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Blowing up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity. Part II: $N \geqslant 4$ ☆

Solutions concentrées pour un problème elliptique de Neumann avec non-linéarité sous- ou sur-critique. II: $N \geqslant 4$

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Abstract

We consider the sub- or supercritical Neumann elliptic problem $-\Delta u + \mu u = u^{\frac{N+2}{N-2}+\varepsilon}$, $u > 0$ in Ω; $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, Ω being a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 4$, $\mu > 0$ and $\varepsilon \neq 0$ a small number. We show that for $\varepsilon > 0$, there always exists a solution to the slightly supercritical problem, which blows up at the most curved part of the boundary as ε goes to zero. On the other hand, for $\varepsilon < 0$, assuming that the domain is not convex, there also exists a solution to the slightly subcritical problem, which blows up at the least curved part of the domain.

Résumé

Ω étant un domaine borné régulier de $\mathbb{R}^N$, $N \geqslant 4$, on considère le problème elliptique de Neumann $-\Delta u + \mu u = u^{\frac{N+2}{N-2}+\varepsilon}$, $u > 0$ dans Ω ; $\frac{\partial u}{\partial n} = 0$ sur $\partial\Omega$, où $\mu > 0$ est un paramètre fixé. On montre que pour $\varepsilon > 0$ assez petit, le problème admet une solution non-constante, qui se concentre quand ε tend vers zéro en un point de la frontière où la courbure moyenne est maximum. En supposant que le domaine n'est pas convexe, on montre aussi, pour $\varepsilon < 0$ assez proche de zéro, l'existence d'une solution non-constante, qui se concentre quand ε tend vers zéro en un point de la frontière où la courbure moyenne est minimum.

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1. Introduction

In this paper we consider the nonlinear Neumann elliptic problem

$$(P_{q,\mu})\quad \begin{cases} -\Delta u + \mu u = u^q, \quad u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < q < +\infty$, $\mu > 0$ and Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geqslant 4$.

Eq. $(P_{q,\mu})$ arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer–Meinhardt system in biological pattern formation [12,27], or for parabolic equations in chemotaxis, e.g. Keller–Segel model [24].

When q is subcritical, i.e. $q < \frac{N+2}{N-2}$, Lin, Ni and Takagi proved that the only solution, for small μ, is the constant one, whereas nonconstant solutions appear for large μ [24] which blow up, as μ goes to infinity, at one or several points. The least energy solution blows up at a boundary point which maximizes the mean curvature of the frontier [29,30]. Higher energy solutions exist which blow up at one or several points, located on the boundary [8,13,22, 42,18], in the interior of the domain [5,7,10,11,15,20,40,43], or some of them on the boundary and others in the interior [17]. (A good review can be found in [27].) In the critical case, i.e. $q = 5$, Zhu [44] proved that, for convex domains, the only solution is the constant one for small μ (see also [41]). For large μ, nonconstant solutions exist [1,35]. As in the subcritical case the least energy solution blows up, as μ goes to infinity, at a unique point which maximizes the mean curvature of the boundary [3,28]. Higher energy solutions have also been exhibited, blowing up at one [2,36,32,14] or several boundary points [26,37,38,16]. The question of interior blow-up is still open. However, in contrast with the subcritical situation, at least one blow-up point has to lie on the boundary [33].

Very few is known about the supercritical case, save the uniqueness of the radial solution on a ball for small μ [23]. In [27], Ni raised the following conjecture.

Conjecture. *For any exponent $q > 1$, and μ large, there always exists a* nonconstant *solution to $(P_{q,\mu})$.*

Our aim, in this paper, is to continue our study [34] on the problem for fixed μ, when the exponent q is close to the critical one, i.e. $q = \frac{N+2}{N-2} + \varepsilon$ and ε is a small nonzero number. Whereas the previous results, concerned with peaked solutions, always assume that μ goes to infinity, we are going to prove that a single interior or boundary peak solution may exist for fixed μ, provided that q is close enough to the critical exponent. In [34], we showed that for $N = 3$, a single interior bubble solution exists for finite μ, as $\varepsilon \to 0$. In this paper, we establish the existence of a single boundary bubble for *any* finite μ and for any smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 4$, provided that $\varepsilon > 0$ is sufficiently small.

Let $H(a)$ denote the boundary mean curvature function at $a \in \partial\Omega$. The following result partially answers Ni's conjecture:

Theorem 1.1. *Suppose that $N \geqslant 4$. Then $(P_{\frac{N+2}{N-2}+\varepsilon,\mu})$ has a nontrivial solution, for $\varepsilon > 0$ close enough to zero, which blows up as ε goes to zero at a point $a \in \partial\Omega$, such that $H(a) = \max_{P\in\partial\Omega} H(P)$.*

In the case of $\varepsilon < 0$, i.e. slightly subcritical case, we then have the following theorem.

Theorem 1.2. *Assume that $N \geqslant 4$ and Ω is not convex. Then $(P_{\frac{N+2}{N-2}+\varepsilon,\mu})$ has a nontrivial solution, for $\varepsilon < 0$ close enough to zero, which blows up as ε goes to zero at a point $a \in \partial\Omega$, such that $H(a) = \min_{P\in\partial\Omega} H(P)$.*

Remark. Theorem 1.2 agrees with the following result of Gui and Lin: in [14], it is proved that if there exists a sequence of single boundary blowing up solutions u_{ε_i} to $P_{\frac{N+2}{N-2}+\varepsilon_i,\mu}$ with $\varepsilon_i \leqslant 0$, then necessarily, u_{ε_i} blows up at a boundary point $a \in \partial\Omega$ such that $H(a) \leqslant 0$ and a is a critical point of H. Here we have established a partial converse to [14].

A similar slightly supercritical Dirichlet problem

$$(Q_\varepsilon)\quad \begin{cases} -\Delta u = u^{\frac{N+2}{N-2}+\varepsilon^2}, & u>0 \quad \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega \end{cases}$$

has been studied in [9], where the existence of solutions with two bubbles in domains with a small hole is established, provided that ε is small. It is interesting to note that, here, and also in [34], we have no condition on the domain, in the slightly supercritical Neumann case.

The scheme of the proof is similar to [34] (see also [9]). However, we use a different framework – i.e. weighted Sobolev spaces – to treat the case $N \geqslant 4$. In the next section, we define a two-parameters set of approximate solutions to the problem, and we look for a true solution in a neighborhood of this set. Considering in Section 3 the linearized problem at an approximate solution, and inverting it in suitable functional spaces, the problem reduces to a finite dimensional one, which is solved in Section 4. Some useful facts and computations are collected in Appendix.

2. Some preliminaries

2.1. Approximate solutions and rescaling

For sake of simplicity, we consider in the following the supercritical case, i.e. we assume that $\varepsilon > 0$. The subcritical case may be treated exactly in the same way. For normalization reasons, we consider throughout the paper the equation

$$-\Delta u + \mu u = \alpha_N u^{\frac{N+2}{N-2}+\varepsilon}, \quad u>0, \tag{2.1}$$

instead of the original one, where $\alpha_N = N(N-2)$. The solutions are identical, up to the multiplicative constant $(\alpha_N)^{-\frac{N-2}{4+(N-2)\varepsilon}}$. We recall that, according to [6], the functions

$$U_{\lambda,a}(x) = \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{N-2}{2}}}, \quad \lambda>0,\ a\in\mathbb{R}^N, \tag{2.2}$$

are the only solutions to the problem

$$-\Delta u = \alpha_N u^{\frac{N+2}{N-2}}, \quad u>0 \quad \text{in } \mathbb{R}^N.$$

As $a \in \partial\Omega$ and λ goes to infinity, these functions provide us with approximate solutions to the problem that we are interested in. However, in view of the additional linear term μu which occurs in $(P_{\frac{N+2}{N-2}+\varepsilon,\mu})$, the approximation needs to be improved.

Integral estimates (see Appendix) suggest to make the additional *a priori* assumption that λ behaves as $1/\varepsilon$ as ε goes to zero. Namely, we set

$$\lambda = \frac{1}{\Lambda\varepsilon}, \qquad \frac{1}{\delta'} < \Lambda < \delta' \tag{2.3}$$

with δ' some strictly positive number. Now, fix $a \in \partial\Omega$. We define $V_{\Lambda,a,\mu,\varepsilon} = V$ satisfying

$$\begin{cases} -\Delta V + \mu V = \alpha_N U_{\frac{1}{\Lambda\varepsilon},a}^{\frac{N+2}{N-2}} & \text{in } \Omega,\\ \frac{\partial V}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

The $V_{\Lambda,a,\mu,\varepsilon}$'s are the suitable approximate solutions in the neighborhood of which we shall find a true solution to the problem. In order to make further computations easier, we proceed to a rescaling. We set

$$\Omega_\varepsilon = \frac{\Omega}{\varepsilon}$$

and define in Ω_ε the functions

$$W_{\Lambda,\xi,\mu,\varepsilon}(x) = \varepsilon^{\frac{N-2}{2}} V_{\Lambda,a,\mu,\varepsilon}(\varepsilon x), \quad \xi = \frac{a}{\varepsilon}. \tag{2.5}$$

$W_{\Lambda,\xi,\mu,\varepsilon} = W$ satisfies

$$\begin{cases} -\Delta W + \mu\varepsilon^2 W = \alpha_N U_{\frac{1}{\Lambda},\xi}^{\frac{N+2}{N-2}} & \text{in } \Omega_\varepsilon, \\ \frac{\partial W}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{2.6}$$

and, since $U_{\frac{1}{\Lambda},\xi} \geqslant C\epsilon^{N-2}$ and $\Delta W \geqslant 0$ at a minimum point of W in the closure of Ω

$$W \geqslant C\epsilon^N \quad \text{in } \overline{\Omega}. \tag{2.7}$$

Another fact that we shall use later is the following: observe that $\partial_\Lambda W$ satisfies

$$\begin{cases} -\Delta(\partial_\Lambda W) + \mu\varepsilon^2 \partial_\Lambda W = \alpha_N \partial_\Lambda \big(U_{\frac{1}{\Lambda},\xi}^{\frac{N+2}{N-2}}\big) & \text{in } \Omega_\varepsilon, \\ \frac{\partial(\partial_\Lambda W)}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Since $|\partial_\Lambda(U_{\frac{1}{\Lambda},\xi}^{\frac{N+2}{N-2}})| \leqslant CU_{\frac{1}{\Lambda},\xi}^{\frac{N+2}{N-2}}$, by comparison principle we obtain

$$|\partial_\Lambda W| \leqslant CW. \tag{2.8}$$

The same holds for $\partial_\xi W$ instead of $\partial_\Lambda W$.

Finding a solution to $(P_{\frac{N+2}{N-2}+\varepsilon,\mu})$ in a neighborhood of the functions $V_{\Lambda,a,\mu,\varepsilon}$ is equivalent, through the following rescaling

$$u(x) \to \varepsilon^{-\frac{2(N-2)}{4+(N-2)\varepsilon}} u\left(\frac{x}{\varepsilon}\right)$$

to solving the problem

$$(P'_{\frac{N+2}{N-2}+\varepsilon,\mu}) \qquad \begin{cases} -\Delta u + \mu\varepsilon^2 u = \alpha_N u^{\frac{N+2}{N-2}+\varepsilon}, \quad u > 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{2.9}$$

in a neighborhood of the functions $W_{\Lambda,\xi,\mu,\varepsilon}$. (From now on, we shall work with $(P'_{\frac{N+2}{N-2}+\varepsilon,\mu})$.) For that purpose, we have to use some local inversion procedure. Namely, we are going to look for a solution to $(P'_{\varepsilon,\mu})$ writing as

$$w = W_{\Lambda,\xi,\mu,\varepsilon} + \omega$$

with ω small and orthogonal at $W_{\Lambda,\xi,\mu,\varepsilon}$, in a suitable sense, to the manifold

$$M = \big\{W_{\Lambda,\xi,\mu,\varepsilon},\ \Lambda \text{ satisfying (2.3)}, \xi \in \partial\Omega_\varepsilon\big\}.$$

The general strategy consists in finding first, using an inversion procedure, a smooth map $(\Lambda,\xi) \mapsto \omega(\Lambda,\xi)$ such that $W_{\Lambda,\xi,\mu,\varepsilon} + \omega(\Lambda,\xi,\mu,\varepsilon)$ solves the problem in an orthogonal space to M. Then, we are left with a finite dimensional problem, for which a solution may be found using the assumptions of the theorems. In the subcritical or critical case, the first step may be performed in H^1 (see e.g. [4,31,32]). However, this approach is not valid any more in the supercritical case, for H^1 does not inject into L^q as $q > \frac{2N}{N-2}$. In [9], a weighted Hölder spaces approach was used. In the present paper, we use weighted Sobolev spaces to reduce the problem to a finite dimensional one.

