in Proposition 2.3. For \(i \in I_1\) and \(j \in \Lambda_{5 \cap I_1 \cup L_i}\), (4.11) is not satisfied and therefore we have

\[
\frac{\partial H}{\partial a_i}(a_i, a_j) \leq \frac{1}{\lambda_{j_0} \lambda_j} \frac{1}{\lambda_i} \frac{1}{\lambda_{j_0} \lambda_j} \frac{1}{\lambda_i} \leq \frac{(c_1 D)^{(n-2)/2}}{d_0^{(3n-4)/2}} \frac{1}{\lambda_1^{n-2}},
\]

where \(D\) is the diameter of \(\Omega\). As in (4.16)

\[
\left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| \leq \frac{1}{\lambda_{j_0} \lambda_j} \frac{1}{\lambda_i} |a_i - a_j|^{-n} \leq \frac{(c_1 D)^{(n-2)/2}}{c_2^{n-1} d_0^{(3n-4)/2}} \frac{1}{\lambda_1^{n-2}}.
\]

We can choose the constants \(c_1\) and \(c_2\) so that

\[
\frac{\partial H}{\partial a_i}(a_i, a_j) \leq \frac{1}{\lambda_{j_0} \lambda_j} |a_i - a_j|^{-n} \leq \frac{(c_1 D)^{(n-2)/2}}{c_2^{n-1} d_0^{(3n-4)/2}} \frac{1}{\lambda_1^{n-2}}.
\]

Using that \(\Lambda\) belongs to \(C(e, \eta)\), thus (4.19) is satisfied. Therefore (4.22) and (4.23) imply

\[
(-\partial J(u), X_6) \geq \frac{1}{2} \frac{c}{\lambda_{j_0}} |\rho'| |\Lambda|^2 + O \left( \sum_{i \in I_1, j \in \Lambda_{5 \cap I_1 \cup L_i}} \varepsilon_{ij} \right) + R_3
\]

\[
\geq \frac{1}{2} \frac{c}{\rho_0} |\Lambda|^2 + O \left( \sum_{i \in I_1, j \in \Lambda_{5 \cap I_1 \cup L_i}} \varepsilon_{ij} \right) + R_3.
\]

Therefore, the lemma follows.

**Corollary 4.11.** – If \(T_2 \cap I_1 = \emptyset\) and, for each \(i\) and \(j\) belonging to \(I_1\), (4.11) is not satisfied, if we have also \(I'_i = \emptyset\), then, for \(m_1\) and \(m_2\) two fixed large constants, the vector-field \(X_6 + m_1 \varepsilon_1 + m_2 \varepsilon_5\) satisfies the condition (i) of Theorem 4.1

