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Hardy–Sobolev critical elliptic equations with boundary singularities

Equations semi-linéaires à exposant de Hardy–Sobolev critique avec singularités à la frontière

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Abstract

Unlike the non-singular case $s = 0$, or the case when 0 belongs to the interior of a domain Ω in $\mathbb{R}^n$ $(n \geqslant 3)$, we show that the value and the attainability of the best Hardy–Sobolev constant on a smooth domain Ω,

$$\mu_s(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2\,dx;\ u \in H_0^1(\Omega) \text{ and } \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} = 1\right\}$$

when $0 < s < 2$, $2^*(s) = \frac{2(n-s)}{n-2}$, and when 0 is on the boundary $\partial\Omega$ are closely related to the properties of the curvature of $\partial\Omega$ at 0. These conditions on the curvature are also relevant to the study of elliptic partial differential equations with singular potentials of the form:

$$-\Delta u = \frac{u^{p-1}}{|x|^s} + f(x,u) \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where f is a lower order perturbative term at infinity and $f(x,0) = 0$. We show that the positivity of the sectional curvature at 0 is relevant when dealing with Dirichlet boundary conditions, while the Neumann problems seem to require the positivity of the mean curvature at 0.

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Résumé

Contrairement au cas non-singulier $s=0$, ou au cas d'une singularité à l'intérieur d' un domaine Ω de $\mathbb{R}^n$ ($n \geqslant 3$), on montre que la valeur de la meilleure constante dans l'inégalité de Hardy–Sobolev sur un domaine régulier,

$$\mu_s(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2\,dx;\ u \in H_0^1(\Omega) \text{ et } \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} = 1\right\}$$

quand $0 < s < 2$, $2^*(s) = \frac{2(n-s)}{n-2}$, et quand 0 appartient à la frontière, est étroitement liée aux propriétés de la courbure de $\partial\Omega$ en 0. Ces mêmes conditions sur la courbure sont aussi pertinentes pour l'existence de solutions d'équations à potentiel singulier de la forme :

$$-\Delta u = \frac{u^{p-1}}{|x|^s} + f(x,u) \quad \text{in } \Omega \subset \mathbb{R}^n,$$

où f est une perturbation d'ordre inférieur à l'infini et $f(x,0)=0$. On montre que la positivité de la courbure sectionelle est suffisante pour l'existence de solutions des problèmes avec conditions de Dirichlet au bord, tandis que pour les problèmes de Neumann, c'est la positivité de la coubure moyenne qui compte.

Keywords: Best Hardy–Sobolev constant; Boundary singularity; Semilinear elliptic equations

1. Introduction

We consider the value of the best Hardy–Sobolev constant [4] on a domain Ω of $\mathbb{R}^n$,

$$\mu_s(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2\,dx;\ u \in H_0^1(\Omega) \text{ and } \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} = 1\right\} \tag{1}$$

and the corresponding ground state solutions for

$$\begin{cases} -\Delta u = \dfrac{|u|^{2^*(s)-2}u}{|x|^s} & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

when $n \geqslant 3$, $0 < s < 2$, and $2^*(s) = \frac{2(n-s)}{n-2}$. Unlike the non-singular case and assuming 0 is on the boundary of the domain Ω, we show that these problems are closely connected to the curvature of the boundary $\partial\Omega$ at 0. This is in sharp contrast with the non-singular context $s=0$, or when 0 belongs to the interior of a domain Ω in $\mathbb{R}^n$, where it is well known that $\mu_s(\Omega) = \mu_0(\mathbb{R}^n)$ for any domain Ω and that $\mu_s(\Omega)$ is never attained unless $\mathrm{cap}(\mathbb{R}^n \setminus \Omega) = 0$.

The case when $\partial\Omega$ has a cusp at 0 has already been shown by Egnell [7] to be quite different from the non-singular setting. Indeed, by considering open cones of the form $\mathcal{C} = \{x \in \mathbb{R}^n;\ x = r\theta, \theta \in D \text{ and } r > 0\}$ where D is a connected domain of the unit sphere S^{n-1} of $\mathbb{R}^n$, Egnell showed that $\mu_s(\mathcal{C})$ is actually attained for $0 < s < 2$ even when $\bar{\mathcal{C}} \neq \mathbb{R}^n$.

The case where $\partial\Omega$ is smooth at 0 turned out to be also interesting as the curvature at 0 gets to play an important role. Indeed, we shall show that the positivity of the sectional curvature at 0 is needed for problems with Dirichlet boundary conditions, while the Neumann problems require the positivity of the mean curvature at 0.

More precisely, assume that the principal curvatures $\alpha_1, \ldots, \alpha_{n-1}$ of $\partial\Omega$ at 0 are finite. The boundary $\partial\Omega$ near the origin can then be represented (up to rotating the coordinates if necessary) by:

$$x_n = h(x') = \frac{1}{2}\sum_{i=1}^{n-1} \alpha_i x_i^2 + \mathrm{o}\big(|x'|^2\big),$$

where $x' = (x_1, \ldots, x_{n-1}) \in B(0, \delta) \cap \{x_n = 0\}$ for some $\delta > 0$ where $B(0, \delta)$ is the ball in $\mathbb{R}^n$ centered at 0 with radius δ.

If we assume the principal curvatures at 0 to be negative, that is $\max_{1 \leqslant i \leqslant n-1} \alpha_i < 0$, then the sectional curvature at 0 is positive and therefore $\partial\Omega$ – viewed as an $(n-1)$-Riemannian submanifold of $\mathbb{R}^n$ – is strictly convex at 0 [9]. The latter property means that there exists a neighborhood U of 0 in $\partial\Omega$, such that the whole of U lies on one side of a hyperplane H that is tangent to $\partial\Omega$ at 0 and $U \cap H = \{0\}$. In our context, we specify the orientation of $\partial\Omega$ in such a way that the normal vectors of $\partial\Omega$ are pointing inward towards the domain Ω. The above curvature condition then amounts to a notion of strict local convexity of $\mathbb{R}^n \setminus \Omega$ at 0. Indeed, setting

$$P_{\gamma,\delta} = \left\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1\colon\ x_n > \gamma(x_1^2 + \cdots + x_{n-1}^2)\right\} \cap B(0, \delta),$$

then, with the above orientation of $\partial\Omega$, the condition that the principal curvatures are negative, yields the existence of $\delta > 0$ and $\gamma < 0$ such that $P_{\gamma,\delta} \subset \Omega$, up to a rotation. If the principal curvatures of $\partial\Omega$ are only non-positive on a neighborhood of 0, then we simply have that $P_{0,\delta} \subset \Omega$ for some $\delta > 0$. The following result will be established in Sections 2 and 3.

Theorem 1.1. *Let Ω be a C^2-smooth domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$, then $\mu_s(\Omega) \leqslant \mu_s(\mathbb{R}^n_+)$. Moreover,*

1) *If $T(\Omega) \subset \mathbb{R}^n_+$ for some rotation T (in particular, if Ω is convex, or if Ω is star-shaped around 0), then $\mu_s(\Omega) = \mu_s(\mathbb{R}^n_+)$ and it is not attained unless Ω is a half-space.*
2) *On the other hand, when $n \geqslant 4$, and if the principal curvatures of $\partial\Omega$ at 0 are negative (i.e., if $\max_{1 \leqslant i \leqslant n-1} \alpha_i < 0$), then $\mu_s(\Omega) < \mu_s(\mathbb{R}^n_+)$, the best constant $\mu_s(\Omega)$ is attained in $H_0^1(\Omega)$ and (2) has a positive solution on Ω.*

The “global convexity” assumption on Ω in 1) can be contrasted with the hypothesis on the principal curvature in 2) which, as discussed above, can be seen as a condition of *local strict concavity of the boundary at* 0 when viewed from the interior of Ω. However, we shall see that the latter is not a necessary condition for the existence of solution for Eq. (2), since we will exhibit domains Ω where $\mu_s(\Omega) < \mu_s(\mathbb{R}^n_+)$, even though $\partial\Omega$ is “flat at zero”. Such an analysis is relevant to the study of elliptic partial differential equations with singular potentials of the form

$$-\Delta u = \frac{u^{p-1}}{|x|^s} + f(x, u) \quad \text{in } \Omega \subset \mathbb{R}^n,$$

under both Dirichlet and Neumann boundary conditions. Here f is to be seen as a lower order perturbative term at infinity and $f(x, 0) = 0$. We shall see that in both Neumann and Dirichlet problems, our existence results depend on conditions on the curvature of the boundary near 0. The following two statements summarize the situation. Slightly more general results will be established later.

In the following Dirichlet problem, the same concavity condition around the origin will play a key role.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary and consider the Dirichlet problem*

$$\begin{cases} -\Delta u = \dfrac{|u|^{2^*(s)-2}u}{|x|^s} + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

for $0 < s < 2$. Assume that $0 \in \partial\Omega$ and that the principal curvatures of $\partial\Omega$ are non-positive in a neighborhood of 0. If $n \geqslant 4$ and if $0 < \lambda < \lambda_1$ (the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$), then (3) has a positive solution.

For the Neumann problem, it is the positivity of the mean curvature at 0 that is needed.

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary and consider the Neuman problem*

$$\begin{cases} -\Delta u = \dfrac{|u|^{2^*(s)-2}u}{|x|^s} + \lambda u & \text{in } \Omega, \\ D_\gamma u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

for $0 < s < 2$*. Assume that* $0 \in \partial\Omega$ *and that the mean curvature of* $\partial\Omega$ *at 0 is positive* (*i.e.,* $\sum_{i=1}^{n-1} \alpha_i > 0$). *If* $n \geqslant 3$ *and* $\lambda < 0$, *then* (4) *has one positive solution.*

Remark 1.4. As expected, the variational methods used in this paper lead to weak solutions. However, since the nonlinearities $g(x,u)$ we consider, satisfy $|g(x,u)| \leqslant C(1 + |u|^{2^*(s)-1})$ on any bounded domain Ω' such that $0 \notin \overline{\Omega'}$, regularity theory and the strong maximum principle can be applied in Ω' (cf. [12], [17, Appendix B]). Therefore, a non-negative solution $u \in H_0^1(\Omega)$ to (3) is necessarily C^∞ on Ω. It satisfies $u(x) > 0$ for every $x \in \Omega$, but may have a singularity at 0. The same remark applies to equations with subcritical perturbation terms as well as to the corresponding Neumann problem.

2. Best Hardy–Sobolev constants

The best Hardy–Sobolev constant of a domain $\Omega \subset \mathbb{R}^n$ ($n \geqslant 3$) is defined as:

$$\mu_s(\Omega) := \inf\left\{ \int_\Omega |\nabla u|^2 \, dx;\ u \in H_0^1(\Omega) \text{ and } \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} = 1 \right\} \tag{5}$$

where $0 \leqslant s < 2$, $2^*(s) = \frac{2(n-s)}{n-2}$.

In the non-singular case $s = 0$, this is nothing but the best Sobolev constant of Ω and it is well known that $\mu_0(\Omega) = \mu_0(\mathbb{R}^n)$ for any domain Ω and that $\mu_0(\Omega)$ is never attained unless $\mathrm{cap}(\mathbb{R}^n \setminus \Omega) = 0$.

Similar results hold in the singular case ($0 < s < 2$) provided 0 belongs to the interior of the domain Ω. Indeed, as noticed by several authors [11], the best constant in the Hardy–Sobolev inequality is not attained on those domain Ω containing 0 and satisfying $\mathrm{cap}(\mathbb{R}^n \setminus \Omega) \neq 0$, while it is attained on $\mathbb{R}^n$ by functions of the form

$$y_a(x) = \frac{(a \cdot (n-s)(n-2))^{\frac{n-2}{2(2-s)}}}{(a + |x|^{2-s})^{\frac{n-2}{2-s}}} \tag{6}$$

for some $a > 0$. Moreover, the functions y_a are the only positive radial solution to

$$-\Delta u = \frac{u^{2^*(s)-1}}{|x|^s} \quad \text{in } \mathbb{R}^n, \tag{7}$$

hence, by denoting $\mu_s := \mu_s(\mathbb{R}^n)$, we have:

$$\mu_s \left(\int_{\mathbb{R}^n} \frac{|y_a|^{2^*(s)}}{|x|^s} \right)^{\frac{2}{2^*(s)}} = \|\nabla y_a\|_2^2 = \int_{\mathbb{R}^n} \frac{|y_a|^{2^*(s)}}{|x|^s} = \mu_s^{\frac{n-s}{2-s}}. \tag{8}$$

In this section, we deal with the more interesting case when 0 *belongs to the boundary of the domain* Ω. We shall see that the situation is completely different as it very much depends on the smoothness and the curvature of the boundary at 0.

