The preventive effect of the convection and of the diffusion in the blow-up phenomenon for parabolic equations

L’effet préventif de la convection et de la diffusion dans les phénomènes d’explosion pour des équations paraboliques

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Abstract

The goal of this paper is to investigate the role of the gradient term and of the diffusion coefficient in the preventing of the blow-up of the solution for semilinear and quasilinear parabolic problems.

Résumé

Le but de cet article est de voir comment le terme de gradient et le coefficient de diffusion empêchent l’explosion des solutions des problèmes semilinéaires et quasilinéaires paraboliques.

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1. Introduction and main results

In the present paper we consider the following convective diffusion equations

\[ u_t + a(t, x) \cdot \nabla u = \Delta u + \lambda u^p + f(t, x), \quad (0.1) \]

\[ u_t + a_i(t, x)u_{x_i} = \Delta u + \lambda u^p + f(t, x), \quad (0.2) \]
and equation
\[ u_t + a(t, x)|\nabla u|^q = \Delta u + \lambda u^p + f(t, x) \] (0.3)
in the domain \( Q_T = \Omega \times (0, T), \quad \Omega \subset \mathbb{R}^n \) coupled with conditions
\[ u(0, x) = u_0(x), \quad u(t, x)|_{S_T} = 0, \quad S_T = \partial \Omega \times (0, T). \] (0.4)
Here \( \lambda \) and \( q \) are positive constants, \( q_i \) and \( p \) are positive integer, \( a_i(t, x), a(t, x) \) and \( f(t, x) \) are given functions, \( a(t, x) = (a_1(t, x), \ldots, a_n(t, x)) \), \( a(t, x) \cdot \nabla u = a_i u_{x_i} \) and
\[ a_i u_{x_i} = \sum_{i=1}^{n} a_i u_i |x|. \]
Without loss of generality assume that domain \( \Omega \) lies in the strip \( |x_1| \leq l_1 \). Suppose that \( u_0(x)|_{\partial \Omega} = 0 \) and \( \max |u_0(x)| \leq K_1 \), (0.5)
where \( K_1 \) is some positive constant.
Let us formulate the results concerning the preventive effect of the gradient terms.

**Theorem 1.** Assume that
\[ a_1(t, x) \geq \lambda (2l_1)^p K_1^{p-1} + \frac{\max_{Q_T} |f(t, x)|}{K_1^{q_1}} \]
or
\[ a_1(t, x) \leq -\lambda (2l_1)^p K_1^{p-1} - \frac{\max_{Q_T} |f(t, x)|}{K_1^{q_1}}. \]
If condition (0.5) is fulfilled then the solution of problem (0.1), (0.4) remains bounded for all \( t > 0 \) and
\[ \max_{\overline{Q_T}} |u(t, x)| \leq 2K_1 l_1. \] (0.6)
**Theorem 2.** Suppose that (0.5) is fulfilled.
(i) If \( q_1 \) is odd and
\[ a_1(t, x) \geq \lambda (2l_1)^p K_1^{p-q_1} + \frac{\max_{Q_T} |f(t, x)|}{K_1^{q_1}} \]
or
\[ a_1(t, x) \leq -\lambda (2l_1)^p K_1^{p-q_1} - \frac{\max_{Q_T} |f(t, x)|}{K_1^{q_1}}. \]
then the solution of problem (0.2), (0.4) remains bounded for all \( t > 0 \) and (0.6) holds.
If in addition \( q_1 > p \), then estimate (0.6) holds with \( a_1(t, x) \geq \alpha \) or \( a_1(t, x) \leq -\alpha \) for any strictly positive number \( \alpha \).
(ii) If \( q_1 \) and \( p \) are even and
\[ a_1(t, x) \geq \lambda (2l_1)^p K_1^{p-q_1} + \frac{\max_{Q_T} f}{K_1^{q_1}}, \quad f \geq 0, \quad u_0 \geq 0, \]
then the solution of problem (0.2), (0.4) remains bounded for all \( t > 0 \) and
\[ 0 \leq u(t, x) \leq 2K_1 l_1. \] (0.7)
If in addition \( q_1 > p \), then estimate (0.7) is valid for \( a_1(t, x) > \alpha \) for any positive \( \alpha \).
Theorem 3. Suppose that \( p \) is even and
\[
a(t, x) \geq \lambda (2l_1)^p K_1^{p-q} + \frac{\max_{Q_T} f(t, x)}{K_1^q}, \quad f \geq 0, \ u_0 \geq 0.
\]
If (0.5) is fulfilled, then the solution of problem (0.3), (0.4) remains bounded for all \( t > 0 \) and (0.7) holds.
For \( q > p \) this estimate holds with \( a(t, x) > \alpha \) for any positive \( \alpha \).