**Proof.** – The proof of this corollary is similar to the proof of Corollary 4.9.

**Proof of Theorem 4.1.** – The vector field \(Z\) required in Theorem 4.1 will be defined by a convex combination of all these cases defined above. Combining Corollaries 4.3,
4.6, 4.9, 4.11 and Lemma 4.4, we easily derive (i), (iii), (iv), (v) and (vi) of Theorem 4.1. Now, we are going to prove claim (ii). We follow the argument of [4] and [9]. Let
\[ u = \sum_{i=1}^{n} \alpha_i P_{\delta_i} \text{ and } \bar{u} = u + \bar{\nu}. \]
Thus
\[ (-\partial J(\bar{u}), Z + \frac{\partial \bar{\nu}}{\partial (\alpha_i, a_i, \lambda_i)}(Z)) = (-\partial J(\bar{u}), Z) - (\partial J(\bar{u}), \frac{\partial \bar{\nu}}{\partial (\alpha_i, a_i, \lambda_i)}(Z)). \]
We need to prove that the terms contained \( \bar{\nu} \) are \( o(\sum \varepsilon_{kr}^{(n-1)/(n-2)} + \sum (\lambda_i d_i)^{1-n}) \). We consider \( (\partial J(\bar{u}), \partial \bar{\nu}/\partial (\alpha_i, a_i, \lambda_i)(Z)) \). Let
\[ E = \text{span}_{i} \{ P_{\delta_i}, \partial P_{\delta_i}/\partial \lambda_i, \partial P_{\delta_i}/\partial a_i \}, \]
\[ F = \text{span}_{i} \{ \partial P_{\delta_i}/\partial \lambda_i, \partial P_{\delta_i}/\partial a_i \}. \]
and \( Q_E \) (respectively \( Q_F \)) be the orthogonal projection onto \( E \) (respectively \( F \)). Since \( \bar{\nu} \) extremizes \( J(\sum \alpha_i P_{\delta_i} + v) \) in the \( v \)-space, \( \partial J(\bar{u}) \) belongs to \( E \). Thus
\[ (\partial J(\bar{u}), \frac{\partial \bar{\nu}}{\partial (\alpha_i, a_i, \lambda_i)}(Z)) = (Q_E(\partial J(\bar{u})), Q_E(\frac{\partial \bar{\nu}}{\partial (\alpha_i, a_i, \lambda_i)}(Z))). \]
Since \( \bar{\nu} \) satisfies \( (V_0) \) then \( (\bar{\nu}, P_{\delta_i}) = 0 \) and therefore
\[ \left( \frac{\partial \bar{\nu}}{\partial (\alpha_i, a_i, \lambda_i)}(Z), P_{\delta_i} \right) = -\left( \bar{\nu}, \frac{\partial P_{\delta_i}}{\partial (\alpha_i, a_i, \lambda_i)}(Z) \right) = 0. \]
Thus \( Q_E(\partial \bar{\nu}/\partial (\alpha_i, a_i, \lambda_i)(Z)) = Q_F(\partial \bar{\nu}/\partial (\alpha_i, a_i, \lambda_i)(Z)) \) and therefore
\[ \left( \partial J(\bar{u}), \frac{\partial \bar{\nu}}{\partial (\alpha_i, a_i, \lambda_i)}(Z) \right) = \left( Q_F(\partial J(\bar{u})), Q_F(\frac{\partial \bar{\nu}}{\partial (\alpha_i, a_i, \lambda_i)}(Z)) \right). \]
By Proposition A.2 of [4], we have, for each \( \phi \in F \),
\[ \left( Q_F\left( \frac{\partial \bar{\nu}}{\partial (\alpha_i, a_i, \lambda_i)}(Z), \phi \right), \right) \leq |\bar{\nu}| |Z| |\phi|. \]
Thus
\[ \left( Q_F(\partial J(\bar{u})), Q_F\left( \frac{\partial \bar{\nu}}{\partial (\alpha_i, a_i, \lambda_i)}(Z) \right) \right) \leq |\bar{\nu}| |Z| \sup_{\phi \in F} \left( \frac{(\partial J(\bar{u}), \phi)}{|\phi|} \right). \]
Using Propositions 2.2 and 2.3, we get
\[ \sup_{\phi \in F} \left( \frac{(\partial J(\bar{u}), \phi)}{|\phi|} \right) = O\left( \sum \varepsilon_{ij} + \sum \frac{1}{(\lambda_i d_i)^{n-2}} + |\bar{\nu}| \right). \]
Using the estimate of \( \bar{\nu} \) (see Lemma 3.1) and that \( |Z| \leq c \), we derive
\[ \left( \partial J(\bar{u}), \frac{\partial \bar{\nu}}{\partial (\alpha_i, a_i, \lambda_i)}(Z) \right) = o\left( \sum \varepsilon_{kr}^{\frac{n+1}{n-2}} + \sum \frac{1}{(\lambda_i d_i)^{n-1}} \right). \]
We, now, left for \( (\partial J(\bar{u}), Z) \). Observe that
\[
(\partial J(\vec{u}), Z) = 2J(\vec{u})\left(\vec{u}, Z\right) - J(\vec{u}) \frac{\lambda^{n+2}}{n+2} \int \vec{u} \frac{\lambda^{n+2}}{n+2} Z
\]
\[
= 2J(u)\left(u, Z\right) - J(u) \frac{\lambda^{n+2}}{n+2} \int u \frac{\lambda^{n+2}}{n+2} Z
\]
\[
+ O\left(|\vec{u}|\left(\sum \varepsilon_{kr} + \sum \frac{1}{(\lambda, d_i)^{n-2}} + |\vec{u}|\right)\right)
\]
\[
= \left(\partial J(u), Z\right) - 2\frac{n}{n+2} \int u \frac{\lambda^{n+2}}{n+2} \int u \frac{\lambda^{n+2}}{n+2} Z + O(|\vec{u}|^2)
\]
\[
+ O\left(\int |\vec{u}| \frac{\lambda^{n+2}}{n+2}|Z|\right) + O\left(|\vec{u}|\left(\sum \varepsilon_{kr} + \sum \frac{1}{(\lambda, d_i)^{n-2}} + |\vec{u}|\right)\right).
\]

Observe that, for \(\phi_i\) equal to \(\lambda_i^{-1} \partial P \delta_i / \partial a_i\) or \(\lambda_i \partial P \delta_i / \partial \lambda_i\),
\[
\int u \frac{\lambda^{n+2}}{n+2} \phi_i = \alpha_i \int P \frac{\lambda^{n+2}}{n+2} \phi_i \nu + O\left(\sum_{j \neq i} \left(\int \delta_j \frac{\lambda^{n+2}}{n+2} |\nu| \delta_i + \int \delta_i \frac{\lambda^{n+2}}{n+2} \delta_j |\nu|\right)\right)
\]
\[
\leq \int P \frac{\lambda^{n+2}}{n+2} \phi_i \nu + c|\nu| \left(\sum_{j \neq k \leq \delta_i} \left(\int \delta_j \frac{\lambda^{n+2}}{n+2} \delta_k \frac{\lambda^{n+2}}{n+2}\right)\right) \frac{\lambda^{n+2}}{n+2}.
\]

Observe that, for \(n \geq 6\) and \(k \neq j\),
\[
\int_{\delta_j \leq \delta_k} (\delta_j \frac{\lambda^{n+2}}{n+2}) \leq \int (\delta_k \delta_j) \frac{\lambda^{n+2}}{n+2} = O(\varepsilon_{kj} \log(\varepsilon_{kj}))
\]
and for \(n \leq 5\),
\[
\left[\int (\delta_k \delta_j) \frac{\lambda^{n+2}}{n+2}\right] \leq \left[\int (\delta_k \delta_j) \frac{\lambda^{n+2}}{n+2} \delta_k \frac{\lambda^{n+2}}{n+2}\right] \frac{\lambda^{n+2}}{n+2} = O\left(\varepsilon_{kj} (\log(\varepsilon_{kj})) \frac{n+2}{n+2}\right).
\]

Using the same argument than (3.2), we have
\[
\int P \frac{\lambda^{n+2}}{n+2} \phi_i \nu \leq c|\nu| \left(\int \frac{1}{(\lambda, d_i)^{(n+2)/2}} + (if n = 6) \log(\lambda, d_i) \frac{\lambda^4}{(\lambda, d_i)^4} + (if n \leq 5) \frac{1}{(\lambda, d_i)^{n-2}}\right).
\]