2.2. Boundary deformations

Fix $a \in \partial\Omega$. We introduce a boundary deformation which strengthens the boundary near a. Without loss of generality, we may assume that $a = 0$ and after rotation and translation of the coordinate system we may assume that the inward normal to $\partial\Omega$ at a is the direction of the positive x_N-axis. Denote $x' = (x_1, \dots, x_{N-1})$, $B'(\delta) = \{x' \in \mathbb{R}^{N-1}\colon |x'| < \delta\}$, and $\Omega_1 = \Omega \cap B(a, \delta)$, where $B(a, \delta) = \{x \in \mathbb{R}^N\colon |x - a| < \delta\}$.

Then, since $\partial\Omega$ is smooth, we can find a constant $\delta > 0$ such that $\partial\Omega \cap B(a, \delta)$ can be represented by the graph of a smooth function $\rho_a : B'(\delta) \to R$, where $\rho_a(0) = 0$, $\nabla\rho_a(0) = 0$, and

$$\Omega \cap B(a,\delta) = \{(x', x_N) \in B(a,\delta)\colon x_N > \rho_a(x')\}. \tag{2.10}$$

Moreover, we may write

$$\rho_a(x') = \frac{1}{2}\sum_{i=1}^{N-1} k_i x_i^2 + O(|x|^3). \tag{2.11}$$

Here k_i, $i = 1, \dots, N-1$, are the principal curvatures at a. Furthermore, the average of the principal curvatures of $\partial\Omega$ at a is the mean curvature $H(a) = \frac{1}{N-1}\sum_{i=1}^{N-1} k_i$. To avoid clumsy notations, we drop the index a in ρ.

On $\partial\Omega \cap B(a,\delta)$, the normal derivative $n(x)$ writes as

$$n(x) = \frac{1}{\sqrt{1 + |\nabla'\rho|^2}}(\nabla'\rho, -1) \tag{2.12}$$

and the tangential derivatives are given by

$$\frac{\partial}{\partial\tau_{i,x}} = \frac{1}{\sqrt{1 + |\partial\rho/\partial x_i|^2}}\left(0, \dots, 1, \dots, \frac{\partial\rho}{\partial x_i}\right), \qquad i = 1, \dots, N-1. \tag{2.13}$$

When there is no confusion, we also drop the dependence of $\partial/\partial\tau_{i,x}$ on x.

2.3. Expansion of V and W

In Appendix (Lemma A.1), we derive the following asymptotic expansion of V: For $N \geqslant 4$, we have the expansion

$$V = U_{\frac{1}{\Lambda\varepsilon}, a} - (\Lambda\varepsilon)^{\frac{4-N}{2}}\varphi_0\left(\frac{x-a}{\Lambda\varepsilon}\right) + O\left(\varepsilon^{\frac{6-N}{2}}|\ln\varepsilon|^m\right) \tag{2.14}$$

where φ_0 solves some linear problem and $m = 1$ for $N = 4$ and $m = 0$ for $N \geqslant 5$. This then implies that

$$W = U_{\frac{1}{\Lambda}, \xi}(x) - \hat\varphi(x) \tag{2.15}$$

where

$$\hat\varphi(x) = \varepsilon\Lambda^{\frac{4-N}{2}}\varphi_0\left(\frac{x-\xi}{\Lambda}\right) + O\left(\varepsilon^2|\ln\varepsilon|^m\right). \tag{2.16}$$

Furthermore, we have the following upper bound

$$\left|\hat\varphi(x)\right| \leqslant \frac{C\varepsilon|\ln, \varepsilon|^n}{(1 + |x - \xi|)^{N-3}}, \quad x \in \Omega_\varepsilon \tag{2.17}$$

where $n = 1$ for $N = 4, 5$ and $n = 0$ for $N \geqslant 6$, whence

$$\left|W(x)\right| \leqslant C(U_{\frac{1}{\Lambda}, \xi})^{1-\tau} \quad \text{in } \Omega_\varepsilon \tag{2.18}$$

where τ is a positive number which can be chosen to be zero as $N \geqslant 6$, and as small as desired as $N = 4, 5$.

3. The finite dimensional reduction

3.1. Inversion of the linearized problem

We first consider the linearized problem at a function $W_{\Lambda,\xi,\mu,\varepsilon}$, and we invert it in an orthogonal space to M. From now on, we omit for sake of simplicity the indices in the writing of $W_{\Lambda,\xi,\mu,\varepsilon}$. Equipping $H^1(\Omega_\varepsilon)$ with the scalar product

$$(u,v)_\varepsilon = \int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v + \mu\varepsilon^2 uv)$$

orthogonality to the functions

$$Y_0 = \frac{\partial W}{\partial \Lambda}, \quad Y_i = \frac{\partial W}{\partial \tau_i}, \quad 1 \leqslant i \leqslant N-1, \tag{3.1}$$

in that space is equivalent, setting

$$Z_0 = -\Delta \frac{\partial W}{\partial \Lambda} + \mu\varepsilon^2 \frac{\partial W}{\partial \Lambda}, \quad Z_i = -\Delta \frac{\partial W}{\partial \tau_i} + \mu\varepsilon^2 \frac{\partial W}{\partial \tau_i}, \quad 1 \leqslant i \leqslant N-1 \tag{3.2}$$

to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle \cdot,\cdot \rangle$, to the functions Z_i, $0 \leqslant i \leqslant N-1$. Then, we consider the following problem : h being given, find a function ϕ which satisfies

$$\begin{cases} -\Delta\phi + \mu\varepsilon^2\phi - \alpha_N(\frac{N+2}{N-2}+\varepsilon)W^{\frac{4}{N-2}+\varepsilon}\phi = h + \sum_i c_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon, \\ 0 \leqslant i \leqslant N-1, \quad \langle Z_i, \phi \rangle = 0 \end{cases} \tag{3.3}$$

for some numbers c_i.

Existence and uniqueness of ϕ will follow from an inversion procedure in suitable functional spaces. For $N=3$, the weighted Hölder spaces in [9] or [34] work well. For $N \geqslant 4$, we use a weighted Sobolev approach which seems more suitable in treating the large dimensions case. (Special attention is needed for the case $N=4$.) Similar approach has been used in [39] in dealing with a slightly supercritical exponent problem.

Let $\mathcal{U}$ be an open set in $\mathbb{R}^N$ and $\xi \in \mathcal{U}$. For $1 < t < +\infty$, a nonnegative integer l, and a real number β, we define a weighted Sobolev norm

$$\|\phi\|_{W^{l,t}_\beta(\mathcal{U})} = \sum_{|\alpha|=0}^{l} \left\| \langle x-\xi\rangle^{\beta+|\alpha|} \partial^\alpha \phi \right\|_{L^t(\mathcal{U})}$$

where $\langle x-\xi\rangle = (1+|x-\xi|^2)^{\frac{1}{2}}$. When $l=0$, we denote $W^{0,t}_\beta(\mathcal{U})$ as $L^t_\beta(\mathcal{U})$.

Let f be a function in Ω_ε. We define the following two weighted Sobolev norms

$$\|f\|_* = \|f\|_{W^{2,t}_\beta(\Omega_\varepsilon)}$$

and

$$\|f\|_{**} = \|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)}.$$

We choose t and β such that

$$N < t < +\infty, \quad \frac{N-2}{2} + \frac{N(N-2)}{4t} < \beta < \frac{N}{t'} - 2 \tag{3.4}$$

where t' is the conjugate exponent of t, i.e., $\frac{1}{t}+\frac{1}{t'}=1$. (It is easily checked that such a choice of t and β is always possible.) Since $t>N$, by Sobolev embedding theorem, we have

$$\left|\nabla\phi(x)\right|+\left|\phi(x)\right|\leqslant C\langle x-\xi\rangle^{-\beta}\|\phi\|_*,\quad \forall x\in\Omega_\varepsilon. \tag{3.5}$$

We recall the following result:

Lemma 3.1 (Corollary 1 of [25]). *The integral operator*

$$Tu(x)=\int\limits_{\mathbb{R}^N}\frac{u(y)}{|x-y|^{N-2}}\,\mathrm{d}y$$

is a bounded operator from $L^t_{\beta+2}(\mathbb{R}^N)$ to $L^t_\beta(\mathbb{R}^N)$, provided that $-\frac{N}{t}<\beta<\frac{N}{t'}-2$.

We are also in need of the following lemma, whose proof is given in the Appendix:

Lemma 3.2. *Let $f\in L^t_{\beta+2}(\Omega_\varepsilon)$ and u satisfy*

$$-\Delta u+\mu\varepsilon^2u=f\quad\text{in }\Omega_\varepsilon,\qquad \frac{\partial u}{\partial n}=0\quad\text{on }\partial\Omega_\varepsilon.$$

Then we have

$$\left|u(x)\right|\leqslant C\int\limits_{\Omega_\varepsilon}\frac{|f(y)|}{|x-y|^{N-2}}\,\mathrm{d}y \tag{3.6}$$

and

$$\|u\|_*\leqslant C\|f\|_{**}. \tag{3.7}$$

The main result of this subsection is:

Proposition 3.1. *There exists $\varepsilon_0>0$ and a constant $C>0$, independent of ε and ξ, Λ satisfying* (2.3), *such that for all $0<\varepsilon<\varepsilon_0$ and all $h\in L^t_{\beta+2}(\Omega_\varepsilon)$, problem* (3.3) *has a unique solution $\phi\equiv L_\varepsilon(h)$. Besides,*

$$\left\|L_\varepsilon(h)\right\|_*\leqslant C\|h\|_{**},\qquad |c_i|\leqslant C\|h\|_{**}. \tag{3.8}$$

Moreover, the map $L_\varepsilon(h)$ is C^1 with respect to Λ,ξ and the $W^{2,t}_\beta(\Omega_\varepsilon)$-norm, and

$$\left\|D_{(\Lambda,\xi)}L_\varepsilon(h)\right\|_*\leqslant C\|h\|_{**}. \tag{3.9}$$

Proof. The argument follows closely the ideas in [9] and [34]. We repeat it since we use a different norm. The proof relies on the following result:

Lemma 3.3. *Assume that ϕ_ε solves* (3.3) *for $h=h_\varepsilon$. If $\|h_\varepsilon\|_{**}$ goes to zero as ε goes to zero, so does $\|\phi_\varepsilon\|_*$.*

Proof of Lemma 3.3. Arguing by contradiction, we may assume that $\|\phi_\varepsilon\|_*=1$. Multiplying the first equation in (3.3) by Y_j and integrating in Ω_ε we find

$$\sum_i c_i\langle Z_i,Y_j\rangle=\left\langle-\Delta Y_j+\mu\varepsilon^2Y_j-\alpha_N\left(\frac{N+2}{N-2}+\varepsilon\right)W^{\frac{4}{N-2}+\varepsilon}Y_j,\phi_\varepsilon\right\rangle-\langle h_\varepsilon,Y_j\rangle.$$

On one hand we check, in view of the definition of Z_i, Y_j

$$\langle Z_0,Y_0\rangle=\|Y_0\|^2_\varepsilon=c_0+o(1),\quad \langle Z_i,Y_i\rangle=\|Y_i\|^2_\varepsilon=c_1+o(1),\quad 1\leqslant i\leqslant N-1 \tag{3.10}$$

where c_0, c_1 are strictly positive constants, and

$$\langle Z_i, Y_j\rangle = o(1), \quad i \neq j. \tag{3.11}$$

On the other hand, in view of the definition of Y_j and W, straightforward computations yield

$$\left\langle -\Delta Y_j + \mu\varepsilon^2 Y_j - \alpha_N\left(\frac{N+2}{N-2}+\varepsilon\right) W^{\frac{4}{N-2}+\varepsilon} Y_j, \phi_\varepsilon \right\rangle = o\big(\|\phi_\varepsilon\|_*\big)$$

and

$$\langle h_\varepsilon, Y_j\rangle = O\big(\|h_\varepsilon\|_{**}\big).$$

Consequently, inverting the quasi diagonal linear system solved by the c_i's, we find

$$c_i = O\big(\|h_\varepsilon\|_{**}\big) + o\big(\|\phi_\varepsilon\|_*\big). \tag{3.12}$$

In particular, $c_i = o(1)$ as ε goes to zero.