Let us give a simple physical interpretation of Theorem 1. Let \( u(t, x) \) be temperature and \( a = (a_1, a_2, a_3) \) velocity field. If the velocity is big enough at least in one direction (for example \( x_1 \) or \( -x_1 \)), then the convective transfer (from the left if \( a_1 \) is positive or from the right if negative) in this direction brings sufficient cold substance from the boundary, so as not to allow the term \( u^p \) to blow-up the temperature.

Eq.(0.3) without gradient term (i.e. when \( a \equiv 0 \)) was investigated by many authors and there is extensive literature on this subject (see, for example, [12] and the references there). It is well known that the phenomenon of blowing up of the solution may occur in this case, i.e. \( |u(t, x^*)| \rightarrow +\infty \) when \( t \rightarrow t^* \) at least for one \( x^* \in \Omega \). In [3] the authors introduce the gradient term \( |\nabla u|^q \) in order to investigate the effect of this term on global existence or non-existence of the solution of the Dirichlet problem. Later the influence of the gradient term in the blow up phenomenon for Eq. (0.3) was studied in [4–8,14–16]. The main issue of these works was to determine for which \( p \) and \( q \) the blow-up in finite time occurs and for which the solution remains bounded. Roughly speaking it turns out that blow-up in finite time may occur if and only if \( p > q \). For more details see [15]. So we can conclude that the gradient term controls the source term in the sense that there is no blow up if \( p < q \). From Theorem 3 of the present paper it follows that if the coefficient \( a \) is big enough, then the gradient term controls the source term even in the case when the power of the gradient term is less then the power of the source term \( p > q \). If \( p < q \), then Theorem 3 guarantees the boundedness of the solution for any positive \( a \). Moreover, as it follows from Theorems 1, 2 the presence of the derivative only in one direction \( x_{i_0} \) can “hold” the maximum of the solution if the coefficient \( a_{i_0} \) is big enough. The interpretation presented above gives some idea why this happens. Application of Eq. (0.3) was given in [13].

Convective diffusion equations with blow up term \( u^p \) were studied in [1,2,10]. Different cases of blow up of the solution were investigated there.

The prevent of blow-up of the solution can be also obtained by taking sufficiently big diffusion (heat conductivity) coefficient at least in one direction. To demonstrate this we consider the following equation
\[
u_t = \kappa u_{x_1x_1} + \Delta' u + \lambda u^p + f(t, x), \quad (0.8)
\]
coupled with conditions (0.4), here \( \Delta' u \equiv \sum_{i=2}^n u_{x_ix_i} \).

Theorem 4. Suppose that
\[
\kappa \geq 3\lambda l_1^{p+1}(2K_1)^{p-1} + \frac{3l_1}{2K_1} \max_{Q_T} |f(t, x)|.
\]
Then the solution of problem (0.8), (0.4) is bounded for all \( t > 0 \) and
\[
\max_{Q_T} |u(t, x)| \leq 2K_1l_1.
\]

The physical interpretation of Theorem 4 is that if the heat conductivity at least in one direction is sufficiently big, then the heat flow through the boundary in that direction is big enough to prevent the unbounded growth of the temperature. Let us mention here that for the equation
\[
u_t = \kappa |u|^p u_{x_1x_1} + \Delta' u + \lambda u^p + f(t, x) \quad (0.9)
\]
the boundedness of the solution of problem (0.9), (0.4) for any positive \( \kappa \) follows immediately from [19]. Here combining the method proposed in [19] with the proof of Theorems 1–3 of the present paper we obtain similar result for problem (0.8), (0.4).