Since \(Z\) belongs to \(F\) thus
\[
\int u \frac{\lambda^{n+2}}{n+2} Z = O\left(\sum \varepsilon_{kr} + \sum \frac{1}{(\lambda, d_i)^{n-1}}\right).
\]
Since \(|Z|(x) \leq c \sum \delta_i(x)\), thus
\[
\int_{u \leq \vec{u}} |\vec{u}| \frac{\lambda^{n+2}}{n+2}|Z| \leq c \int_{u \leq \vec{u}} |\vec{u}| \frac{\lambda^{n+2}}{n+2} u + \sum_i \int_{u \leq \vec{u}} |\vec{u}| \frac{\lambda^{n+2}}{n+2} (\delta_i - P \delta_i)
\]
\[
\leq c \left(|\vec{u}| \frac{\lambda^{n+2}}{n+2} + \sum_i \frac{1}{(\lambda, d_i)^{n}}\right).
\] (4.24)
The proof of (ii) is therefore completed. □

Before ending this section, we give a new proof, based on Theorem 1.2 and 4.1, of the formula for the difference of topology of the critical points at infinity of the functional $J$. This formula was proved in [8].

Proof of Proposition 1.6. – From Proposition 1.5, we have the only critical points at infinity for $J$ are \( \sum \delta(x_i, \infty) \) where $X = (x_1, \ldots, x_p)$ satisfies $\rho(X) > 0$ and $\rho'(X) = 0$. Moreover, Theorem 1.2 gives the normal form of $J$ in the neighborhood of these critical points at infinity. Thus, the homology required is the product of the homologies defined by each variables. Therefore, using the fact that in the $\alpha$-space is an unique maximum and in the $\lambda$-space, there is an unique minimum, our proposition follows. □

5. Proof of Theorem 1.3

We consider now the Morse Lemma when there is a solution $w$ of (P). The proof of the Theorem 1.3 is similar to Theorem 1.2 and it relies also on the existence of a pseudo-gradient $Z'$ which satisfies a proposition similar to the Theorem 4.1. We have the following main result which the proof is given at the end of this section.

THEOREM 5.1. – There exists a pseudogradient $Z'$ so that the following holds: there is a constant $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i P \delta_i + \alpha_0 (w + h)$ in $V(p, \varepsilon, w)$ such that

(i) \[
( - \partial J(u), Z') \geq c \sum_{i=1}^p \left( \lambda_i^{-n/2} + (\lambda_i d_i)^{1-n} \right) + c \sum_{k \neq i} \varepsilon_k |h|^2,
\]

(ii) \[
( - \partial J(u + \bar{v}), Z' + \frac{\partial \bar{v}}{\partial (\alpha_i, a_i, \lambda_i, h)} (Z')) \geq c \left( \sum_{i=1}^p \left( \lambda_i^{-n/2} + (\lambda_i d_i)^{1-n} \right) + \sum_{k \neq i} \varepsilon_k |h|^2 \right).
\]

This pseudogradient satisfies the (P.S.) condition and it increases the least distance to the boundary along any flow line.

First, we immediately derive from claim (ii) of this theorem, that $\partial J$ cannot be zero in $V(p, \varepsilon, w)$ and hence the Proposition 1.6 follows.

Now, we state the following lemma, which the proof may be deduce from [4], pp. 354–355.

LEMMA 5.2. – There is a $C^1$-map which to each $(\alpha, a, \lambda)$ such that $\sum_{i=1}^p \alpha_i P \delta_{(a, \lambda)} + \alpha_0 w$ belongs to $V(p, \varepsilon, w)$ associates $\bar{r} = \bar{r}(\alpha, a, \lambda)$ satisfying

\[
J \left( \sum \alpha_i P \delta_i + \alpha_0 w + \bar{r} \right) = \max \left\{ J \left( \sum \alpha_i P \delta_i + \alpha_0 (w + h) \right), \ h \in T_w(W_u(w)) \right\}.
\]
We now prove Theorem 1.3.

Proof of Theorem 1.3. – Arguing as in Appendix 2 of [4], we immediately derive from Theorem 5.1 that for each \( u = \sum_{i=1}^{p} \alpha_i P \delta_i + \alpha_0 (w + h) \) belongs to \( V(p, \varepsilon, w) \) we can find a change of variables \((a, \lambda, h) \rightarrow (\tilde{a}, \tilde{\lambda}, \tilde{h})\) such that

\[
J \left( \sum_{i=1}^{p} \alpha_i P \delta_i + \alpha_0 (w + h) + \tilde{v} \right) = J \left( \sum_{i=1}^{p} \alpha_i P \delta_i + \alpha_0 (w + \tilde{h}) \right).
\]

From Lemma 5.2, we deduce that there is a change of variables \( h - \bar{h} \rightarrow H \) such that

\[
J \left( \sum_{i=1}^{p} \alpha_i P \delta_i + \alpha_0 (w + h) \right) = J \left( \sum_{i=1}^{p} \alpha_i P \delta_i + \alpha_0 (w + \bar{h}) \right) - |H|^2.
\]

Arguing as in the proof of Lemma 3.1, we obtain the following estimate

\[
|\bar{h}| = O \left( \sum \lambda_i^{-\frac{n-2}{2}} \right).
\]

Thus, by the same argument used to prove Theorem 1.2, we can find another change of variables \((a, \lambda) \rightarrow (\tilde{a}, \tilde{\lambda})\) such that

\[
J \left( \sum_{i=1}^{p} \alpha_i P \delta_i + \alpha_0 w + \bar{h} \right) = J \left( \sum_{i=1}^{p} \alpha_i P \delta_i + \alpha_0 w \right).
\]

The proof of the Theorem 1.3 is therefore completed under Theorem 5.1. \(\square\)

We, now, left on the proof of Theorem 5.1.

