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Ann. I. H. Poincaré – AN 22 (2005) 543–555
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Pointwise curvature estimates for F-stable hypersurfaces

Estimations ponctuelles de la courbure des hypersurfaces F-stables

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Received 2 July 2004; accepted 6 October 2004

Available online 7 April 2005

Abstract

We consider immersed hypersurfaces in euclidean $\mathbb{R}^{n+1}$ which are stable with respect to an elliptic parametric functional of the form $\mathcal{F}(X) = \int_M F(N)\,\mathrm{d}\mu$. We prove a pointwise curvature estimate provided that $n \leqslant 5$ and F is sufficiently close to the area integrand. This extends the pointwise curvature estimates of Schoen, Simon and Yau [Acta Math. 134 (1975) 275] for stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ and of Simon [Math. Z. 154 (1977) 265] for minimizers of $\mathcal{F}$. Our result follows from an integral curvature estimate and a generalized Simons inequality that were established recently [Calc. Var. Partial Differential Equations (2004), DOI: 10.1007/S00526-004-0306-5], together with a Moser type iteration argument.

Résumé

Nous considérons des hypersurfaces dans l'espace euclidien $\mathbb{R}^{n+1}$ qui sont stables par rapport à une fonctionnelle paramétrique elliptique de la forme $\mathcal{F}(X) = \int_M F(N)\,\mathrm{d}\mu$. Nous démontrons une estimation ponctuelle de la courbure sous l'hypothèse $n \leqslant 5$ et F est suffisamment proche de l'intégrande aire. Ce résultat étend l'estimation ponctuelle de la courbure obtenue par Schoen, Simon et Yau [Acta Math. 134 (1975) 275] pour les hypersurfaces minimales stables de $\mathbb{R}^{n+1}$ et par Simon [Math. Z. 154 (1977) 265] pour les minimiseurs de F. Une estimation intégrale de la courbure, une inégalité de Simons généralisée, établie récemment dans [Calc. Var. Partial Differential Equations (2004), DOI : 10.1007/S00526-004-0306-5], ainsi qu'un argument itératif de type Moser nous permet d'obtenir cette estimation ponctuelle.

MSC: 53C42; 49Q10; 35J60

1. Introduction

If u is a solution of the two-dimensional minimal surface equation defined over the disk $B_R(x_0) := \{x \in \mathbb{R}^2\colon |x - x_0| < R\}$, then the principal curvatures κ_1, κ_2 of the corresponding graph can be estimated by

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2004.10.005

$$(\kappa_1^2+\kappa_2^2)(x_0)\leqslant\frac{C}{R^2}$$

with a universal constant C. This is the well-known curvature estimate of Heinz [11] which appears as a quantitative version of Bernstein's celebrated theorem [1]. In fact, by letting $R\to\infty$ one deduces that every entire solution of the minimal surface equation in $\mathbb{R}^2$ has to be an affine linear function.

There are several important extensions and variants of these results, see e.g. [7] and [10]. In particular, Schoen [18] has proved an analogue of Heinz's estimate for stable minimal surfaces in $\mathbb{R}^3$, and again this estimate implies a Bernstein result.

In higher dimensions Schoen, Simon and Yau [19] have obtained an integral curvature estimate for stable minimal hypersurfaces immersed in a Riemannian manifold $\mathcal{N}^{n+1}$, and in the particular case where $\mathcal{N}$ is the euclidean $\mathbb{R}^{n+1}$ this estimate leads to a pointwise curvature estimate up to $n\leqslant 5$. Improvements for $n\leqslant 6$ have been made by Simon [21] and Schoen, Simon [20] using regularity theory for minimal hypersurfaces and methods from geometric measure theory, respectively.