The case when $\partial\Omega$ is not smooth at 0 has been well analysed by Egnell [7]. Starting with the case where Ω is a half-space $\mathbb{R}^n_+$ or more generally an open cone of the form $\mathcal{C} = \{x \in \mathbb{R}^n;\ x = r\theta, \theta \in D \text{ and } r > 0\}$ where D is a connected domain on the unit sphere S^{n-1} of $\mathbb{R}^n$, Egnell [7] showed that $\mu_s(\mathcal{C})$ is actually attained for $0 < s < 2$ even when $\bar{\mathcal{C}} \neq \mathbb{R}^n$, and therefore there exists a positive solution for

$$\begin{cases} -\Delta u = \dfrac{u^{2^*(s)-1}}{|x|^s} & \text{in } \mathcal{C}, \\ u = 0 & \text{on } \partial\mathcal{C}, \\ u(x) = \mathrm{o}(|x|^{2-n}) & \text{as } |x| \to \infty \text{ in } \mathcal{C}. \end{cases} \tag{9}$$

A consequence of Egnell's result is that $\mu_s(\mathcal{C}) \neq \mu_s(\mathbb{R}^n)$ whenever $\mathbb{R}^n \backslash \mathcal{C}$ is non-negligible. For otherwise, we can find a $u \in H_0^1(\mathcal{C})$, $u \geqslant 0$ in $\mathcal{C}$, which attains $\mu_s(\mathbb{R}^n)$. Such a solution u satisfies

$$-\Delta u = \lambda \frac{|u|^{2^*(s)-2}u}{|x|^s}$$

in $\mathbb{R}^n$, where $\lambda > 0$ is a Lagrange multiplier. By the strong maximum principle $u > 0$ in $\mathbb{R}^n$, which is a contradiction. One obtains in particular that, $\mu_s(\mathbb{R}^n_+) > \mu_s(\mathbb{R}^n)$, and more generally that

$$\mu_s(\mathcal{C}_1) > \mu_s(\mathcal{C}_2), \tag{10}$$

whenever $\mathcal{C}_i$ are cones such that $\mathcal{C}_1 \subset \mathcal{C}_2$.

The main ingredient in this analysis comes from the fact that the quantities $||\nabla u||_{L^2(\mathbb{R}^n)}$ and $\int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s}\,dx$ are invariant under scaling $u(x) \mapsto r^{(n-2)/2}u(rx)$. This means that whenever $0 \in \partial\Omega$, we have $\mu_s(\Omega) = \mu_s(\lambda\Omega)$ for any $\lambda > 0$. It is also clear that μ_s is invariant under rotations. These observations combined with the fact that $\mu_s(\Omega_1) \geqslant \mu_s(\Omega_2)$ if $\Omega_1 \subseteq \Omega_2$, yield that the best constant for any finite cone (that is, the intersection of an infinite cone with a bounded connected open set) is the same as the best constant for the corresponding infinite cone.

In the sequel, we deal with the distinct and more interesting case where 0 *is a smooth point of the boundary of the domain* Ω as stated in Theorem 1.1. In contrast to Egnell's result on pointed cones, we have in particular the following examples which give a totally different picture when the "cones" are smooth at 0.

Proposition 2.1. *Assume $n \geqslant 4$ and define, for each $\gamma \in \mathbb{R}$, the open paraboloid*

$$P_\gamma = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}\colon x_n > \gamma|x'|^2\}.$$

1) *If $\gamma \geqslant 0$, then $\mu_s(P_\gamma) = \mu_s(\mathbb{R}^n_+)$.*
2) *If $\gamma < 0$, then $\mu_s(P_\gamma) = \mu_s(\mathbb{R}^n)$.*

It follows that $\mu_s(P_\gamma)$ is not attained unless $P_\gamma = \mathbb{R}^n$ or $\mathbb{R}^n_+$.

Proof. (1) If $\gamma \geqslant 0$, then $P_\gamma \subset \mathbb{R}^n_+$ and obviously $\mu_s(\Omega) \geqslant \mu_s(\mathbb{R}^n_+)$. We shall prove below that the reverse inequality $\mu_s(\Omega) \leqslant \mu_s(\mathbb{R}^n_+)$ holds whenever $\partial\Omega$ is smooth at 0.

For (2), notice that for $\lambda > 0, \lambda P_\gamma = P_{\frac{\gamma}{\lambda}}$. On the other hand, if $\gamma < 0$, then $M := \mathbb{R}^n \backslash \{x = (0, x_n);\ x_n \leqslant 0\} = \bigcup_{0<\lambda<1} \lambda P_\gamma$. Choose $u_\varepsilon \in C_0^\infty(M)$, such that $\int_M \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}\,dx = 1$, and $\int_M |\nabla u_\varepsilon|^2 \leqslant \mu_s(M) + \varepsilon$. There exists $\delta > 0$, such that for all $\lambda < \delta$, $u_\varepsilon \in C_0^\infty(P_{\frac{\gamma}{\lambda}})$, which implies that $\mu_s(P_{\frac{\gamma}{\lambda}}) \leqslant \mu_s(M) + \varepsilon$. It leads immediately to $\inf_\lambda \mu_s(P_{\frac{\gamma}{\lambda}}) \leqslant \mu_s(M)$. Since $\inf_\lambda \mu_s(P_{\frac{\gamma}{\lambda}}) = \mu_s(P_\gamma)$ by scaling invariance, we have $\mu_s(P_\gamma) = \mu_s(M)$. That $\mu_s(\mathbb{R}^n) = \mu_s(M)$ follows from the fact that $M = \mathbb{R}^n \setminus L$ where $L = \{x = (0, x_n); x_n \leqslant 0\}$ is a 1-dimensional subspace of $\mathbb{R}^n$, whose capacity is zero as soon as $n \geqslant 4$ ([14], p. 397). $\square$

Behind these examples lies a more general phenomenon summarized in Theorem 1.1 whose proof will be given in various parts throughout this section. First, we prove that $\mu_s(\Omega) \leqslant \mu_s(\mathbb{R}^n_+)$. Note that $\mu_s(\mathbb{R}^n_+) = \mu_s(B_\delta)$ for all $\delta > 0$, where

$$B_\delta = \{x = (x', x_n) \in \mathbb{R}^n_+;\ |x'|^2 + (x_n - \delta)^2 < \delta^2\}.$$

Indeed, since $B_\delta \subset \mathbb{R}^n_+$, we have that $\mu_s(\mathbb{R}^n_+) \leqslant \mu_s(B_\delta)$ for all $\delta > 0$, hence $\mu_s(\mathbb{R}^n_+) \leqslant \inf_\delta \mu_s(B_\delta)$. On the other hand, choose $u_\varepsilon \in C_0^\infty(R^n_+)$, such that $\int_{\mathbb{R}^n_+} \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}\,dx = 1$, and $\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 \leqslant \mu_s(\mathbb{R}^n_+) + \varepsilon$. There exists $m_\varepsilon \in \mathbf{N}$, such that for all $m > m_\varepsilon$, $u_\varepsilon \in C_0^\infty(B_m)$, which implies that $\mu_s(B_m) \leqslant \mu_s(\mathbb{R}^n_+) + \varepsilon$. This leads immediately to $\inf_\delta \mu_s(B_\delta) \leqslant \mu_s(\mathbb{R}^n_+)$, hence to equality. Since $\lambda B_\delta = B_{\lambda\delta}$ for all $\lambda, \delta > 0$, we get the conclusion from scaling invariance. Now by the smoothness assumption on the domain Ω, there exists – modulo a rotation – a ball $B_\varepsilon \subseteq \Omega$ centered at $(0, \varepsilon)$. This means that $\mu_s(\Omega) \leqslant \mu_s(B_\varepsilon) = \mu_s(\mathbb{R}^n_+)$. Assertion (1) of Theorem 1.1 is then obtained by monotonicity and by the rotation invariance of $\mu_s(\Omega)$.

Theorem 2.2. *If the principal curvatures of $\partial\Omega$ at 0 are negative, and if $n \geqslant 4$, then $\mu_s(\Omega) < \mu_s(\mathbb{R}^n_+)$.*

As seen in the introduction, if the principal curvatures of $\partial\Omega$ at 0 are negative, then there is $\gamma < 0$ and $\delta > 0$ such that the set

$$P_{\gamma,\delta} = \left\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}\colon\ x_n > \gamma(x_1^2 + \cdots + x_{n-1}^2)\right\} \cap B(0, \delta),$$

is included in Ω, up to a rotation. We also note that if the principal curvatures of $\partial\Omega$ are non-positive on a neighborhood of 0, then $P_{0,\delta} \subset \Omega$.

By Egnell's result [7], the problem

$$\begin{cases} -\Delta u = \dfrac{|u|^{2^*(s)-2}u}{|x|^s} & \text{in } \mathbb{R}^n_+, \\ u \in H_0^1(\mathbb{R}^n_+), & u > 0 \end{cases} \tag{11}$$

has a positive solution ϕ, which, up to a multiplier, also attains the best constant $\mu_s(\mathbb{R}^n_+)$. We may assume that $\phi \in H_0^1(\mathbb{R}^n_+)$, that $\int_{\mathbb{R}^n} \frac{|\phi|^{2^*(s)}}{|x|^s} = 1$, and $||\nabla\phi||_2^2 = \mu_s(\mathbb{R}^n_+)$. We shall also extend ϕ to all of $\mathbb{R}^n$ by letting it equal 0 on the complement of $\mathbb{R}^n_+$. For these extremal functions, there holds the following estimates (see ([7], or appendix in [13]):

$$\left|\phi(x)\right| \leqslant \frac{C}{|x|^{n-2}} \quad \text{and} \quad \left|\nabla\phi(x)\right| \leqslant \frac{C}{|x|^{n-1}}, \quad \forall x \neq 0. \tag{12}$$

To prove the theorem, it is sufficient to find a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega |\nabla u|^2\,dx \Big/ \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\right)^{\frac{2}{2^*(s)}} < \mu_s(\mathbb{R}^n_+).$$

Following Jannelli and Solimini [13], we shall "bend", cut-off and rescale ϕ, to get it into Ω while still controlling its various norms. Indeed, denote $x' = (x_1, \ldots, x_{n-1}, 0)$, while $x = x' + x_n e_n$.

For any $\sigma > 0$, the change of variables $\theta_\sigma(x) = x - \frac{\gamma}{\sigma}|x'|^2 e_n$ is measure-preserving, in other words, if J_{θ_σ} is the Jacobian matrix related to θ_σ, then $|\det(J_{\theta_\sigma})| = 1$. Define the bending $\phi^{(\sigma)}(x) = \phi(\theta_\sigma(x))$. By direct computations, we know that for sufficiently large $\sigma > 0$,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\phi^{(\sigma)}(x)|^{2^*(s)}}{|x|^s} &= \int_{\mathbb{R}^n} \frac{|\phi^{2^*(s)}(x)|}{|\theta_\sigma^{-1}(x)|^s}\,dx = \int_{\mathbb{R}^n_+} \frac{|\phi^{2^*(s)}(x)|}{|x + \frac{\gamma}{\sigma}|x'|^2 e_n|^s}\,dx \\ &= \int_{\mathbb{R}^n_+} \frac{\phi^{2^*(s)}(x)}{|x|^s}\,dx - s\frac{\gamma}{\sigma}\int_{\mathbb{R}^n_+} \frac{\phi^{2^*(s)}(x)|x'|^2 x_n}{|x|^{s+2}}\,dx + \mathrm{o}(\sigma^{-1}) \\ &= 1 - C_1\frac{\gamma}{\sigma} + \mathrm{o}(\sigma^{-1}), \end{aligned} \tag{13}$$

where $C_1 > 0$ is independent of the "curvature" γ and the scaling factor σ. Here we used a Taylor expansion and the fact that

$$\int_{\mathbb{R}^n_+} \frac{\phi^{2^*(s)}(x)|x'|^2 x_n}{|x|^{s+2}}\,dx \leqslant \int_{\mathbb{R}^n_+\cap\{0<x_n\leqslant 1\}} \frac{\phi^{2^*(s)}(x)}{|x|^s}\,dx + \int_{|x|>1} \frac{\phi^{2^*(s)}(x)}{|x|^{s-1}}\,dx < +\infty,$$

by the estimate on ϕ given in (12).

Consider now the functional

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^n_+} |\nabla v|^2\,dx - \frac{1}{2^*(s)}\int_{\mathbb{R}^n_+} \frac{|v|^{2^*(s)}}{|x|^s}\,dx.$$

By a variant of Pohozaev identity [8,13], one has

$$\frac{d}{d\varepsilon}\big[I_0\big(\phi\big(x-\varepsilon k|x'|^2 e_n\big)\big)\big]_{\varepsilon=0} = \frac{k}{2}\int_{\{x_n=0\}} |\nabla\phi|^2|x'|^2 = C_2 k,$$

where $C_2 := \frac{1}{2}\int_{\{x_n=0\}} |\nabla\phi|^2|x'|^2 > 0$, again which is independent of γ and σ.

Therefore, for sufficiently large $\sigma > 0$, we have

$$I_0(\phi^{(\sigma)}) = \frac{1}{2}\mu_s(\mathbb{R}^n_+) - \frac{1}{2^*(s)} + C_2\frac{\gamma}{\sigma} + \mathrm{o}(\sigma^{-1}). \tag{14}$$

Combining (14) with (13), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \big|\nabla\phi^{(\sigma)}\big|^2 &= 2I_0(\phi^{(\sigma)}) + \frac{2}{2^*}\int_{\mathbb{R}^n} \frac{|\phi^{(\sigma)}(x)|^{2^*(s)}}{|x|^s}\\ &= \int_{\mathbb{R}^n} |\nabla\phi|^2 + 2\left(C_2 - \frac{1}{2^*(s)}C_1\right)\frac{\gamma}{\sigma} + \mathrm{o}\left(\frac{1}{\sigma}\right)\\ &= \mu_s(\mathbb{R}^n_+) + 2\left(C_2 - \frac{1}{2^*(s)}C_1\right)\frac{\gamma}{\sigma} + \mathrm{o}\left(\frac{1}{\sigma}\right).\end{aligned}$$

Note that for $\gamma = 0$, we have $\phi^{(\sigma)} = \phi$, which means that there is no any error term in the above estimates.

Define now a cut-off function ψ_σ, such that $\psi_\sigma \equiv 1$ for $|x| \leqslant \frac{1}{2}\delta\sigma$ and $\psi_\sigma \equiv 0$ for $|x| \geqslant \delta\sigma$, ψ_σ is radially symmetric, and $|\psi'_\sigma(r)| \leqslant C\frac{1}{\sigma}$.