We would like to emphasize that all results of the present paper mentioned above can be easily extended to the case where the term \( \lambda u^p \) is substituted by function \( Q(u) \). The restrictions on \( Q(u) \) are formulated in Section 4, in particular besides \( \lambda u^p \) we can take \( Q(u) = e^u \).

The paper is organized as follows. In Section 2 we give the proofs of Theorems 1–3. Section 3 deals with problem (0.8), (0.4). In Section 4 we substitute \( \lambda u^p \) by \( Q(u) \). In the last section we discuss the existence theorems. For problems (0.1), (0.4) and (0.8), (0.4) the existence of a classical solution follows from the \( L_\infty \) estimate of the solution under some assumptions on the smoothness of the coefficients. The same is valid for problems (0.2), (0.4) and (0.3), (0.4), if \( q_1 \leq 2 \) and \( q \leq 2 \) correspondingly (see [9,11]). We formulate conditions which guarantee the existence in the special case of classical solution for \( q_1 > 2 \) for problem (0.2), (0.4). Concerning the global solvability of problem (0.3), (0.4) for \( q > 2 \) we refer to [8,14,16].

2. Proof of Theorem 1–3

**Proof of Theorem 1.** Suppose that

\[
a_1(t, x) > \lambda (2l_1)^p K_1^{p-1} + \frac{\max_{\overline{Q_T}} |f(t, x)|}{K_1},
\]

(1.1)

Introduce the auxiliary equation:

\[
u_t - \Delta \nu = -a(t, x) \cdot \nabla \nu + f_p(\nu) + f(t, x) \quad \text{in} \quad Q_T,
\]

(1.2)

where

\[
f_p(z) = \begin{cases} \lambda z^p, & \text{for} \ |z| \leq 2K_1l_1, \\ \lambda (2K_1l_1)^p, & \text{for} \ z > 2K_1l_1, \\ \lambda (-2K_1l_1)^p, & \text{for} \ z < -2K_1l_1. \end{cases}
\]

The goal is to obtain the estimate \( |\nu(t, x)| \leq 2K_1l_1 \) for the solution of problem (1.2), (0.4). If such estimate takes place, then Eqs. (1.2) and (0.1) coincide and as a consequence the solution of problem (0.1), (0.4) will be bounded.

Consider function \( v(t, x) \equiv u(t, x) - h(x_1) \), where \( h(x_1) = K_1(l_1 + x_1) \). One can easily see that

\[
v_t - \Delta v = -a(t, x) \cdot \nabla u + f_p(\nu) + f(t, x).
\]

(1.3)

If the function \( v(t, x) \) attains maximum at the point \( N \in \overline{Q_T} \setminus \Gamma \) (\( \Gamma \) is the parabolic boundary of the domain \( Q_T \)), then at this point we have \( \nabla v = 0 \), i.e. \( u_{x_1} = h' = K_1, \ u_{x_i} = 0, \) for \( i > 2 \) and hence

\[
v_t - \Delta v|_N = -a_1(N)K_1 + f_p(\nu(N)) + f(N) \\
< -\lambda (2l_1)^p K_1^{p-1} + \max_{\overline{Q_T}} |f(t, x)| + \lambda (2K_1l_1)^p + f(N) \leq 0.
\]

This contradicts the assumption that at \( N \) we have maximum of \( v \). Here we use the inequality \( f_p(\nu) \leq \lambda (2K_1l_1)^p \).

On \( \Gamma \) the function \( v(t, x) \) is non-positive. In fact, due to (0.5), \( v(0, x) = u_0(x) - K_1(l_1 + x_1) \leq 0 \) and from (0.4) \( v|_{\partial T} = -h|_{\partial T} \leq 0 \). Hence \( v \leq 0 \) in \( \overline{Q_T} \) and

\[
u(t, x) \leq K_1(l_1 + x_1) \leq 2K_1l_1.
\]