Since $\|\phi_\varepsilon\|_* = 1$, elliptic theory shows that along some subsequence, $\tilde\phi_\varepsilon(x) = \phi_\varepsilon(x-\xi)$ converges uniformly in any compact subset of $\mathbb{R}^N_+$ to a nontrivial solution of

$$-\Delta\tilde\phi = \alpha_N \frac{N+2}{N-2} U_{\tilde\Lambda,0}^{\frac{4}{N-2}}\tilde\phi$$

for some $\tilde\Lambda > 0$. Moreover, $\tilde\phi \in L^t_\beta(\mathbb{R}^N)$. A bootstrap argument (see e.g. Proposition 2.2 of [39]) implies $|\tilde\phi(x)| \leqslant C/|x|^{N-2}$. As a consequence, $\tilde\phi$ writes as

$$\tilde\phi = \alpha_0 \frac{\partial U_{\tilde\Lambda,0}}{\partial\tilde\Lambda} + \sum_{i=1}^{N-1} \alpha_i \frac{\partial U_{\tilde\Lambda,0}}{\partial a_i}$$

(see [31]). On the other hand, equalities $\langle Z_i, \phi_\varepsilon\rangle = 0$ provide us with the equalities

$$\int_{\mathbb{R}^N_+} -\Delta\frac{\partial U_{\tilde\Lambda,0}}{\partial\tilde\Lambda}\tilde\phi = \int_{\mathbb{R}^N_+} U_{\tilde\Lambda,0}^{\frac{4}{N-2}} \frac{\partial U_{\tilde\Lambda,0}}{\partial\tilde\Lambda}\tilde\phi = 0,$$

$$\int_{\mathbb{R}^N_+} -\Delta\frac{\partial U_{\tilde\Lambda,0}}{\partial a_i}\tilde\phi = \int_{\mathbb{R}^N_+} U_{\tilde\Lambda,0}^{\frac{4}{N-2}} \frac{\partial U_{\tilde\Lambda,0}}{\partial a_i}\tilde\phi = 0, \quad 1 \leqslant i \leqslant N-1.$$

As we have also

$$\int_{\mathbb{R}^N_+} \left|\nabla\frac{\partial U_{\tilde\Lambda,0}}{\partial\tilde\Lambda}\right|^2 = c_0 > 0, \quad \int_{\mathbb{R}^N_+} \left|\nabla\frac{\partial U_{\tilde\Lambda,0}}{\partial a_i}\right|^2 = c_1 > 0, \quad 1 \leqslant i \leqslant N-1,$$

and

$$\int_{\mathbb{R}^N_+} \nabla\frac{\partial U_{\tilde\Lambda,0}}{\partial\tilde\Lambda}.\nabla\frac{\partial U_{\tilde\Lambda,0}}{\partial a_i} = \int_{\mathbb{R}^N_+} \nabla\frac{\partial U_{\tilde\Lambda,0}}{\partial a_j}.\nabla\frac{\partial U_{\tilde\Lambda,0}}{\partial a_i} = 0, \quad i \neq j,$$

the α_j's solve a homogeneous quasi diagonal linear system, yielding $\alpha_j = 0$, $0 \leqslant \alpha_j \leqslant N-1$, and $\tilde\phi = 0$. So $\phi_\varepsilon(x-\xi) \to 0$ in $C^1_{\mathrm{loc}}(\Omega_\varepsilon)$. Now, since

$$\left|\langle x-\xi\rangle^{\beta+2} W^{\frac{4}{N-2}+\varepsilon}\phi_\varepsilon\right|^t \leqslant C\|\phi_\varepsilon\|_*^t \langle x-\xi\rangle^{(2-(4+(N-2)\varepsilon)(1-\tau))t} \in L^1(\mathbb{R}^N),$$

(using (2.18)), by the Dominated Convergence Theorem we obtain

$$\int_{\Omega_\varepsilon} \left|\langle x-\xi\rangle^{\beta+2} W^{\frac{4}{N-2}+\varepsilon}\phi_\varepsilon\right|^t = o(1) \quad \text{i.e.} \quad \|W^{\frac{4}{N-2}+\varepsilon}\phi_\varepsilon\|_{**} = o(1).$$

On the other hand, from (2.6), (3.2) and the definition of U, we know that

$$\langle x-\xi\rangle^{\beta+2}|Z_i| \leqslant C\langle x-\xi\rangle^{\beta-N} \in L^t(\mathbb{R}^N).$$

Applying Lemma 3.2 we obtain

$$\|\phi_\varepsilon\|_* \leqslant C\|W^{\frac{4}{N-2}+\varepsilon}\phi_\varepsilon\|_{**} + C\|h_\varepsilon\|_{**} + C\sum_i |c_i|\|Z_i\|_{**} = o(1)$$

that is, a contradiction.

Proof of Proposition 3.1 completed. We set

$$H = \left\{\phi \in H^1(\Omega_\varepsilon),\ \langle Z_i,\phi\rangle = 0,\ 0 \leqslant i \leqslant N-1\right\}$$

equipped with the scalar product $(\cdot,\cdot)_\varepsilon$. Problem (3.3) is equivalent to finding $\phi \in H$ such that

$$(\phi,\theta)_\varepsilon = \left\langle \alpha_N\left(\frac{N+2}{N-2}+\varepsilon\right)W^{\frac{4}{N-2}+\varepsilon}\phi + h, \theta\right\rangle, \quad \forall \theta \in H$$

that is

$$\phi = T_\varepsilon(\phi) + \tilde{h} \tag{3.13}$$

$\tilde{h}$ depending linearly on h, and T_ε being a compact operator in H. Fredholm's alternative ensures the existence of a unique solution, provided that the kernel of $\mathrm{Id} - T_\varepsilon$ is reduced to 0. We notice that any $\phi_\varepsilon \in \mathrm{Ker}(\mathrm{Id} - T_\varepsilon)$ solves (3.3) with $h = 0$. Thus, we deduce from Lemma 3.3 that $\|\phi_\varepsilon\|_* = o(1)$ as ε goes to zero. As $\mathrm{Ker}(\mathrm{Id} - T_\varepsilon)$ is a vector space, $\mathrm{Ker}(\mathrm{Id} - T_\varepsilon) = \{0\}$. The inequalities (3.8) follow from Lemma 3.3 and (3.12). This completes the proof of the first part of Proposition 3.1.

The smoothness of L_ε with respect to Λ and ξ is a consequence of the smoothness of T_ε and $\tilde{h}$, which occur in the implicit definition (3.13) of $\phi \equiv L_\varepsilon(h)$, with respect to these variables. Inequalities (3.9) are obtained differentiating (3.3), writing the derivatives of ϕ with respect to Λ and ξ as a linear combination of the Z_i' and an orthogonal part, and estimating each term using the first part of the proposition – see [9,19] for detailed computations. □

3.2. The reduction

Let

$$S_\varepsilon(u) = -\Delta u + \mu\varepsilon^2 u - \alpha_N u_+^{\frac{N+2}{N-2}+\varepsilon}$$

where $u_+ = \max(0,u)$. Then (2.9) is equivalent to

$$S_\varepsilon(u) = 0 \quad \text{in } \partial\Omega_\varepsilon, \qquad u_+ \not\equiv 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_\varepsilon \tag{3.14}$$

for if u satisfies (3.14), the Maximum Principle ensures that $u > 0$ in Ω_ε and (2.9) is satisfied. Observe that

$$S_\varepsilon(W+\phi) = -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - \alpha_N(W+\phi)_+^{\frac{N+2}{N-2}+\varepsilon}$$

may be written as

$$S_\varepsilon(W+\phi) = -\Delta\phi + \mu\varepsilon^2\phi - \left(\frac{N+2}{N-2}+\varepsilon\right)\alpha_N W^{\frac{4}{N-2}+\varepsilon}\phi - R^\varepsilon - \alpha_N N_\varepsilon(\phi) \tag{3.15}$$

with

$$N_\varepsilon(\phi) = (W+\phi)_+^{\frac{N+2}{N-2}+\varepsilon} - W^{\frac{N+2}{N-2}+\varepsilon} - \left(\frac{N+2}{N-2}+\varepsilon\right) W^{\frac{4}{N-2}+\varepsilon}\phi, \tag{3.16}$$

$$R^\varepsilon = \Delta W - \mu\varepsilon^2 W + \alpha_N W^{\frac{N+2}{N-2}+\varepsilon} = \alpha_N\big(W^{\frac{N+2}{N-2}+\varepsilon} - U_{\frac{1}{\Lambda},\xi}^{\frac{N+2}{N-2}}\big). \tag{3.17}$$

We first have:

Lemma 3.4. *There exists C, independent of ξ, Λ satisfying* (2.3), *such that*

$$\|R^\varepsilon\|_{**} \leqslant C\varepsilon, \qquad \big\|D_{(\Lambda,\xi)}R^\varepsilon\big\|_{**} \leqslant C\varepsilon.$$

Proof. According to (2.15) and (2.18), $W = U + O(\varepsilon U^{\frac{N-3}{N-2}(1-\tau)})$ uniformly in Ω_ε (where τ is a positive number which is either zero, or may be chosen as small as desired). Consequently, noticing that $U \geqslant C\varepsilon^{N-2}$ in Ω_ε, C independent of ε, easy computations yield

$$R^\varepsilon = O\big(\varepsilon U^{\frac{N+2}{N-2}(1-\tau')}|\ln U| + \varepsilon U^{\frac{N+1}{N-2}(1-\tau'')}\big) \tag{3.18}$$

uniformly in Ω_ε whence, using (3.4)

$$\begin{aligned}\|R^\varepsilon\|_{**} &= \big\|\langle x-\xi\rangle^{\beta+2}\big(U^{\frac{N+2}{N-2}} - W^{\frac{N+2}{N-2}+\varepsilon}\big)\big\|_{L^t(\Omega_\varepsilon)}\\ &\leqslant C\varepsilon\big\|\langle x-\xi\rangle^{\beta+2}\big(U^{\frac{N+2}{N-2}(1-\tau')}|\ln U| + U^{\frac{N+1}{N-2}(1-\tau'')}\big)\big\|_{L^t(\Omega_\varepsilon)} \leqslant C\varepsilon.\end{aligned}$$

The first estimate of the lemma follows. The other ones are obtained in the same way, differentiating (3.17) and estimating each term as previously. □

We consider now the following nonlinear problem: finding ϕ such that, for some numbers c_i

$$\begin{cases} -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - \alpha_N(W+\phi)_+^{\frac{N+2}{N-2}+\varepsilon} = \sum_i c_i Z_i \quad \text{in } \Omega_\varepsilon,\\ \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega_\varepsilon,\\ 0 \leqslant i \leqslant N-1, \quad \langle Z_i, \phi\rangle = 0. \end{cases} \tag{3.19}$$

The first equation in (3.19) writes as

$$-\Delta\phi + \mu\varepsilon^2\phi - \left(\frac{N+2}{N-2}+\varepsilon\right)\alpha_N W^{\frac{4}{N-2}+\varepsilon}\phi = \alpha_N N_\varepsilon(\phi) + R^\varepsilon + \sum_i c_i Z_i \tag{3.20}$$

for some numbers c_i. We now obtain some estimates concerning N_ε.

