A minimization problem associated with elliptic systems of FitzHugh–Nagumo type

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Abstract

We consider a minimization problem associated with the elliptic systems of FitzHugh–Nagumo type and prove that the minimizer of this minimization problem has not only a boundary layer, but also may oscillate in a set of positive measure.

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Résumé

Nous étudions des solutions d’énergie minimale pour l’équation de FitzHugh–Nagumo. Nous prouvons que ces solutions ont plusieurs transitions rapides si la diffusion est petite.

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1. Introduction

In this paper, we consider the following problem:

\[ \begin{cases} -\varepsilon^2 \Delta u = f(u) - v, & \text{in } \Omega, \\ -\Delta v + \gamma v = \delta u, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial \Omega, \end{cases} \]  

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\varepsilon$ is a parameter, $\gamma$ and $\delta$ are nonnegative constants, $f(t)$ is $C^1$-function in $\mathbb{R}^1$ satisfying the following conditions:

- $f_1$ There are $0 < \tau_1 < \tau_2$ such that $f(\tau_1) < 0$, $f(\tau_2) > 0$, $f'(t) < 0$ if $t \in (-\infty, \tau_1) \cup (\tau_2, +\infty)$, and $f'(t) > 0$ if $t \in (\tau_1, \tau_2)$. Moreover, $f'(t) \to +\infty$ as $t \to -\infty$, $f'(t) \to -\infty$ as $t \to +\infty$. 

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Let $I_{-1} = (-\infty, \tau_1)$, $I_0 = (\tau_1, \tau_2)$, and $I_1 = (\tau_2, +\infty)$. By $(f_1)$, $f(t)$ has exactly three zero points $a_i \in I_i$, $i = -1, 0, 1$. We assume that

$$(f_2) \int_{a_{i-1}}^{a_i} f(s) \, ds > 0.$$ 

Typical examples satisfying $(f_1)$ and $(f_2)$ include $f(t) = t(a-t)(t-1)$, $a \in (0, \frac{1}{2})$; and $f_c(t) = f(t-c)$, $c > 0$.

System (1.1) is a modification of the FitzHugh–Nagumo equation which arises in studies on the physiological phenomenon of nerve conduction. This system has been studied among others by DeFigueiredo, Mitidieri, Troy [10,14,15], Lazer and McKenna [16], Reinecke and Sweers [18–21]. Existence results in [18–20] are in some sense analogies of the results for the scalar case [10]. Numerical results in [21] suggest that (1.1) should have other types of solutions. The aim of this paper is to prove that for suitably large $\delta > 0$, (1.1) has solutions, which either oscillate around a constant in a compact subset of $\Omega$, or have a sharp interior layer. These solutions are local minimum of the corresponding functional. We know that for the autonomous scalar equation ($\delta = 0$), the minimizer does not have interior layer. See for example [5–7].

For each $u \in H^1_0(\Omega)$, let $G_{\gamma} u$ be the unique solution of the following problem:

$$
\begin{align*}
-\Delta v + \gamma v &= u, & & \text{in } \Omega, \\
v &= 0, & & \text{on } \partial \Omega.
\end{align*}
$$

Then we see (1.1) is equivalent to the following nonlocal elliptic problem:

$$
\begin{align*}
-\varepsilon^2 \Delta u + \delta G_{\gamma} u &= f(u), & & \text{in } \Omega, \\
u &= 0, & & \text{in } H^1_0(\Omega).
\end{align*}
$$

(1.2)

The energy associated with (1.2) is

$$I(u) = \frac{1}{2} \int_{\Omega} (|Du|^2 + \delta G_{\gamma} u) - \int_{\Omega} F(u), \quad u \in H^1_0(\Omega).$$

(1.3)

It is easy to see from \( \int_{\Omega} u G_{\gamma} u = \int_{\Omega} (|DG_{\gamma} u|^2 + \gamma |G_{\gamma} u|^2) \geq 0 \), that $I(u)$ is bounded from below in $H^1_0(\Omega)$ and $I(u)$ is weakly lower semicontinuous in $H^1_0(\Omega)$. So the following problem has a minimizer:

$$\inf \{ I(u): u \in H^1_0(\Omega) \}. \quad (1.4)$$

In this paper, we will analyse the profile of the global minimizer of (1.4) for $\varepsilon > 0$ small. Before we state our results, we give some notation.

Let $u = h_+(v)$, $v \in f(I_1)$, be the inverse function of $v = f(u)$ restricted to $I_1$; and let $u = h_-(v)$, $v \in f(I_{-1})$, be the inverse function of $v = f(u)$ restricted to $I_{-1}$.

Let

$$j(\alpha) \colonequals \int_{h_- (\alpha)}^{h_+ (\alpha)} (f(s) - \alpha) \, ds.$$ 

(1.5)

By $(f_1)$, we see that $j'(\alpha) = h_-(\alpha) - h_+(\alpha) < 0$. Thus by $(f_2)$, there is a unique $\alpha_0 > 0$ such that $j(\alpha_0) = 0$, $j(\alpha) > 0$ if $\alpha < \alpha_0$, and $j(\alpha) < 0$ if $\alpha > \alpha_0$.

We extend $h_+(v)$ continuously into $v \in (f(\tau_2), +\infty)$ in such a way that $h_+(v)$ is decreasing. Then since $h_+(v)$ is decreasing, it is easy to see that the following problem has a unique solution $v_2$:

$$
\begin{align*}
-\Delta v + \gamma v &= \delta h_+(v), & & \text{in } \Omega, \\
v &= 0, & & \text{on } \partial \Omega,
\end{align*}
$$

(1.6)
Moreover, by using the maximum principle, we can deduce easily that $v_{b_1} < v_{b_2}$ if $\delta_1 < \delta_2$. By the comparison theorem, it is easy to see that $\max_{x \in \Omega} v_\delta(x) \to +\infty$ as $\delta \to +\infty$. So, there is a unique $b_0 > 0$, such that $\max_{x \in \Omega} v_{b_0}(x) = a_0$. It is easy to check that $b_0 > \gamma a_0 / h_+(a_0)$.

Define

$$h(v) = \begin{cases} h_+(v), & \text{if } v < a_0; \\ h_-(v), & \text{if } v > a_0. \end{cases}$$

Consider

$$\begin{cases} -\Delta v + \nu v \in [\delta h(v + 0), \delta h(v - 0)], & \text{in } \Omega, \\ v \in H^1_0(\Omega). \end{cases}$$

Then, the above problem has a solution, which is the global minimum of the corresponding functional. Besides, (1.7) has exactly one solution because $h(v)$ is decreasing. This is easy to prove but also follows from monotone operator theory as in [4]. Note that if $\delta \leq \delta_0$, the solution of (1.7) is the solution of (1.6) and vice versa. Let $v$ be the solution of (1.7). It is easy to see that if $\delta > \delta_0$, the set $\{x \in \Omega: v(x) \geq a_0\}$ has nonzero measure. In the following, we denote

$$S = \{x \in \Omega: v(x) < a_0\}.$$

Note that $S = \Omega$ if $0 \leq \delta < \delta_0$ and $\Omega \setminus S \neq \emptyset$ if $\delta > \delta_0$.

**Theorem 1.1.** Suppose that $h_-(a_0) \leq 0$. Let $u_\epsilon$ be a global minimizer of (1.4) and let $v_\epsilon = \delta G_\nu u_\epsilon$. Then $v_\epsilon \to v$ in $C^{1,\gamma}(\Omega)$, for any $\sigma \in (0, 1)$, where $v$ is the solution of (1.7). Moreover, we have

(i) if $0 \leq \delta < \delta_0$, then $u_\epsilon \to h_+(v)$ uniformly in any compact subset of $\Omega$ as $\epsilon \to 0$;

(ii) if $\delta = \delta_0$, then $\{x: v(x) = a_0\} = \Omega \setminus S$ and the measure of the set $\{x: v(x) = a_0\}$ is zero. Moreover, $u_\epsilon \to h_+(v)$ uniformly in any compact subset of $\Omega$ as $\epsilon \to 0$;

(iii) if $\delta > \delta_0$, then $\{x: v(x) = a_0\} = \Omega \setminus S$ and the measure of the set $\{x: v(x) = a_0\}$ is positive. Moreover, $u_\epsilon \to h_+(v)$ uniformly in any compact subset of $\Omega$ as $\epsilon \to 0$, $u_\epsilon \to \gamma a_0 / \delta$ weak in $L^\infty(\Omega \setminus S)$ as $\epsilon \to 0$, but $u_\epsilon$ does not converges almost everywhere to $\gamma a_0 / \delta$ as $\epsilon \to 0$ for any subsequence, and for any $\theta > 0$ small,

$$m\left\{x: v(x) = a_0, u_\epsilon(x) \notin \left(h_-(a_0) - \theta, h_-(a_0) + \theta\right) \cup \left(h_+(a_0) - \theta, h_+(a_0) + \theta\right)\right\} \to 0$$

as $\epsilon \to 0$, where $mS$ denotes the measure of the set $S$.