By direct computations, we know

$$\begin{aligned}\int_{\mathbb{R}^n} \big|\nabla(\phi^{(\sigma)}\psi_\sigma)\big|^2 &= \int_{\mathbb{R}^n} |\nabla\phi^{(\sigma)}|^2\psi_\sigma^2 + 2\int \phi^{(\sigma)}\psi_\sigma\nabla\phi^{(\sigma)}\cdot\nabla\psi_\sigma + \int |\phi^{(\sigma)}|^2|\nabla\psi_\sigma|^2\\ &= \int_{\mathbb{R}^n} |\nabla\phi^{(\sigma)}|^2 + \int_{\mathbb{R}^n} |\nabla\phi^{(\sigma)}|^2(\psi_\sigma^2 - 1) + \int |\phi^{(\sigma)}|^2|\nabla\psi_\sigma|^2 + 2\int \phi^{(\sigma)}\psi_\sigma\nabla\phi^{(\sigma)}\cdot\nabla\psi_\sigma.\end{aligned}$$

From (12), there holds

$$\int |\nabla\phi^{(\sigma)}|^2(1-\psi_\sigma^2)\,dx \leqslant \int_{|x|\geqslant\frac{1}{2}\delta\sigma} |\nabla\phi^{(\sigma)}|^2 \leqslant C\int_{\frac{1}{2}\delta\sigma}^{+\infty} \frac{r^{n-1}}{r^{2n-2}}\,dr = \mathrm{O}(\sigma^{2-n}),$$

$$\int |\nabla \psi_\sigma|^2 (\phi^{(\sigma)})^2 \leqslant C \int_{\frac{1}{2}\delta\sigma}^{\delta\sigma} \frac{r^{n-1}}{r^{2n-2}}\, dr = \mathrm{O}(\sigma^{2-n}).$$

For $\int \phi^{(\sigma)} \psi_\sigma \nabla \phi^{(\sigma)} \cdot \nabla \psi_\sigma$ we have a similar estimate. Hence for $n \geqslant 4$,

$$\int_{\mathbb{R}^n} \left|\nabla (\phi^{(\sigma)} \psi_\sigma)\right|^2 = \mu_s(\mathbb{R}^n_+) + 2\left(C_2 - \frac{1}{2^*(s)} C_1\right) \frac{\gamma}{\sigma} + \mathrm{o}(\sigma^{-1}).$$

Similarly,

$$\int \frac{|\phi^{(\sigma)} \psi_\sigma|^{2^*(s)}}{|x|^s} = \int \frac{|\phi^{(\sigma)}|^{2^*(s)}}{|x|^s} + \int \frac{|\phi^{(\sigma)}|^{2^*(s)} (1 - \psi_\sigma^{2^*(s)})}{|x|^s}.$$

From the estimate (12), since $s < 2$, we know that

$$\int \frac{|\phi^{(\sigma)}|^{2^*(s)} (1 - \psi_\sigma^{2^*(s)})}{|x|^s} \leqslant C \int_{\frac{1}{2}\delta\sigma}^{\infty} \frac{r^{n-1}}{r^{2^*(s)(n-2)+s}}\, dr = C \int_{\frac{1}{2}\delta\sigma}^{\infty} r^{s-n-1}\, dr = \mathrm{O}(\sigma^{s-n}) = \mathrm{o}(\sigma^{-1}).$$

It follows that

$$\int \frac{|\phi^{(\sigma)} \psi_\sigma|^{2^*(s)}}{|x|^s} = 1 - C_1 \frac{\gamma}{\sigma} + \mathrm{o}(\sigma^{-1}).$$

Set now

$$\phi_\sigma(x) \equiv \sigma^{n/2^*} \phi^{(\sigma)}(\sigma x) \psi_\sigma(\sigma x)$$

and note that $\operatorname{supp}(\phi_\sigma) \subset \overline{P_{\gamma,\delta}} \subset \overline{\Omega}$ for every $\sigma > 0$. Since $\|\nabla u\|_{L^2(\mathbb{R}^n)}$ and $\int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s}$ are invariant under the scaling $u(x) \mapsto r^{\frac{n-2}{2}} u(rx)$, the following estimates then hold:

$$\int_\Omega |\nabla \phi_\sigma|^2 = \mu_s(\mathbb{R}^n_+) + 2\left(C_2 - \frac{1}{2^*(s)} C_1\right) \frac{\gamma}{\sigma} + \mathrm{o}\left(\frac{1}{\sigma}\right), \tag{15}$$

$$\int_\Omega \frac{|\phi_\sigma|^{2^*(s)}}{|x|^s} = 1 - C_1 \frac{\gamma}{\sigma} + \mathrm{o}\left(\frac{1}{\sigma}\right). \tag{16}$$

Now we claim that for σ large enough,

$$\int_\Omega |\nabla \phi_\sigma|^2\, dx \Big/ \left(\int_\Omega \frac{|\phi_\sigma|^{2^*(s)}}{|x|^s}\right)^{\frac{2}{2^*(s)}} < \mu_s(\mathbb{R}^n_+).$$

From the estimates (15) and (16), the above is equivalent to:

$$\mu_s(\mathbb{R}^n_+) + 2\left(C_2 - \frac{C_1}{2^*(s)}\right) \frac{\gamma}{\sigma} + \mathrm{o}\left(\frac{1}{\sigma}\right) < \mu_s(\mathbb{R}^n_+)\left(1 - \frac{2C_1}{2^*(s)} \frac{\gamma}{\sigma}\right) + \mathrm{o}\left(\frac{1}{\sigma}\right),$$

which, in view of the negativity of γ, reduces to verifying that

$$2C_2 > \frac{2C_1}{2^*(s)} \left(1 - \mu_s(\mathbb{R}^n_+)\right).$$

It is therefore sufficient to show that $\mu_s(\mathbb{R}^n_+) > \mu_s(\mathbb{R}^n) > 1$, which is done in the following lemma.

Lemma 2.3. *For $n \geqslant 4$, we have $\mu_s(\mathbb{R}^n) > 1$.*

Proof. By the Hardy, Sobolev and Hölder inequalities, for any $u \in H^1(\mathbb{R}^n)$, $u \neq 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s} &= \int_{\mathbb{R}^n} \frac{|u|^s}{|x|^s} \cdot |u|^{2^*(s)-s} \leqslant \left(\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2}\right)^{\frac{s}{2}} \left(\int_{\mathbb{R}^n} |u|^{(2^*(s)-s)\frac{2}{2-s}}\right)^{\frac{2-s}{2}} \\
&= \left(\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2}\right)^{\frac{s}{2}} \left(\int_{\mathbb{R}^n} |u|^{2^*}\right)^{\frac{2-s}{2}} \leqslant \left[\mu_2(\mathbb{R}^n)\right]^{\frac{-s}{2}} \left(\int_{\mathbb{R}^n} |\nabla u|^2\right)^{\frac{s}{2}} \left[\mu_0(\mathbb{R}^n)\right]^{-\frac{2^*}{2}\cdot\frac{2-s}{2}} \left(\int_{\mathbb{R}^n} |\nabla u|^2\right)^{\frac{2^*}{2}\cdot\frac{2-s}{2}} \\
&= \left[\mu_2(\mathbb{R}^n)\right]^{\frac{-s}{2}} \left[\mu_0(\mathbb{R}^n)\right]^{-\frac{2^*}{2}\cdot\frac{2-s}{2}} \left(\int_{\mathbb{R}^n} |\nabla u|^2\right)^{\frac{n-s}{n-2}},
\end{aligned}$$

which implies

$$\int_{\mathbb{R}^n} |\nabla u|^2\,dx \Big/ \left(\int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s}\right)^{\frac{2}{2^*(s)}} \geqslant \left[\mu_2(\mathbb{R}^n)\right]^{\frac{s}{2^*(s)}} \cdot \left[\mu_0(\mathbb{R}^n)\right]^{\frac{n(2-s)}{2(n-s)}}.$$

By minimizing over u, we get

$$\mu_s(\mathbb{R}^n) \geqslant \left[\mu_2(\mathbb{R}^n)\right]^{\frac{s}{2^*(s)}} \cdot \left[\mu_0(\mathbb{R}^n)\right]^{\frac{n(2-s)}{2(n-s)}}.$$

Since $n \geqslant 4$, the Hardy constant $\mu_2(\mathbb{R}^n) = (\frac{n-2}{2})^2 \geqslant 1$ and the optimal Sobolev constant

$$\mu_0(\mathbb{R}^n) = \frac{1}{4}(\omega_n)^{2/n}\left[n(n-2)\right] > 1. \qquad \square$$

Exterior domains. The "strict concavity of Ω at 0" (implied by the strict negativity of the principal curvatures of $\partial\Omega$ at 0) is not necessary for the existence of the solution to (2), since there are domains Ω that are flat at 0, yet satisfying $\mu_s(\Omega) < \mu_s(\mathbb{R}^n_+)$. These examples are based on the following observations:

Proposition 2.4. *If Ω is an exterior domain with $0 \in \partial\Omega$, then $\mu_s(\Omega) = \mu_s(\mathbb{R}^n)$.*

Indeed, the hypothesis means that $\mathbb{R}^n \backslash \overline{\Omega}$ is connected and bounded. In this case, we have $\mathbb{R}^n \backslash \{0\} = \bigcup_{0<\lambda<1} \lambda\Omega$. Because $C_0^\infty(\mathbb{R}^n \backslash \{0\})$ is dense in $H^1(\mathbb{R}^n)$ for $n \geqslant 2$ (cf. [5, Lemma 2.2]), we also have $\mu_s(\mathbb{R}^n) = \mu_s(\mathbb{R}^n \backslash \{0\})$. Combining these two facts with scaling invariance, yields easily that $\mu_s(\Omega) = \mu_s(\mathbb{R}^n)$.

The above remark allows the construction of various interesting examples. Indeed, let Ω_0 be any exterior domain with $0 \in \partial\Omega$ and define $\Omega_r := \Omega_0 \cap B(0, r)$, where $B(0, r)$ is the standard Euclidean ball with radius $r > 0$, centered at 0. Obviously $\partial\Omega_r$ is smooth at 0 and $\mu_s(\Omega_{r_1}) \leqslant \mu_s(\Omega_{r_2})$ if $r_1 > r_2$. We have the following

Proposition 2.5. *There exists $r_0 \geqslant 0$ such that $r \to \mu_s(\Omega_r)$ is left-continuous and strictly decreasing on $(r_0, +\infty)$. In particular, $\mu_s(\mathbb{R}^n) < \mu_s(\Omega_r) < \mu_s(\mathbb{R}^n_+)$ for all $r \in (r_0, +\infty)$.*

Proof. Using similar arguments as above (scaling invariance and approximation of smooth functions), combined with the smoothness assumption on $\partial\Omega_0$, one can easily observe that:

$$\mu_s(\Omega_0) = \inf_{r>0} \mu_s(\Omega_r) \quad \text{and} \quad \mu_s(\mathbb{R}^n_+) = \sup_{r>0} \mu_s(\Omega_r).$$

Now we claim that for all $r > 0$, $\mu_s(\Omega_r) > \mu_s(\mathbb{R}^n)$. Indeed otherwise, by Corollary 3.2, there is some $r^* > 0$, such that $\mu_s(\Omega_{r^*}) = \mu_s(\mathbb{R}^n)$ is attained by some function $u \in H_0^1(\Omega_{r^*})$ with $u \geqslant 0$. In other words, $\mu_s(\mathbb{R}^n)$ is also attained by this function u, hence u satisfies the corresponding Euler–Lagrange equation in the whole space, while by the Strong Maximum Principle, we know $u > 0$ in $\mathbb{R}^n$, which is a contradiction.

The argument for the left-continuity of $\mu_s(\Omega_r)$ goes like this: For a fixed $r > 0$ and arbitrarily small $\varepsilon > 0$, one can always choose a function $u \in C_0^\infty(\Omega_r)$, such that $\int_\Omega |\nabla u|^2\,dx \leqslant \mu_s(\Omega_r) + \varepsilon$, and $\int_{\Omega_r} \frac{|u|^{2^*(s)}}{|x|^s}\,dx = 1$. Since $\operatorname{supp}(u)$ is compact, the distance $\operatorname{dist}(\partial B(0,r), \operatorname{supp}(u)) =: \delta > 0$. It follows that $\operatorname{supp}(u) \subset \Omega_{r'}$, where $r - \delta < r' < r$, hence $\mu_s(\Omega_{r'}) \leqslant \mu_s(\Omega_r) + \varepsilon$, for $r - \delta < r' < r$, which means that $\mu_s(\Omega_r)$ is left-continuous. This implies that there must be some $r > 0$, such that $\mu_s(\mathbb{R}^n_+) > \mu_s(\Omega_r) > \mu_s(\mathbb{R}^n)$. Now define $r_0 := \inf\{r > 0;\ \mu_s(\mathbb{R}^n) < \mu_s(\Omega_r) < \mu_s(\mathbb{R}^n_+)\}$.

It is clear that for every $r > r_0$, $\mu_s(\mathbb{R}^n) < \mu_s(\Omega_r) < \mu_s(\mathbb{R}^n_+)$. Suppose now there exist $r_2 > r_1 > r_0$, but $\mu_s(\Omega_{r_1}) = \mu_s(\Omega_{r_2})$. Using Corollary 3.2, there exists a nonnegative function $u_1 \in H_0^1(\Omega_{r_1})$, where $\mu_s(\Omega_{r_2})$ is attained. Hence u_1 satisfies the corresponding Euler–Lagrange equation in Ω_{r_2}, and again this violates the Strong Maximum Principle, hence the strict monotonicity.