Now let us consider function \( \tilde{v}(t, x) \equiv u(t, x) + h(x_1) \). We have

\[
\tilde{v}_t - \Delta \tilde{v} = -a(t, x) \cdot \nabla u + f_p(\nu) + f(t, x).
\]
If the function $\tilde{v}(t, x)$ attains minimum at the point $N_1 \in \bar{Q}_T \setminus \Gamma$ then at this point we have $\nabla \tilde{v} = 0$, i.e. $u_{x_1} = -h' = -K_1$, $u_{x_i} = 0$ for $i \geq 2$ and hence
\[
\tilde{v}_t - \Delta \tilde{v}|_{N_1} = a(N_1)K_1 + f_p(u(N_1)) + f(N_1) > \lambda (2l_1)^p K_1^p + \max_{\bar{Q}_T} f(t, x) - \lambda (2K_1l_1)^p + f(N_1) \geq 0.
\]
This contradicts the assumption that $\tilde{v}$ attains minimum at $N_1$. Here we use the inequality $f_p(u) \geq -\lambda (2K_1l_1)^p$. It is clear that on $\Gamma$ the function $\tilde{v}$ is non-negative. In fact, due to (0.5) $\tilde{v}(0, x) = u_0(x) + K_1(l_1 + x_1) \geq 0$ and from (0.4) $\tilde{v}|_{\partial \Omega_T} = h|_{\partial \Omega_T} \geq 0$, hence $\tilde{v} \geq 0$ in $\bar{Q}_T$ and
\[
u(t, x) \geq -K(l_1 + x_1) \geq -2K_1l_1.
\]
For $a_1 > \lambda (2l_1)^p K_1^{-p} + K_1^{-1} \max_{\bar{Q}_T} |f|$ Theorem 1 is proved.

Suppose now that $a_1 \geq \lambda (2l_1)^p K_1^{-p} + K_1^{-1} \max_{\bar{Q}_T} |f|$. Substitute function $v$ in (1.3) by $v_1 = ve^{-t}$. For $v_1$ we have
\[
v_{1t} + v_1 - \Delta v_1 = e^{-t}(-a(t, x) \cdot \nabla u + f_p(u) + f(t, x)).
\]
Function $v_1$ cannot attain positive maximum at the point $N \in \bar{Q}_T \setminus \Gamma$ because in this point the left side is strictly positive and the right side non-positive. On $\Gamma$ the function $v_1$ is non-positive, hence $v_1 \leq 0$ in $\bar{Q}_T$ and $u \leq K_1(l_1 + x_1) \leq 2K_1l_1$.

Similarly considering $\tilde{v}_1 = \tilde{v}e^{-t}$ instead of $\tilde{v}$ we obtain the needed estimate from below.

The case $a_1(l, x) \leq -\lambda (2l_1)^p K_1^{-p} - K_1^{-1} \max_{\bar{Q}_T} |f|$ can be treated in the same way with the only difference in the choice of the barrier. Here instead of $h(x_1) = K_1(l_1 + x_1)$ we must take $\tilde{h}(x_1) = K_1(l_1 - x_1)$.

Theorem 1 is proved. \(\square\)

**Proof of Theorem 2.** If $q_1$ is odd, then the proof of this theorem is similar to the previous one. In fact, consider auxiliary equation
\[
u_t - \Delta \nu = -a(t, x)u_{x_1} + f_p(u) + f(t, x) \quad \text{in} \quad \bar{Q}_T, \tag{1.4}
\]
where $f_p$ is the same as in the proof of Theorem 1. For the function $v(t, x) = u(t, x) - h(x_1)$, where $h(x_1) = K_1(l_1 + x_1)$ we have
\[
u_t - \Delta \nu = -a(t, x)u_{x_1} + f_p(u) + f(t, x).
\]
If the function $v(t, x)$ attains maximum at the point $N \in \bar{Q}_T \setminus \Gamma$, then at this point we have $\nabla v = 0$, i.e. $u_{x_1} = h' = -K_1$, $u_{x_i} = 0$ for $i \geq 2$ and hence
\[
u_t - \Delta \nu|_{N} = -a_1(N)K_1^{q_1} + f_p(u(N)) + f(N) < -\lambda (2l_1)^p K_1^p + \max_{\bar{Q}_T} f(t, x) + \lambda (2K_1l_1)^p + f(N) \leq 0.
\]
This contradicts the assumption that at $N$ we have maximum of $v$. Taking into account that on $\Gamma$ the function $v(t, x)$ is non-positive we conclude that $v(t, x) \leq 0$ in $\bar{Q}_T$ and hence
\[
u(t, x) \leq K_1(l_1 + x_1) \leq 2K_1l_1.
\]
Function $\tilde{v}(t, x) = u(t, x) + h(x_1)$ satisfies the equation
\[
\tilde{v}_t - \Delta \tilde{v} = -a_1(t, x)u_{x_1} + f_p(u) + f(t, x).
\]
If $\tilde{v}(t, x)$ attains minimum at the point $N_1 \in \bar{Q}_T \setminus \Gamma$, then at this point we have $\nabla \tilde{v} = 0$, i.e. $u_{x_1} = -h' = -K_1$, $u_{x_i} = 0$ for $i \geq 2$ and hence
\[ \tilde{v}_t - \Delta \tilde{v} |_{N_1} = a_1(N_1) K_1^{q_1} f_p(u(N_1)) + f(N_1) \\
> \lambda (2l_1)^p K_1^p + \max_{\partial \Gamma} |f(t, x)| - \lambda (2K_1 l_1)^p + f(N_1) \geq 0. \]