**Theorem 1.2.** Suppose that $h_-(a_0) > 0$. Let $u_\epsilon$ be a global minimizer of (1.4), and let $v_\epsilon = \delta G_\nu u_\epsilon$. Then $v_\epsilon \to v$ in $C^{1,\gamma}(\Omega)$, for any $\sigma \in (0, 1)$, where $v$ is the solution of (1.7). Moreover, we have

(i) if $0 \leq \delta < \delta_0$, then $u_\epsilon \to h_+(v)$ uniformly in any compact subset of $\Omega$ as $\epsilon \to 0$;

(ii) if $\delta = \delta_0$, then $\{x: v(x) = a_0\} = \Omega \setminus S$ and the measure of the set $\{x: v(x) = a_0\}$ is zero. Moreover, $u_\epsilon \to h_+(v)$ uniformly in any compact subset of $\Omega$ as $\epsilon \to 0$;

(iii) if $\delta > \delta_1 = \max(\delta_0, \gamma a_0 / h_-(a_0))$, then the measure of the set $\{x: v(x) = a_0\}$ is positive, and $u_\epsilon \to h_+(v)$ uniformly in any compact subset of $\Omega$ as $\epsilon \to 0$;

(iv) if $\delta_0 < \gamma a_0 / h_-(a_0)$ and $\delta \in (\delta_0, \gamma a_0 / h_-(a_0))$, then $\{x: v(x) = a_0\} = \Omega \setminus S$ and the measure of the set $\{x: v(x) = a_0\}$ is positive. Moreover, $u_\epsilon \to h_+(v)$ uniformly in any compact subset of $\Omega$ as $\epsilon \to 0$, $u_\epsilon \to \gamma a_0 / \delta$ weak in $L^\infty(\Omega \setminus S)$ as $\epsilon \to 0$, but $u_\epsilon$ does not converges almost everywhere to $\gamma a_0 / \delta$ as $\epsilon \to 0$ for any subsequence, and for any $\theta > 0$ small,

$$m\left\{x: v(x) = a_0, u_\epsilon(x) \notin \left(h_-(a_0) - \theta, h_-(a_0) + \theta\right) \cup \left(h_+(a_0) - \theta, h_+(a_0) + \theta\right)\right\} \to 0$$

as $\epsilon \to 0$.
are either the constants

Using the results in [1–3, 11], we can easily classify all the bounded solutions in (1.8) if 

Moreover, \( u_\varepsilon \to h_+(v) \) uniformly in any compact subset of \( S \) as \( \varepsilon \to 0 \), \( u_\varepsilon \to h_-(0) \) in measure in \( \Omega \setminus S \) as \( \varepsilon \to 0 \).

If \( f(u) = u(a - u)(a - 1) \), \( 0 < a < \frac{1}{2} \), then \( h_-(a_0) < 0 \). Thus we see from Theorem 1.1 that for \( \delta > \delta_0 \), the minimizer of (1.4) has a boundary layer, and it oscillates wildly around the constant \( \gamma a_0/\delta \) in the set \( \Omega \setminus S \).

Moreover, for any \( T \subset \Omega \setminus S \) which has positive measure, the portion in \( T \) where \( u_\varepsilon \) is close to \( h_+(a_0) \) has measure close to \( (\gamma a_0^2 - h_-(a_0))/(h_+(a_0) - h_-(a_0))m(T) \), while in most of the rest part of \( T \), \( u_\varepsilon \) is close to \( h_-(a_0) \).

If we translate \( f(t) \) to the right suitably, we see from Theorem 1.2 that for \( \delta > \delta_1 \), the minimizer of (1.4) not only has a boundary layer, but also has an interior layer near the measure-zero set \( \{ x: v(x) = a_0 \} \).

Noting that \( \delta_0 \) only depends on \( h_+(v) \) for \( v \leq a_0 \), we can easily give examples where \( (f_1) \) and \( (f_2) \) are satisfied and \( \delta_0 > \gamma a_0/\delta_0 \), and examples where \( (f_1) \) and \( (f_2) \) are satisfied and \( \delta_0 < \gamma a_0/h_-(a_0) \). In the first case, we only need to construct \( f \), such that \( h_-(a_0) \) is very close to \( h_+(a_0) \), while in the second case, we only need to construct \( f \), such that \( h_-(a_0) > 0 \) is very small.

We are not able to prove the uniform convergence of \( u_\varepsilon \) on any compact subset of \( \Omega \) if \( \delta = \delta_0 \). It is not clear whether the convergence in (v) of Theorem 1.2 can be replaced by uniform convergence in any compact subset of \( \Omega \setminus S \).

To have a better understanding of the profile of a global minimizer \( u_\varepsilon \) of (1.3), we can blow up \( u_\varepsilon \) at any point \( x_0 \in \partial \Omega \) and obtain good asymptotic of \( u_\varepsilon \) near the boundary. Roughly speaking, \( u_\varepsilon (x) \) depends mainly on \( d(x, \partial \Omega) \) if \( d(x, \partial \Omega) \leq R \varepsilon \) for any \( R > 0 \). In other words, \( u_\varepsilon \) transits from 0 to \( h_+(0) \) in the inward normal direction of the boundary. See Proposition 3.5 in Section 3. On the other hand, if we blow up \( u_\varepsilon \) at a point \( x_0 \in \{ x: v(x) = a_0 \} \), we will encounter the following variant of the De Giorgi conjecture [9]:

\[
\begin{align*}
-\Delta w &= f(w) - a_0, & \text{in } R^N, \\
J(w, A) &\leq J(w + \varphi, A), & \forall \varphi \in H^1_0(A), \\
\end{align*}
\]

where \( A \) is any bounded open set in \( R^N \),

\[
J(w, A) = \int_A \left( \frac{1}{2} |Dw|^2 - (F(w) - a_0w) \right).
\]

Using the results in [1–3, 11], we can easily classify all the bounded solutions in (1.8) if \( N = 2, 3 \). These solutions are either the constants \( h_\pm(a_0) \), or the ODE solution. See the discussion in Section 2. As an application of this result to the analysis of the behaviour of \( u_\varepsilon \) in \( \{ x: v(x) = a_0 \} \), we see that if \( N = 2, 3 \), then \( u_\varepsilon \) transits from \( h_+(a_0) \) to \( h_-(a_0) \) mainly in one direction in a neighbourhood of \( x_0 \in \{ x: v(x) = a_0 \} \) of order \( \varepsilon \), although the direction can change rapidly with \( x_0 \). For other phase transition problems which lead to the De Giorgi conjecture, the readers can refer to [17, 22].

Our next result shows that for some \( \delta > \delta_0 \), \( I_\varepsilon(u) \) has a local minimizer which behaves quite well in the interior of \( \Omega \).

\textbf{Theorem 1.3.} Let \( \delta > \delta_0 \) be the number such that \( \max_{x \in \Omega} v_\delta(x) = f(\tau_2) \), where \( v_\delta \) is the solution of (1.6) with \( \delta = \delta_\delta \). Suppose that \( \delta \in (\delta_0, \delta) \). Then there is an \( \varepsilon_0 > 0 \), such that for \( \varepsilon \in (0, \varepsilon_0) \), (1.1) has a solution \( (\bar{u}_\varepsilon, \bar{v}_\varepsilon) \), satisfying

(i) \( \bar{v}_\varepsilon \to \bar{v} \) in \( C^{1,\sigma}(\Omega) \), for any \( \sigma \in (0, 1) \), where \( \bar{v} \) is the solution of (1.6);

(ii) \( \bar{u}_\varepsilon \to h_+(\bar{v}) \) uniformly in any compact subset of \( \Omega \);

(iii) \( \bar{u}_\varepsilon \) is a local minimizer of \( I_\varepsilon(u) \).
Solutions of the same type as in Theorem 1.3 were obtained in [21] by using a bifurcation theorem. In the result of [21], \( \delta \) is a parameter depending on \( \varepsilon \). In [21], numerical analysis suggests that (1.1) with \( f(u) = u(u-a)(1-u), \ a \in (0, \frac{1}{2}) \), have a solution which has an interior layer. Our result here shows that the number of the interior layers of the global minimizer will increase as \( \varepsilon \) tends to 0 in this case. On the other hand, since \( \bar{u}_\varepsilon \) is a local minimum, we can attach a peak solution to this local minimum to get a new solution. We shall discuss this problem in a forthcoming paper. It is worth pointing out that the solution obtained by attaching a peak solution to the local minimum \( \bar{u}_\varepsilon \) converges to \( h^+(v) \) in \( L^p(\Omega) \), \( \forall \ p > 1 \), as \( \varepsilon \to 0 \), but it does not converge to \( h^+(v) \) uniformly in any compact subset of \( \Omega \). Thus for the solutions of (1.1), \( L^p \) convergence does not imply uniform convergence.

This paper is arranged as follows. In Section 2, we prove Theorems 1.1 and 1.2. Section 3 contains the proof of Theorem 1.3.