Remark 2.6. In the above situation, both cases $r_0 > 0$ and $r_0 = 0$ could happen. Indeed,

(a) If $\mathbb{R}^n \setminus \overline{\Omega_0} = B(0, r^*) \cap \mathbb{R}^n_+$, then $r_0 \geqslant r^*$. Notice that in this case, we have $\mu_s(\Omega_r) < \mu_s(\mathbb{R}^n_+)$ whenever $r > r_0$, and therefore there exists a solution to (2), though $\partial\Omega$ is flat near 0.
(b) If $\mathbb{R}^n \setminus \overline{\Omega_0} = B_\delta := \{x = (x', x_n) \in \mathbb{R}^n\colon\ (x_n - \delta)^2 + |x'|^2 < \delta^2\}$, then $r_0 = 0$.

3. Blow-up analysis and attainability of best constants

In this section, we show that some aspects of the well known blow-up techniques are still valid in our context. The novelties here – when there is a singularity at $0 \in \partial\Omega$ – are the fact that the energies are not translation invariant, and that the limiting case is the half-space $\mathbb{R}^n_+$ as opposed to all of $\mathbb{R}^n$. Consider the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u^{q-1} + \dfrac{|u|^{p-2}u}{|x|^s} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $0 \in \bar{\Omega}$, $0 < s < 2 < n$, $p = 2^*(s)$ and $2 \leqslant q < 2^*(0) = 2n/(n-2)$. Here $\lambda > 0$, if $q > 2$, but we can take $\lambda \in \mathbb{R}$, if $q = 2$.

The following discussion applies to the case where $0 \in \Omega$ and also to the case where $0 \in \partial\Omega$, a boundary that is smooth near the origin. The "limiting problem" will be:

$$\begin{cases} -\Delta u = \dfrac{|u|^{p-2}u}{|x|^s} & \text{on } M, \\ u(x) \to 0 & \text{as } \|x\| \to \infty, \end{cases} \tag{18}$$

where

$$M = \begin{cases} \mathbb{R}^n, & \text{if } 0 \in \Omega, \\ \mathbb{R}^n_+, & \text{if } 0 \in \partial\Omega. \end{cases}$$

The energy functional for (17) is well defined on $H_0^1(\Omega)$ by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{q}\int |u|^q - \frac{1}{p}\int_\Omega \frac{|u|^p}{|x|^s}\,dx,$$

while (18) corresponds to the functional I_0

$$I_0(u)=\frac{1}{2}\int_M |\nabla u|^2-\frac{1}{p}\int_M \frac{|u|^p}{|x|^s}\,dx$$

defined on $D^{1,2}(M)$, which is the closure of $C_0^\infty(M)$ under the norm $\|u\|_{D^{1,2}(M)}=\int_M|\nabla u|^2$.

In view of Egnell's result, both limiting problems have a solution corresponding to a critical point of I_0. The following is a direct extension of the known case when $s=0$, established by Struwe.

Theorem 3.1. *Suppose $(u_m)_m$ is a sequence in $H_0^1(\Omega)$ that satisfies $I_\lambda(u_m)\to c$ and $I'_\lambda(u_m)\to 0$ strongly in $H^{-1}(\Omega)$ as $m\to\infty$. Then, there is an integer $k\geqslant 0$, a solution U^0 of* (17) *in $H_0^1(\Omega)$, solutions $U^1,\dots,U^k$ of* (18) *in $D^{1,2}(M)\subset D^{1,2}(\mathbb{R}^n)$, sequences of radii $r_m^1,\dots,r_m^k>0$ such that for some subsequence $m\to\infty$, $r_m^k\to 0$ and*

1) *$u_m\to U^0$ weakly in $H_0^1(\Omega)$,*
2) *$\|U_m-U^0-\sum_{j=1}^k (r_m^j)^{\frac{2-n}{2}}U^j((r_m^j)^{-1}\cdot)\|\to 0$, where $\|\cdot\|$ is the norm in $D^{1,2}(\mathbb{R}^n)$,*
3) *$\|U_m\|^2\to\sum_{j=0}^k\|U^j\|^2$,*
4) *$I_\lambda(U_m)\to I_\lambda(U^0)+\sum_{j=1}^k I_0(U^j)$.*

Recalling that a functional I is said to have the *Palais–Smale condition at level* c (P-S)$_c$, if any sequence $(u_m)_m$ in $H_0^1(\Omega)$ that satisfies $I_\lambda(u_m)\to c$ and $I'_\lambda(u_m)\to 0$ in $H^{-1}(\Omega)$ as $m\to\infty$, is necessarily relatively compact in $H_0^1(\Omega)$, we can immediately deduce from the above theorem that I_λ satisfies (P-S)$_c$ for any $c<\frac{2-s}{2(n-s)}\mu_s(M)^{\frac{n-s}{2-s}}$. This implies the following:

Corollary 3.2. *Suppose that $0\in\partial\Omega$ and that $\partial\Omega$ is smooth near the origin.*

1) *If $\mu_s(\Omega)<\mu_s(\mathbb{R}^n_+)$, then $\mu_s(\Omega)$ is attained.*
2) *If the principal curvatures of $\partial\Omega$ at 0 are negative, and if $n\geqslant 4$, then there is a positive solution to* (2).

Proof. The above theorem yields that I_0 satisfies (P-S)$_c$ for any $c<\frac{2-s}{2(n-s)}\mu_s(\mathbb{R}^n_+)^{\frac{n-s}{2-s}}$.
If $\mu_s(\Omega)<\mu_s(\mathbb{R}^n_+)$, then

$$\beta:=\inf_{p\in P}\sup_{t\in[0,1]} I_0\big(p(t)\big)=\frac{2-s}{2(n-s)}\mu_s(\Omega)^{\frac{n-s}{2-s}}<\frac{2-s}{2(n-s)}\mu_s(\mathbb{R}^n_+)^{\frac{n-s}{2-s}},$$

where

$$P=\big\{p\in C^0\big([0,1]:\ H_0^1(\Omega)\big):\ p(0)=0,\, I_0\big(p(1)\big)\leqslant 0\big\}.$$

That $\beta=\frac{2-s}{2(n-s)}\mu_s(\Omega)^{\frac{n-s}{2-s}}$ can be proved using the similar argument for $s=0$ [17, p. 178]. The mountain pass theorem yields a sequence $u_k\in H_0^1(\Omega)$ such that

$$I_0(u_k)\to\frac{2-s}{2(n-s)}\mu_s(\Omega)^{\frac{n-s}{2-s}}\quad\text{and}\quad dI_0(u_k)\to 0\quad\text{in } H^{-1}(\Omega).$$

The (P-S) condition yields that $u_k\to u$ in $H_0^1(\Omega)$, $I_0(u)=\frac{2-s}{2(n-s)}\mu_s(\Omega)^{\frac{n-s}{2-s}}$ and $dI_0(u)=0$; that is $\int_\Omega|\nabla u|^2=\int_\Omega\frac{|u|^p}{|x|^s}$, so that

$$\sup_{t>0}I_0(tu)=\sup_{t>0}\left\{\frac{t^2}{2}\int_\Omega|\nabla u|^2-\frac{t^p}{p}\int_\Omega\frac{|u|^p}{|x|^s}\right\}=I_0(u).$$

But

$$\sup_{t>0} I_0(tu) = \frac{2-s}{2(n-s)}\left[\int_\Omega |\nabla u|^2 \Big/ \left(\int_\Omega \frac{|u|^p}{|x|^s}\right)^{2/p}\right]^{\frac{n-s}{2-s}},$$

which implies that

$$\int_\Omega |\nabla u|^2 \Big/ \left(\int_\Omega \frac{|u|^p}{|x|^s}\right)^{2/p} = \mu_s(\Omega)$$

is attained at u.

For 2), it is enough to combine assertion 1) with Theorem 2.2. □

The proof of Theorem 3.1 requires several lemmas, some of which are quite standard, like the following Brezis–Lieb type lemma ([2], when $s = 0$).

Lemma 3.3. *Assume $\{u_m\} \subset H_0^1(\Omega)$ is such that $u_m \to u$ a.e. on Ω and $u_m \to u$ weakly in $H_0^1(\Omega)$. Then,*

1) $\int_\Omega \frac{|u_m|^p}{|x|^s} - \int_\Omega \frac{|u_m-u|^p}{|x|^s} \to \int_\Omega \frac{|u|^p}{|x|^s}$ *as $n \to \infty$.*
2) $\int_\Omega |\nabla u_m|^2 - \int_\Omega |\nabla u_m - \nabla u|^2 \to \int_\Omega |\nabla u|^2$ *as $n \to \infty$.*
3) *If $u_m \to u$ weakly in $D^{1,2}(\mathbb{R}^n)$, then*

$$\frac{|u_m|^{p-2}u_m}{|x|^s} - \frac{|u_m-u|^{p-2}(u_m-u)}{|x|^s} \to \frac{|u|^{p-2}u}{|x|^s}$$

in $H^{-1}(\mathbb{R}^n)$.

Proof. The first two assertions are standard. Here is a proof of 3). By the mean value theorem, we have

$$\left|\frac{|u_m|^{p-2}u_m}{|x|^s} - \frac{|u_m-u|^{p-2}(u_m-u)}{|x|^s}\right| \leqslant (p-1)\big[|u_m| + |u|\big]^{p-2}\frac{|u|}{|x|^s}.$$

For $R > 0$ and $w \in D(\mathbb{R}^n)$, we get from Hölder's inequality:

$$\begin{aligned}
&\left|\int_{|x|>R} \left(\frac{|u_m|^{p-2}}{|x|^s}u_m - \frac{|u_m-u|^{p-2}}{|x|^s}(u_m-u)\right)w\right| \\
&\quad \leqslant C \int_{|x|>R} \frac{|u_m|^{p-2} + |u|^{p-2}}{|x|^s}|u||w| \\
&\quad = C\int \frac{|u_m|^{p-2}|u||w|}{|x|^{\frac{s(p-2)}{p}+\frac{s}{p}+\frac{s}{p}}} + C\int \frac{|u|^{p-2}|u||w|}{|x|^{\frac{s(p-2)}{p}+\frac{s}{p}+\frac{s}{p}}} \\
&\quad \leqslant C\left[\left(\int_{|x|>R} \frac{|u_m|^p}{|x|^s}\right)^{\frac{p-2}{p}} + \left(\int_{|x|>R} \frac{|u|^p}{|x|^s}\right)^{\frac{p-2}{p}}\right]\cdot\left(\int \frac{|u|^p}{|x|^s}\right)^{1/p}\cdot\left(\int \frac{|w|^p}{|x|^s}\right)^{1/p} \\
&\quad \leqslant C\|w\|\left(\int \frac{|u|^p}{|x|^s}\right)^{1/p}.
\end{aligned}$$

Here we have used the Hardy–Sobolev inequality:

$$\left(\int_\Omega \frac{|w|^p}{|x|^s}\right)^{1/p} \leqslant C\|w\|.$$

We also have that

$$\left|\int_{|x|>R} \frac{|u|^{p-2}u}{|x|^s} w\right| \leqslant \int_{|x|>R} \frac{|u|^{p-1}}{|x|^{s\cdot(1-\frac{1}{p})}} \cdot \frac{w}{|x|^{\frac{s}{p}}}\, dx \leqslant \left(\int_{|x|>R} \frac{|u|^p}{|x|^s}\right)^{\frac{p-1}{p}} \cdot \left(\int \frac{|w|^p}{|x|^s}\right)^{1/p}$$
$$\leqslant C\|w\|\left(\int \frac{|u|^p}{|x|^s}\right)^{1/p}.$$

By the dominated convergence theorem, for every $\varepsilon > 0$, there exists $R > 0$ and $k > 0$ such that for all $m > k$, we have

$$\left|\int_{|x|>R} \left(\frac{|u_m|^{p-2}u}{|x|^s} - \frac{|u_m - u|^{p-2}(u_m - u)}{|x|^s} - \frac{|u|^{p-2}}{|x|^s}\right) w\right| \leqslant \varepsilon\|w\|.$$

As in [11, Lemma 4.3], we have on $B(0, R)$,

$$\int_{|x|<R} \frac{|u_m - u|^{p-2}(u_m - u)}{|x|^s} w \to 0 \quad \text{as } m \to \infty$$

and

$$\int_{|x|<R} \frac{|u_m|^{p-2}u_m w}{|x|^s} \to \int_{|x|<R} \frac{|u|^{p-2}}{|x|^s} w \quad \text{as } m \to \infty.$$

Hence

$$\int_{|x|<R} \left(\frac{|u_m|^{p-2}}{|x|^s} u_m - \frac{|u_m - u|^{p-2}}{|x|^s}(u_m - u)\right) w \underset{m\to\infty}{\longrightarrow} \int_{|x|<R} \frac{|u|^{p-2}u}{|x|^s} w,$$

which completes the proof. □

Lemma 3.4. *Consider $(u_m)_m$ in $H_0^1(\Omega)$ such that $I_\lambda(u_m) \to c$, and $dI_\lambda(u_m) \to 0$ in $H^{-1}(\Omega)$. For $(r_m) \in (0, \infty)$ with $r_m \to 0$, assume that the rescaled sequence $v_m(x) := r_m^{(n-2)/n} u_m(r_m x)$ is such that $v_m \to v$ weakly in $D^{1,2}(\mathbb{R}^n)$ and $v_m \to v$ a.e. on $\mathbb{R}^n$.*

Then, $dI_0(v) = 0$ and the sequence

$$w_m(x) := u_m(x) - r_m^{\frac{2-n}{2}} v\left(\frac{x}{r_m}\right)$$

satisfies $I_0(w_m) \to c - I_0(v)$, $dI_0(w_m) \to 0$ in $H^{-1}(\Omega)$ and $\|w_m\|^2 = \|u_m\|^2 - \|v\|^2 + \mathrm{o}(1)$.