Proof of Theorem 5.1. – The new fact here is the terms \((-w(a)/\lambda^{(n-2)/2})\) and \(|h|^2\) where \(w\) is the solution of \((P)\) and \(h\) belongs to \(T_w(W_w(w))\). By Proposition 2.4, we get

\[
(-\partial J(u), h) \geq c|h|^2 + O \left( \sum \frac{1}{\lambda_i^{n-2}} \right).
\]

We need to add another vector-field on the variables \((a_i, \lambda_i)\). It will be defined as in Theorem 4.1. Let \(d_0 > 0\) be a small constant. We divide the set \(\{1, \ldots, p\}\) into \(T_1 \cup T_2 \cup T_3\) where:

- \(T_1 = \{i \text{ s.t. } i \text{ satisfies } (4.1) \text{ and } d_i < d_0\}\),
- \(T_2 = \{i \text{ s.t. } i \text{ does not satisfy } (4.1) \text{ and } d_i < d_0\}\),
- \(T_3 = \{i \text{ s.t. } d_i \geq d_0\}\).

In \(T_2 \cup T_3\), we order the \(\lambda_i\)'s: \(\lambda_{i_1} \leq \lambda_{i_2} \leq \cdots \leq \lambda_{i_s}\). On these variables, we define a vector-field as follows

\[
X'_1 = -\sum_{k=1}^{s} 2^k \alpha_{i_k} \lambda_{i_k} \frac{\partial P \delta_{i_k}}{\partial \lambda_{i_k}}.
\]

By Proposition 2.5, we derive, as in (4.2)
\[ (-\partial J(u), X'_i) \geq \sum_{k=1}^{\frac{n}{2}} \frac{w(a_{ik})}{\lambda_i^{n-2k/2}} - \sum_{j \neq i} 2^k a_i a_j \frac{\partial^2 \varepsilon_{ij}}{\partial \lambda_i} (1 + o(1)) + O\left(\frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}}\right) \]

\[ - \frac{n-2}{2} \sum_{k} 2^k a_i a_j \frac{H(a_j, a_{ik})}{(\lambda_i \lambda_j)^{(n-2)/2}} (1 + o(1)) + R_4, \]

where \( R_4 \) is the remainder term in Proposition 2.5, it satisfies

\[ R_4 = o\left(|h|^2 + \sum \frac{1}{\lambda_k^{n/2}} + \sum \varepsilon_{kr}\right) + R_2. \]

Observe that

- for \( i \in T_3 \), \( w(a_i) > c(d_i) \) and therefore

\[ \frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}} = O\left(\frac{w(a_i)}{\lambda_i^{(n-2)/2}}\right) \quad \text{and} \quad \sum_j H(a_i, a_j) = O\left(\frac{w(a_i)}{\lambda_i^{(n-2)/2}}\right). \]

- for \( i \in T_2 \), we have \( w(a_i) \geq c d_i \). Therefore:

\[ \frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}} = o\left(\frac{d_i}{\lambda_i^{(n-2)/2}}\right) = o\left(\frac{w(a_i)}{\lambda_i^{(n-2)/2}}\right). \]

Observe that, for \( i \in T_2 \cup T_3 \) and \( j \in T_1 \), using (4.3) and (4.5), thus we get as in (4.6)

\[ (-\partial J(u), X'_i) \geq c \sum_{i \in T_2 \cup T_3} \left[ \frac{1}{\lambda_i^{n/2}} + \sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^{n/2}} \right] + R_4. \]

We need to add some more terms in our lower-bound. Thus, we will define another vector field using the set \( T_1 \).

First, we order the \( \lambda_i d_i \)'s for \( i \in T_1 \) (for simplicity, we will assume that \( T_1 = \{1, \ldots, q\} \) : \( \lambda_1 d_1 \leq \lambda_2 d_2 \leq \cdots \leq \lambda_q d_q \) and we define the set

\[ I_1 = \{1\} \cup \{i \in T_1 \text{ s.t. } \forall k \leq i \ c_1 \lambda_k d_k \leq \lambda_{k-1} d_{k-1}\}, \]

where \( c_1 \) is a fixed small constant. For \( i \in I_1 \), we introduce

\[ L_i = \{j \in T_1 \text{ s.t. } c_2 \max(d_i, d_j) \geq |a_i - a_j|\}, \]

where \( c_2 \) is a fixed small constant. We define the following two vector-fields using the variables \( a_i \)'s and \( \lambda_i \)'s, for \( i \in T_1 \)

\[ X'_2 = - \sum_{i \in T_1} \frac{1}{(\lambda_i d_i)^{n/2}} \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \quad \text{and} \quad X'_3 = \sum_{i \in T_1} a_i \frac{\partial P \delta_i}{\partial a_i} \left( -\frac{n_i}{\lambda_i} \right) \]

where \( \lambda_{i_0} = \max(\lambda_i, i \in I_1) \). The vector-field \( X'_2 \) satisfies
\[
(\partial J(u), X'_2) = 2J(u) \sum_{i \in T_1} \frac{c_i^2}{\lambda_i d_i} \left[ \frac{n - 2}{2} \left( \alpha_0 \frac{w(a_i)}{\lambda_i^{(n-2)/2}} - \alpha_i \frac{H(a_i, a_i)}{\lambda_i^{n-2}} \right) + \omega(1) \right] + R_4 + O \left( \frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}} + \sum_{j \neq i} \epsilon_{ij} \right).
\]

Using the fact that \(w(a_i) \geq cd_i\) and, by (4.1), \(\sum \epsilon_{ij} = O\left(\sup (\lambda_k d_k)^{n-1}\right)\), we derive
\[
(\partial J(u), X'_2) \geq c \sum_{i \in T_1} \frac{1}{\lambda_i^{n/2}} + O \left( \sup \left( \frac{1}{\lambda_i^{n-1}} \right) \right) + R_4.
\]