Lemma 3.5. *Assume that $N \geqslant 4$ and* (3.4) *holds. There exist $\varepsilon_1 > 0$, independent of Λ, ξ, and C, independent of ε, Λ, ξ, such that for $|\varepsilon| \leqslant \varepsilon_1$, and $\|\phi\|_* \leqslant 1$*

$$\big\|N_\varepsilon(\phi)\big\|_{**} \leqslant C\|\phi\|_*^{\min(2,\frac{N+2}{N-2}+\varepsilon)} \tag{3.21}$$

and, for $\|\phi_i\|_* \leqslant 1$

$$\big\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\big\|_{**} \leqslant C\big(\max\big(\|\phi_1\|_*, \|\phi_2\|_*\big)\big)^{\min(1,\frac{4}{N-2}+\varepsilon)}\|\phi_1-\phi_2\|_*. \tag{3.22}$$

Proof. The argument is similar to Lemma 3.1 and Proposition 3.5 of [39]. For the convenience of the reader, we include a proof here. We deduce from (3.16) that

$$\begin{cases} \left|N_\varepsilon(\phi)\right| \leqslant C\left(W^{\frac{6-N}{N-2}+\varepsilon}|\phi|^2+|\phi|^{\frac{N+2}{N-2}+\varepsilon}\right) & \text{if } N \leqslant 6, \\ \left|N_\varepsilon(\phi)\right| \leqslant C|\phi|^{\frac{N+2}{N-2}+\varepsilon} & \text{if } N \geqslant 7. \end{cases} \tag{3.23}$$

Using (3.4) and (3.5) we have

$$\left\||\phi|^{\frac{N+2}{N-2}+\varepsilon}\right\|_{**} = \left(\int_{\Omega_\varepsilon}\left(\langle x-\xi\rangle^{\beta+2}|\phi|^{\frac{N+2}{N-2}+\varepsilon}\right)^t\right)^{\frac{1}{t}}$$

$$\leqslant C\|\phi\|_*^{\frac{N+2}{N-2}+\varepsilon}\left(\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{t(\beta+2-(\frac{N+2}{N-2}+\varepsilon)\beta)}\right)^{\frac{1}{t}} \leqslant C\|\phi\|_*^{\frac{N+2}{N-2}+\varepsilon}.$$

For $N=4,5,6$, using also (2.18), and noticing that W^ε is bounded since W is bounded and satisfies (2.7)), we have

$$\left\|W^{\frac{6-N}{N-2}+\varepsilon}|\phi|^2\right\|_{**} = \left(\int_{\Omega_\varepsilon}\left(\langle x-\xi\rangle^{\beta+2}W^{\frac{6-N}{N-2}+\varepsilon}|\phi|^2\right)^t\right)^{\frac{1}{t}}$$

$$\leqslant C\|\phi\|_*^2\left(\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{(2-\beta+(N-6)(1-\tau))t}\right)^{\frac{1}{t}} \leqslant C\|\phi\|_*^2$$

whence (3.21). Concerning (3.22), we write

$$N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2) = \partial_\eta N_\varepsilon(\eta)(\phi_1-\phi_2)$$

for some $\eta = x\phi_1+(1-x)\phi_2$, $x\in[0,1]$. From

$$\partial_\eta N_\varepsilon(\eta) = \left(\frac{N+2}{N-2}+\varepsilon\right)\left((W+\eta)_+^{\frac{4}{N-2}+\varepsilon}-W^{\frac{4}{N-2}+\varepsilon}\right)$$

we deduce

$$\begin{cases} \left|\partial_\eta N_\varepsilon(\eta)\right| \leqslant C\left(W^{\frac{6-N}{N-2}+\varepsilon}|\eta|+|\eta|^{\frac{4}{N-2}+\varepsilon}\right) & \text{if } N \leqslant 6, \\ \left|\partial_\eta N_\varepsilon(\eta)\right| \leqslant C|\eta|^{\frac{4}{N-2}+\varepsilon} & \text{if } N \geqslant 7 \end{cases} \tag{3.24}$$

whence (3.22), using as previously (3.4) and (3.5). □

We state now the following result:

Proposition 3.2. *There exists C, independent of ε and ξ, Λ satisfying* (2.3), *such that for small ε problem* (3.19) *has a unique solution $\phi = \phi(\Lambda,\xi,\mu,\varepsilon)$ with*

$$\|\phi\|_* \leqslant C\varepsilon. \tag{3.25}$$

Moreover, $(\Lambda,\xi)\mapsto\phi(\Lambda,\xi,\mu,\varepsilon)$ is C^1 with respect to the $W_\beta^{2,t}(\Omega_\varepsilon)$-norm, and

$$\left\|D_{(\Lambda,\xi)}\phi\right\|_* \leqslant C\varepsilon. \tag{3.26}$$

Proof. Following [9], we consider the map A_ε from $\mathcal{F} = \{\phi \in H^1 \cap W^{2,t}_\beta(\Omega_\varepsilon)\colon \|\phi\|_* \leqslant C_0\varepsilon\}$ to $H^1 \cap W^{2,t}_\beta(\Omega_\varepsilon)$ defined as

$$A_\varepsilon(\phi) = L_\varepsilon\big(\alpha_N N_\varepsilon(\phi) + R^\varepsilon\big).$$

Here C_1 is a large number, to be determined later, and L_ε is give by Proposition 3.1. We remark that finding a solution ϕ to problem (3.19) is equivalent to finding a fixed point of A_ε. One the one hand we have, for $\phi \in \mathcal{F}$ and ε small enough

$$\big\|A_\varepsilon(\phi)\big\|_* \leqslant \big\|L_\varepsilon\big(N_\varepsilon(\phi)\big)\big\|_* + \big\|L_\varepsilon(R^\varepsilon)\big\|_* \leqslant \big\|N_\varepsilon(\phi)\big\|_{**} + C\varepsilon \leqslant 2C\varepsilon$$

with C independent of C_0, implying that A_ε sends $\mathcal{F}$ into itself, if we choose $C_0 = 2C$. On the other hand A_ε is a contraction. Indeed, for ϕ_1 and ϕ_2 in $\mathcal{F}$, we write

$$\big\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\big\|_* \leqslant C\big\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\big\|_{**} \leqslant C\varepsilon^{\min(1,\frac{4}{N-2})}\|\phi_1 - \phi_2\|_* \leqslant \frac{1}{2}\|\phi_1 - \phi_2\|_*$$

by Lemma (3.5). Contraction Mapping Theorem implies that A_ε has a unique fixed point in $\mathcal{F}$, that is problem (3.19) has a unique solution ϕ such that $\|\phi\|_* \leqslant C_0\varepsilon$.

In order to prove that $(\Lambda, \xi) \mapsto \phi(\Lambda, \xi)$ is C^1, we remark that setting for $\eta \in \mathcal{F}$

$$B(\Lambda, \xi, \eta) \equiv \eta - L_\varepsilon\big(\alpha_N N_\varepsilon(\eta) + R^\varepsilon\big)$$

ϕ is defined as

$$B(\Lambda, \xi, \phi) = 0. \tag{3.27}$$

We have

$$\partial_\eta B(\Lambda, \xi, \eta)[\theta] = \theta - \alpha_N L_\varepsilon\big(\theta\,(\partial_\eta N_\varepsilon)(\eta)\big).$$

Using Proposition 3.1, (3.5), (3.24) and (3.4) we obtain for $N \geqslant 7$

$$\begin{aligned}\big\|L_\varepsilon\big(\theta(\partial_\eta N_\varepsilon)(\eta)\big)\big\|_* &\leqslant C\big\|\theta(\partial_\eta N_\varepsilon)(\eta)\big\|_{**} \leqslant C\big\|\langle x-\xi\rangle^{-\beta}(\partial_\eta N_\varepsilon)(\eta)\big\|_{**}\|\theta\|_* \\ &\leqslant C\big\|\langle x-\xi\rangle^2|\eta|^{\frac{4}{N-2}+\varepsilon}\big\|_{L^t(\Omega_\varepsilon)}\|\theta\|_* \leqslant C\|\eta\|_*^{\frac{4}{N-2}+\varepsilon}\|\theta\|_* \\ &\leqslant C\varepsilon^{\frac{4}{N-2}}\|\theta\|_*\end{aligned}$$

and, proceeding in the same way, using also (2.18), we find as $N = 4, 5, 6$

$$\big\|L_\varepsilon\big(\theta(\partial_\eta N_\varepsilon)(\eta)\big)\big\|_* \leqslant C\varepsilon\|\theta\|_*.$$

Therefore we can write, for any $N \geqslant 4$

$$\big\|L_\varepsilon\big(\theta(\partial_\eta N_\varepsilon)(\eta)\big)\big\|_* \leqslant C\varepsilon^{\min(1,\frac{4}{N-2})}\|\theta\|_*.$$

Consequently, $\partial_\eta B(\Lambda, \xi, \phi)$ is invertible in $W^{2,t}_\beta(\Omega_\varepsilon)$ with uniformly bounded inverse. Then, the fact that $(\Lambda, \xi) \mapsto \phi(\Lambda, \xi)$ is C^1 follows from the fact that $(\Lambda, \xi, \eta) \mapsto L_\varepsilon(N_\varepsilon(\eta))$ is C^1 and the implicit functions theorem.

Finally, let us show how estimates (3.26) may be obtained. Derivating (3.27) with respect to Λ, we have

$$\partial_\Lambda\phi = \big(\partial_\eta B(\Lambda, \xi, \phi)\big)^{-1}\big(\alpha_N(\partial_\Lambda L_\varepsilon)\big(N_\varepsilon(\phi)\big) + \alpha_N L_\varepsilon\big((\partial_\Lambda N_\varepsilon)(\phi)\big) + \partial_\Lambda\big(L_\varepsilon(R^\varepsilon)\big)\big)$$

whence, according to Proposition 3.1

$$\begin{aligned}\|\partial_\Lambda\phi\|_* &\leqslant C\big(\big\|(\partial_\Lambda L_\varepsilon)\big(N_\varepsilon(\phi)\big)\big\|_* + \big\|\big(L_\varepsilon(\partial_\Lambda N_\varepsilon)(\phi)\big)\big\|_* + \big\|\big(\partial_\Lambda\big(L_\varepsilon(R^\varepsilon)\big)\big)\big\|_*\big) \\ &\leqslant C\big(\big\|N_\varepsilon(\phi)\big\|_{**} + \big\|(\partial_\Lambda N_\varepsilon)(\phi)\big\|_{**} + \big\|\big(\partial_\Lambda\big(L_\varepsilon(R^\varepsilon)\big)\big)\big\|_*\big).\end{aligned}$$

From (3.21) and (3.25) we know that

$$\left\|N_\varepsilon(\phi)\right\|_{**} \leqslant C\varepsilon^{\min(2,\frac{N+2}{N-2})}.$$

Concerning the next term, we notice that according to the definition (3.16) of N_ε and the boundedness of W^ε

$$\begin{aligned}
&\left|(\partial_\Lambda N_\varepsilon)(\phi)\right| \\
&\quad= \left(\frac{N+2}{N-2}+\varepsilon\right)\left|(W+\phi)_+^{\frac{4}{N-2}+\varepsilon} - W^{\frac{4}{N-2}+\varepsilon} - \left(\frac{4}{N-2}+\varepsilon\right)W^{\frac{6-N}{N-2}+\varepsilon}\phi\right||\partial_\Lambda W| \\
&\quad\leqslant C\left[W^{\frac{4}{N-2}}|\phi| \text{ if } N \geqslant 7;\ W^{\frac{4}{N-2}}|\phi| + W|\phi|^{\frac{4}{N-2}+\varepsilon} \text{ if } N \leqslant 6\right] \\
&\quad\leqslant C\Big[\langle x-\xi\rangle^{-4(1-\tau)-\beta}\|\phi\|_* \text{ if } N \geqslant 7; \\
&\qquad \langle x-\xi\rangle^{-4(1-\tau)-\beta}\|\phi\|_* + \langle x-\xi\rangle^{-(N-2)(1-\tau)-\frac{4}{N-2}\beta}\|\phi\|_*^{\frac{4}{N-2}+\varepsilon} \text{ if } N \leqslant 6\Big]
\end{aligned}$$

where we used successively the fact that $W > 0$ (see (2.7)) and $|\partial_\Lambda W| \leqslant CW$ (see (2.8)), inequality (3.5) and $W \leqslant CU^{1-\tau} \leqslant C\langle x-\xi\rangle^{-(N-2)(1-\tau)}$.