2. The profile of the global minimizers

Let us recall that \( G_\gamma u \) is the solution of
\[
\begin{align*}
-\Delta v + \gamma v &= u, \quad \text{in } \Omega, \\
v &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]
It is easy to check that there is \( C > 0 \), such that \( |G_\gamma u|_\infty \leq C |u|_\infty \).

Lemma 2.1. There is a constant \( C > 0 \), such that for any solution \( (u_\varepsilon, v_\varepsilon) \) of (1.1), we have \( |u_\varepsilon|_\infty, |v_\varepsilon|_\infty \leq C \).

Proof. Let \( x_0 \in \Omega \) be a maximum point of \( u_\varepsilon \). Then
\[
0 \leq -\varepsilon^2 \Delta u_\varepsilon(x_0) = f(u_\varepsilon(x_0)) - v_\varepsilon(x_0) \leq f(u_\varepsilon(x_0)) + Cu_\varepsilon(x_0).
\]
But \( f(u)/u \to -\infty \), as \( u \to +\infty \). Thus we see from the above relation that \( u_\varepsilon(x_0) \leq C' \). Similarly, we can prove \( \min_{x \in \Omega} u_\varepsilon \geq -C' \). \( \square \)

Let \( u_\varepsilon \) be a minimizer of (1.4), \( v_\varepsilon = \delta G_\gamma u_\varepsilon \). By Lemma 2.1, \( u_\varepsilon \) is bounded in \( L^\infty(\Omega) \). From
\[
-\Delta v_\varepsilon + \gamma v_\varepsilon = \delta u_\varepsilon, \quad \text{in } \Omega,
\]
we see that \( v_\varepsilon \) is bounded in \( W^{2,p}(\Omega) \) for and \( p > 1 \). Thus we assume that up to a subsequence,
\[
v_\varepsilon \to v \quad \text{in } C^{1,\sigma}(\Omega),
\]
for any \( \sigma \in (0, 1) \).

Lemma 2.2. Let \( u_\varepsilon \) be a minimizer of (1.4), \( v_\varepsilon = \delta G_\gamma u_\varepsilon \). Then
\[
u_\varepsilon \to \begin{cases} h^+(v), & \text{uniformly in any compact subset of } \{x: 0 < v(x) < \alpha_0\}; \\ h^-(v), & \text{uniformly in any compact subset of } \{x: v(x) > \alpha_0\}. \end{cases}
\]

Proof. For any small \( \tau > 0 \), let \( \eta > 0 \) be small enough, such that
\[
|v_\varepsilon(x) - v(x_0)| < \tau, \quad \forall x \in B_\eta(x_0).
\]
Let \( M > 0 \) be a large constant satisfying \( M \geq \max_{x \in \partial \Omega} |u_\varepsilon| \) for all \( \varepsilon > 0 \). Consider
\[
\inf\{J_{\varepsilon, +}(u): u \in H^1(B_\eta(x_0)), u = -M \text{ on } \partial B_\eta(x_0)\},
\]
(2.2)
where
\[ J_{\epsilon,+}(u) = \frac{\epsilon^2}{2} \int_{B_{\eta}(x_0)} |Du|^2 - \int_{B_{\eta}(x_0)} (F(u) - (v(x_0) + 2\tau)u). \]

Let \( w_{\epsilon,+} \) be a minimizer of (2.2). Then
\[-\epsilon^2 \Delta w_{\epsilon,+} + w_{\epsilon,+} = f(w_{\epsilon,+}) - (v(x_0) + 2\tau).\]

Thus similar to the proof of Lemma 2.1, we know that \( |w_{\epsilon,+}| \leq C \) for some \( C > 0 \), independent of \( \epsilon, \eta > 0 \) small.

We claim that \( u_{\epsilon} \geq w_{\epsilon,+} \).

Let \( S_{\epsilon} = \{ x : w_{\epsilon,+} > u_{\epsilon}, x \in B_{\eta}(x_0) \} \). Since \( w_{\epsilon,+} < u_{\epsilon} \) if \( |x - x_0| = \eta \), we see \( S_{\epsilon} \subset B_{\eta}(x_0) \). Let
\[ \varphi_{\epsilon} = \begin{cases} w_{\epsilon,+} - u_{\epsilon}, & x \in S_{\epsilon}, \\ 0, & x \in \Omega \setminus S_{\epsilon}. \end{cases} \]

Then \( \varphi_{\epsilon} \in H^1_0(\Omega) \) and \( \varphi_{\epsilon} \geq 0 \). Thus, we have
\[ \begin{align*}
0 & \leq I_\epsilon(u_{\epsilon} + \varphi_{\epsilon}) - I_\epsilon(u_{\epsilon}) \\
& = I^*_\epsilon(u_{\epsilon} + \varphi_{\epsilon}) - I^*_\epsilon(u_{\epsilon}) + \frac{\delta}{2} \int_\Omega \left( u_{\epsilon} + \varphi_{\epsilon} \right) G_{\gamma}(u_{\epsilon} + \varphi_{\epsilon}) - u_{\epsilon} G_{\gamma} u_{\epsilon}
\end{align*} \]
\[ = I^*_\epsilon(u_{\epsilon} + \varphi_{\epsilon}) - I^*_\epsilon(u_{\epsilon}) + \int_\Omega \varphi_{\epsilon} \psi_{\epsilon} + \frac{\delta}{2} \int_\Omega \varphi_{\epsilon} G_{\gamma} \psi_{\epsilon}, \quad (2.3) \]

where
\[ I^*_\epsilon(u) = \frac{\epsilon^2}{2} \int_{B_{\eta}(x_0)} |Du|^2 - \int_{B_{\eta}(x_0)} F(u). \]

On the other hand, we have
\[ \begin{align*}
0 & \leq J_{\epsilon,+}(w_{\epsilon,+} - \varphi_{\epsilon}) - J_{\epsilon,+}(w_{\epsilon,+}) \\
& = I^*_\epsilon(w_{\epsilon,+} - \varphi_{\epsilon}) - I^*_\epsilon(w_{\epsilon,+}) - \int_{S_{\epsilon}} (v(x_0) + 2\tau) \varphi_{\epsilon}
\end{align*} \]
\[ = I^*_\epsilon(u_{\epsilon}) - I^*_\epsilon(u_{\epsilon} + \varphi_{\epsilon}) + \int_{S_{\epsilon}} (v(x_0) + 2\tau) \varphi_{\epsilon}
\]
\[ = I_{\epsilon}(u_{\epsilon}) - I_{\epsilon}(u_{\epsilon} + \varphi_{\epsilon}) + \frac{\delta}{2} \int_{S_{\epsilon}} \varphi_{\epsilon} G_{\gamma} \varphi_{\epsilon} - \int_{S_{\epsilon}} (v(x_0) + 2\tau - v_{\epsilon}) \varphi_{\epsilon}
\]
\[ \leq \frac{\delta}{2} \int_{S_{\epsilon}} \varphi_{\epsilon} G_{\gamma} \varphi_{\epsilon} - \int_{S_{\epsilon}} (v(x_0) + 2\tau - v_{\epsilon}) \varphi_{\epsilon}. \quad (2.4) \]

Noting that \( v(x_0) + 2\tau - v_{\epsilon} > \tau \) if \( x \in B_{\eta}(x_0) \), we obtain
\[ \tau \int_{S_{\epsilon}} \varphi_{\epsilon} \leq \int_{S_{\epsilon}} (v(x_0) + 2\tau - v_{\epsilon}) \varphi_{\epsilon} \leq \frac{\delta}{2} \int_{\Omega} \varphi_{\epsilon} G_{\gamma} \varphi_{\epsilon}. \quad (2.5) \]

Since \( |\varphi_{\epsilon}| \leq 2C \), we have
\[ |G_{\gamma} \varphi_{\epsilon}|_{L^\infty(\Omega)} \leq C |\varphi_{\epsilon}|_{L^p(\Omega)} \leq C \eta^{N/p}. \]
for $p > \frac{N}{2}$. So
\[
\tau \int_{S_t} \varphi_{\epsilon} \leq C \eta^{N/p} \int_{S_t} \varphi_{\epsilon}.
\]
Thus, we see that if $\eta > 0$ small, we obtain $\varphi_{\epsilon} = 0$. So we have proved that $w_{\epsilon, +} \leq u_{\epsilon}$.

Similarly, consider
\[
\inf\{J_{\epsilon, -}(u): u \in H^1(B_{\eta}(x_0)), u = M\text{ on } \partial B_{\eta}(x_0)\},
\]
where
\[
J_{\epsilon, -}(u) = \frac{\epsilon^2}{2} \int_{B_{\eta}(x_0)} |Du|^2 - \int_{B_{\eta}(x_0)} (F(u) - (v(x_0) - 2\tau)u).
\]
Let $w_{\epsilon, -}$ be a minimizer of (2.6). Then we have $u_{\epsilon} \leq w_{\epsilon, -}$.