Proof. Easy computations yield the dilation invariance:

$$\|v_m\|^2 = \int_{\mathbb{R}^n} \left|\nabla\left(r_m^{\frac{n-2}{2}} u_m(r_m x)\right)\right|^2 dx = \int_{\mathbb{R}^n} |\nabla u_m|^2\, dx = \|u_m\|^2,$$
$$\int_{\mathbb{R}^n} \frac{|v_m|^p}{|x|^s} = \int_{\mathbb{R}^n} r_m^{n-s} \frac{|u_m(r_m x)|^p}{|x|^s}\, dx = \int_{\mathbb{R}^n} \frac{|u_m|^p}{|x|^s}\, dx,$$

therefore $I_0(v_m) = I_0(u_m)$, i.e., the functional I_0 is invariant under dilation. Since $v_m \to v$ in $D^{1,2}(\mathbb{R}^n)$, it is clear that

$$\begin{aligned}\|w_m\|^2 &= \langle \nabla w_m, \nabla w_m\rangle_{L^2(\mathbb{R}^n)} = \langle \nabla v_m - \nabla v, \nabla v_m - \nabla v\rangle \\ &= \|v_m\|^2 + \|v\|^2 - 2\langle \nabla v, \nabla v_m\rangle = \|v_m\|^2 - \|v\|^2 + \mathrm{o}(1) \\ &= \|u_m\|^2 - \|u\|^2 + \mathrm{o}(1).\end{aligned}$$

Since $v_m \to v$ weakly in $D^{1,2}(\mathbb{R}^n)$, $dI_0(u_m) \to 0$ in $H^{-1}(\Omega)$, Lemma 3.3 leads to

$$I_0(w_m) = I_0(v_m) - I(v) + \mathrm{o}(1) = I_0(u_m) - I_0(v) + \mathrm{o}(1) = c - I_0(v) + \mathrm{o}(1).$$

Since $r_m \to 0$, we have $dI_0(v) = 0$, and again by Lemma 3.3, we finally obtain

$$dI_0(w_m) = dI_0(u_m) - dI_0\left(r_m^{\frac{2-n}{2}} v\left(\frac{\cdot}{r_m}\right)\right) + \mathrm{o}(1) = \mathrm{o}(1). \qquad \square$$

We also need the following:

Lemma 3.5. *If $u \in D^{1,2}(\mathbb{R}^n)$ and $v \in C_0^\infty(\mathbb{R}^n)$, then*

$$\int \frac{v^2|u|^p}{|x|^s} \leqslant \mu_s(\mathbb{R}^n)^{-1}\left(\int_{\operatorname{supp} v} \frac{|u|^p}{|x|^s}\right)^{\frac{p-2}{p}} \int \left|\nabla(vu)\right|^2.$$

Proof. By Hölder's inequality,

$$\int \frac{v^2|u|^p}{|x|^s} = \int \frac{|u|^{p-2}}{|x|^{s(1-\frac{2}{p})}} \cdot \frac{|uv|^2}{|x|^{\frac{2s}{p}}} \leqslant \left(\int_{\operatorname{supp} v} \frac{|u|^p}{|x|^s}\right)^{1-\frac{2}{p}} \cdot \left(\int \frac{|uv|^p}{|x|^s}\right)^{\frac{2}{p}}.$$

Now apply the Hardy–Sobolev inequality. $\square$

Proof of Theorem 3.1. Let $u_m \to u$ be in $H_0^1(\Omega)$ such that $I_\lambda(u_m) \to c$, and $dI_\lambda(u_m) \to 0$ in $H^{-1}(\Omega)$. That such a (P-S)-sequence is bounded, is well known and can be found in [11, Lemma 4.4]. Note that when $q = 2$, λ can be chosen to be any real number. There exists therefore a subsequence, still denoted by (u_m) such that for some $U_0 \in H_0^1(\Omega)$, $u_m \to U_0$ weakly in $H_0^1(\Omega)$ and $\nabla u_m \to \nabla U_0$ a.e. An easy consequence of Lemma 3.3 is that $dI_\lambda(U_0) = 0$. Moreover, the sequence $u_m^1 := u_m - U_0$ satisfies

$$(*) \qquad \begin{cases} \|u_m^1\|^2 = \|u_m\|^2 - \|U_0\|^2 + \mathrm{o}(1), \\ dI_0(u_m^1) \to 0 \quad \text{in } H^{-1}(\Omega), \\ I_0(u_m^1) \to c - I_\lambda(U_0). \end{cases}$$

Case (1): If $u_m^1 \to 0$ in $L^p(\Omega, |x|^{-s}\,dx)$, then $\langle dI_0(u_m^1), u_m^1\rangle = \int_\Omega |\nabla u_m^1|^2 - \int_\Omega \frac{|u_m|^p}{|x|^s} \to 0$, since $dI_0(u_m^1) \to 0$. It follows that $u_m^1 \to 0$ in $H_0^1(\Omega)$, and we are done.

Case (2): If $u_m^1 \not\to 0$ in $L^p(\Omega, |x|^{-s}\,dx)$, then from

$$\left\langle dI_0(u_m^1), u_m^1\right\rangle = \int_\Omega |\nabla u_m^1|^2 - \int_\Omega \frac{|u_m|^p}{|x|^s} = \mathrm{o}\left(\|u_m^1\|\right)$$

and

$$\int_\Omega |\nabla u_m^1|^2 \geqslant \mu_s(\mathbb{R}^n)\left(\int_\Omega \frac{|u_m^1|^p}{|x|^s}\right)^{2/p},$$

we have

$$\left(\int_{\Omega} \frac{|u_m^1|^p}{|x|^s}\right)^{1-\frac{2}{p}} > \frac{\mu_s}{2},$$

for large n and we may therefore assume that

$$\int_{\mathbb{R}^n} \frac{|u_m^1|^p}{|x|^s} > \delta \quad \text{for some } 0 < \delta < \left(\frac{\mu_s}{2}\right)^{\frac{n-s}{2-s}}.$$

Define an analogue of Levy's concentration function,

$$Q_m(r) = \int_{B(0,r)} \frac{|u_m^1|^p}{|x|^s}.$$

Since $Q_m(0) = 0$ and $Q(\infty) > \delta$, there exists a sequence $r_m^1 > 0$ such that for each n,

$$\delta = \int_{B(0,r_m^1)} \frac{|u_m^1|^p}{|x|^s}.$$

Define $v_m^1(x) := (r_m^1)^{(n-2)/n} u_m^1(r_m^1 x)$. Since $\|v_m^1\| = \|u_m^1\|$ is bounded, we may assume $v_m^1 \to U_1$ in $D^{1,2}(\mathbb{R}^n)$ weakly, $v_m^1 \to U_1$ a.e. on $\mathbb{R}^n$ and $\delta = \int_{B(0,1)} \frac{|v_m^1|^p}{|x|^s}\,dx$. We now show that $U_1 \not\equiv 0$.

Define $\Omega_m = \frac{1}{r_m^1}\Omega$, and let $f_m \in H_0^1(\Omega)$ be such that for any $h \in H_0^1(\Omega)$, we have $\langle dI_0(u_m^1), h\rangle = \int_\Omega \nabla f_m \cdot \nabla h$. Then $g_m(x) := (r_m^1)^{(n-2)/2} f_m(r_m x)$ satisfies $\int_{\Omega_m} |\nabla g_m|^2 = \int_\Omega |\nabla f_m|^2$ and $\langle dI_0(v_m^1), h\rangle = \int_{\Omega_m} \nabla g_m \cdot \nabla h$ for any $h \in H_0^1(\Omega_m)$.

If $U_1 \equiv 0$, then $v_m^1 \to 0$ in $L^p_{1\text{oc}}\big(B(0,1), |x|^{-s}\,dx\big)$. Choosing $h \in C_0^\infty(\mathbb{R}^n)$ such that supp $h \subset B(0,1)$, we get from Lemma 3.5,

$$\begin{aligned}
\int_{B(0,1)} \left|\nabla(hv_m^1)\right|^2 &= \int_{B(0,1)} \nabla v_m^1 \cdot \nabla(h^2 v_m^1) + \mathrm{o}(1) = \int \frac{h^2|v_m^1|^p}{|x|^s} + \int \nabla g_m \cdot \nabla(h^2 v_m^1) + \mathrm{o}(1) \\
&\leqslant \mu_s(\mathbb{R}^n)^{-1} \cdot \left(\int_{B(0,1)} \frac{|u_m^1|^p}{|x|^s}\right)^{1-\frac{p}{2}} \int \left|\nabla(hv_m^1)\right|^2 + \mathrm{o}(1) \\
&= \mu_s(\mathbb{R}^n)^{-1} \delta^{\frac{2-s}{n-s}} \int \left|\nabla(hv_m^1)\right|^2 + \mathrm{o}(1) \leqslant \frac{1}{2} \int \left|\nabla(hv_m^1)\right|^2 + \mathrm{o}(1).
\end{aligned}$$

Hence $\nabla v_m^1 \to 0$ in $L^2_{1\text{oc}}(B(0,1))$ and $v_m^1 \to 0$ in $L^p(B(0,1), |x|^{-s}\,dx)$, which contradicts the fact that $\int_{B(0,1)} \frac{|v_m^1|^p}{|x|^s} = \delta > 0$. Thus we have proved that $U_1 \not\equiv 0$.

Since Ω is bounded, we can assume that $r_m^1 \to r_\infty^1 \geqslant 0$. If $r_\infty^1 > 0$, the fact that $u_m^1 \to 0$ weakly in $H_0^1(\Omega)$ will imply that $v_m^1(x) := (r_m^1)^{(n-2)/2} u_m^1(r_m x) \to 0$ weakly in $D^{1,2}(\mathbb{R}^n)$, which contradicts that $U_1 \not\equiv 0$, and therefore $r_m^1 \to 0$.

By $(*)$ and Lemma 3.4, $dI_0(U_1) = 0$, and U_1 is a weak solution of

$$\begin{cases} -\Delta u = \dfrac{|u|^{p-2}u}{|x|^s} & \text{in } M, \\ u \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $M=\mathbb{R}^n$ if $0\in\Omega$ and where $M=\mathbb{R}^n_+$ if $0\in\partial\Omega$. Indeed, to show the latter case, we can assume without loss of generality that $\partial\mathbb{R}^n_+=\{x_n=0\}$ is tangent to $\partial\Omega$ at 0, and that $-e_n=(0,\dots,-1)$ is the outward normal to $\partial\Omega$ at that point. For any compact $K\subset\mathbb{R}^n_-$, we have for m large enough, that $\frac{\Omega}{r^1_m}\cap K=\emptyset$, as $r^1_m\to 0$. Since $\operatorname{supp} v^1_m\subset\frac{\Omega}{r^1_m}$ and $v^1_m\to U_1$ a.e. in $\mathbb{R}^n$, it follows that $U_1=0$ a.e. on K, and therefore $\operatorname{supp} U_1\subset\mathbb{R}^n_+$.

The sequence $u^2_m(x):=u^1_m(x)-(r^1_m)^{(2-n)/2}U_1(x/r^1_m)$ also satisfies

$$\|u^2_m\|^2=\|u_m\|^2-\|U_0\|^2-\|U_1\|^2+\mathrm{o}(1),$$
$$I_0(u^2_m)\to c-I_\lambda(U_0)-I_0(U_1),$$
$$dI_0(u^2_m)\to 0\quad\text{in } H^{-1}(\Omega).$$

Moreover, any nontrivial critical point u of I_0 on $H^1_0(M)$ satisfies

$$\mu_s(M)\left(\int_M\frac{|u|^p}{|x|^s}\right)^{\frac{2}{p}}\leqslant\int_M|\nabla u|^2=\int_M\frac{|u|^p}{|x|^s},$$

so that

$$I_0(u)=\left(\frac12-\frac1p\right)\int_M\frac{|u|^p}{|x|^s}\geqslant c^*:=\frac{2-s}{2(n-s)}\mu_s(M)^{\frac{n-2}{2-s}}.$$

By iterating the above procedure, we construct similarly sequences (U_j), (r^j_m) with the above properties. Since for every $j\geqslant 1$, $I_0(U_j)\geqslant c^*$, the iteration must necessarily terminate after a finite number of steps. □

Remark 3.6. This type of blow-up result also holds for domains Ω with a conic singularity at 0. More precisely, consider an infinite open cone of the form $\mathcal{C}=\{x\in\mathbb{R}^n;\ x=r\theta, \theta\in D \text{ and } r>0\}$ where D is a connected domain of the unit sphere S^{n-1} of $\mathbb{R}^n$, and assume the domain Ω satisfies $B_r\cap\Omega=\mathcal{C}\cap(\Omega\cap B_r)$ for every ball B_r (centered at 0) with radius $r<r_0$, where r_0 is some positive number (i.e., Ω has a conic singularity at 0), then Theorem 3.1 remains true, with M – in this case – being the corresponding infinite cone $\mathcal{C}$.

Behind our analysis, is the fact that (P-S)-sequences either converge or concentrate at 0. This is due to the fact that the embedding $H^1_0(\Omega')\hookrightarrow L^{2^*(s)}(\Omega',|x|^{-s}dx)$ is compact whenever $0\notin\overline{\Omega'}$, which means there are no bubbles away from the origin. The following corollary can also be obtained by combining Corollary 3.2 with Egnell's analysis, which imply that $\lim_{r\to 0^+}\mu_s(\Omega\cap B_r)=\mu_s(\mathbb{R}^n_+)$.

Corollary 3.7. *Suppose that* $0\in\partial\Omega$ *and that* $\partial\Omega$ *is* C^2 *at* 0. *If* $\mu_s(\Omega)$ *is not attained, then there exists* $r_0>0$ *such that* $\Omega\cap B_{r_0}\neq\emptyset$ *and* $\mu_s(\Omega)=\mu_s(\Omega\cap B_r)$ *for every* $r\in(0,r_0)$.