As (3.4) ensures that $\langle x-\xi\rangle^{-4(1-\tau)-\beta}$, and $\langle x-\xi\rangle^{-(N-2)(1-\tau)-\frac{4}{N-2}\beta}$ for $N \leqslant 6$, are in $L^t_{\beta+2}(\mathbb{R}^N)$ (provided that τ is chosen small enough), (3.25) yields

$$\left\|(\partial_\Lambda N_\varepsilon)(\phi)\right\|_{**} \leqslant C\varepsilon.$$

From Proposition 3.1 we deduce the estimate for the last term

$$\left\|\partial_\Lambda\big(L_\varepsilon(R^\varepsilon)\big)\right\|_* \leqslant C\|R^\varepsilon\|_{**} \leqslant C\varepsilon$$

and finally

$$\|\partial_\Lambda\phi\|_* \leqslant C\varepsilon.$$

This concludes the proof of Proposition 3.2. (The first derivatives of ϕ with respect to ξ may be estimated in the same way, but this is not needed here.) □

3.3. Coming back to the original problem

We introduce the following functional defined in $H^1(\Omega_\varepsilon) \cap W^{2,t}_\beta(\Omega_\varepsilon)$

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon}\left(|\nabla u|^2 + \mu\varepsilon^2 u^2\right) - \frac{\alpha_N}{2N/(N-2)+\varepsilon}\int_{\Omega_\varepsilon} u_+^{\frac{2N}{N-2}+\varepsilon} \tag{3.28}$$

whose nontrivial critical points are solutions to $(P'_{\frac{N+2}{N-2}+\varepsilon,\mu})$. Setting

$$I_\varepsilon(\Lambda, a) \equiv J_\varepsilon(W_{\Lambda,a} + \phi_{\varepsilon,\Lambda,a}) \tag{3.29}$$

we have:

Proposition 3.3. *The function $u = W + \phi$ is a solution to problem $(P'_{\frac{N+2}{N-2}+\varepsilon,\mu})$ if and only if (Λ, a) is a critical point of I_ε.*

Proof. We notice that $u = W + \phi$ being a solution to $(P'_{\frac{N+2}{N-2}+\varepsilon,\mu})$ is equivalent to being a critical point of J_ε. It is also equivalent to the cancellation of the c_i's in (3.19) or, in view of (3.10), (3.11)

$$J'_\varepsilon(W+\phi)[Y_i] = 0, \quad 0 \leqslant i \leqslant N-1. \tag{3.30}$$

On the other hand, we deduce from (3.29) that $I'_\varepsilon(\Lambda, a) = 0$ is equivalent to the cancellation of $J'_\varepsilon(W + \phi)$ applied to the derivatives of $W + \phi$ with respect to Λ and ξ. According to the definition (3.1) of the Y_i's, Lemma 3.4 and Proposition 3.2 we have

$$\frac{\partial(W+\phi)}{\partial\Lambda} = Y_0 + y_0, \quad \frac{\partial(W+\phi)}{\partial\xi_j} = Y_j + y_j, \quad 1 \leqslant j \leqslant N-1,$$

with $\|y_i\|_* = o(1)$, $0 \leqslant i \leqslant N-1$. Writing

$$y_i = y'_i + \sum_{j=0}^{N-1} a_{ij} Y_j, \quad \langle y'_i, Z_j\rangle = (y'_i, Y_j)_\varepsilon = 0, \quad 0 \leqslant i, j \leqslant N-1,$$

and

$$J'_\varepsilon(W+\phi)[Y_i] = \alpha_i$$

it turns out that $I'_\varepsilon(\Lambda, a) = 0$ is equivalent, since $J'_\varepsilon(W+\phi)[\theta] = 0$ for $\langle\theta, Z_j\rangle = (\theta, Y_j)_\varepsilon = 0$, $0 \leqslant j \leqslant N-1$, to

$$\big(\mathrm{Id} + [a_{ij}]\big)[\alpha_i] = 0.$$

As $a_{ij} = O(\|y_i\|_*) = o(1)$, we see that $I'_\varepsilon(\Lambda, a) = 0$ means exactly that (3.30) is satisfied. □

4. Proofs of Theorems 1.1 and 1.2

In view of Proposition 3.3 we have, for proving the theorem, to find critical points of I_ε. We establish first a C^1-expansion of I_ε.

4.1. Expansion of I_ε

Proposition 4.1. *There exist A, B, C, strictly positive constants such that*

$$I_\varepsilon(\Lambda, a) = A - B\Lambda\varepsilon H(a) + \frac{(N-2)^2}{4} A\varepsilon \ln\Lambda + \varepsilon\left(C + \frac{(N-2)^2}{4N} A\right) + \varepsilon\sigma_\varepsilon(\Lambda, a)$$

with σ_ε and $\partial_\Lambda\sigma_\varepsilon$ going to zero as ε goes to zero, uniformly with respect to Λ satisfying (2.3).

Proof. In Appendix, we shall prove

$$J_\varepsilon(W) = A - B\Lambda\varepsilon H(a) + \frac{(N-2)^2}{4} A\varepsilon \ln\Lambda + \varepsilon\left(C + \frac{(N-2)^2}{4N} A\right) + o(\varepsilon). \tag{4.1}$$

Then it remains to show that

$$I_\varepsilon(\Lambda, a) - J_\varepsilon(W+\phi) = o(\varepsilon). \tag{4.2}$$

Actually, in view of (3.29), a Taylor expansion and the fact that $J'_\varepsilon(W+\phi)[\phi] = 0$ yield

$$I(\Lambda, a) - J_\varepsilon(W) = J_\varepsilon(W+\phi) - J_\varepsilon(W) = -\int_0^1 J''_\varepsilon(W + t\phi)[\phi, \phi]t\,\mathrm{d}t$$

$$= -\int_0^1 \left(\int_{\Omega_\varepsilon} \left(|\nabla\phi|^2 + \mu\varepsilon^2\phi^2 - \alpha_N\left(\frac{N+2}{N-2} + \varepsilon\right)(W + t\phi)_+^{\frac{4}{N-2}+\varepsilon}\phi^2 + R^\varepsilon\phi\right)\right)t\,\mathrm{d}t$$

$$= -\int_0^1 \left(\alpha_N \int_{\Omega_\varepsilon} \left(N_\varepsilon(\phi)\phi + \left(\frac{N+2}{N-2}+\varepsilon\right)\left[W^{\frac{4}{N-2}+\varepsilon} - (W+t\phi)_+^{\frac{4}{N-2}+\varepsilon}\right]\phi^2\right)\right) t\,\mathrm{d}t$$
$$-\frac{1}{2}\int_{\Omega_\varepsilon} R^\varepsilon \phi.$$

The first term can be estimated as follows. Using (3.23), (3.5), (3.4) and Proposition 3.2, we have, for $N \geqslant 7$

$$\left|\int_{\Omega_\varepsilon} N_\varepsilon(\phi)\phi\right| \leqslant C\|\phi\|_*^{\frac{2N}{N-2}+\varepsilon} \int_{\Omega_\varepsilon} \langle x-\xi\rangle^{-\beta(\frac{2N}{N-2}+\varepsilon)} \leqslant C\varepsilon^{\frac{2N}{N-2}}.$$

In the same way we obtain for $N = 4, 5, 6$, in view of (3.23) and (2.18)

$$\left|\int_{\Omega_\varepsilon} N_\varepsilon(\phi)\phi\right| \leqslant C\varepsilon^{\frac{2N}{N-2}} + C\|\phi\|_*^3 \int_{\Omega_\varepsilon} \langle x-\xi\rangle^{-3\beta-(6-N)(1-\tau)} \leqslant C\varepsilon^3$$

whence finally, for any $N \geqslant 4$

$$\left|\int_{\Omega_\varepsilon} N_\varepsilon(\phi)\phi\right| \leqslant C\varepsilon^{\min(3,\frac{2N}{N-2})}. \tag{4.3}$$

For the second term, the same arguments as previously yield

$$\int_{\Omega_\varepsilon} \left|W^{\frac{4}{N-2}+\varepsilon} - (W+t\phi)_+^{\frac{4}{N-2}+\varepsilon}\right|\phi^2 \leqslant C\int_{\Omega_\varepsilon}\left(W^{\frac{4}{N-2}+\varepsilon}|\phi|^2 + |\phi|^{2+\frac{4}{N-2}+\varepsilon}\right)$$
$$\leqslant C\left(\|\phi\|_*^2 \int_{\Omega_\varepsilon} \langle x-\xi\rangle^{-2\beta-4(1-\tau)} + \|\phi\|_*^{2+\frac{4}{N-2}+\varepsilon}\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-\beta(2+\frac{4}{N-2}+\varepsilon)}\right)$$

whence, using again (3.4)

$$\int_{\Omega_\varepsilon} \left|W^{\frac{4}{N-2}+\varepsilon} - (W+t\phi)_+^{\frac{4}{N-2}+\varepsilon}\right|\phi^2 \leqslant C\varepsilon^2. \tag{4.4}$$

Concerning the last term, we remark that according to (3.18)

$$R^\varepsilon \leqslant C\varepsilon\langle x-\xi\rangle^{-(N+1)(1-\tau)}$$

uniformly in Ω_ε. Therefore

$$\int_{\Omega_\varepsilon} |R^\varepsilon\phi| \leqslant C\varepsilon\|\phi\|_* \int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-(N+1)-\beta}$$

yielding, through Proposition 3.2

$$\int_{\Omega_\varepsilon} |R^\varepsilon\phi| \leqslant C\varepsilon^2. \tag{4.5}$$

The desired result follows from (4.3), (4.4) and (4.5). The same estimate holds for the first derivative with respect to Λ, obtained similarly with more delicate computations – see Proposition 3.4 of [19]. □

4.2. Proofs of Theorem 1.1 and Theorem 1.2 completed

We first prove Theorem 1.1 through a max-min argument. Since Ω is smooth and bounded, $\max_{P\in\partial\Omega} H(P) = \gamma > 0$. For $\delta < \gamma$, we define

$$(\partial\Omega)_\delta = \{a \in \partial\Omega \text{ s.t. } H(a) > \delta\},$$

and

$$\hat{I}_\varepsilon(\Lambda, a) = \frac{A - I_\varepsilon(\Lambda, a)}{B\varepsilon} + \frac{1}{B}\left(C + \frac{(N-2)^2}{4N}A\right). \tag{4.6}$$

By Proposition 4.1, we have the following asymptotic expansion for $\hat{I}_\epsilon(\Lambda, a)$:

$$\hat{I}_\varepsilon(\Lambda, a) = \Lambda H(a) - \alpha \ln \Lambda - \tilde{\sigma}_\varepsilon(\Lambda, a) \tag{4.7}$$

with

$$\alpha = \frac{(N-2)^2}{4B}A > 0 \quad \text{and} \quad \tilde{\sigma}_\varepsilon(\Lambda, a) = o(1), \quad \partial_\Lambda \tilde{\sigma}_\varepsilon(\Lambda, a) = o(1) \quad \text{as } \varepsilon \to 0.$$

We set

$$\Sigma_0 = \left\{(\Lambda, a) \,\Big|\, \frac{c_1}{2} < \Lambda < \frac{2}{c_1},\ a \in (\partial\Omega)_{\gamma_0}\right\} \tag{4.8}$$

where c_1 is a small number, to be chosen later, and $0 < \gamma_0 < \gamma$. We define also

$$B = \left\{(\Lambda, a) \,\Big|\, c_1 \leqslant \Lambda \leqslant \frac{1}{c_1},\ a \in (\partial\Omega)_{\gamma_1}\right\}, \qquad B_0 = \{c_1\} \times (\partial\Omega)_{\gamma_1} \cup \left\{\frac{1}{c_1}\right\} \times (\partial\Omega)_{\gamma_1}$$

where $\gamma_0 < \gamma_1 < \gamma$. (Here we choose, for γ_1 close enough to γ, a contractible component of $(\partial\Omega)_{\gamma_1}$ so that B is contractible.)