By a result of [6,7], we know
\[
w_{\epsilon, +} \to \begin{cases} h^+(v(x_0) + 2\tau), & \text{if } v(x_0) + 2\tau < a_0; \\ h^-(v(x_0) + 2\tau), & \text{if } v(x_0) + 2\tau > a_0; \end{cases}
\]
and
\[
w_{\epsilon, -} \to \begin{cases} h^+(v(x_0) - 2\tau), & \text{if } v(x_0) - 2\tau < a_0; \\ h^-(v(x_0) - 2\tau), & \text{if } v(x_0) - 2\tau > a_0; \end{cases}
\]
uniformly on any compact subset of $B_{\eta}(x_0)$. Thus this lemma follows from $w_{\epsilon, +} \leq u_{\epsilon} \leq w_{\epsilon, -}$. □

**Lemma 2.3.** Let $u_{\epsilon}$ be a minimizer of (1.4), $v_{\epsilon} = \delta G_\gamma u_{\epsilon}$. Then
\[
m\{x: v(x) = a_0, u_{\epsilon}(x) \notin (h_-(a_0) - \theta, h_+(a_0) + \theta) \cup (h_+(a_0) - \theta, h_-(a_0) + \theta)\} \to 0
\]
as $\epsilon \to 0$, where $mS$ denotes the measure of the set $S$.

**Proof.** Let $x_0 \in \Omega$ and let $C_r(x_0)$ be the cube with side $r$, centred at $x_0$, with sides parallel to the axes. For any small $\eta > 0$, we may assume that $\epsilon > 0$ is small enough such that $C_{\epsilon+r}(x_0) \in \Omega$. Define
\[
\bar{u}_{\epsilon}(x) = \begin{cases} u_{\epsilon}(x), & x \in \Omega \setminus C_{\epsilon+r}(x_0); \\ h^-(v_{\epsilon}(x')), & x \in C_{\epsilon+r}(x_0) \setminus C_\eta(x_0); \\ h^-(v_{\epsilon}(x)), & x \in C_\eta(x_0), \end{cases}
\]
where $x' = t_{\eta, \epsilon}(x - x_0)/|x - x_0| \in \partial C_\eta(x_0)$ and $x'' = t_{\eta, \epsilon}''(x - x_0)/|x - x_0| \in \partial C_{\eta+r}(x_0)$. Then
\[
0 \leq l(\bar{u}_{\epsilon}) - l(u_{\epsilon})
\]
\[
= \frac{1}{2} \int_{\Omega} (|D\bar{u}_{\epsilon}|^2 - |Du_{\epsilon}|^2) + \frac{\delta}{2} \int_{\Omega} (\bar{u}_{\epsilon} G_\gamma \bar{u}_{\epsilon} - u_{\epsilon} G_\gamma u_{\epsilon}) - \int_{\Omega} (F(\bar{u}_{\epsilon}) - F(u_{\epsilon})).
\]
\[
= I_1 + I_2 - I_3.
\]
Noting that $u_{\epsilon}$ satisfies $-\Delta u_\epsilon = \epsilon^{-2}(f(u_{\epsilon}) - v_{\epsilon})$, using Theorem 2.10 and Theorem 4.5 in [13], we see
\[
\epsilon |Du_{\epsilon}(x)| \leq C |u_{\epsilon}|_{L^\infty(B_\eta(x_0))} + C \epsilon^2 |\epsilon^{-2}(f(u_{\epsilon}) - v_{\epsilon})|_{L^\infty(B_\eta(x_0))}.
\]
In particular, $\epsilon |Du_{\epsilon}| \leq C$ if $d(x, \partial\Omega) \geq 2\epsilon$. Thus it is easy to check that $\epsilon |D\bar{u}_{\epsilon}| \leq C$. As a result,
\begin{align*}
I_1 &= \frac{1}{2}\epsilon^2 \int_{C_{\varepsilon}(x_0)} (|D\bar{u}_\varepsilon|^2 - |Du_\varepsilon|^2) \leq \frac{1}{2}\epsilon^2 \int_{C_{\varepsilon}(x_0)} |D\bar{u}_\varepsilon|^2
\leq Cm (C_{\varepsilon+\eta}(x_0) \setminus C_\eta(x_0)) + \frac{1}{2}\epsilon^2 \int_{C_\eta(x_0)} |Dh_\varepsilon-v_\varepsilon|^2 \leq C(\varepsilon\eta^{N-1} + \varepsilon^2\eta^N). \tag{2.8}
\end{align*}

On the other hand, we have

\begin{align*}
I_2 &= \int_\Omega (\bar{u}_\varepsilon - u_\varepsilon)v_\varepsilon + \frac{\delta}{2}\int_\Omega (\bar{u}_\varepsilon - u_\varepsilon)G_\gamma(\bar{u}_\varepsilon - u_\varepsilon) = I_4 + I_5, \tag{2.9}
\end{align*}

and

\begin{align*}
I_4 &= \int_{C_{\varepsilon+\eta}(x_0)} (\bar{u}_\varepsilon - u_\varepsilon)v_\varepsilon \\
&= O(m(C_{\varepsilon+\eta}(x_0) \setminus C_\eta(x_0))) + \int_{C_\eta(x_0)} (\bar{u}_\varepsilon - u_\varepsilon)v_\varepsilon \\
&= \int_{C_\eta(x_0)} (h_\varepsilon - u_\varepsilon)v_\varepsilon + O(\varepsilon\eta^{-1}). \tag{2.10}
\end{align*}

Let \( G_\gamma(x, y) \) be the Green’s function of \(-\Delta + \gamma\) with Dirichlet boundary condition. Then \( G_\gamma(x, y) \leq \frac{C}{|x-y|^{N-2}} \). For any \( x \in C_{\varepsilon+\eta}(x_0) \), we have

\begin{align*}
|G_\gamma(\bar{u}_\varepsilon - u_\varepsilon)(x)| &= \left| \int_\Omega G_\gamma(x, y)(\bar{u}_\varepsilon(y) - u_\varepsilon(y)) dy \right| \\
&= \left| \int_{C_{\varepsilon+\eta}(x_0)} G_\gamma(x, y)(\bar{u}_\varepsilon(y) - u_\varepsilon(y)) dy \right| \\
&\leq C \int_{C_{\varepsilon+\eta}(x_0)} \frac{1}{|x-y|^{N-2}} dy \leq C(\varepsilon + \eta)^2.
\end{align*}

So

\begin{align*}
I_5 &= \frac{\delta}{2} \int_{C_{\varepsilon+\eta}(x_0)} (\bar{u}_\varepsilon - u_\varepsilon)G_\gamma(\bar{u}_\varepsilon - u_\varepsilon) = O((\varepsilon + \eta)^{N+2}). \tag{2.11}
\end{align*}

For \( I_3 \), we have

\begin{align*}
I_3 &= \int_{C_{\varepsilon+\eta}(x_0)} (F(\bar{u}_\varepsilon) - F(u_\varepsilon)) = \int_{C_\eta(x_0)} (F(\bar{u}_\varepsilon) - F(u_\varepsilon)) + O(\varepsilon\eta^{-1}). \tag{2.12}
\end{align*}

Combining (2.7)–(2.12), we obtain

\begin{align*}
\int_{C_\eta(x_0)} ((h_\varepsilon - u_\varepsilon)v_\varepsilon - (F(h_\varepsilon - u_\varepsilon)) + O(\varepsilon\eta^{N-1} + (\varepsilon + \eta)^{N+2}) \geq 0. \tag{2.13}
\end{align*}
Thus
\[
\int_{C_N(x_0)} \left( (F(h_-(v_\varepsilon)) - h_-(v_\varepsilon)v_\varepsilon) - (F(u_\varepsilon) - u_\varepsilon v_\varepsilon) \right) \leq O(\varepsilon \eta^{N-1} + (\varepsilon + \eta)^{N+2}). \tag{2.14}
\]

Since \( v = 0 \) on \( \partial \Omega \), we see \( \{ x : v(x) = \alpha_0 \} \) is a compact subset of \( \Omega \). Thus we can choose \( C_\eta(x_j), j \in J \), where \( J \) contains finite number of points, such that, \( C_\eta(x_i) \cap C_\eta(x_j) = \emptyset, \forall i \neq j \), the set \( \{ C_\eta(x_j), j \in J \} \) covers \( \{ x : v(x) = \alpha_0 \} \). It is easy to see that the number of such cubes is at most \( C^N/\eta^N \) for some large constant \( C > 0 \) independent on \( N \). Hence, from (2.14), we obtain
\[
\int_{v(x)=\alpha_0} \left( (F(h_-(v_\varepsilon)) - h_-(v_\varepsilon)v_\varepsilon) - (F(u_\varepsilon) - u_\varepsilon v_\varepsilon) \right) \leq C \frac{\varepsilon \eta^{N-1} + (\varepsilon + \eta)^{N+2}}{\eta^N}.
\]