This contradicts the assumption that \( \tilde{v}(t, x) \) attains minimum at \( N_1 \). Taking into account that on \( \Gamma \) the function \( \tilde{v}(t, x) \) is non-negative we conclude that \( \tilde{v} \geq 0 \) in \( \bar{Q}_T \) and consequently
\[ u(t, x) \geq -K_1(l_1 + x_1) \geq -2K_1 l_1. \]

For even \( q_1 \), the estimate \( u(t, x) \leq 2K_1 l_1 \) can be obtained similarly. In order to obtain the estimate from the below we need to assume that \( p \) is even number too. For even \( p \) one can easily see that the solution of problem (1.4), (0.4) cannot attain negative minimum in \( \bar{Q}_T \setminus \Gamma \), and due to the fact that \( u_0(x) \geq 0 \) we obtain the needed estimate from the below.

If \( q_1 > p \), then for any \( \alpha > 0 \) we can select \( K_1 \) sufficiently big (without changing \( u_0(x) \)) so that
\[ \frac{\lambda (2l_1)^p}{K_1^{q_1-p}} + \frac{\max_{\partial \Gamma} |f(t, x)|}{K_1^{q_1}} \leq \alpha. \]

Theorem 2 is proved. \( \square \)

**Proof of Theorem 3.** It is similar to the proof of Theorem 2. In fact, consider auxiliary equation
\[ u_t - \Delta u = -a(t, x) |u|^q + f_p(u) + f(t, x) \text{ in } Q_T, \tag{1.5} \]

where \( f_p \) is the same as in the proof of Theorem 1. For the function \( v(t, x) = u(t, x) - h(x_1) \), where \( h(x_1) = K_1(l_1 + x_1) \) we have
\[ \tilde{v}_t - \Delta v = -a(t, x) |u|^q + f_p(u) + f(t, x). \]

At the point \( N \in \bar{Q}_T \setminus \Gamma \) of maximum of \( v \) similarly to the previous cases we have
\[ \tilde{v}_t - \Delta v |_{N} = -a(N) K_1^q + f_p(u(N)) + f(N) \tag{1.6} \]
\[ < -\lambda (2l_1)^p K_1^p - \max_{\partial \Gamma} |f(t, x)| + \lambda (2K_1 l_1)^p + f(N) \leq 0. \tag{1.7} \]

This contradicts the assumption that at \( N \) we have maximum of \( v \). Taking into account that on \( \Gamma \) the function \( v \) is non-positive we conclude that \( v \leq 0 \) in \( \bar{Q}_T \) and hence \( u \leq 2K_1 l_1. \)

For even \( p \) one can easily see that the solution of problem (1.5), (0.4) cannot attain negative minimum in \( \bar{Q}_T \setminus \Gamma \) and due to the fact that \( u_0(x) \geq 0 \) we obtain the needed estimate from the below.