It is trivial to see that $B_0 \subset B \subset \Sigma_0$, B_0, B are closed and B is connected. Let Γ be the class of continuous functions $\varphi : B \to \Sigma_0$ with the property that $\varphi(y) = y$ for all $y \in B_0$. Define the max-min value c as

$$c = \max_{\varphi\in\Gamma} \min_{y\in B} \hat{I}_\varepsilon(\varphi(y)). \tag{4.9}$$

We now show that c defines a critical value. To this end, we just have to verify the following two conditions

(H1) $\min_{y\in B_0} \hat{I}_\varepsilon(\varphi(y)) > c$, $\forall \varphi \in \Gamma$;
(H2) For all $y \in \partial\Sigma_0$ such that $\hat{I}_\epsilon(y) = c$, there exists τ_y a tangent vector to $\partial\Sigma_0$ at y such that

$$\partial_{\tau_y} \hat{I}_\epsilon(y) \neq 0.$$

Suppose (H1) and (H2) hold. Then standard deformation argument ensures that the max-min value c is a (topologically nontrivial) critical value for $\hat{I}_\varepsilon(\Lambda, a)$ in Σ_0.

To check (H1) and (H2), we write $\varphi(y) = (\varphi_1(y), \varphi_2(y))$ where $\varphi_1(y) \in [\frac{c_1}{2}, \frac{2}{c_1}]$ and $\varphi_2(y) \in (\partial\Omega)_{\gamma_0}$.

Since $\varphi|_{B_0} = \mathrm{id}$, B is contractible and φ is continuous, necessarily there is some y in B such that $H(\varphi_2(y)) = \gamma$. Then, in view of (4.7)

$$c \geqslant d_0 := \min\{\hat{I}_\varepsilon(\Lambda, a),\ H(a) = \gamma,\ \Lambda > 0\} = \alpha - \alpha \ln \alpha + \alpha \ln \gamma + o(1).$$

Now, let $(\Lambda_0, a_0) \in B$ be such that $H(a_0) = \gamma$, $\Lambda_0 = \frac{\alpha}{\gamma}$ (c_1 being chosen small enough so that $\Lambda_0 \in [c_1, \frac{1}{c_1}]$). We note that $\hat{I}_\varepsilon(\Lambda_0, a_0) = d_0 + o(1)$. For any $\varphi \in \Gamma$, φ_1 is a continuous function from B to $[\frac{c_1}{2}, \frac{2}{c_1}]$ such that $[c_1, \frac{1}{c_1}] \subset \varphi_1(B)$. Thus, there exists $y_0 \in B$ such that $\varphi_1(y_0) = \Lambda_0$, whence

$$\min_{y\in B} \hat{I}_\varepsilon\big(\varphi(y)\big) \leqslant \hat{I}_\varepsilon\big(\Lambda_0, \varphi_2(y_0)\big) \leqslant \frac{\alpha}{\gamma} H\big(\varphi_2(y_0)\big) - \alpha\ln\alpha + \alpha\ln\gamma + o(1) \leqslant d_0 = o(1).$$

As a consequence

$$c = d_0 + o(1) = \alpha - \alpha\ln\alpha + \alpha\ln\gamma + o(1). \tag{4.10}$$

For $y \in B_0$, we have $\varphi_1(y) = c_1$ or $\varphi_1(y) = \frac{1}{c_1}$. In the first case, we have $\hat{I}_\varepsilon(y) = c_1 H(\varphi_2(y)) - \alpha\ln c_1 + o(1) > \alpha\ln\frac{1}{c_1} + o(1) > 2d_0 > c$, provided c_1 is small enough. In the latter case, we have $\hat{I}_\varepsilon(y) = \frac{1}{c_1} H(\varphi_2(y)) + \alpha\ln c_1 + o(1) > \frac{\gamma_1}{c_1} + \alpha\ln c_1 + o(1) > 2d_0 > c$, provided again c_1 is small enough. So (H1) is verified.

To check (H2), we observe that $\partial(\Sigma_0) = (\{\frac{c_1}{2}\} \times (\partial\Omega)_{\gamma_0}) \cup (\{\frac{2}{c_1}\} \times (\partial\Omega)_{\gamma_0}) \cup ([c_1, \frac{1}{c_1}] \times (\partial(\partial\Omega)_{\gamma_0}))$. Let $y = (y_1, y_2) \in \partial\Sigma_0$ be such that $\hat{I}_\varepsilon(y) = c$.

On $(\{\frac{c_1}{2}\} \times (\partial\Omega)_{\gamma_0}) \cup (\{\frac{2}{c_1}\} \times (\partial\Omega)_{\gamma_0})$, previous arguments show that $\hat{I}_\varepsilon(y) > c$ as c_1 is chosen sufficiently small. On $([c_1, \frac{1}{c_1}] \times (\partial((\partial\Omega)_{\gamma_0}))$, taking $\tau_y = \frac{\partial}{\partial\Lambda}$, we obtain

$$\partial_{\tau_y} \hat{I}_\epsilon(y) = H(y_2) - \frac{\alpha}{\Lambda} + o(1) \neq 0$$

since $\partial_{\tau_y} \hat{I}_\epsilon(y) = 0$ would yield $\Lambda H(y_2) = \alpha + o(1)$, and

$$\hat{I}_\varepsilon(y) = \alpha - \alpha\ln\alpha + \alpha\ln H\big(\varphi_2(y)\big) + o(1) = \alpha - \alpha\ln\alpha + \alpha\ln\gamma_0 + o(1).$$

Then, (4.10) shows that $\hat{I}_\varepsilon(y) < c$, a contradiction to the assumption. So (H2) is also verified.

In conclusion, we proved that for ε small enough, c is a critical value, i.e. a critical point $(\Lambda_\varepsilon, a_\varepsilon) \in \Sigma_0$ of $\hat{I}_\varepsilon$ exists. Let $u_\varepsilon = W_{\Lambda_\varepsilon,\xi_\varepsilon,\mu,\varepsilon} + \phi_{\Lambda_\varepsilon,\xi_\varepsilon,\mu,\varepsilon}$. u_ε is a nontrivial solution to the problem

$$-\Delta u + \mu\varepsilon^2 u = u_+^{\frac{N+2}{N-2}+\varepsilon} \quad \text{in } \Omega_\varepsilon; \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Then, the strong maximum principle shows that $u_\varepsilon > 0$ in Ω_ε. The fact that u_ε blows up, as ε goes to zero, at a point a such that $H(a) = \max_{P\in\partial\Omega} H(P)$, follows from the construction of u_ε. This concludes the proof of Theorem 1.1.

In the case of $\varepsilon < 0$, we have

$$\hat{I}_\varepsilon(\Lambda, a) = \Lambda H(a) + \alpha\ln(\Lambda) - \tilde{\sigma}_\varepsilon(\Lambda, a).$$

We assume that Ω is nonconvex. Similarly as before, we define

$$(\partial\Omega)_\delta = \big\{a \in \partial\Omega \mid H(a) < -\delta\big\}$$

where $0 < \delta < \gamma = -\min_{a\in\partial\Omega} H(a) > 0$, and

$$\Sigma_0 = \left\{(\Lambda, a) \,\middle|\, \frac{c_1}{2} \leqslant \Lambda \leqslant \frac{2}{c_1},\ a \in (\partial\Omega)_{\gamma_0}\right\},$$

$$B = \left\{(\Lambda, a) \,\middle|\, c_1 \leqslant \Lambda \leqslant \frac{1}{c_1},\ a \in (\partial\Omega)_{\gamma_1}\right\}, \qquad B_0 = \{c_1\} \times (\partial\Omega)_\gamma \cup \left\{\frac{1}{c_1}\right\} \times (\partial\Omega)_{\gamma_1}$$

with $\gamma_0 < \gamma_1 < \gamma$.

Let Γ be the class of continuous functions $\varphi : B \to \Sigma_0$ with the property that $\varphi(y) = y$ for all $y \in B_0$. We define the min-max value c as

$$c = \min_{\varphi\in\Gamma} \max_{y\in B} \hat{I}_\varepsilon\big(\varphi(y)\big).$$

Arguing as previously, we find that c is a critical point of $\hat{I}_\varepsilon$. This proves Theorem 1.2.

Appendix

A.1. Error estimates

We recall that, according to the definition of $V_{\Lambda,a,\mu,\varepsilon}$ in Section 2

$$V_{\Lambda,a,\mu,\varepsilon}(x) = U_{\frac{1}{\Lambda\varepsilon},a}(x) - \varphi_{\Lambda,a,\mu,\varepsilon} \tag{A.1}$$

with $\varphi_{\Lambda,a,\mu,\varepsilon}$ satisfying

$$\begin{cases} -\Delta\varphi_{\Lambda,a,\mu,\varepsilon} + \mu\varphi_{\Lambda,a,\mu,\varepsilon} = \mu U_{\frac{1}{\Lambda\varepsilon},a} & \text{in } \Omega, \\ \dfrac{\partial\varphi_{\Lambda,a,\mu,\varepsilon}}{\partial n} = \dfrac{\partial U_{\frac{1}{\Lambda\varepsilon},a}}{\partial n} & \text{on } \partial\Omega. \end{cases} \tag{A.2}$$

This subsection is devoted to an expansion of $\varphi_{\Lambda,a,\mu,\varepsilon}$.

We recall that, through space translation and rotation, we assume that $a = 0$ and Ω is given, in a neighborhood of a, by (2.10) and (2.11). We introduce an auxiliary function φ_0: let φ_0 be such that

$$\begin{cases} \Delta\varphi_0 = 0 & \text{in } \mathbb{R}^N_+ = \{(x', x_N), x_N > 0\}, \\ \dfrac{\partial\varphi_0}{\partial x_N} = \frac{N-2}{2} \dfrac{\sum_{i=1}^{N-1} k_i x_i^2}{(1+|x|^2)^{\frac{N}{2}}} & \text{on } \partial\mathbb{R}^N_+, \\ \varphi_0(x) \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{A.3}$$

Using Green's representation, φ_0 writes as

$$\varphi_0(x) = \frac{1}{\omega_{N-1}} \sum_{i=1}^{N-1} k_i \int_{\mathbb{R}^{N-1}} \frac{y_i^2}{(1+|y'|^2)^{\frac{N}{2}}} \frac{1}{|x-y'|^{N-2}} \, \mathrm{d}y' \tag{A.4}$$

where ω_{N-1} denotes the measure of the unit sphere in $\mathbb{R}^N$. From (A.4) we deduce that

$$\left|\varphi_0(x)\right| \leqslant \frac{C}{(1+|x|)^{N-3}} \tag{A.5}$$

and

$$\left|\nabla\varphi_0(x)\right| \leqslant \frac{C}{(1+|x|)^{N-2}}, \qquad \left|D^2\varphi_0(x)\right| \leqslant \frac{C}{(1+|x|)^{N-1}}. \tag{A.6}$$

Definition. From now on, we consider φ_0 as a smooth continuation in $\mathbb{R}^N$ of the previous function defined in $\mathbb{R}^N_+$, such that (A.5), (A.6) hold in whole $\mathbb{R}^N$.

We state:

Lemma A.1. *For $N \geqslant 4$, we have the expansion*

$$\varphi_{\Lambda,a,\mu,\varepsilon}(x) = (\Lambda\varepsilon)^{\frac{4-N}{2}} \varphi_0\left(\frac{x-a}{\Lambda\varepsilon}\right) + O\left(\varepsilon^{\frac{6-N}{2}} |\ln\varepsilon|^m\right) \tag{A.7}$$

with $m = 1$ for $N = 4$ and $m = 0$ for $N \geqslant 5$. Moreover,

$$\left|\varphi_{\Lambda,a,\mu,\varepsilon}(x)\right| \leqslant C \frac{\varepsilon^{\frac{4-N}{2}} |\ln\varepsilon|^n}{(1+|(x-a)/(\Lambda\varepsilon)|)^{N-3}} \quad \textit{and} \quad \left|\varphi_{\Lambda,a,\mu,\varepsilon}(x)\right| \leqslant C\left(U_{\frac{1}{\Lambda\varepsilon},a}(x)\right)^{1-\tau} \tag{A.8}$$

with $n = 1$ and $\tau > 0$ is any small fixed number for $N = 4, 5$, $n = 0$ and $\tau = 0$ for $N \geqslant 6$.