So for any \( \eta > 0 \),
\[
\int_{v(x)=\alpha_0} \left( (F(h_-(\alpha_0)) - h_-(\alpha_0)\alpha_0) - (F(u_\varepsilon) - u_\varepsilon \alpha_0) \right) \leq C \frac{\varepsilon \eta^{N-1} + (\varepsilon + \eta)^{N+2}}{\eta^N} + o_\varepsilon(1).
\]

That is,
\[
\int_{v(x)=\alpha_0} \int_{u_\varepsilon} h_-(\alpha_0) \left( f(\tau) - \alpha_0 \right) d\tau \leq C \frac{\varepsilon \eta^{N-1} + (\varepsilon + \eta)^{N+2}}{\eta^N} + o_\varepsilon(1). \tag{2.15}
\]

Note that
\[
\int_x h_-(\alpha_0) \geq c_0 > 0,
\]
if \( s \notin (h_-(\alpha_0) - \theta, h_-(\alpha_0) + \theta) \cup (h_+(\alpha_0) - \theta, h_+(\alpha_0) + \theta) \), and \( \int_x h_-(\alpha_0)(f(\tau) - \alpha_0) \geq 0 \) for all \( s \), (2.15) yields
\[
m\{ x : v(x) = \alpha_0, u_\varepsilon(x) \notin \{ h_-(\alpha_0) - \theta, h_-(\alpha_0) + \theta \} \cup \{ h_+(\alpha_0) - \theta, h_+(\alpha_0) + \theta \} \} \rightarrow 0 \tag{2.16}
\]
as \( \varepsilon \to 0 \) for every \( \theta > 0 \) small. \( \Box \)

**Lemma 2.4.** Let \( u_\varepsilon \) be a minimizer of (1.4), \( v_\varepsilon = \delta G_{\gamma}u_\varepsilon \). Then \( v_\varepsilon \to v \) in \( C^{1,\sigma}(\Omega) \) for any \( \sigma \in (0, 1) \), and \( v \) is a solution of (1.7).

**Proof.** Since \( u_\varepsilon \) is bounded in \( L^\infty(\Omega) \), we may assume that up to a subsequence, there is a \( u \in L^\infty(\Omega) \), such that
\[
u_\varepsilon \to u, \text{ weak* in } L^\infty(\Omega).
\]

By Lemmas 2.2 and 2.3, we see \( u = h_+(v) \) if \( x \in \{ x : v(x) < \alpha_0 \} \), \( u = h_-(v) \) if \( x \in \{ x : v(x) > \alpha_0 \} \), and \( u \in [h_-(\alpha_0), h_+(\alpha_0)] \) if \( x \in \{ x : v(x) = \alpha_0 \} \). Thus, \( v \) satisfies
\[
\begin{cases}
-\Delta v + \gamma v \in [\delta h(v-0), \delta h(v+0)], & \text{in } \Omega, \\
v = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where \( h(v) = h_+(v) \) if \( v < \alpha_0 \), \( h(v) = h_-(v) \) if \( v > \alpha_0 \). \( \Box \)

Before we prove Theorems 1.1 and 1.2, we need the following lemma:

**Lemma 2.5.** There is a \( \delta_0 > 0 \), such that if \( \delta < (0, \delta_0) \), the solution \( v \) of (1.6) satisfies \( \max_{x \in \Omega} v(x) < \alpha_0 \); if \( \delta > \delta_0 \), the solution \( v \) of (1.6) satisfies \( \max_{x \in \Omega} v(x) > \alpha_0 \).
Suppose that \( \delta_1 < \delta_2 \), then the solutions \( v_{\delta_1} \) and \( v_{\delta_2} \) of (1.6) corresponding to \( \delta = \delta_1 \) and \( \delta = \delta_2 \) respectively satisfy \( v_{\delta_1} < v_{\delta_2} \). On the other hand, suppose that \( \max_{x \in \Omega} v_{\delta} \leq \alpha_0 \) for \( \delta \to +\infty \). Since

\[
-\Delta v_{\delta} + \gamma v_{\delta} = \delta h^+(v_{\delta}) \geq \delta h^+(\alpha_0),
\]

we see \( v_{\delta} \geq c_0 \delta e \), for some constant \( c_0 > 0 \), where \( e \) is the first eigenfunction of \(-\Delta + \gamma \) with Dirichlet condition. This is a contradiction.

Let

\[
\delta_0 = \inf \left\{ \delta : \max_{x \in \Omega} v_{\delta} > \alpha_0 \right\}.
\]

Then \( \delta_0 \in (0, +\infty) \) and \( \delta_0 \) is the number we need. \( \square \)

Remark 2.6. It is easy to see from \(-\Delta v(x_0) > 0\) at any maximum point of \( v \) that \( \delta_0 > \gamma \alpha_0 / h_+(\alpha_0) \).

Proof of Theorem 1.1. If \( \delta \in (0, \delta_0) \), it follows from Lemma 2.5 that the solution \( v \) of (1.7) satisfies \( v < \alpha_0 \). Thus (i) follows from Lemma 2.2.

If \( \delta = \delta_0 \), then \( \max_{x \in \Omega} v_{\delta} = \alpha_0 \). Suppose that \( m\{ x : v(x) = \alpha_0 \} > 0 \). Then we have \( \delta_0 = \gamma \alpha_0 / h_+(\alpha_0) \). This is a contradiction to Remark 2.6. Thus \( m\{ x : v(x) = \alpha_0 \} = 0 \) and (ii) follows from Lemma 2.2.

Suppose that \( \delta > \delta_0 \). Since \( h(t) \leq 0 \) if \( t > \alpha_0 \), we see that the solution \( v_{\delta} \) of (1.7) satisfies \( v_{\delta}(x) \leq \alpha_0 \) for all \( x \in \Omega \). Now we claim that

\[
m\{ x : v_{\delta}(x) = \alpha_0 \} > 0.
\]

Suppose that \( m\{ x : v_{\delta}(x) = \alpha_0 \} = 0 \). Then we see that \( v_{\delta} \) is also the solution of (1.6) and \( v_{\delta} \leq \alpha_0 \). This is a contradiction to the definition of \( \delta_0 \).

Suppose that \( u_{\varepsilon} \to \gamma \alpha_0 / \delta \) almost everywhere in \( \{ x : v_{\delta}(x) = \alpha_0 \} \). Then

\[
m\left\{ x : \left| u_{\varepsilon}(x) - \frac{\gamma \alpha_0}{\delta} \right| \geq \tau \right\} \to 0
\]
as \( \varepsilon \to 0 \), for any \( \tau > 0 \). This is a contradiction to Lemma 2.3 and Remark 2.6. Thus, (iii) follows from Lemmas 2.2, 2.3 and 2.4. \( \square \)

Proof of Theorem 1.2. The proofs of (i) and (ii) of this theorem are exactly the same as those in Theorem 1.1.

Suppose that \( \delta > \gamma \alpha_0 / h_-(\alpha_0) \). We claim that

\[
m\{ x : v_{\delta}(x) = \alpha_0 \} = 0.
\]

Suppose that \( m\{ x : v_{\delta}(x) = \alpha_0 \} > 0 \). Then we have

\[
\gamma \alpha_0 = \delta u(x), \quad \text{for almost every } x \in \{ x : v_{\delta}(x) = \alpha_0 \}.
\]

So \( u(x) = \gamma \alpha_0 / \delta < h_-(\alpha_0) \). This is a contradiction to \( u(x) \in [h_-(\alpha_0), h_+(\alpha_0)] \) for almost every \( x \in \{ x : v_{\delta}(x) = \alpha_0 \} \). Thus (iii) follows from Lemma 2.2.

Now we consider the case \( \delta_0 < \gamma \alpha_0 / h_-(\alpha_0) \).

Suppose that \( \delta \in (\delta_0, \gamma \alpha_0 / h_-(\alpha_0)) \). We claim that \( \max_{x \in \Omega} v(x) = \alpha_0 \). In fact, since \( \delta h_-(\alpha_0) - \gamma \alpha_0 \leq 0 \) and \( h_-(t) \) is decreasing for \( t > \alpha_0 \), we see that \( \delta h_-(t) - \gamma t < 0 \) if \( t > \alpha_0 \). Suppose that \( \max_{x \in \Omega} v(x) > \alpha_0 \) and let \( x_0 \in \Omega \) satisfy \( v(x_0) = \max_{x \in \Omega} v(x) > \alpha_0 \). Then \( v \) is \( C^1 \) in a small neighbourhood of \( x_0 \). But

\[
0 \leq -\Delta v(x_0) = \delta h_-(v(x_0)) - \gamma v(x_0) < 0.
\]

So we get a contradiction.
Since\[ \delta \in (\delta_0, \gamma_0) \] we can prove (iv) in a similar way as in the proof of (iii) of Theorem 1.1.