Note that Theorem 1.1 implies that $\mu_s(\Omega)$ is not attained whenever Ω is star-shaped around 0, and therefore there is no ground-state solution for (2). The following standard Pohozaev-type identity, gives a stronger result:

Proposition 3.8. *If the domain* Ω *is star-shaped around* 0, *then problem*

$$\begin{cases}-\Delta u=\dfrac{|u|^{2^*(s)-2}u}{|x|^s} & \text{in } \Omega,\\ u\in H^1_0(\Omega)\end{cases}\tag{19}$$

has no non-trivial solution.

Proof. The assumption Ω is star-shaped around 0 simply means that $x \cdot \gamma > 0$ on $\partial\Omega \setminus \{0\}$, where γ is the outward unit normal to $\partial\Omega$. Multiply Eq. (19) by $x \cdot \nabla u$ on both sides and integrate by parts, we obtain

$$\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 x \cdot \gamma \, d\sigma + \frac{n-2}{2}\int_{\Omega} |\nabla u|^2 \, dx = \frac{n-s}{2^*(s)}\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx.$$

On the other hand, multiplying the equation by u and integrating, we have

$$\int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx.$$

Combining these two identities, one gets $\int_{\partial\Omega} |\nabla u|^2 x \cdot \gamma \, d\sigma = 0$, which concludes the proposition. □

Remark 3.9. Unlike the case $s = 0$, we can have solutions to (2) for star-shaped domains. Indeed, consider a bean-shaped domain with vertex at 0. Since the principal curvatures are strictly negative at 0, there exists a solution to (2). Note that this is not contradictory to Proposition 3.8, since the domain is not star-shaped at 0, though it is star-shaped at some other point.

4. Least energy solution to the perturbed Dirichlet problems

Throughout this section, we assume that Ω is a bounded domain in $\mathbb{R}^n$ and that $0 \in \partial\Omega$, $\partial\Omega$ is Lipschitz continuous, $\partial\Omega$ is C^2 at the origin. Consider the functional

$$I_q(v) = \int_{\Omega} \left[\frac{1}{2}|\nabla v|^2 - \frac{1}{2^*(s)}\frac{v_+^{2^*(s)}}{|x|^s} - \frac{\lambda}{q}|v|^q\right] dx$$

on $H_0^1(\Omega)$, where $2 \leqslant q < 2^* := \frac{2n}{n-2}$.

We shall deal first with the case of linear perturbations (see [3] when $s = 0$).

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary and consider the Dirichlet problem*

$$\begin{cases} -\Delta u = \dfrac{|u|^{2^*(s)-2}u}{|x|^s} + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{20}$$

for $0 < s < 2$ and $n \geqslant 4$. Assume that $0 \in \partial\Omega$ and that $\partial\Omega$ is C^2-smooth at 0. If $\partial\Omega$ has non-positive principal curvatures on a neighborhood of 0 (in particular, if $\partial\Omega$ has negative principal curvatures at 0), then for any $0 < \lambda < \lambda_1$, (20) has a positive solution.

Proof. The results of the last section give that I_q satisfies the Palais–Smale condition $(\text{P-S})_c$ for any

$$c < \frac{2-s}{2(n-s)}\mu_s(\mathbb{R}_+^n)^{\frac{n-s}{2-s}}.$$

So, we need to find a critical level below that threshold, for the functional

$$I(v) = \int_{\Omega} \left[\frac{1}{2}|\nabla v|^2 - \frac{\lambda}{2}|v|^2 - \frac{1}{2^*(s)}\frac{v_+^{2^*(s)}}{|x|^s}\right] dx$$

on the space $H_0^1(\Omega)$.

To use a mountain-pass argument, note that since $\lambda < \lambda_1$, then 0 is clearly a strict local minimum for I. The condition on the curvature at 0 implies that – modulo a rotation – there is some $P_{\gamma,\delta} \subset \Omega$, where $\gamma \leqslant 0$ and $\delta > 0$. Since $\mu_s(\Omega) \leqslant \mu_s(\mathbb{R}^n_+)$, we only need to consider two cases:

Case 1. $\mu_s(\Omega) < \mu_s(\mathbb{R}^n_+)$.

By Corollary 3.2, there exists then a function $w \in H^1_0(\Omega)$, such that $\int_\Omega |\nabla w|^2 = \mu_s(\Omega)$ and $\int_\Omega \frac{|w|^{2^*(s)}}{|x|^s}\,dx = 1$. Without loss of generality we can assume that w is nonnegative by replacing w with $|w|$. Since λ is positive, we have the following inequality:

$$\sup_{t>0} I(tw) < \sup_{t>0} J(tw), \quad \text{where } J(v) = \int_\Omega \left\{ \frac{1}{2}|\nabla v|^2 - \frac{1}{2^*(s)} \frac{v_+^{2^*(s)}}{|x|^s} \right\} dx.$$

Since

$$\sup_{t>0} J(tw) = \frac{2-s}{2(n-s)} \mu_s(\Omega)^{\frac{n-s}{2-s}},$$

the conclusion follows.

Case 2. $\mu_s(\Omega) = \mu_s(\mathbb{R}^n_+)$.

This means that $\gamma = 0$ in view of Theorem 2.2. In this case, we will closely follow the strategy used in Theorem 2.2 where we start from an extremal function $\phi \in H^1_0(\mathbb{R}^n_+)$, and through cutting and scaling, we get a test function ϕ_σ on Ω, whose various norms are controllable perturbations of those of ϕ. Note that bending is not required here, therefore we only need to pay the cost of the scaling and of the cut-off.

As mentioned in Theorem 2.2, the decays estimates on ϕ and ψ are: $|\phi(x)| \sim C/|x|^{n-2}$, $|\nabla\phi(x)| \sim C/|x|^{n-1}$ and $|\nabla\psi_\sigma(x)| \sim C/\sigma$. Since no bending is required, direct computations similar to those in Theorem 2.2, show that

$$\int_{|x| \geqslant \frac{1}{2}\delta\sigma} |\nabla\phi^\sigma|^2 |\psi_\sigma|^2 \sim \frac{C}{\sigma^{n-2}},$$

$$\int_{|x| \geqslant \frac{1}{2}\delta\sigma} |\phi^\sigma|^2 |\nabla\psi_\sigma|^2 \sim \frac{C}{\sigma^{n-2}},$$

$$\int_{|x| \geqslant \frac{1}{2}\delta\sigma} \frac{|\phi^\sigma|^{2^*(s)}}{|x|^s} \sim \frac{C}{\sigma^{n-s}};$$

here and below C represents various positive constants, which are independent of σ. We therefore have the following estimates:

$$\int_\Omega |\nabla\phi_\sigma|^2 = \mu_s(\mathbb{R}^n_+) + \mathrm{O}\left(\frac{1}{\sigma^{n-2}}\right),$$

$$\int_\Omega |\phi_\sigma|^q = C\sigma^{\frac{q(n-2)}{2}-n} + \mathrm{o}(\sigma^{\frac{q(n-2)}{2}-n}),$$

$$\int_\Omega \frac{|\phi_\sigma|^{2^*(s)}}{|x|^s} = 1 + \mathrm{O}\left(\frac{1}{\sigma^{n-s}}\right).$$

For $2 \leqslant q < 2^*$, we obtain

$$\int_\Omega |\phi_\sigma|^q = \int_{\mathbb{R}^n} \left|\sigma^{\frac{n}{2^*}}\phi^{(\sigma)}(\sigma x)\psi_\sigma(\sigma x)\right|^q dx$$
$$= \sigma^{\frac{q(n-2)}{2}-n} \int_{\mathbb{R}^n} \left|\phi^{(\sigma)}(x)\psi_\sigma(x)\right|^q dx$$
$$= C\sigma^{\frac{q(n-2)}{2}-n} + \mathrm{o}(\sigma^{\frac{q(n-2)}{2}-n}).$$

Notice that when $q = 2$, the order of σ is -2 and the above estimates, combined with the assumption $\mu_s(\Omega) = \mu_s(\mathbb{R}^n_+)$ give, for $n \geqslant 5$,

$$I(t\phi_\sigma) = \frac{t^2}{2}\int_\Omega |\nabla\phi_\sigma|^2 - \frac{\lambda t^2}{2}\int_\Omega |\phi_\sigma|^2 - \frac{t^{2^*(s)}}{2^*(s)}\int_\Omega \frac{|\phi_\sigma|^{2^*(s)}}{|x|^s}$$
$$= \frac{t^2}{2}\left(\mu_s(\Omega) + \mathrm{O}\left(\frac{1}{\sigma^{n-2}}\right)\right) - t^{2^*(s)}\left(\frac{1}{2^*(s)} + \mathrm{O}\left(\frac{1}{\sigma^{n-s}}\right)\right) - \lambda Ct^2\frac{1}{\sigma^2} + \mathrm{o}\left(\frac{1}{\sigma^2}\right).$$

Since $\lambda > 0$, then for σ large, the minimum is attained in a uniformly bounded interval, and it is easy to see that $\sup_{t>0} I(t\phi_\sigma)$ achieves its maximum at t_M, where

$$t_M = \mu_s(\Omega)^{\frac{1}{2^*(s)-2}} - C\sigma^{-2} + \mathrm{o}(\sigma^{-2}).$$

Substituting the value into the expression of $I(t\phi_\sigma)$ and noticing that t_M is bounded when $\sigma \to \infty$, it eventually leads to

$$\sup_{t>0} I(t\phi_\sigma) = \frac{2-s}{2(n-s)}\mu_s(\Omega)^{\frac{n-s}{2-s}} - C\sigma^{-2} + \mathrm{o}(\sigma^{-2}),$$

where $C > 0$ is independent of σ. From the above identity we can see that for sufficiently large σ,

$$\sup_{t>0} I(t\phi_\sigma) < \frac{2-s}{2(n-s)}\mu_s(\Omega)^{\frac{n-s}{2-s}},$$

and we are done.

The case $n = 4$ could be treated similarly, with the help of the stronger estimate

$$\int_\Omega |\phi_\sigma|^2 \sim C\sigma^{-2}\int_{\frac{1}{2}\delta\sigma}^{\delta\sigma} \frac{dr}{r} \sim \frac{\log\sigma}{\sigma^2}.$$

Adopting the similar strategy as in the case $s = 0$ [17], one can argue that the mountain-pass solution must be of one sign, say, nonnegative. Then the maximum principle concludes its positivity. □

Now we deal with the Dirichlet problem with a non-linear perturbative term.

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Assume also that $0 \in \partial\Omega$ and that $\partial\Omega$ is C^2-smooth at 0. If $n \geqslant 4$, then equation*

$$\begin{cases} -\Delta u = \dfrac{|u|^{2^*(s)-2}u}{|x|^s} + \lambda u|u|^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{21}$$

with $\lambda > 0$ has one positive solution under one of the following conditions:

1) $\frac{2+2^*}{2} < q < 2^*$, *where* $2^* = 2^*(0) = \frac{2n}{n-2}$,

2) $2 < q < 2^*$ *and* $\partial\Omega$ *has non-positive principal curvatures in a neighborhood of* 0.

Proof. The idea again is to try to find a critical point for the functional

$$I_q(v) = \int_\Omega \left[\frac{1}{2}|\nabla v|^2 - \frac{1}{2^*(s)}\frac{v_+^{2^*(s)}}{|x|^s} - \frac{\lambda}{q}|v|^q\right] dx$$

in $H_0^1(\Omega)$ through a mountain-pass argument, by using that I_q satisfies (P-S)$_c$ for any $c < \frac{2-s}{2(n-s)}\mu_s(\Omega)^{\frac{n-s}{2-s}}$. As above, we need to deal with two cases.

Case 1. $\mu_s(\Omega) < \mu_s(\mathbb{R}_+^n)$.

As before, there exists by Corollary 3.2, a positive function $w \in H_0^1(\Omega)$, such that $\int_\Omega |\nabla w|^2 = \mu_s(\Omega)$ and $\int_\Omega \frac{|w|^{2^*(s)}}{|x|^s}\,dx = 1$. Since λ is positive, we have:

$$\sup_{t>0} I_q(tw) < \sup_{t>0} J(tw), \quad \text{where } J(v) = \int_\Omega \left\{\frac{1}{2}|\nabla v|^2 - \frac{1}{2^*(s)}\frac{v_+^{2^*(s)}}{|x|^s}\right\} dx,$$

while $\sup_{t>0} J(tw) = \frac{2-s}{2(n-s)}\mu_s(\Omega)^{\frac{n-s}{2-s}}$.

Case 2. $\mu_s(\Omega) = \mu_s(\mathbb{R}_+^n)$.