Proof. We first remark that the second inequality in (A.8) is a straightforward consequence of the first one. Next, we decompose

$$\varphi = \varphi^1 + \varphi^2$$

where φ^1 satisfies

$$\begin{cases} -\Delta\varphi^1_{\Lambda,a,\mu,\varepsilon} + \mu\varphi^1_{\Lambda,a,\mu,\varepsilon} = 0 & \text{in } \Omega,\\ \dfrac{\partial\varphi^1_{\Lambda,a,\mu,\varepsilon}}{\partial n} = \dfrac{\partial U_{\frac{1}{\Lambda\varepsilon},a}}{\partial n} & \text{on } \partial\Omega \end{cases}$$

and φ^2 satisfies

$$\begin{cases} -\Delta\varphi^2_{\Lambda,a,\mu,\varepsilon} + \mu\varphi^2_{\Lambda,a,\mu,\varepsilon} = \mu U_{\frac{1}{\Lambda\varepsilon},a} & \text{in } \Omega,\\ \dfrac{\partial\varphi^2_{\Lambda,a,\mu,\varepsilon}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us estimate φ^2 first. Let

$$\hat\varphi^j(x) = \varepsilon^{\frac{N-2}{2}}\varphi^j(\varepsilon x).$$

Then $\hat\varphi^2$ satisfies

$$\begin{cases} -\Delta\hat\varphi^2_{\Lambda,a,\mu,\varepsilon} + \mu\varepsilon^2\hat\varphi^2_{\Lambda,a,\mu,\varepsilon} = \mu\varepsilon^2 U_{\frac{1}{\Lambda},\xi} & \text{in } \Omega_\varepsilon,\\ \dfrac{\partial\hat\varphi^2_{\Lambda,a,\mu,\varepsilon}}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Inequality (3.6) of Lemma 3.2 provides us with

$$\left|\hat\varphi^2(x)\right| \leqslant C\varepsilon^2 \int_{\Omega_\varepsilon} \frac{U_{\frac{1}{\Lambda},\xi}}{|x-y|^{N-2}}\,\mathrm{d}y \leqslant C\varepsilon^2 \int_{\Omega_\varepsilon} \frac{dy}{(1+|y-\xi|)^{N-2}|x-y|^{N-2}}$$

whence

$$\left|\hat\varphi^2(x)\right| \leqslant C\frac{\varepsilon^2|\ln\varepsilon|^m}{(1+|x-\xi|)^{N-4}}$$

with $m = 1$ for $N = 4$ and $m = 0$ for $N \geqslant 5$. (For $N \geqslant 5$, see Lemma 2.3 of [21].) Consequently

$$\varphi^2(x) = O\big(\varepsilon^{\frac{6-N}{2}}|\ln\varepsilon|^m\big) \quad \text{and} \quad \left|\varphi^2(x)\right| \leqslant C\frac{\varepsilon^{\frac{4-N}{2}}|\ln\varepsilon|^m}{(1+|(x-a)/(\Lambda\varepsilon)|)^{N-3}}.$$

This finishes the estimate for φ^2. Next we estimate φ^1. To this end, we write

$$\varphi^1_{\Lambda,a,\mu,\varepsilon} = (\Lambda\varepsilon)^{\frac{4-N}{2}}\varphi_0\left(\frac{x-a}{\Lambda\varepsilon}\right) + \varphi^3_{\Lambda,a,\mu,\varepsilon}(x) + \varphi^4_{\Lambda,a,\mu,\varepsilon}(x)$$

where $\varphi^3_{\Lambda,a,\mu,\varepsilon}$ satisfies

$$\begin{cases} -\Delta\varphi^3_{\Lambda,a,\mu,\varepsilon} + \mu\varphi^3_{\Lambda,a,\mu,\varepsilon} = 0 & \text{in } \Omega,\\ \dfrac{\partial\varphi^3_{\Lambda,a,\mu,\varepsilon}}{\partial n} = \dfrac{\partial U_{\frac{1}{\Lambda\varepsilon},a}}{\partial n} - \dfrac{\partial}{\partial n}\left((\Lambda\varepsilon)^{\frac{4-N}{2}}\varphi_0\left(\dfrac{x-a}{\Lambda\varepsilon}\right)\right) & \text{on } \partial\Omega \end{cases}$$

and $\varphi^4_{\Lambda,a,\mu,\varepsilon}$ satisfies

$$\begin{cases} -\Delta\varphi^4_{\Lambda,a,\mu,\varepsilon}+\mu\varphi^4_{\Lambda,a,\mu,\varepsilon}=(\Delta-\mu)\left((\Lambda\varepsilon)^{\frac{4-N}{2}}\varphi_0\left(\frac{x-a}{\Lambda\varepsilon}\right)\right) & \text{in } \Omega,\\ \dfrac{\partial\varphi^4_{\Lambda,a,\mu,\varepsilon}}{\partial n}=0 & \text{on } \partial\Omega. \end{cases}$$

The estimate for φ^4 is similar to that of φ^2. Namely, in view of (A.3) and (A.4), inequality (3.6) of Lemma 3.2 gives

$$\begin{aligned} \left|\hat\varphi^4(x)\right| &\leqslant C\varepsilon^3\left(\frac{1}{\varepsilon^2}\int\limits_{\Omega_\varepsilon\setminus\mathbb{R}^N_+}\frac{\mathrm{d}y}{(1+|y-\xi|)^{N-1}|x-y|^{N-2}}+\int\limits_{\Omega_\varepsilon}\frac{\mathrm{d}y}{(1+|y-\xi|)^{N-3}|x-y|^{N-2}}\,\mathrm{d}y\right)\\ &\leqslant C\varepsilon^3\left(\frac{1}{\varepsilon(1+|x-\xi|)^{N-3}}+\frac{|\ln\varepsilon|^p}{(1+|x-\xi|)^{N-5}}\right) \end{aligned}$$

with $p=1$ for $N=5$ and $p=0$ for $N\neq 5$, whence

$$\varphi^4(x)=O(\varepsilon^{\frac{6-N}{2}})\quad\text{and}\quad \left|\varphi^4(x)\right|\leqslant C\frac{\varepsilon^{\frac{4-N}{2}}|\ln\varepsilon|^p}{(1+|(x-a)/(\Lambda\varepsilon)|)^{N-3}}.$$

It only remains to estimate φ^3. For $x\in\partial\Omega\cap B(a,\delta)$, we consider the following change of variable (still assuming $a=0$)

$$\Lambda\varepsilon y'=x',\quad \Lambda\varepsilon y_N=x_N-\rho(x').$$

According to the definition of U and (2.12), we have

$$\begin{aligned} \frac{\partial U_{\frac{1}{\Lambda\varepsilon},a}}{\partial n}(x)&=-(N-2)(\Lambda\varepsilon)^{\frac{N-2}{2}}\frac{\langle x-a,n\rangle}{((\Lambda\varepsilon)^2+|x-a|^2)^{\frac{N}{2}}}\\ &=-\frac{N-2}{2}\frac{(\Lambda\varepsilon)^{\frac{N-2}{2}}}{((\Lambda\varepsilon)^2+|x-a|^2)^{\frac{N}{2}}}\left(\sum_{i=1}^{N-1}k_ix_i^2+O\big(|x'|^3\big)\right)\\ &=-\frac{N-2}{2}\frac{(\Lambda\varepsilon)^{\frac{2-N}{2}}}{(1+|y'|^2)^{\frac{N}{2}}}\left(\sum_{i=1}^{N-1}k_iy_i^2+O\big(\varepsilon|y'|^3\big)\right) \end{aligned}$$

and, using (A.3) and (A.6)

$$\begin{aligned} \frac{\partial}{\partial n}\left((\Lambda\varepsilon)^{\frac{4-N}{2}}\varphi_0\left(\frac{x-a}{\Lambda\varepsilon}\right)\right)&=(\Lambda\varepsilon)^{\frac{2-N}{2}}\left(\nabla'\varphi_0\left(\frac{x-a}{\Lambda\varepsilon}\right)\cdot\nabla'\rho(x)-\frac{\partial\varphi_0}{\partial x_N}\left(\frac{x-a}{\Lambda\varepsilon}\right)\right)\\ &=-\frac{N-2}{2}\frac{(\Lambda\varepsilon)^{\frac{2-N}{2}}}{(1+|y'|^2)^{\frac{N}{2}}}\sum_{i=1}^{N-1}k_iy_i^2+O\left(\frac{\varepsilon^{\frac{4-N}{2}}|y'|}{(1+|y'|)^{N-2}}\right). \end{aligned}$$

Therefore

$$\frac{\partial\hat\varphi^3}{\partial n_x}(x)=\varepsilon^{\frac{N}{2}}\frac{\partial\varphi^3}{\partial n_{\varepsilon x}}(\varepsilon x)=O\left(\frac{\varepsilon^2|x'|}{(1+|x'|)^{N-2}}\right)\quad\text{for } x\in\partial\Omega_\varepsilon\cap B\left(a,\frac{\delta}{\varepsilon}\right).\tag{A.9}$$

On the other hand we have clearly, from (A.6) and the definition of U

$$\frac{\partial\hat\varphi^3}{\partial n}(x)=O(\varepsilon^{N-1})\quad\text{for } x\in\partial\Omega_\varepsilon\cap B^c\left(a,\frac{\delta}{\varepsilon}\right).\tag{A.10}$$

Then, standard elliptic theory shows that $\hat{\varphi}^3 = O(\varepsilon^2)$ uniformly in Ω_ε, whence $\varphi^3(x) = O(\varepsilon^{\frac{6-N}{2}})$ uniformly in Ω. Moreover, (A.9) and (A.10) lead, through Green's representation, to the estimate

$$\left|\hat{\varphi}^3(x)\right| \leqslant C\frac{\varepsilon^2}{(1+|x-\xi|)^{N-4}}$$

whence

$$\left|\varphi^3(x)\right| \leqslant C\frac{\varepsilon^{\frac{4-N}{2}}}{(1+|(x-a)/(\Lambda\varepsilon)|)^{N-3}}.$$

This concludes the proof of Lemma A.1. □

A.2. Integral estimates

Omitting, for sake of simplicity, the indices $\Lambda, a, \mu, \varepsilon$, we state:

Proposition A.1. $N \geqslant 4$. *Assuming that Λ satisfies* (2.3)*, we have the uniform expansions as ε goes to zero*

$$J_\varepsilon(W) = A - B\Lambda|\varepsilon|H(a) + \frac{(N-2)^2A}{4}\varepsilon\ln\Lambda + \left(C + \frac{(N-2)^2A}{4N}\right)\varepsilon + O(\varepsilon^{2-\tau}),$$

$$\frac{\partial J_\varepsilon}{\partial\Lambda}(W) = \frac{(N-2)^2A\varepsilon}{4\Lambda} - BH(a)|\varepsilon| + O(\varepsilon^{2-\tau})$$

with

$$A = (N-2)\int_{\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} \qquad C = -\frac{(N-2)^2}{2}\int_{\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}}\ln U_{1,0} > 0 \tag{A.11}$$

and

$$B = \frac{(N-2)^2}{N-3}\int_{\partial\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}}|y|^2. \tag{A.12}$$

Proof. For sake of simplicity, we assume that $\varepsilon > 0$ (the computations are equivalent as $\varepsilon < 0$). In view of (A.2) and (2.15), we write

$$\int_{\Omega_\varepsilon}\left(|\nabla W|^2 + \mu\varepsilon^2W^2\right) = \int_{\Omega_\varepsilon}(-\Delta W + \mu\varepsilon^2W)W = \int_{\Omega_\varepsilon}\alpha_N U^{\frac{N+2}{N-2}}W = \alpha_N\int_{\Omega_\varepsilon}U^{\frac{2N}{N-2}} - \alpha_N\int_{\Omega_\varepsilon}U^{\frac{N+2}{N-2}}\hat{\varphi}.$$

with $U = U_{\frac{1}{\Lambda},\xi}$. On the other hand

$$\begin{aligned}\int_{\Omega_\varepsilon} W^{\frac{2N}{N-2}+\varepsilon} &= \int_{\Omega_\varepsilon} W^{\frac{2N}{N-2}} + \int_{\Omega_\varepsilon} W^{\frac{2N}{N-2}}(W^\varepsilon - 1)\\ &= \int_{\Omega_\varepsilon}(U-\hat{\varphi})^{\frac{2N}{N-2}} + \varepsilon\int_{\Omega_\varepsilon}(U-\hat{\varphi})^{\frac{2N}{N-2}}\ln(U-\hat{\varphi}) + O\big(\varepsilon^2|\ln\varepsilon|\big)\\ &= \int_{\Omega_\varepsilon} U^{\frac{2N}{N-2}} - \frac{2N}{N-2}\int_{\Omega_\varepsilon} U^{\frac{N+2}{N-2}}\hat{\varphi} + \varepsilon\int_{\Omega_\varepsilon}(U-\hat{\varphi})^{\frac{2N}{N-2}}\ln(U-\hat{\varphi}) + O\big(\varepsilon^2|\ln\varepsilon|\big).\end{aligned}$$