Finally, if $\delta = \delta_1 = \gamma \alpha_0 / (h_0 + (\alpha_0))$, then $u_\varepsilon \to h_-(\alpha_0)$ weak* in $L^\infty(\Omega \setminus S)$, which, together with Lemma 2.3, gives $u_\varepsilon \to h_-(\alpha_0)$ in measure in $\Omega \setminus S$. \quad \Box

Before we close this section, we discuss briefly the local behaviour of $u_\varepsilon$ in a small neighbourhood of $x_0 \in \{ x : v(x) = \alpha_0 \}$.

Let $w_\varepsilon(y) = u_\varepsilon(\varepsilon y + x_0)$, then $w_\varepsilon$ satisfies

\[ -\Delta w = f(w) - \alpha_0, \quad \text{in } R^N, \]

\[ J(w, A) \leq J(w + \varphi, A), \quad \forall \varphi \in H^1_0(A), \]

where $A$ is any bounded open set in $R^N$, $J(w, A) = \int_A \left( \frac{1}{2} |Dw|^2 - (F(w) - \alpha_0 w) \right)$. If $N = 2, 3$, then either $w = h_-(\alpha_0)$, or $w = h_+(\alpha_0)$, or $w(y) = w_0(a, y)$ for some $a \in S^{N-1}$, where $w_0$ is a solution of

\[ -w_0'' = f(w_0) - \alpha_0, \quad w_0' > 0, \quad \text{in } R^1. \]

**Proof.** It is easy to see that

\[ -\Delta w = f(w) - \alpha_0, \quad \text{in } R^N. \]

On the other hand, for any bounded open set $A$ in $R^N$, and $\varphi \in H^1_0(A)$, we have

\[ I(u_\varepsilon) \leq I(u_\varepsilon + \varphi_\varepsilon), \]

where $\varphi_\varepsilon(x) = \varphi((x - x_0) / \varepsilon)$. Thus

\[ -\int |Du_\varepsilon|^2 - \int F(u_\varepsilon + \varphi_\varepsilon) + \int \psi_\varepsilon v_\varepsilon + \frac{\delta}{2} \int \psi_\varepsilon G_{\gamma} \psi_\varepsilon. \]

That is,

\[ -\int |Du_\varepsilon|^2 - \int F(u_\varepsilon + \varphi_\varepsilon) + \int \psi_\varepsilon v_\varepsilon + \frac{\delta}{2} \int \psi_\varepsilon G_{\gamma} \psi_\varepsilon. \quad (2.17) \]

Since $|G_{\gamma} \psi_\varepsilon|_{L^\infty(\Omega)} \to 0$ as $\varepsilon \to 0$, we have

\[ \left| \int \psi_\varepsilon G_{\gamma} \psi_\varepsilon \right| \leq |G_{\gamma} \psi_\varepsilon|_{L^\infty(\Omega)} \int |\psi_\varepsilon| = o(\varepsilon^N). \]

Letting $\varepsilon \to 0$ in (2.17), we obtain

\[ -\int F(w) \leq -\int |Dw|^2 - \int F(w + \varphi) + \int \varphi_0. \]
That is $J(w, A) \leq J(w + \varphi, A)$.

It is easy to see that $J(w, A) \leq J(w + \varphi, A)$ implies

$$
\int_{B_R(0)} |Dw|^2 \leq CR^{N-1}, 
$$

(2.18)

for any $R > 0$, where $C > 0$ is some constant independent of $R$. See for example [2].

On the other hand, $J(w, A) \leq J(w + \varphi, A)$ implies

$$
\int_{RN} (|D\varphi|^2 - f'(w)\varphi^2) \geq 0, \quad \forall \varphi \in C_0^\infty(R^N),
$$

(2.19)

which will give that the following problem have a positive solution $\xi$:

$$
-\Delta \xi - f'(w)\xi = 0, \quad \text{in } RN.
$$

See for example [3,11]. Thus, using (2.18), we see that if $N = 2, 3$, there is a constant $C_i$, such that

$$
\frac{\partial w}{\partial x_i} = C_i \xi.
$$

See [2,3].

If $C_i = 0$, $i = 1, \ldots, N$, then $w = C$. Thus $f(C) - \alpha_0 = 0$. But from (2.19), we see $f'(C) \leq 0$. Thus $C = h(\alpha_0)$.

If $C_i \neq 0$ for some $i$, then $\partial w/\partial x_j = C'_j \partial w/\partial x_i$, $j = 1, \ldots, N$. Thus the result follows. \qed

Remark 2.8. The second part in Proposition 2.7 is a direct consequence of the results in [2,3,11]. This fact was observed in [12].

3. The existence of local minimizer

In Section 2, we have proved that if $\delta > \delta_0$, the global minimizer of (1.4) will either oscillate around a constant in an open set of positive measure, or have an interior jump. In this section, we shall prove that there exists a $\tilde{\delta} > \delta_0$, such that (1.1) has a solution, which is a local minimizer of $I_\delta(u)$ and just has a boundary layer.

Let $\tilde{\delta} > 0$ be the constant, such that the solution $v_{\tilde{\delta}}$ of (1.6) satisfies

$$
f(\tau_2) = \max_{x \in \Omega} v_{\tilde{\delta}}(x).
$$

Then $\delta_0 < \tilde{\delta}$.

Suppose that $\delta \in (\delta_0, \tilde{\delta})$. Let $v_{\delta}$ be the solution of (1.6). Then we have

$$
\max_{x \in \Omega} v_{\delta}(x) \in (\alpha_0, f(\tau_0)).
$$

Let $A = \{x \in \Omega: v_{\delta}(x) \geq \alpha_0\}$, where $v_{\delta}$ is the solution of (1.6). Then $A$ is a compact subset of $\Omega$. Let $\theta > 0$ be so small that $A_\theta = \{x: d(x, A) \leq \theta\} \subset \Omega$.

We denote by $g(u)$ an extension of $f(u)$, $u \geq \tau_2$, into $(-\infty, \tau_2)$ in such a way that $g(u) \in C^1(R^1)$ and $g(u)$ is decreasing. Let

$$
\tilde{f}(x, u) = (1 - 1_{A_\theta}) f(u) + 1_{A_\theta} g(u),
$$

where $1_S = 1$ if $x \in S$, $1_S = 0$ if $x \notin S$.

Consider the following problem

$$
\inf \{ J_\delta(u), u \in H_0^1(\Omega) \},
$$

(3.1)
where
\[ J_\varepsilon(u) = \frac{1}{2} \int_\Omega (|Du|^2 + uG_\gamma u) - \int_\Omega \bar{F}(x, u), \]
and \( \bar{F}(x, u) = \int_u^0 \bar{f}(x, \tau) d\tau \).

Let \( u = k(v) \) be the inverse function of \( v = g(u) \).

Let \( \bar{u}_\varepsilon \) be a minimizer of (3.1), \( \bar{v}_\varepsilon = \delta G_\gamma \bar{u}_\varepsilon \). Then, \( \bar{u}_\varepsilon \) is uniformly bounded and \( \bar{v}_\varepsilon \) is bounded in \( W^{2,p}(\Omega) \) for any \( p > 1 \). Thus we have
\[ \bar{v}_\varepsilon \to \bar{v}, \quad \text{in } C^{1,\sigma}(\Omega), \]
for any \( \sigma \in (0, 1) \). Similar to Lemmas 2.2 and 2.3, we have

**Lemma 3.1.**
\[ \bar{u}_\varepsilon \to \begin{cases} k(\bar{v}), & \text{uniformly in any compact subset of } \text{int}(A_\theta); \\ h_+(\bar{v}), & \text{uniformly in any compact subset of } \{ x: 0 < \bar{v}(x) < \alpha_0 \} \cap (\Omega \setminus A_\theta); \\ h_-(\bar{v}), & \text{uniformly in any compact subset of } \{ x: \bar{v}(x) > \alpha_0 \} \cap (\Omega \setminus A_\theta), \end{cases} \]

**Lemma 3.2.**
\[ m \{ x: x \in \Omega \setminus A_\theta, \bar{v}(x) = \alpha_0, \bar{u}_\varepsilon(x) \notin \{ h_-(\alpha_0) - \bar{\theta}, h_-(\alpha_0) + \bar{\theta} \} \cup \{ h_+(\alpha_0) - \bar{\theta}, h_+(\alpha_0) + \bar{\theta} \} \} \to 0 \]
as \( \varepsilon \to 0 \), for any \( \bar{\theta} > 0 \).