Furthermore, \(X'_3\) satisfies, using Proposition 2.6
\[
(-\partial J(u), X'_3) = \sum_{i \in T_1} \frac{c_i^2}{\lambda_i} J(u) \sum_{i \in T_1} \alpha_i^2 - \frac{\partial H(a_i, a_i)}{\partial a_i} - \frac{\partial w(a_i)}{\partial n_i} (1 + o(1)) + R_5.
\]

For \(i \in I_1\) and \(j \in L_i\), as in Lemma 4.5, we can find a vector-field \(X'_4(i, j)\) satisfying
\[
(-\partial J(u), X'_4(i, j)) \geq c \epsilon_{ij} + R_4 + O \left( \sum_{k \in T_3} (\epsilon_{ki} + \epsilon_{kj}) \right)
\]
and therefore, we can define
\[
X'_4 = \sum_{i \in I_1, j \in L_i} X'_4(i, j).
\]

Thus
\[
(-\partial J(u), X'_4) \geq c \sum_{i \in I_1, j \in L_i} \epsilon_{ij} + R_4 + O \left( \sum_{k \in T_3} \epsilon_{kr} \right).
\]

We can choose \(m_1 > m_2 > m_3\), three fixed large constants, so that the vector-field \(Z'\), required in Theorem 5.1, will be defined as follows
\[
Z' = m_1 X'_1 + m_2 X'_4 + m_3 X'_3 + X'_2 + \eta
\]
it satisfies the following required estimate

\[ (-\partial J(u),Z') \geq c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{\sigma/2}} + \frac{1}{(\lambda_i d_i)^{n-1}} + \sum_{k \neq r} \epsilon_{kr} \right). \]

Therefore the proof of (i) is completed. Using the estimate of $\bar{v}$ (see Lemma 3.2), the proof of (ii) follows as in Theorem 4.1. Thus Theorem 5.1 is completed. \(\Box\)

Acknowledgements

The authors would like to thank Professor Abbas Bahri for bringing to our attention this problem and for helpful discussions and comments.

Appendix A

In this appendix, we collect the estimates of the different integral quantities which occur in the paper. These estimates are originally introduced by Bahri [3] and Bahri and Coron [6]. For the proof, we refer the interested reader to [3,6,12]. In this appendix, we suppose that $\lambda_i d_i$ is large enough and $\epsilon_{ij}$ is small enough. We have the following estimates

**Lemma A.1.** –

\[ |P_{\delta}|^2 = S_n - c'_1 \frac{H(a,a)}{\lambda^{n-2}} + O\left( \frac{\log(\lambda d)}{(\lambda d)^n} \right) \]

where $c'_1 = c_0 c'_2$, $c'_2 = \int_{\Omega} dx \frac{d}{(1 + |x|^2)^{(n-2)/2}}$ and $c_0$ is defined in (1.1).

**Lemma A.2.** –

\[ \int_{\Omega} P_{\delta} \frac{\partial}{\partial \lambda} P_{\delta} = S_n - 2n \frac{2}{n-2} c'_1 \frac{H(a,a)}{\lambda^{n-2}} + O\left( \frac{\log(\lambda d)}{(\lambda d)^n} \right). \]

**Lemma A.3.** – For $i \neq j$

\[ (P_{\delta i}, P_{\delta j}) = c'_i \left( \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right) + O\left( \epsilon_{ij}^{-1} \log(\epsilon_{ij}^{-1}) + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} \right). \]

**Lemma A.4.** – For $i \neq j$

\[ \int_{\Omega} P_{\delta i} \frac{\partial}{\partial \lambda} P_{\delta j} = (P_{\delta i}, P_{\delta j}) + O\left( \epsilon_{ij}^{-1} \log(\epsilon_{ij}^{-1}) + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} \right). \]

**Lemma A.5.** –

\[ \left( P_{\delta}, \frac{\partial P_{\delta}}{\partial \lambda} \right) = \frac{n-2}{2} c'_1 \frac{H(a,a)}{\lambda^{n-2}} + O\left( \frac{\log(\lambda d)}{(\lambda d)^n} \right). \]
LEMMA A.6. –
\[
\int P\frac{\partial^2 P}{\partial \lambda^2} = 2 \left( P, \frac{\partial P}{\partial \lambda} \right) + O\left( \frac{\log(\lambda d)}{(\lambda d)^n} \right).
\]

LEMMA A.7. – For \( i \neq j \)
\[
\left( P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right) = c'_i \left( \lambda_i, \frac{\partial \varepsilon_{ij}}{\partial a_i} + \frac{n-2}{2} \frac{H(a_i, a_j)}{\lambda_i \lambda_j^{(n-2)/2}} \right)
+ O\left( \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{ij}^\infty \log(\varepsilon_{ij}^{-1}) \right).
\]

LEMMA A.8. – For \( i \neq j \)
\[
\int P\frac{\partial^2 P}{\partial \lambda^2} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = \left( P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right) + O\left( \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{ij}^\infty \log(\varepsilon_{ij}^{-1}) \right).
\]

LEMMA A.9. – For \( i \neq j \)
\[
n + 2 - \frac{2}{n - 2} \int P\delta_j \left( P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right) = \left( P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right) + O\left( \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{ij}^\infty \log(\varepsilon_{ij}^{-1}) \right).
\]

LEMMA A.10. –
\[
\left( P\delta, \frac{\partial P\delta}{\partial a} \right) = -c'_1 \lambda \frac{\partial H}{\partial a(a, a)} + O\left( \frac{1}{(\lambda d)^n} \right).
\]

LEMMA A.11. –
\[
\int P\frac{\partial^2 P}{\partial \lambda^2} \lambda \frac{\partial P\delta}{\partial a} = 2 \left( P, \frac{\partial P\delta}{\partial \lambda} \right) + O\left( \frac{\log(\lambda d)}{(\lambda d)^n} \right).
\]