The validity of this expansion can be verified by Lebesgue's Dominated Convergence Theorem and the fact that $|W-U| \leqslant C\varepsilon|\ln\varepsilon|^n U_{\frac{1}{\Lambda},a}^{\frac{N-3}{N-2}}$ (see the first inequality in (A.8) and similar arguments in Section 5 of [34]). Note also that

$$\int_{\Omega_\varepsilon} (U-\hat{\varphi})^{\frac{2N}{N-2}} \ln(U-\hat{\varphi}) = -\frac{N-2}{2}\ln\Lambda \int_{\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} + \int_{\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} \ln U_{1,0} + O(\varepsilon^{1-\tau}).$$

Then, according to the definition (3.28) of J_ε and $\alpha_N = N(N-2)$

$$J_\varepsilon(W) = \left((N-2) + \frac{(N-2)^3}{4N}\varepsilon\right)\int_{\Omega_\varepsilon} U^{\frac{2N}{N-2}} + \frac{N(N-2)}{2}\int_{\Omega_\varepsilon} U^{\frac{N+2}{N-2}}\hat{\varphi}$$
$$+ \frac{(N-2)^3}{4}\varepsilon\ln\Lambda\int_{\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} - \varepsilon\frac{(N-2)^2}{2}\int_{\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}}\ln U_{1,0} + O(\varepsilon^{2-\tau}) \tag{A.13}$$

noticing (see estimates below), that $\int_{\Omega_\varepsilon} U^{\frac{2N}{N-2}} = O(1)$ and $\int_{\Omega_\varepsilon} U^{\frac{N+2}{N-2}}\hat{\varphi} = O(\varepsilon^{1-\tau})$. We observe that

$$\int_{\Omega_\varepsilon} U^{\frac{2N}{N-2}} = \int_{\mathbb{R}^N_+} U_{\frac{1}{\Lambda},0}^{\frac{2N}{N-2}}\left(y', y_N + \frac{\rho(\varepsilon y')}{\varepsilon}\right) + O(\varepsilon^{2-\tau})$$
$$= \int_{\mathbb{R}^N_+} U_{\frac{1}{\Lambda},0}^{\frac{2N}{N-2}}(y', y_N) + \int_{\mathbb{R}^N_+} \frac{\partial U_{\frac{1}{\Lambda},0}^{\frac{2N}{N-2}}}{\partial y_N}(y', y_N)\left(\frac{\rho(\varepsilon y')}{\varepsilon}\right) + O(\varepsilon^{2-\tau})$$

whence

$$\int_{\Omega_\varepsilon} U^{\frac{2N}{N-2}} = \int_{\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} - \frac{1}{2}\Lambda\varepsilon H(a)\int_{\partial\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}}|y|^2\,\mathrm{d}y + O(\varepsilon^{2-\tau}). \tag{A.14}$$

On the other hand, in view of the expansion of $\varphi_{\Lambda,a,\mu,\varepsilon}$ in Lemma A.1, we also have

$$\alpha_N\int_{\Omega_\varepsilon} U^{\frac{N+2}{N-2}}\hat{\varphi}_{\Lambda,a,\mu,\varepsilon} = \Lambda\varepsilon\alpha_N\int_{\Omega_\varepsilon} U_{1,0}^{\frac{N+2}{N-2}}\varphi_0 + O(\varepsilon^{2-\tau}) = \Lambda\varepsilon\alpha_N\int_{\mathbb{R}^N_+} U_{1,0}^{\frac{N+2}{N-2}}\varphi_0 + O(\varepsilon^{2-\tau})$$
$$= \Lambda\varepsilon\int_{\mathbb{R}^N_+}(-\Delta U_{1,0}\varphi_0 + U_{1,0}\Delta\varphi_0) + O(\varepsilon^{2-\tau}) = \Lambda\varepsilon\int_{\partial\mathbb{R}^N_+}\left(-\frac{\partial\varphi_0}{\partial y_N}U_{1,0}\right) + O(\varepsilon^{2-\tau})$$
$$= -\Lambda\varepsilon\frac{N-2}{2}\sum_{j=1}^{N-1}k_j\int_{\partial\mathbb{R}^N_+} U_{1,0}\frac{y_j^2}{(1+|y|^2)^{\frac{N}{2}}} + O(\varepsilon^{2-\tau}).$$

Therefore

$$\alpha_N\int_{\Omega_\varepsilon} U^{\frac{N+2}{N-2}}\hat{\varphi}_{\Lambda,a,\mu,\varepsilon} = -\Lambda\varepsilon\frac{N-2}{2}H(a)\int_{\partial\mathbb{R}^N_+}\frac{|y|^2}{(1+|y|^2)^{N-1}} + O(\varepsilon^{2-\tau}). \tag{A.15}$$

Substituting (A.14) and (A.15) into (A.13), we obtain

$$J_\varepsilon(W) = A - B^* \Lambda \varepsilon H(a) + \frac{(N-2)^2}{4} A\varepsilon \ln \Lambda + \varepsilon\left(\frac{(N-2)^2}{4N} A + C\right) + O(\varepsilon^{2-\tau})$$

where A, C are given in (A.11) and

$$B^* = \frac{N-2}{2} \int_{\partial\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} |y|^2 + \frac{N-2}{4} \int_{\partial\mathbb{R}^N_+} \frac{|y|^2}{(1+|y|^2)^{N-1}}.$$

To make the proof of Proposition A.1 complete, it only remains to show that $B^* = B$ defined by (A.12). In fact, it is easily seen that

$$\int_{\partial\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} |y|^2 = \omega_{N-2} \int_0^\infty \frac{r^N}{(1+r^2)^N}\,\mathrm{d}r = \frac{N-3}{2(N-1)} \omega_{N-2} \int_0^\infty \frac{r^N}{(1+r^2)^{N-1}}\,\mathrm{d}r$$

where ω_{N-2} is the area of the unit sphere in R^{N-1}. The last equality follows from simple integration by parts. Then, we can rewrite B^* as

$$B^* = B = \frac{(N-2)^2}{N-3} \int_{\partial\mathbb{R}^N_+} U_{1,0}^{\frac{2N}{N-2}} |y|^2.$$

The expansions for the derivatives of J_ε are obtained exactly in the same way. □

A.3. *Proof of Lemma 3.2*

We prove (3.6) first. Through scaling, we may assume that $\varepsilon = 1$. Let $G(x, y)$ be the Green's function satisfying

$$-\Delta G(x, y) + \mu G(x, y) = \delta_y \quad \text{in } \Omega, \qquad \frac{\partial G(x, y)}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Then we have for $x \in \Omega$,

$$u(x) = \int_\Omega G(x, y) f(y)\,\mathrm{d}y.$$

So it is enough to show that there exists a constant C, independent of x and y, such that

$$\left|G(x, y)\right| \leqslant \frac{C}{|x-y|^{N-2}}.$$

To this end, we decompose G in two parts:

$$G(x, y) = K\left(|x-y|\right) + H(x, y)$$

where $K(|x-y|)$ is the singular part of G and $H(x, y)$ is the regular part of G. Certainly we have $|K(|x-y|)| \leqslant \frac{C}{|x-y|^{N-2}}$. It remains to show that

$$\left|H(x, y)\right| \leqslant \frac{C}{|x-y|^{N-2}}. \tag{A.16}$$

Note that, if $d(x, \partial\Omega) > d_0 > 0$ or $d(y, \partial\Omega) > d_0 > 0$, then $|H(x, y)| \leqslant C$ and hence (A.16) also holds. So we just need to estimate $H(x, y)$ for $d(x, \partial\Omega)$ and $d(y, \partial\Omega)$ small. Let $y \in \Omega$ be such that $d = d(y, \partial\Omega)$ is small. So there exists a unique point $\bar{y} \in \partial\Omega$ such that $d = |y - \bar{y}|$. Without loss of generality, we may assume $\bar{y} = 0$ and the outer normal at $\bar{y}$ is pointing toward x_N-direction. Let y^* be the reflection point $y^* = (0, \ldots, 0, -d)$ and consider the following auxiliary function

$$H^*(x, y) = K\big(|x - y^*|\big).$$

Then H^* satisfies $\Delta H^* - \mu H^* = 0$ in Ω and on $\partial\Omega$

$$\frac{\partial}{\partial n}\big(H^*(x, y)\big) = -\frac{\partial}{\partial n}\big(K\big(|x - y|\big)\big) + O\left(\frac{1}{d^{N-3}}\right).$$

Hence we derive that

$$H(x, y) = -H^*(x, y) + O\left(\frac{1}{d^{N-3}}\right)$$

which proves (A.16) for $x, y \in \Omega$. This implies that for $x \in \Omega$

$$|u(x)| \leqslant C \int_{\Omega} \frac{1}{|x - y|^{N-2}} \big|f(y)\big| \, dy. \tag{A.17}$$

If $x \in \partial\Omega$, we consider a sequence of points $x_i \in \Omega, x_i \to x \in \partial\Omega$ and take the limit in (A.17). Lebesgue's Dominated Convergence Theorem applies and (3.6) is proved.

We turn now to the proof of (3.7). By Lemma 3.1, we have

$$\|u\|_{L^t_\beta(\Omega_\varepsilon)} \leqslant C\|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)}$$

hence

$$\|\varepsilon^2 u\|_{L^t_{\beta+2}(\Omega_\varepsilon)} \leqslant C\|u\|_{L^t_\beta(\Omega_\varepsilon)} \leqslant C\|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)}.$$

By a usual transformation and extension (as done in Step 2 of Proof of Theorem 2.1 in [30]) and interpolation, one can show that

$$\|u\|_{W^{2,t}_\beta(B_{\delta/\varepsilon}(\xi))} \leqslant C\|\varepsilon^2 u\|_{L^t_{\beta+2}(\Omega_\varepsilon)} + C\|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)} \leqslant C\|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)}, \tag{A.18}$$

where δ is a small fixed constant. Next we take a cut-off function $\chi(x)$ such that $\chi(x) = 1$ for $|x| \leqslant \frac{\delta}{2}$ and $\chi(x) = 0$ for $|x| > \delta$, and we consider the function

$$u^1(x) = u(y)\big(1 - \chi(\varepsilon y - \xi)\big)$$

which satisfies

$$-\Delta_x u^1 + \mu\varepsilon^2 u^1 = 2\varepsilon\nabla_y u.\nabla_x\chi + \varepsilon^2 u\Delta_x\chi + f(1 - \chi)$$

in $\tilde{\Omega} = \Omega\backslash\{|x - a| < \delta\}$. Applying the elliptic regularity theory, we obtain

$$\|u^1\|_{W^{2,t}(\tilde{\Omega})} \leqslant C\big\|2\varepsilon\nabla_y u\nabla_x\chi + \varepsilon^2 u\Delta_x\chi + f(1 - \chi)\big\|_{L^t(\tilde{\Omega})}$$

whence, taking account of (A.18)

$$\|u^1\|_{W^{2,t}_\beta(\Omega_\varepsilon\backslash B_{\frac{\delta}{\varepsilon}}(\xi))} \leqslant C\|f\|_{L^t(\tilde{\Omega})} + C\varepsilon^{\beta+2}\|f\|_{L^t_{\beta+2}(\Omega_\varepsilon)}. \tag{A.19}$$

Combining (A.18) and (A.19), we obtain (3.7). □

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