The proofs of Lemmas 3.1 and 3.2 are exactly the same as those of Lemmas 2.2 and 2.3, and thus we omit them. Define
\[ \bar{k}(x, v) = (1 - 1_{A_\theta})h(v) + 1_{A_\theta}k(v). \]
Then, from Lemmas 3.1 and 3.2, we have

**Lemma 3.3.** \( \bar{v} \) satisfies
\[
\begin{cases}
-\Delta \bar{v} + \gamma \bar{v} \in [\delta \bar{k}(x, \bar{v} + 0), \delta \bar{k}(x, \bar{v} - 0)], & \text{in } \Omega, \\
\bar{v} = 0, & \text{on } \partial \Omega. 
\end{cases}
\quad (3.2)
\]

For each fixed \( x \), \( \bar{k}(x, \bar{v}) \) is decreasing in \( \bar{v} \), thus it is easy to see that the solution of (3.2) is unique. Now we are ready to prove the following result:

**Proposition 3.4.** Suppose that \( \delta \in (\delta_0, \bar{\delta}) \). Let \( \bar{u}_\varepsilon \) be a minimizer of (3.1), \( \bar{v}_\varepsilon = \delta G_\gamma \bar{u}_\varepsilon \). Then
\[ \bar{u}_\varepsilon \to h_+(\bar{v}), \quad \text{uniformly in any compact subset of } \Omega, \]
and \( \bar{v}_\varepsilon \to \bar{v} \) in \( C^{1,\sigma}(\Omega) \), where \( \bar{v} \) is the solution of (1.6).

**Proof.** First we prove that \( \bar{v} \) is the solution of (1.6). Because the solution of (3.2) is unique, to prove that \( \bar{v} \) satisfies (1.6), we only need to prove that the solution \( v \) of (1.6) also satisfies (3.2).

Since \( \delta \in (\delta_0, \bar{\delta}) \), we know the solution \( v \) of (1.6) satisfies \( \max_{x \in \Omega} v(x) \in (\alpha_0, f(\tau_2)) \). Thus, \( \hat{k}(x, v) = k(v) = h_+(v) \) if \( x \in A_\theta \). On the other hand, \( v < \alpha_0 \) if \( x \in \Omega \setminus A_\theta \). Thus \( \hat{k}(x, v) = h(v) = h_+(v) \) if \( x \in \Omega \setminus A_\theta \). Hence, \( v \) is the solution of (3.2) and
\[ \{ x: v(x) \geq \alpha_0 \} \cap (\Omega \setminus A_\theta) = \emptyset. \]
In view of Lemma 3.1, to prove Proposition 3.4, it remains to prove that for any \( x_0 \in \partial A_\theta \),
\[ \bar{u}_\varepsilon \to h_+(\bar{v}), \quad \text{uniformly in } B_{\delta(x_0)}. \]

The proof of this claim is similar to that in Lemma 2.2. The only change here is that we need to use that minimizer of the following problem to control \( \bar{u}_\varepsilon \):
\[
\inf \left\{ \frac{\varepsilon^2}{2} \int_{B_\varepsilon(x_0)} |Du|^2 - \int_{B_\varepsilon(x_0)} (F(x,u) - v_0u): u \in H^1(B_{\eta}(x_0)), \ u = C \text{ on } \partial B_{\eta}(x_0) \right\}, \tag{3.3}
\]
where \( v_0 \in (0, \alpha_0) \) is a constant.

It is easy to check that the minimizer \( w_\varepsilon \) of (3.3) satisfies \( w_\varepsilon \to h_+(v_0) \) uniformly in \( B_{\delta(x_0)} \). Noting that \( v_\varepsilon(x) < \alpha_0 \) for any \( x \in \partial A_\theta \), we can now prove that \( \bar{u}_\varepsilon \to h_+(\bar{v}) \), uniformly in \( B_{\delta(x_0)} \) exactly in the same way as in Lemma 2.2. \( \square \)

The following result gives the asymptotic behaviour of the minimizer of (3.1) near the boundary.

**Proposition 3.5.** Let \( \bar{u}_\varepsilon \) be the minimizer of (3.1) (or (1.3)). Let \( U_\varepsilon(y) = \bar{u}_\varepsilon(\varepsilon y + x_0), x_0 \in \partial \Omega \), then \( U_\varepsilon(y) \to U(y) \) as \( \varepsilon \to 0 \) in \( C^2_{\text{loc}}(\mathbb{R}^N) \) (after suitably translating and rotating the coordinate systems), and \( U \) is the unique solution of
\[
\begin{cases}
-\Delta U = f(U), & \text{in } \mathbb{R}^N_+ , \\
0 \leq U \leq h_+(0), & \text{in } \mathbb{R}^N_+, \\
U = 0, & \text{on } x_N = 0, \\
U(x',x_N) \to h_+(0), & \text{as } x_N \to +\infty, \text{uniformly for } x' \in \mathbb{R}^{N-1}.
\end{cases}
\tag{3.4}
\]

**Proof.** In fact, since \( U_\varepsilon \) satisfies
\[
-\Delta U_\varepsilon = f(U_\varepsilon) - \bar{v}_\varepsilon(\varepsilon y + x_0),
\]
\( U_\varepsilon \) is bounded in \( L^\infty \) and \( \bar{v}_\varepsilon(\varepsilon y + x_0) \to 0 \) as \( \varepsilon \to 0 \) uniformly for bounded \( y \), we see that
\[
U_\varepsilon(y) \to U(y) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^N_+),
\]
as \( \varepsilon \to 0 \), and \( U(y) \) satisfies
\[
\begin{cases}
-\Delta U = f(U), & \text{in } \mathbb{R}^N_+ , \\
U = 0, & \text{on } x_N = 0.
\end{cases}
\]

Now we prove \( U(x',x_N) \to h_+(0) \), as \( x_N \to +\infty, \text{uniformly for } x' \in \mathbb{R}^{N-1} \). To prove this, we only need to prove that for any \( \tau > 0 \) small, there exists \( R_0 > 0 \) large, such that
\[
|\bar{u}_\varepsilon(x + \varepsilon Rv) - h_+(0)| < \tau, \tag{3.5}
\]
for all \( x \in \partial \Omega, \ R \geq R_0, \ \varepsilon \in (0, \varepsilon_R) \), where \( v \) is the unit inward normal of \( \partial \Omega \) at \( x, \varepsilon_R > 0 \) is a small constant depending on \( R \).

For any \( x \in \partial \Omega \), let \( x_\varepsilon = x + \varepsilon Rv \). Consider the following problem:
\[
\inf \left\{ \frac{\varepsilon^2}{2} \int_{B_{\varepsilon R}(x_\varepsilon)} |D\bar{w}|^2 - \int_{B_{\varepsilon R}(x_\varepsilon)} (F(\bar{w}) - \eta \bar{w}): \bar{w} \in H^1(B_{\varepsilon R}(x_\varepsilon)), \ \bar{w} = C \text{ on } \partial B_{\varepsilon R}(x_\varepsilon) \right\}, \tag{3.6}
\]
where \( |\eta| > 0 \) is a small constant and \( C \) is a constant.
Let \( w(y) = \bar{w}(\varepsilon R y + x_\varepsilon) \). Then (3.6) becomes
\[
\inf \left\{ \frac{1}{R^2} \int_{B_1(0)} |D\bar{w}|^2 - \int_{B_1(0)} (F(w) - \eta w) : \bar{w} \in H^1(B_1(0)), \ w = C, \text{ on } \partial B_1(0) \right\} .
\] (3.7)

Let \( w_R \) be the minimizer of (3.7). Then there is a \( R_0 > 0 \) large, such that
\[
|w_R(y) - h_+(\eta)| < \tau,
\]
for all \( R > R_0, \ y \in B_1(0) \). Thus, the minimizer \( \bar{w}_\varepsilon \) of (3.6) satisfies
\[
|\bar{w}_\varepsilon(y) - h_+(\eta)| < \tau, \tag{3.8}
\]
for all \( R > R_0, \ y \in B_{1/2}(x_\varepsilon) \).

Now for each \( R > R_0 \), we choose \( \varepsilon_R > 0 \) small, such that \( \varepsilon R < \theta \) for \( \varepsilon \in (0, \varepsilon_R) \), where \( \theta > 0 \) is a suitably small constant. Let \( \bar{w}_{\varepsilon,-} \) be the minimizer of
\[
\inf \left\{ \frac{1}{2} \int_{B_{\varepsilon R}(x_\varepsilon)} |D\bar{w}|^2 - \int_{B_{\varepsilon R}(x_\varepsilon)} (F(\bar{w}) - \eta \bar{w}) : \bar{w} \in H^1(B_{\varepsilon R}(x_\varepsilon)), \ \bar{w} = \bar{C} \text{ on } \partial B_{\varepsilon R}(x_\varepsilon) \right\} .
\] (3.9)

and let \( \bar{w}_{\varepsilon,+} \) be the minimizer of
\[
\inf \left\{ \frac{1}{2} \int_{B_{\varepsilon R}(x_\varepsilon)} |D\bar{w}|^2 - \int_{B_{\varepsilon R}(x_\varepsilon)} (F(\bar{w}) + \eta \bar{w}) : \bar{w} \in H^1(B_{\varepsilon R}(x_\varepsilon)), \ \bar{w} = -\bar{C}, \text{ on } \partial B_{\varepsilon R}(x_\varepsilon) \right\} .
\] (3.10)

where \( \eta > 0 \) is a small constant and \( \bar{C} > 0 \) is a large constant. Similar to the proof of Lemma 2.2, we know that if \( \theta > 0 \) is suitably small, then
\[
\bar{w}_{\varepsilon,-} < u_\varepsilon < \bar{w}_{\varepsilon,+}, \quad \forall y \in B_{1/2}(x_\varepsilon).
\]

On the other hand, it follows from (3.8) that
\[
\left| \bar{w}_{\varepsilon,+} - h_+(\eta) \right|, \left| \bar{w}_{\varepsilon,-} - h_+(\eta) \right| < \tau.
\]

Thus (3.5) follows.