If \( q > p \), then as in previous case for any \( \alpha > 0 \) we can select \( K_1 \) sufficiently big such that \( \lambda (2l_1)^p K_1^{p-q} \leq \alpha. \)

Theorem 3 is proved. \( \square \)

3. The influence of the diffusion

**Proof of Theorem 4.** Suppose that
\[ \kappa > 3l_1^{p+1} (2K_1)^{p-1} + \frac{3l_1}{2K_1} \max_{\partial \Gamma} |f|. \]

Construct the auxiliary problem using the cut function \( f_p \) introduced in Section 2. Consider equation
\[ u_t - \kappa u_{x_1} = \Delta u = f_p(u) + f(t, x), \tag{3.1} \]

coupled with conditions (0.4). As in the proof of Theorem 1, here the goal is to establish the estimate \( |u| \leq 2K_1 l_1 \) for the solution of problem (3.1), (0.4). Such estimate implies the coincidence of Eqs. (3.1) and (0.8) and consequently the estimate \( |u| \leq 2K_1 l_1 \) is valid for the solution of problem (0.8), (0.6) as well.
Introduce the barrier function \( h(x_1) \):

\[
h(x_1) \equiv \frac{-K_1}{3l_1} x_1^2 + \frac{5}{3} K_1 x_1 + 2K_1 l_1.
\]

Such choice is stipulated by the necessity the barrier function to satisfy the following properties:

\[
h(-l_1) = 0, \quad h(0) = 2K_1 l_1, \quad h'(x_1) \geq K_1, \quad -\kappa h'' > \lambda (2K_1 l_1)^p + \sup |f|.
\]

For \( v(t, x) \equiv u(t, x) - h(x_1) \) we have

\[
v_t - \kappa v_{x_1 x_1} - \Delta v = \varphi_\varepsilon(u) + f(t, x) + \kappa h''(x_1) \equiv \varphi_\varepsilon(u) + f(t, x) - \frac{2K_1}{3l_1} (3.2)
\]

\[
< \varphi_\varepsilon(u) + f(t, x) - \lambda (2K_1 l_1)^p - \max |f(t, x)| \leq 0. (3.3)
\]

Thus function \( v \) cannot attain maximum in \( Q_T \setminus \Gamma \). On \( \Gamma \) we have \( v \leq 0 \). Hence \( u(t, x) \leq h(x_1) \). Consequently

\[ u(t, x) \leq h(0) = 2K_1 l_1. \]

Similarly we prove that \( \tilde{v} \equiv u - h(-x_1) \leq 0 \), i.e. \( u(t, x) \leq h(-x_1) \). Consequently

\[ u(t, x) \geq -h(0) = -2K_1 l_1. \]

Similarly to the proof of Theorem 1, introducing function \( v_1 = ve^{-t} \), we can prove that Theorem 4 is valid with \( \kappa \geq 3l_1^{p+1} (2K_1)^{p-1} + 3l_1 (2K_1)^{-1} \max_{Q_T} |f| \).

Theorem 4 is proved. □

4. Arbitrary source

In this section we consider arbitrary function \( Q(u) \) instead of \( \lambda u^p \). Assume that \( Q \) satisfies the following inequality

\[
-Q(2l_1 K_1) \leq Q(z) \leq Q(2l_1 K_1) \quad \text{for} \quad |z| \leq 2l_1 K_1. \quad (4.1)
\]

We restrict ourselves by the generalization of Theorem 1. Other results can be generalized exactly in the same manner.

Consider equation

\[
u_t + a(t, x) \cdot \nabla u = \Delta u + Q(u) + f(t, x), (4.2)\]

coupled with conditions (0.4).

**Theorem 5.** Assume that \( Q \) satisfies (4.1) and

\[
a_1(t, x) \geq Q(2l_1 K_1) K_1^{-1} + \frac{\max_{Q_T} |f(t, x)|}{K_1}
\]

or

\[
a_1(t, x) \leq -Q(2l_1 K_1) K_1^{-1} - \frac{\max_{Q_T} |f(t, x)|}{K_1}.
\]