Again, as in Theorems 2.2 and 4.1, from an extremal function $\phi \in H_0^1(\mathbb{R}_+^n)$, one gets through bending, cutting-off and scaling, a function ϕ_σ on Ω, with the following estimates:

$$\int_\Omega |\nabla\phi_\sigma|^2 = \mu_s(\mathbb{R}_+^n) + \mathrm{O}\left(\frac{\gamma}{\sigma}\right), \tag{22}$$

$$\int_\Omega |\phi_\sigma|^q = C\sigma^{\frac{q(n-2)}{2}-n} + \mathrm{o}(\sigma^{\frac{q(n-2)}{2}-n}), \quad \text{for } 2 \leqslant q < 2^*, \tag{23}$$

$$\int_\Omega \frac{|\phi_\sigma|^{2^*(s)}}{|x|^s} = 1 + \mathrm{O}\left(\frac{\gamma}{\sigma}\right). \tag{24}$$

Now we estimate the mountain-pass value. By (22)–(24), and the assumption $\mu_s(\Omega) = \mu_s(\mathbb{R}_+^n)$, we obtain

$$\begin{aligned} I(t\phi_\sigma) &= \frac{t^2}{2}\int_\Omega |\nabla\phi_\sigma|^2 - \frac{\lambda t^q}{q}\int_\Omega |\phi_\sigma|^q - \frac{t^{2^*(s)}}{2^*(s)}\int_\Omega \frac{|\phi_\sigma|^{2^*(s)}}{|x|^s} \\ &= \frac{t^2}{2}\left(\mu_s(\Omega) + \mathrm{O}\left(\frac{\gamma}{\sigma}\right)\right) - \frac{t^{2^*(s)}}{2^*(s)}\left(1 + \mathrm{O}\left(\frac{\gamma}{\sigma}\right)\right) - C\lambda t^q\sigma^{\frac{q(n-2)}{2}-n} + \mathrm{o}(\sigma^{\frac{q(n-2)}{2}-n}). \end{aligned}$$

In part 1) since $-1 < \frac{q(n-2)}{2} - n < 0$, $\frac{\gamma}{\sigma} = \mathrm{o}(\sigma^{\frac{q(n-2)}{2}-n})$, and $\sup_{t>0} I(t\phi_\sigma)$ achieves its maximum at t_M on a uniformly bounded interval when σ large, where

$$t_M = \mu_s(\Omega)^{\frac{1}{2^*(s)-2}} - C\sigma^{\frac{q(n-2)}{2}-n} + \mathrm{o}(\sigma^{\frac{q(n-2)}{2}-n}).$$

Substituting the value into the expression of $I(t\phi_\sigma)$ and noticing that t_M is bounded when $\sigma \to \infty$, this eventually leads to

$$\sup_{t>0} I(t\phi_\sigma) = \frac{2-s}{2(n-s)}\mu_s(\Omega)^{\frac{n-s}{2-s}} + \mathrm{O}(\gamma\sigma^{-1}) - C\lambda\sigma^{\frac{q(n-2)}{2}-n} + \mathrm{o}(\sigma^{\frac{q(n-2)}{2}-n}).$$

Hence for σ is sufficiently large, without any restriction on γ, the range of q in 1) guarantees that

$$\sup_{t>0} I(t\phi_\sigma) < \frac{2-s}{2(n-s)}\mu_s(\Omega)^{\frac{n-s}{2-s}}.$$

In part 2) now we only need to deal with $\gamma = 0$ (since $\gamma < 0$ belongs to case 1, which has been discussed). As in the proof of Theorem 4.1, no more bending is required, therefore we only need to pay the cost of the cut-off and scaling, hence we have

$$\int_\Omega |\nabla\phi_\sigma|^2 = \mu_s(\Omega) + \mathrm{O}\left(\frac{1}{\sigma^{n-2}}\right),$$

$$\int_\Omega |\phi_\sigma|^q = C\sigma^{\frac{q(n-2)}{2}-n} + \mathrm{o}(\sigma^{\frac{q(n-2)}{2}-n}),$$

$$\int_\Omega \frac{|\phi_\sigma|^{2^*(s)}}{|x|^s} = 1 + \mathrm{O}\left(\frac{1}{\sigma^{n-s}}\right).$$

We require $\frac{q(n-2)}{2} - n > -n + 2$, hence the conditions $q > 2$ and $n \geqslant 4$ are sufficient. $\square$

5. The Neumann problem

When $\partial\Omega \in C^2$, it is easy to see that the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega, |x|^{-s}\,dx)$ is continuous, where p is the Hardy–Sobolev exponent. Just as in the non-singular case [1,15], problem (4) has a variational structure. It is easy to check that the positive solution of (4) corresponds to the nonzero critical points of the functional

$$J(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - \frac{1}{2^*(s)}\frac{u_+^{2^*(s)}}{|x|^s} - \frac{1}{2}\lambda u^2\right] dx$$

defined on $H^1(\Omega)$ and the norm $\|u\|_{H^1(\Omega)} := \|\nabla u\|_{L^2} + \|u\|_{L^2}$ is equivalent to

$$|u|_H = \left(\int_\Omega \left(|\nabla u|^2 + \lambda u^2\right) dx\right)^{\frac{1}{2}}.$$

The relative compactness of Palais–Smale sequences can easily be adapted from [19] where the case $s = 0$ is considered. One then obtain the following:

Lemma 5.1. *Let (u_j) be a sequence in $H^1(\Omega)$ such that $J(u_j) \to c$ and $J'(u_j) \to 0$ in $H^{-1}(\Omega)$ as $j \to \infty$. If the level*

$$c < \frac{2-s}{4(n-s)}\mu_s(\mathbb{R}^n)^{\frac{n-2}{2-s}}, \tag{25}$$

then there is a non-zero $u \in H^1(\Omega)$ such that $J(u) \leqslant c$ and $J'(u) = 0$.

The rest of the proof of Theorem 1.3 consists of finding a least energy solution to (4) below that threshold. Since the boundary $\partial\Omega$ is C^2, and the mean curvature of $\partial\Omega$ at 0 is positive, the boundary near the origin can be represented (up to rotating the coordinates if necessary) by:

$$x_n = h(x') = \frac{1}{2}\sum_{i=1}^{n-1}\alpha_i x_i^2 + \mathrm{o}\left(|x'|^2\right),$$

where $x' = (x_1, \dots, x_{n-1}) \in D(0,\delta)$ for some $\delta > 0$ where $D(0,\delta) = B(0,\delta) \cap \{x_n = 0\}$. Here $\alpha_1, \dots, \alpha_{n-1}$ are the principal curvatures of $\partial\Omega$ at 0 and the mean curvature $\sum_{i=1}^{n-1} \alpha_i > 0$. Set

$$u_\varepsilon(x) = \varepsilon^{\frac{n-2}{2(n-s)}} \left(\varepsilon + |x|^{2-s}\right)^{\frac{2-n}{2-s}}.$$

Theorem 5.2. *Under the above assumptions, problem* (4) *possesses a positive solution, provided* $n \geqslant 3$.

Proof. For notational convenience, we denote $2^*(s)$ by p throughout the proof. The solutions of (4) corresponds to the nonzero critical points of the functional

$$J(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - \frac{1}{p}\frac{u_+^p}{|x|^s} - \frac{1}{2}\lambda u^2\right] dx.$$

Set

$$c = \inf_{\psi \in \Psi} \sup_{t \in (0,1)} J\big(\psi(t)\big),$$

the mountain-pass level, where $\Psi = \{\psi \in C([0,1], H^1(\Omega));\ \psi(0) = 0,\ J(\psi(1)) \leqslant 0\}$. We also set

$$c^* = \inf_{u \in H(\Omega)} \Big\{\sup_{t>0} J(tu);\ u \geqslant 0,\ u \not\equiv 0\Big\}.$$

It is easy to see that $c \leqslant c*$. In view of Lemma 5.1, we need to prove

$$c^* < \frac{2-s}{4(n-s)} \mu_s^{\frac{n-s}{2-s}}.$$

We claim that

$$Y_\varepsilon = \sup_{t>0} J(tu_\varepsilon) < \frac{2-s}{4(n-s)} \mu_s^{\frac{n-s}{2-s}} \tag{26}$$

for $\varepsilon > 0$ sufficiently small. Denote

$$K_1(\varepsilon) = \int_\Omega |\nabla u_\varepsilon|^2, \quad K_2(\varepsilon) = \int_\Omega \frac{|u_\varepsilon|^p}{|x|^s}\, dx \quad \text{and} \quad g(x') = \frac{1}{2}\sum_{i=1}^{n-1} \alpha_i x_i^2.$$

The proof is divided into two cases.

Case 1. $n \geqslant 4$. One then has

$$\begin{aligned} K_1(\varepsilon) &= \int_{R_+^n} |\nabla u_\varepsilon|^2\, dx - \int_{D(0,\delta)} dx' \int_0^{h(x')} |\nabla u_\varepsilon|^2\, dx_n + \mathrm{O}(\varepsilon^{\frac{n-2}{2-s}}) \\ &= \frac{1}{2}K_1 - \int_{R^{n-1}} dx' \int_0^{g(x')} |\nabla u_\varepsilon|^2\, dx_n - \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |\nabla u_\varepsilon|^2\, dx_n + \mathrm{O}(\varepsilon^{\frac{n-2}{2-s}}), \end{aligned}$$

where

$$K_1 = \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\, dx = (n-2)^2 \int_{\mathbb{R}^n} \frac{|y|^{(2-2s)}}{(1+|y|^{2-s})^{\frac{2(n-s)}{2-s}}}\, dy, \tag{27}$$

which is independent of ε. Observing that

$$\begin{aligned}
\mathrm{I}(\varepsilon) &:= \int_{\mathbb{R}^{n-1}} dx' \int_0^{g(x')} |\nabla u_\varepsilon|^2\, dx_n \\
&= (n-2)^2 \varepsilon^{\frac{n-2}{2-s}} \int_{R^{n-1}} dx' \int_0^{g(x')} \frac{|x|^{2-2s} dx_n}{(\varepsilon + |x|^{2-s})^{\frac{2(n-s)}{2-s}}} \\
&= (n-2)^2 \int_{\mathbb{R}^{n-1}} dy' \int_0^{g(y')\varepsilon^{\frac{1}{2-s}}} \frac{|y|^{2-2s} dy_n}{(1+|y|^{2-s})^{\frac{2(n-s)}{2-s}}}.
\end{aligned} \tag{28}$$

We note that

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{1}{2-s}} \mathrm{I}(\varepsilon) = (n-2)^2 \int_{\mathbb{R}^{n-1}} \frac{|x'|^{2-2s} g(x')}{(1+|x'|^{2-s})^{\frac{2(n-s)}{2-s}}}\, dx' = \left(\sum_{i=1}^{n-1} \alpha_i\right) A,$$

where

$$A := \frac{(n-2)^2}{2} \int_{\mathbb{R}^{n-1}} \frac{|x'|^{2-2s} |x_j|^2}{(1+|x'|^{2-s})^{\frac{2(n-s)}{2-s}}}\, dx' = \frac{(n-2)^2}{2(n-1)} \int_{\mathbb{R}^{n-1}} \frac{|x'|^{4-2s}}{(1+|x'|^{2-s})^{\frac{2(n-s)}{2-s}}}\, dx' > 0.$$

In view of the curvature assumption, this implies

$$\mathrm{I}(\varepsilon) > 0 \quad \text{and} \quad \mathrm{I}(\varepsilon) = \mathrm{O}(\varepsilon^{\frac{1}{2-s}}).$$

Moreover,

$$\begin{aligned}
\mathrm{I}_1(\varepsilon) &:= \left| \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |\nabla u_\varepsilon|^2\, dx_n \right| \\
&= \left| (n-2)^2 \varepsilon^{\frac{n-2}{2-s}} \int_{D(0,\delta)} dx_n \int_{g(x')}^{h(x')} \frac{|x|^{2-2s}}{(\varepsilon + |x'|^{2-s})^{\frac{2(n-s)}{2-s}}}\, dx' \right| \\
&\leqslant C(n-2)^2 \varepsilon^{\frac{n-2}{2-s}} \int_{D(0,\delta)} \frac{|h(x') - g(x')|\, dx'}{(\varepsilon + |x'|^{2-s})^{\frac{2(n-s)}{2-s}}},
\end{aligned}$$

where C depends only on δ, n.

Since $h(x') = g(x') + \mathrm{o}(|x'|^2)$, it follows that $\forall \sigma > 0, \exists C(\sigma) > 0$ such that

$$\bigl| h(x') - g(x') \bigr| \leqslant \sigma |x'|^2 + C(\sigma) |x'|^{\frac{5}{2}}$$

and

$$\mathrm{I}_1(\varepsilon) \leqslant C \varepsilon^{\frac{n-2}{2-s}} \int_{D(0,\delta)} \frac{\sigma |x'|^2 + C(\sigma)|x'|^{\frac{5}{2}}}{(\varepsilon + |x'|^{2-s})^{\frac{2(n-s)}{2-s}}}\, dx' \leqslant C \varepsilon^{\frac{1}{2-s}} \bigl(\sigma + C(\sigma) \varepsilon^{\frac{1}{2(2-s)}} \bigr),$$

which implies

$$\mathrm{I}_1(\varepsilon) = \mathrm{o}(\varepsilon^{\frac{1}{2-s}}) \quad \text{as } \varepsilon \to 0.$$

Thus we obtain

$$K_1(\varepsilon) = \frac{1}{2}K_1 - \mathrm{I}(\varepsilon) + \mathrm{o}(\varepsilon^{\frac{1}{2-s}}). \tag{29}$$

On the other hand,

$$\begin{aligned} K_2(\varepsilon) &= \int\limits_{\mathbb{R}^n_+} \frac{u_\varepsilon^p}{|x|^s}\,dx - \int\limits_{D(0,\delta)} dx' \int\limits_0^{h(x')} \frac{u_\varepsilon^p}{|x|^s}\,dx_n + \mathrm{O}(\varepsilon^{\frac{n-s}{2-s}}), \\ &= \frac{1}{2}K_2 - \int\limits_{R^{n-1}} dx' \int\limits_0^{g(x')} \frac{u_\varepsilon^p}{|x|^s}\,dx_n - \int\limits_{D(0,\delta)} dx' \int\limits_{g(x')}^{h(x')} \frac{u_\varepsilon^p}{|x|^s}\,dx_n + \mathrm{O}(\varepsilon^{\frac{n-s}{2-s}}), \end{aligned}$$

where

$$\begin{aligned} K_2 &= \int\limits_{\mathbb{R}^n} \frac{u_\varepsilon^p}{|x|^s} = \int\limits_{\mathbb{R}^n} \frac{\varepsilon^{\frac{p(n-2)}{2(n-s)}}}{|x|^s(\varepsilon + |x|^{2-s})^{\frac{p(n-2)}{2-s}}} \\ &= \varepsilon^{\frac{n-s}{2-s}} \int\limits_{\mathbb{R}^n} \frac{dx}{|x|^s(\varepsilon + |x|^{2-s})^{\frac{2(n-s)}{2-s}}} \\ &= \int\limits_{\mathbb{R}^n} \frac{dy}{|y|^s(1 + |y|^{2-s})^{\frac{2(n-s)}{2-s}}}. \end{aligned}$$