LEMMA A.12. – For \( i \neq j \)
\[
\left( P\delta_j, \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \right) = -c'_1 \lambda \frac{\partial H}{\partial a(a_i, a_j)} + c'_1 \lambda \frac{\partial \varepsilon_{ij}}{\partial a_i}
+ O\left( \sum_{k=i,j} \frac{1}{(\lambda_k d_k)^n} + \varepsilon_{ij}^\infty \lambda d \delta_{ij} a_i - a_j \right).
\]

LEMMA A.13. – For \( i \neq j \)
\[
\int P\delta^2_j \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} = \left( P\delta_j, \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \right) + O\left( \sum_{k=i,j} \frac{1}{(\lambda_k d_k)^n} + \varepsilon_{ij}^\infty \log(\varepsilon_{ij}^{-1}) \right).
\]

LEMMA A.14. – For \( i \neq j \)
\[ \frac{n+2}{n-2} \int P_{\delta_j} \left( P_{\delta_i}^{\lambda_{ij}} \frac{1}{\partial \lambda} \partial P_{\delta_i} \right) = \left( P_{\delta_j} \frac{1}{\lambda_i} \partial P_{\delta_i} \right) + O \left( \sum_{k=i,j} \frac{1}{(\lambda_k d_k)^n} \right) + O \left( \frac{\varepsilon_{ij}}{\lambda_{ij}} \log (\varepsilon_{ij}^{-1}) \right). \]

Lemma A.15.

\[ \left( w, \lambda \frac{\partial P_{\delta}}{\partial \lambda} \right) = -\frac{n-2}{2} c_2 \frac{w(a)}{\lambda^{(n-2)/2}} \left( 1 + O \left( \frac{1}{(\lambda d)^2} \right) \right) + O \left( \frac{1}{\lambda^{n/2}} + \frac{1}{(\lambda d)^n} \right). \]

Lemma A.16.

\[ \int \left( \frac{n+2}{n-2} P_{\delta}^{2 \lambda_{ij}} \frac{1}{\partial \lambda} \partial P_{\delta} \right) = \left( w, \lambda \frac{\partial P_{\delta}}{\partial \lambda} \right) + O \left( \frac{\lambda^{n/2}}{(\lambda d)^n} \right). \]

Lemma A.17.

\[ \left( w, \frac{1}{\lambda} \frac{\partial P_{\delta}}{\partial a} \right) = -c_1 \frac{Dw(a)}{\lambda^{n/2}} \left( 1 + O \left( \frac{1}{(\lambda d)^2} \right) \right) + O \left( \frac{d}{\lambda^{n/2}} + \frac{1}{(\lambda d)^n} \right). \]

Lemma A.18.

\[ \frac{n+2}{n-2} \int \left( P_{\delta}^{2 \lambda_{ij}} \frac{1}{\lambda} \frac{\partial P_{\delta}}{\partial a} \right) = \left( w, \frac{1}{\lambda} \frac{\partial P_{\delta}}{\partial a} \right) + O \left( \frac{1}{\lambda^{n/2}} \right) + O \left( \frac{(\log(\lambda d))^{n/2}}{(\lambda d)^n} \right). \]

Appendix B

Lemma B.1 [3,12]. For \( \theta = \theta_{(a, \lambda)} = \delta_{(a, \lambda)} - P_{\delta_{(a, \lambda)}} \) and \( H \) the regular part of the Green's function, we have the following estimates:

\[ \theta(x) = \frac{c_0}{\lambda^{(n-2)/2}} H(a, x) + O \left( \frac{1}{\lambda^{(n+2)/2} d^n} \right), \quad \left. \partial \theta \right|_{L^{\frac{2n}{n-2}}} = O \left( \frac{1}{(\lambda d)^{(n-2)/2}} \right), \]

\[ \frac{\partial \theta}{\partial a}(x) = \frac{c_0}{\lambda^{(n-2)/2}} \frac{\partial H}{\partial a}(a, x) + O \left( \frac{1}{\lambda^{(n-2)/2} d^{n+1}} \right), \quad \left. \frac{\partial \theta}{\partial a} \right|_{L^{\frac{2n}{n-2}}} = O \left( \frac{1}{(\lambda d)^{n/2}} \right), \]

\[ \frac{\partial \theta}{\partial \lambda}(x) = -\frac{n-2}{2} \frac{c_0}{\lambda^{(n-2)/2}} H(a, x) + O \left( \frac{1}{\lambda^{(n+1)/2} d^n} \right), \quad \left. \frac{\partial \theta}{\partial \lambda} \right|_{L^{\frac{2n}{n-2}}} = O \left( \frac{1}{(\lambda d)^{(n+1)/2}} \right), \]

where \( d = d(a, \partial \Omega) \) and \( c_0 \) is the constant defined in (1.1).

Lemma B.2 [3,12]. For each \( a \in \Omega \), near the boundary of \( \Omega \), let \( n_a = n \) the outward normal to \( \partial \Omega \) at \( a \)

\[ H(a, a) = (2d)^{2-n} + O(d^{2-n}), \quad H(x, a) \leq c \max(d_x, d_a)^{2-n}, \quad \frac{\partial H}{\partial a}(a, a) = \frac{n-2}{2d^{n-2} d_a} + O \left( \frac{1}{d^{n-1}} \right), \quad \left. \frac{\partial H}{\partial x} \right|_{(x, a)} = c \frac{1}{d_x} H(x, a). \]