Then the solution of problem (4.2), (0.4) remains bounded for all \( t > 0 \) and

\[
\max_{Q_T} |u(t, x)| \leq 2K_1 l_1.
\]
Proof. Without loss of generality assume that
\[ a_1(t,x) > Q(2l_1)K_1^{-1} + \frac{\max_{Q_T}|f(t,x)|}{K_1}. \]
The only difference in comparison with Theorem 1 is in constructing the cut function. Introduce the auxiliary equation:
\[ u_t - \Delta u = -a(t,x) \cdot \nabla u + f_0(u) + f(t,x) \text{ in } Q_T, \] (4.3)
where
\[ f_0(z) = \begin{cases} Q(z), & \text{for } |z| \leq 2K_1l_1, \\ Q(2K_1l_1), & \text{for } z > 2K_1l_1, \\ Q(-2K_1l_1), & \text{for } z < -2K_1l_1. \end{cases} \]
Exactly in the same manner as in the proof of Theorem 1 we show that for the solution of problem (4.3), (0.4) the estimate
\[ |u(t,x)| \leq 2K_1l_1 \] is valid. Consequently (4.3) and (4.2) coincide and as a consequence the solution of problem (4.2), (0.4) is bounded.

The case \( a_1(t,x) \leq -Q(u)K_1^{-1} - \max_{Q_T}|f| \) can be treated in the same way.

Consider now equation
\[ u_t = \kappa u_{x_1} + \Delta' u + Q(u) + f(t,x) \] (4.4)
coupled with conditions (0.4). Analogously we can prove the following theorem.

**Theorem 6.** Assume that \( Q \) satisfies (4.1) and
\[ \kappa \geq \frac{3l_1}{2K_1}Q(2K_1l_1) + \frac{3l_1}{2K_1}\max_{Q_T}|f(t,x)|. \]
Then the solution of problem (4.4), (0.4) remains bounded for all \( t > 0 \) and
\[ \max_{Q_T}|u(t,x)| \leq 2K_1l_1. \]

5. Remarks on the existence of the solution

As it was already mentioned in the introduction, for problem (0.1), (0.4) as well as for (0.2), (0.4) if \( q \leq 2 \) and (0.3), (0.4) if \( q \leq 2 \) the existence of a classical solution follows from the \( L_\infty \) estimate of the solution under some assumptions on the smoothness of the coefficients. Namely functions \( a_i(t,x) \) and \( f(t,x) \) must be Hölder continuous functions (see [9,11]). The a priori estimate of the gradient which is the key step when proving the existence theorem generally speaking fails when \( q_i > 2 \) and \( q > 2 \). Recently there appeared several papers where the condition on no more than quadratic growth of the gradient term is generalized (see [17,18,20]). Unfortunately, for the Dirichlet problem, Eqs. (0.2) and (0.3) do not satisfy assumptions of these papers and we cannot use the approach suggested there. Let us apply here the classical approach that goes back to S.N. Bernstein and involves the preliminary boundary estimates, differentiation of the equation with respect to \( x_i, i = 1, \ldots, n \), followed by multiplication by \( u_{x_i} \) and summation over \( i \). The maximum principle is then applied to the resulting equation for the function \( w = |\nabla u|^2 \).

Consider problem (0.2), (0.4) in domain \( P_T = (0,T) \times P \), where \( P = \{x: |x_i| < l_i, i = 1, \ldots, n\} \). Suppose that \( p \) and \( q_i > 2 \) are even and
\[ a_i \geq \lambda(2l_i)^pK_i^{p-q_i} + \frac{\max_{Q_T}|f|}{K_i}, \quad f \geq 0, \quad u_0 \geq 0, \]
where $|u_{0i}(x)| \leq K_i$, $i = 1, \ldots, n$. From Theorem 2 we conclude that in $P_T$ the following estimates hold
\begin{equation}
0 \leq u(t, x) \leq K_i(l_i + x_i), \quad 0 \leq u(t, x) \leq K_i(l_i - x_i).
\end{equation}
(5.3)
These inequalities imply the boundary gradient estimates:
\begin{equation}
\left| u_{xi}(t, x) \right|_{x_i = \pm l_i} \leq K_i.
\end{equation}

Having these boundary gradient estimates we can easily apply the Bernstein’s approach if $a_i = a_i(t)$, $i = 1, \ldots, n$ to obtain the apriori estimate of $|\nabla u|$ which in turn implies the existence of the classical solution under the smoothness assumptions on $a_i$ and $f$. For more details see [9,11].

References