It is well known (see [11]) that K_1, K_2 satisfy

$$K_1/K_2^{\frac{n-2}{n-s}} = \mu_s := \mu_s(\mathbb{R}^n). \tag{30}$$

Since

$$\mathrm{II}(\varepsilon) := \int\limits_{R^{n-1}} dx' \int\limits_0^{g(x')} \frac{u_\varepsilon^p}{|x|^s}\,dx_n = \int\limits_{\mathbb{R}^{n-1}} dy' \int\limits_0^{\varepsilon^{\frac{1}{2-s}} g(y')} \frac{dy_n}{|y|^s(1 + |y|^{2-s})^{\frac{2(n-s)}{2-s}}}, \tag{31}$$

this implies that

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{1}{2-s}}\,\mathrm{II}(\varepsilon) = \int\limits_{\mathbb{R}^{n-1}} \frac{g(y')\,dy'}{|y'|^s(1 + |y'|^{2-s})^{\frac{2(n-s)}{2-s}}} = \left(\sum_{i=1}^{n-1}\alpha_i\right)B,$$

where

$$B = \frac{1}{2}\int\limits_{\mathbb{R}^{n-1}} \frac{|y_j|^2\,dy'}{|y'|^s(1 + |y'|^{2-s})^{\frac{2(n-s)}{2-s}}} = \frac{1}{2(n-1)}\int\limits_{\mathbb{R}^{n-1}} \frac{|y'|^2\,dy'}{|y'|^s(1 + |y'|^{2-s})^{\frac{2(n-s)}{2-s}}} > 0.$$

It follows from the curvature assumption again, that

$$\mathrm{II}(\varepsilon) > 0 \quad \text{and} \quad \mathrm{II}(\varepsilon) = \mathrm{O}(\varepsilon^{\frac{1}{2-s}}).$$

Similarly,

$$\left|\int\limits_{D(0,\delta)} dx' \int\limits_{g(x')}^{h(x')} \frac{u_\varepsilon^p}{|x|^s}\,dx_n\right| = \mathrm{o}(\varepsilon^{\frac{1}{2-s}}).$$

Therefore,

$$K_2(\varepsilon)=\frac{1}{2}K_2-\mathrm{II}(\varepsilon)+\mathrm{o}(\varepsilon^{\frac{1}{2-s}}). \tag{32}$$

Moreover, careful calculations lead to

$$K_3(\varepsilon):=\int\limits_{\Omega}u_\varepsilon^2=\begin{cases}\mathrm{O}(\varepsilon^{\frac{1}{2-s}}), & n=3,\\ \mathrm{O}(|\varepsilon^{\frac{2}{2-s}}\ln\varepsilon|), & n=4,\\ \mathrm{O}(\varepsilon^{\frac{2}{2-s}}), & n\geqslant 5.\end{cases}$$

Let $t_\varepsilon>0$ be a constant that

$$J(t_\varepsilon u_\varepsilon)=Y_\varepsilon=\sup_{t>0}J(tu_\varepsilon)=\sup_{t>0}\left\{\frac{1}{2}\big(K_1(\varepsilon)+\lambda K_3(\varepsilon)\big)t^2-\frac{1}{p}K_2(\varepsilon)t^p\right\}. \tag{33}$$

For $n\geqslant 4$, $K_3(\varepsilon)=\mathrm{o}(\varepsilon^{\frac{1}{2-s}})$, hence

$$Y_\varepsilon=J(t_\varepsilon u_\varepsilon)\leqslant\sup_{t>0}\left[\frac{1}{2}K_1(\varepsilon)t^2-\frac{1}{p}K_2(\varepsilon)t^p\right]+\mathrm{o}(\varepsilon^{\frac{1}{2-s}})=\frac{2-s}{2(n-s)}\left[\frac{K_1(\varepsilon)}{(K_2(\varepsilon))^{\frac{n-2}{n-s}}}\right]^{\frac{n-s}{2-s}}+\mathrm{o}(\varepsilon^{\frac{1}{2-s}}).$$

We claim that

$$K_1(\varepsilon)/\big(K_2(\varepsilon)\big)^{\frac{n-2}{n-s}}<2^{-\frac{2-s}{n-s}}\mu_s+\mathrm{o}(\varepsilon^{\frac{1}{2-s}})=\frac{1}{2}K_1/\left(\frac{1}{2}K_2\right)^{\frac{n-2}{n-s}}+\mathrm{o}(\varepsilon^{\frac{1}{2-s}}),$$

which will lead to our conclusion.

By (29), (20) and (30), the above is equivalent to

$$\begin{aligned}\left(\frac{1}{2}K_1-\mathrm{I}(\varepsilon)\right)\left(\frac{1}{2}K_2\right)^{\frac{n-2}{n-s}}&<\frac{1}{2}K_1\left(\frac{1}{2}K_2-\mathrm{II}(\varepsilon)+\mathrm{o}(\varepsilon^{\frac{1}{2-s}})\right)^{\frac{n-2}{n-s}}+\mathrm{o}(\varepsilon^{\frac{1}{2-s}})\\ &=\frac{1}{2}K_1\left\{\left(\frac{1}{2}K_2\right)^{\frac{n-2}{n-s}}-\frac{n-2}{n-s}\left(\frac{1}{2}K_2\right)^{\frac{s-2}{n-s}}\mathrm{II}(\varepsilon)\right\}+\mathrm{o}(\varepsilon^{\frac{1}{2-s}}),\end{aligned}$$

i.e., $\lim_{\varepsilon\to 0}\frac{\mathrm{I}(\varepsilon)}{\mathrm{II}(\varepsilon)}>\frac{(n-2)K_1}{(n-s)K_2}$.

From (28) and (31), and using L'Hôpital's rule, we know

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\mathrm{I}(\varepsilon)}{\mathrm{II}(\varepsilon)}=\lim_{\varepsilon\to 0}\frac{\mathrm{I}'(\varepsilon)}{\mathrm{II}'(\varepsilon)}&=(n-2)^2\int\limits_{\mathbb{R}^{n-1}}\frac{|y'|^{2-2s}g(y')\,dy'}{(1+|y'|^{2-s})^{2(n-s)/(2-s)}}\bigg/\int\limits_{R^{n-1}}\frac{g(y')\,dy'}{|y'|^s(1+|y'|^{2-s})^{2(n-s)/(2-s)}}\\ &=(n-2)^2\int\limits_0^\infty\frac{r^{n+2-2s}\,dr}{(1+r^{2-s})^{2(n-s)/(2-s)}}\bigg/\int\limits_0^\infty\frac{r^{n-s}\,dr}{(1+r^{2-s})^{2(n-s)/(2-s)}}.\end{aligned}$$

Integrating by parts, one has for $2\leqslant\beta\leqslant 2(n-s)-1$,

$$\int\limits_0^\infty\frac{r^{\beta-2}\,dr}{(1+r^{2-s})^{\frac{2(n-s)}{2-s}-1}}=\frac{2n-2-s}{\beta-1}\int\limits_0^\infty\frac{r^{\beta-s}\,dr}{(1+r^{2-s})^{\frac{2(n-s)}{2-s}}}.$$

Observing that

$$\int\limits_0^\infty\frac{r^{\beta-s}\,dr}{(1+r^{2-s})^{\frac{2(n-s)}{2-s}}}=\int\limits_0^\infty\frac{r^{\beta-2}\,dr}{(1+r^{2-s})^{\frac{2(n-s)}{2-s}-1}}-\int\limits_0^\infty\frac{r^{\beta-2}\,dr}{(1+r^{2-s})^{\frac{2(n-s)}{2-s}}},$$

hence

$$\int_0^\infty \frac{r^{\beta-s}\,dr}{(1+r^{2-s})^{\frac{2(n-s)}{2-s}}} = \frac{\beta-1}{2n-\beta-1-s}\int_0^\infty \frac{r^{\beta-2}\,dr}{(1+r^{2-s})^{\frac{2(n-s)}{2-s}}}.$$

Therefore one has

$$\lim_{\varepsilon\to 0}\frac{\mathrm{I}(\varepsilon)}{\mathrm{II}(\varepsilon)} = \frac{n+1+s}{n-3}(n-2)^2,$$

and

$$\frac{n-2}{n-s}\frac{K_1}{K_2} = \frac{(n-2)^3}{n-s}\int_0^\infty \frac{r^{n+1-2s}\,dr}{(1+r^{2-s})^{\frac{2(n-s)}{2-s}}}\Big/\int_0^\infty \frac{r^{n-1-s}\,dr}{(1+r^{2-s})^{\frac{2(n-s)}{2-s}}} = \frac{(n-2)^3}{n-s}\frac{n-s}{n-2} = (n-2)^2.$$

We therefore get

$$\frac{\mathrm{I}(\varepsilon)}{\mathrm{II}(\varepsilon)} > \frac{n-2}{n-s}\frac{K_1}{K_2} + \mathrm{o}(1).$$

Case 2. $n=3$. Careful calculations lead to

$$K_1(\varepsilon) \leqslant \frac{1}{2}K_1 - C\varepsilon^{\frac{1}{2-s}}|\ln\varepsilon| + \mathrm{o}(\varepsilon^{\frac{1}{2-s}}) \quad \text{for some } C>0, \tag{34}$$

$$K_2(\varepsilon) = \frac{1}{2}K_2 - \mathrm{O}(\varepsilon^{\frac{1}{2-s}}). \tag{35}$$

Letting $J(t_\varepsilon u_\varepsilon) = Y_\varepsilon = \sup_{t>0} J(tu_\varepsilon)$, we have

$$Y_\varepsilon \leqslant \sup_{t>0}\left[\frac{1}{2}K_1(\varepsilon)t^2 - \frac{1}{p}K_2(\varepsilon)t^p\right] + \mathrm{O}(\varepsilon^{\frac{1}{2-s}}) = \frac{2-s}{2(n-s)}\left[\frac{K_1(\varepsilon)}{(K_2(\varepsilon))^{\frac{n-2}{n-s}}}\right]^{\frac{n-s}{2-s}} + \mathrm{O}(\varepsilon^{\frac{1}{2-s}}).$$

Consequently if

$$K_1(\varepsilon)/\big(K_2(\varepsilon)\big)^{\frac{n-2}{n-s}} < 2^{-\frac{2-s}{n-s}}\mu_s - \mathrm{O}(\varepsilon^{\frac{1}{2-s}}), \tag{36}$$

then (26) follows.

By (34) and (35), (36) reduces to

$$\frac{1}{2}K_1 - C\varepsilon^{\frac{1}{2-s}}|\ln\varepsilon| < 2^{-\frac{2-s}{n-s}}\mu_s\left[\frac{1}{2}K_2 - \mathrm{O}(\varepsilon^{\frac{1}{2-s}})\right]^{\frac{n-2}{n-s}} + \mathrm{O}(\varepsilon^{\frac{1}{2-s}}) = \frac{1}{2}\mu_s K_2^{\frac{n-2}{n-s}} + \mathrm{O}(\varepsilon^{\frac{1}{2-s}}).$$

Since

$$K_1/K_2^{\frac{n-2}{n-s}} = \mu_s,$$

we get (36) immediately. Hence we found a critical point $u\in H^1(\Omega)$ of $J(u)$. Now we show that $u>0$. Because

$$0 = \langle J'(u), u_-\rangle = \int_\Omega \big[|\nabla u_-|^2 - \lambda(u_-)^2\big]\,dx,$$

where $u_- = \min(u,0)$, and $\lambda<0$, we conclude that $u_-\equiv 0$, or $u\geqslant 0$. Since u cannot be constant, $u>0$ by maximum principle. □

Remark 5.3. As noticed in [16] (there $s=0$), if Ω is an exterior domain, the mean curvature at 0 (when seen from inside) is negative, then there exists a least-energy solution. The proof is almost the same as above. While if $\mathbb{R}^n\backslash\overline{\Omega}$ is close to a ball in some sense, then for $\lambda>0$, (4) has no least energy solution.

One may of course replace the nonlinearity in (4) with a more general nonlinear term and obtain similar results. The same arguments also apply to get the following extension of Theorem 5.2.

Theorem 5.4. *Suppose that the mean curvature of $\partial\Omega$ at 0 is positive, then the problem*

$$\begin{cases} -\Delta_p u = \dfrac{|u|^{p^*(s)-1}}{|x|^s} - \lambda u^{p-1} & \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \nu = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega \end{cases} \tag{37}$$

has a solution. Here ν is the outward unit normal to $\partial\Omega$, $\lambda > 0$, $1 < p < n$, $0 < s < p$, $p^*(s) = \frac{p(n-s)}{n-p}$ and where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

Based on the mountain pass solution – found in Theorem 1.3 – and using a suitable form of Ljusternik–Schnirelman [10] theory, one can establish the following theorem. Analogous results in this direction have been obtained, for example, in [6,18] for the Neumann problem when $s = 0$, [11] for the Dirichlet problem when $s > 0$, $0 \in \Omega$.

Theorem 5.5. *Under the same assumptions as in Theorem* 1.3*, Eq.* (4) *also has a sign-changing solution, provided* $n \geqslant 6$.

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