Lemma B.3. Let \( x_1 \) and \( x_2 \) be two points of \( \Omega \) such that \( d_1 \leq d_2 \) and \( c_2 d_2 \leq |x_1 - x_2| \) where \( c_2 \) is a fixed constant. If \( d_1 \) is small enough then \( (\partial G/\partial n_1)(x_1, x_2) \leq 0. \)
Proof. — We argue by contradiction. We assume that there exists a sequence \((x_1^m, x_2^m) \in \Omega^2\) such that \(d_1^m\) tends to zero, \(d_2^m \leq d_2^m, c_2 d_2^m \leq |x_1^m - x_1^m|, (\partial G/\partial n_1)(x_1^m, x_2^m) > 0\). In the case where \(\lim x_1^m \neq \lim x_2^m\), the strong maximum principle yields a contradiction. Suppose, now, that \(\lim x_1^m = \lim x_2^m\). Three cases may occur:

1. \(d_2^m/|x_1^m - x_2^m|\) tends to zero.
2. \(|x_1^m - x_2^m|/d_2^m\) is bounded and \(d_2^m/|x_1^m - x_2^m|\) tends to zero.
3. \(|x_1^m - x_2^m|/d_2^m\) is bounded.

Let us consider the first case. We introduce the transformation \(T\) defined on \(\Omega\) by

\[
T: \Omega \to \bar{\Omega} = T(\Omega) = \frac{x}{|x_1^m - x_2^m|}.
\]

Let \(G\) and \(\tilde{G}\) be the Green’s functionals associated to \(\Omega\) and \(\bar{\Omega}\). We have

\[
G(x, y) = \tilde{G}(\tilde{x}, \tilde{y})/|x_1^m - x_2^m|^{m-2}.
\]

Thus

\[
\frac{\partial G}{\partial n_1}(x_1^m, x_2^m) = \frac{1}{|x_1^m - x_2^m|^{m-1}} \frac{\partial \tilde{G}}{\partial n_1}(\tilde{x}_1^m, \tilde{x}_2^m).
\]

Let \(\tilde{x}_1^m\) be such that \(\tilde{x}_1^m \in \partial \bar{\Omega}\) and \(\tilde{d}_2^m = |\tilde{x}_1^m - \tilde{x}_2^m|\). For a fixed constant \(R (0 < R < 1/2)\), we choose \(y_1 \in \bar{\Omega}\) such that \(B_1(y_1, R)\) is included in \(\bar{\Omega}\) and contains \(\tilde{x}_1^m\). We introduce the functional

\[
v(x) = e^{-\alpha |x - y_1|^2} - e^{-\alpha R^2}
\]

where \(\alpha\) is chosen such that \(\Delta v \geq 0\) on \(B_1 \setminus B_2(y_1, r)\) \((0 < r < R/2)\). The functional \(v\) satisfies

\[
v(x) = 0 \quad \text{on } \partial B_1 \quad \text{and} \quad v(x) > 0 \quad \text{in } B_1.
\]

Let \(A\), a set independent of \(m\), be such that

\[
B_2(y_1, r) \subset A \subset \bar{\Omega}, \quad \tilde{x}_2^m \in A \quad \text{and} \quad \tilde{x}_2^m \in \partial A.
\]

Observe that, for \(x \in \partial B_2\), we have \(|\tilde{x}_2 - x| \geq 1/4\) and for each \(t \in [\tilde{x}_2, \tilde{x}_2], |t - x| \geq 1/4\). Thus

\[
G_A(x, \tilde{x}_2) = G_A(x, \tilde{x}_2) + \frac{\partial G_A}{\partial n_2}(x, \tilde{x}_2)\tilde{d}_2 + O\left(\sup_{t \in [\tilde{x}_2, \tilde{x}_2]} \frac{\partial^2 G_A}{\partial n_2^2}(x, t)\tilde{d}_2^2\right).
\]

Since \(\tilde{x}_2 \in \partial A\) then \(G_A(x, \tilde{x}_2) = 0\). Using Hopf Lemma,

\[
G_A(x, \tilde{x}_2) \geq c(A)\tilde{d}_2 + O(\tilde{d}_2^2) \geq c\tilde{d}_2,
\]

where \(c\) is independent of \(m\). Since \(A \subset \bar{\Omega}\) then

\[
\tilde{G} \geq G_A.
\]
and therefore for each \( x \in \partial B_2 \), \( v(x) \leq M \). Thus, for each \( x \in \partial (B_1 \setminus B_2) \)

\[
\tilde{G}(x, \tilde{x}_2) - \frac{cd_2}{M} v(x) \geq 0.
\]

Observe that

\[
-\Delta \left( \tilde{G}(\cdot, \tilde{x}_2) - \frac{cd_2}{M} v(\cdot) \right) \geq 0 \quad \text{in} \quad B_1 \setminus B_2.
\]

Thus, using the maximum principle, we derive

\[
\tilde{G}(\cdot, \tilde{x}_2) - \frac{cd_2}{M} v(\cdot) \geq 0 \quad \text{in} \quad B_1 \setminus B_2.
\]

The value of this functional at \( \bar{x}_1 \) is zero. Thus

\[
\frac{\partial}{\partial n_1} \left( \tilde{G}(\bar{x}_1, \tilde{x}_2) - \frac{cd_2}{M} v(\bar{x}_1) \right) \leq 0.
\]

Therefore

\[
\frac{\partial \tilde{G}}{\partial n_1}(\bar{x}_1, \tilde{x}_2) \leq \frac{cd_2}{M} \frac{\partial v}{\partial n_1}(\bar{x}_1) \leq -cd_2.
\]

We need to prove this estimate for \( \partial \tilde{G}/\partial n_1(\bar{x}_1, \tilde{x}_2) \). Observe that

\[
\frac{\partial \tilde{G}}{\partial n_1}(\bar{x}_1, \tilde{x}_2) = \frac{\partial \tilde{G}}{\partial n_1}(\bar{x}_1, \tilde{x}_2) + O \left( \sup_{t \in [\bar{x}_1, \tilde{x}_1]} \frac{\partial^2 \tilde{G}}{\partial n_1^2}(t, \tilde{x}_2) dt \right).
\]