It remains to prove that \( 0 \leq U \leq h_+(0) \), in \( \mathbb{R}^N \). For any \( \eta > 0 \) small, we claim that
\[
-\eta < \bar{u}_\varepsilon(x) < h_+(0) + \eta, \quad \forall x \in \{ x : d(x, \partial\Omega) \leq R_\varepsilon \}. \tag{3.11}
\]

Let \( S_\varepsilon = \{ x : u_\varepsilon(x) > h_+(0) + \eta, \ d(x, \partial\Omega) \leq R_\varepsilon \} \). By (3.5), we know that \( S_\varepsilon \cap \{ x : d(x, \partial\Omega) = R_\varepsilon \} = \emptyset \). Define \( w_\varepsilon = u_\varepsilon \) if \( x \in \Omega \setminus S_\varepsilon \), \( w_\varepsilon = h_+(0) + \eta \) if \( x \in S_\varepsilon \). Then \( u_\varepsilon - w_\varepsilon \in H^1_0(\Omega) \). Thus we have
\[
0 \leq J_\varepsilon(w_\varepsilon) - J_\varepsilon(u_\varepsilon)
\]
\[
\leq \int_{S_\varepsilon} \left( F(u_\varepsilon) - F(h_+(0) + \eta) \right) + \frac{\delta}{2} \int_{S_\varepsilon} (u_\varepsilon - w_\varepsilon) G_y u_\varepsilon + \frac{\delta}{2} \int_{S_\varepsilon} (u_\varepsilon - w_\varepsilon) G_y(u_\varepsilon - w_\varepsilon)
\]
\[
\leq \int_{S_\varepsilon} \left( F(u_\varepsilon) - F(h_+(0) + \eta) \right) + \frac{\delta}{2} |G_y u_\varepsilon|_{L^\infty(S_\varepsilon)} \int_{S_\varepsilon} (u_\varepsilon - w_\varepsilon)
\]
\[
+ \frac{\delta}{2} |G_y(u_\varepsilon - w_\varepsilon)|_{L^\infty(S_\varepsilon)} \int_{S_\varepsilon} (u_\varepsilon - w_\varepsilon). \tag{3.12}
\]

Because \( G_y u_\varepsilon \) is small near the boundary of \( \Omega \) and \( S_\varepsilon \subset \{ x : d(x, \partial\Omega) \leq \tau \} \), we see
\[
|G_y u_\varepsilon|_{L^\infty(S_\varepsilon)} \leq \tau(\varepsilon),
\]
where $\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$. On the other hand, we have
\[
|G_\gamma(u_\varepsilon - w_\varepsilon)|_{L^\infty(\Omega)} \leq |u_\varepsilon - w_\varepsilon|_{L^p(\Omega)} \leq C m(S_\varepsilon).
\]
Thus,
\[
0 \leq J_\varepsilon(w_\varepsilon) - J_\varepsilon(u_\varepsilon) \leq \int_{S_\varepsilon} \left( F(u_\varepsilon) - F\left(h_+(0) + \eta\right) \right) + \tau''(\varepsilon) \int_{S_\varepsilon} (u_\varepsilon - w_\varepsilon),
\]
where $\tau''(\varepsilon) \to 0$ as $\varepsilon \to 0$.

But
\[
F\left(h_+(0) + \eta\right) - F(u_\varepsilon) = \int_{u_\varepsilon}^h f(s) \, ds \geq -f\left(h_+(0) + \eta\right)(u_\varepsilon - (h_+(0) + \eta)),
\]
for any $u_\varepsilon > h_+(0) + \eta$. Thus we obtain from (3.13) that
\[
-f\left(h_+(0) + \eta\right) \int_{S_\varepsilon} (u_\varepsilon - (h_+(0) + \eta)) \leq \tau''(\varepsilon) \int_{S_\varepsilon} (u_\varepsilon - (h_+(0) + \eta)).
\]
Thus $S_\varepsilon = \emptyset$. Thus $u_\varepsilon < h_+(0) + \eta$. Similarly, $u_\varepsilon > -\eta$ if $d(x, \partial\Omega) \leq R \varepsilon$. Thus we have proved (3.11). Clearly, $0 \leq U \leq h_+(0)$ in $R^N_\varepsilon$ follows from (3.11). \quad \Box

**Remark 3.6.** The solution of (3.4) is unique and is a function of $x_N$ only. See [7].

**Proposition 3.7.** Suppose that $\delta \in (\delta_0, \delta_\tilde{\delta})$. Let $\tilde{u}_\varepsilon$ be a minimizer of (3.1). Then $u_\varepsilon$ is a local minimizer of (1.4).

**Proof.** We only need to prove
\[
\int_\Omega (\varepsilon^2 |D\phi|^2 + \delta \phi G_\gamma \phi) - \int_\Omega f' (\tilde{u}_\varepsilon) \phi^2 \geq c_0 \int_\Omega \phi^2, \quad \forall \phi \in H^1_0(\Omega),
\]
for some $c_0 > 0$. But
\[
\int_\Omega (\varepsilon^2 |D\phi|^2 + \delta \phi G_\gamma \phi) - \int_\Omega f' (\tilde{u}_\varepsilon) \phi^2 \geq \int_\Omega \varepsilon^2 |D\phi|^2 - \int_\Omega f' (\tilde{u}_\varepsilon) \phi^2,
\]
so the claim follows if we can prove
\[
\inf_{\phi \in H^1_0(\Omega), \phi \neq 0} \frac{\int_\Omega \varepsilon^2 |D\phi|^2 - \int_\Omega f' (\tilde{u}_\varepsilon) \phi^2}{\int_\Omega \phi^2} =: \mu_\varepsilon > 0.
\]

Let $\phi_\varepsilon$ be a minimizer of (3.14). We may choose $\phi_\varepsilon$ such that $\phi_\varepsilon \geq 0$ and $\max_{x \in \Omega} \phi(x) = 1$.

Suppose that $\mu_\varepsilon \to \mu \leq 0$. Let $x_\varepsilon$ be a maximum point of $\phi_\varepsilon$. Suppose that $d(x_\varepsilon, \partial\Omega)/\varepsilon \to +\infty$ as $\varepsilon \to 0$.

Then $|u_\varepsilon(x_\varepsilon) - h_+(v(x_\varepsilon))|$ is small. As a result, $f'(\tilde{u}_\varepsilon(x_\varepsilon)) \leq -c_0 < 0$. Since
\[
-\Delta \phi_\varepsilon - f' (\tilde{u}_\varepsilon) \phi_\varepsilon = \mu_\varepsilon \phi_\varepsilon,
\]
we see that $-f'(\tilde{u}_\varepsilon(x_\varepsilon)) \leq \mu_\varepsilon \to 0$ as $\varepsilon \to 0$. This is impossible. So we have proved that $d(x_\varepsilon, \partial\Omega)/\varepsilon \to c < +\infty$.

Let $\bar{\phi}_\varepsilon(x) = \phi_\varepsilon(x + \tilde{x}_\varepsilon)$, where $\tilde{x}_\varepsilon \in \partial\Omega$ is the point such that $|\tilde{x}_\varepsilon - x_\varepsilon| = d(x_\varepsilon, \partial\Omega)$. Then $\bar{\phi}_\varepsilon$ is bounded in $L^\infty$ and $\bar{\phi}_\varepsilon((x_\varepsilon - \tilde{x}_\varepsilon)/\varepsilon) = 1$. Moreover, $\bar{\phi}_\varepsilon$ satisfies
\[
-\Delta \bar{\phi}_\varepsilon - f' (\tilde{u}_\varepsilon(\varepsilon y + \tilde{x}_\varepsilon)) \bar{\phi}_\varepsilon = \mu_\varepsilon \phi_\varepsilon.
\]
Thus, in view of the boundedness of $\bar{\phi}_\varepsilon$, we may assume up to a subsequence that $\bar{\phi}_\varepsilon \to \bar{\phi}$ in $C^2_{loc}(\mathbb{R}^n_+)$ and $\bar{\phi}$ is a bounded nontrivial solution of
\[
\begin{cases}
-\Delta \bar{\phi} - f'(U)\bar{\phi} = \mu \bar{\phi}, & \text{in } \mathbb{R}^n_+,
\bar{\phi} = 0, & \text{on } \mathbb{R}^N_+.
\end{cases}
\]
where $U$ is the solution of (3.4). This is impossible. See the proof of Lemma 4.2 in [7], or the proof of Proposition 2 in [8].

Proof of Theorem 1.3. Theorem 1.3 follows from Propositions 3.4 and 3.7.

References