For \( t \in [\bar{x}_1, \tilde{x}_1] \), we need to estimate

\[
\frac{\partial^2 \tilde{G}}{\partial n_1^2}(t, \tilde{x}_2) = \frac{\partial^2 \tilde{G}}{\partial n_1^2}(t, \tilde{x}_2) + O \left( \sup_{z \in [\bar{x}_2, \tilde{x}_2]} \frac{\partial^3 \tilde{G}}{\partial n_1^2 \partial n_2}(t, z) dz \right).
\]

Since \( \partial^2 \tilde{G}/\partial n_1^2(t, \tilde{x}_2) = 0 \), a contradiction if we prove that

\[
\frac{\partial^{\alpha+\beta} \tilde{G}}{\partial x^\alpha \partial y^\beta} = O(1) \quad \text{if} \quad |x - y| \geq c.
\]

We interest now to this proof. First, we introduce the sets

\[
B_3(\bar{x}_1, 1/4) \cap \tilde{\Omega} \quad \text{and} \quad B_4(\bar{x}_2, 1/4) \cap \tilde{\Omega}.
\]

For each \( x \in B_3 \cap \tilde{\Omega} \) and \( y \in B_4 \cap \tilde{\Omega} \), \( \tilde{G}(x, y) \) is a harmonic functional in \( B_3 \cap \tilde{\Omega} \). Thus

\[
\tilde{G}(x, y) = - \int_{\partial B_3 \cap \tilde{\Omega}} \frac{\partial G_3}{\partial v}(t, x) \tilde{G}(t, y) dt.
\]
Therefore
\[
\frac{\partial^\alpha}{\partial x^\alpha}(x, y) = - \int_{\partial B_3 \cap \tilde{\Omega}} \tilde{\partial} \frac{\partial^{\alpha+1} G_3}{\partial \nu \partial x^\alpha}(t, x) \tilde{G}(t, y) dt = O(1)
\]
(depending only on \(B_3\)). We have also
\[
\tilde{G}(x, y) = - \int_{\partial B_4 \cap \tilde{\Omega}} \tilde{\partial} \frac{\partial^{\alpha+1} G_4}{\partial \nu}(t, y) \tilde{G}(t, x) dt.
\]
Therefore
\[
\frac{\partial^\alpha + \beta}{\partial x^\alpha \partial y^\beta}(x, y) = - \int_{\partial B_4 \cap \tilde{\Omega}} \tilde{\partial} \frac{\partial^{\alpha+1} G_4}{\partial \nu \partial y^\beta}(t, y) \frac{\partial^{\alpha} \tilde{G}}{\partial x^\alpha}(t, x) dt = O(1)
\]
(depending on \(B_3, B_4\)). The result follows.

We left now for the second case. We introduce the following transformation
\[
T : \Omega \rightarrow \tilde{\Omega} = T(\Omega)
\]
\[
x \rightarrow \tilde{x} = x/d_2
\]
then \(\tilde{d}_2 = 1, \tilde{d}_1 = d_1/d_2 = o(1)\) and \(|\tilde{x}_1 - \tilde{x}_2| = |x_1 - x_2|/d_2 \geq c_2\). As in the first case, we introduce the set \(A\) such that
\[
A \subset \tilde{\Omega}, \quad \tilde{x}_1 \in \partial A, \quad \tilde{x}_1 \in A \quad \text{and} \quad \tilde{x}_2 \in A
\]
\((A\) is a compact set independent of \(m\)). Observe that
\[
\tilde{G} - G_A \geq 0 \quad \text{in} \quad A \quad \text{and} \quad \tilde{G}(\tilde{x}_1, \tilde{x}_2) - G_A(\tilde{x}_1, \tilde{x}_2) = 0.
\]
Thus
\[
\frac{\partial}{\partial n_1} (\tilde{G}(\tilde{x}_1, \tilde{x}_2) - G_A(\tilde{x}_1, \tilde{x}_2)) \leq 0.
\]
Therefore
\[
\frac{\partial \tilde{G}}{\partial n_1}(\tilde{x}_1, \tilde{x}_2) \leq -c
\]
\((c\) is independent of \(m\)).
\[
\frac{\partial \tilde{G}}{\partial n_1}(\tilde{x}_1, \tilde{x}_2) = \frac{\partial \tilde{G}}{\partial n_1}(\tilde{x}_1, \tilde{x}_2) + O\left( \sup_{t \in [\tilde{x}_1, \tilde{x}_2]} \frac{\partial^2 \tilde{G}}{\partial n_1^2}(t, \tilde{x}_2) \tilde{d}_1 \right) \leq -c + O(\tilde{d}_1) \leq -c/2
\]
a contradiction with the assumptions.

The third case is proved in the appendix of [8]. It relies on the following estimates
\[
\frac{\partial}{\partial n_1} \left( \frac{1}{|x_1 - x_2|^{n-2}} \right) = -(n-2) \frac{(x_1 - x_2)n_1}{|x_1 - x_2|^n} = -(n-2) \frac{d_2 - d_1 + o(d_1)}{|x_1 - x_2|^n}
\]
\[ \frac{\partial H}{\partial n_1}(x_1, x_2) = \frac{-(n - 2)}{|x_1 - x_2 + 2d_1n_1|^n} (x_1 - x_2 + 2n_1(x_1 - x_2)n_1 - 2d_1n_1)n_1 + o(d_1) \]
\[ = \frac{(n - 2)(d_1 + d_2)}{|x_1 - x_2|^2 + 4d_1d_2} + o\left(\frac{1}{d_1^n}\right) > 0. \]

REFERENCES