In this paper we consider immersed hypersurfaces $X:M^n\to\mathbb{R}^{n+1}$ with Gauß mapping N and induced surface measure μ which are stable with respect to a parametric functional of the form

$$\mathcal{F}(X)=\int_M F(N)\,\mathrm{d}\mu.$$

The integrand F is of class $C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$ and satisfies the homogeneity condition

$$F(tz)=tF(z)\quad\text{for all } z\in S^n,\ t>0. \tag{1}$$

Moreover, throughout the paper F is assumed to be *elliptic*, i.e.

$$F_{zz}(z)=\left(\frac{\partial^2F}{\partial z^\alpha\partial z^\beta}(z)\right)_{\alpha,\beta=1,\dots,n+1}:z^\perp\to z^\perp$$

is a positive definite endomorphism for all $z\in S^n$, or equivalently

$$\lambda(F):=\inf_{z\in S^n,\ V\in z^\perp\setminus\{0\}}\frac{(\partial^2F/\partial z^\alpha\partial z^\beta)(z)V^\alpha V^\beta}{|V|^2}>0. \tag{2}$$

Clearly, $\mathcal{F}$ generalizes the area functional

$$\mathcal{A}(X)=\int_M \mathrm{d}\mu$$

which is obtained in case $F(z)=A(z):=|z|$ is the *area integrand*.

In a previous paper [25] we have shown that an integral curvature estimate for F-stable hypersurfaces can be proved, whenever $\mathcal{F}$ is sufficiently close to the area functional. To be precise, define the norm

$$\|G\|_{C^k}:=\sup_{z\in S^n}\left(\sum_{|I|\leqslant k}\left|\frac{\partial^{|I|}G}{\partial z^I}(z)\right|^2\right)^{1/2}$$

for an arbitrary integrand $G\in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$. It was proven in [25, Theorem 1.1] that given $n\geqslant 2$ and $p\in(4,4+\sqrt{8/n})$, there exists a constant $\delta(n,p)>0$ with the property that if F is an elliptic integrand satisfying $\|F-A\|_{C^4}<\delta$ and if X is an F-stable hypersurface, then the integral curvature estimate

$$\int_M |S|^p\varphi^p\,\mathrm{d}\mu\leqslant C(n,p,F)\int_M|\nabla\varphi|^p\,\mathrm{d}\mu \tag{3}$$

holds for all nonnegative testfunctions $\varphi \in C_c^\infty(M)$. Here, $|S|$ stands for the length of the Weingarten operator, i.e. $|S|^2 = \kappa_1^2 + \cdots + \kappa_n^2$, where $\kappa_1, \ldots, \kappa_n$ denote the principal curvatures of X.

In this paper we wish to discuss a pointwise curvature estimate for F-stable hypersurfaces. To this end, define

$$\delta_\star(n) := \sup\bigl\{\delta(n,p)\colon\ 4 < p < 4 + \sqrt{8/n},\ p > n\bigr\}$$

for $2 \leqslant n \leqslant 5$. Moreover, let us use the notation

$$\mathcal{B}_R(x_0) := \bigl\{x \in M\colon\ r(x) < R\bigr\}, \quad R > 0,$$

whenever $r : M \to \mathbb{R}$ is a Lipschitz function with $r(x_0) = 0$ and $|\nabla r| \leqslant 1$ μ-a.e.. Our main result is the following:

Theorem 1.1. *Let $2 \leqslant n \leqslant 5$ and let F be an elliptic integrand satisfying $\|F - A\|_{C^4} < \delta_\star(n)$. Suppose X is a stable hypersurface for the parametric functional*

$$\mathcal{F}(X) = \int_M F(N)\,\mathrm{d}\mu.$$

If $\mathcal{B}_R(x_0) \Subset M$ for some point $x_0 \in M$ and radius $R > 0$, and if $\mu(\mathcal{B}_R(x_0)) \leqslant KR^n$, then we have

$$\sup_{\mathcal{B}_{\theta R}(x_0)} |S|^2 \leqslant \frac{C(n, F, K, \theta)}{R^2} \tag{4}$$

for all $\theta \in (0,1)$.

In particular, if we choose $r(x) = d(x, x_0)$, then we obtain the curvature estimate on the usual open balls $B_{\theta R}(x_0)$, and letting $R \to \infty$ we infer the Bernstein result [25]:

Corollary 1.2. *Let $2 \leqslant n \leqslant 5$ and let F be an elliptic integrand with $\|F - A\|_{C^4} < \delta_\star(n)$. Suppose X is a complete connected F-stable hypersurface that satisfies the growth condition*

$$\mu\bigl(B_R(x_0)\bigr) \leqslant KR^n$$

for some point $x_0 \in M$ and some sequence $R \to \infty$. Then $X(M)$ is a hyperplane.

Observe that in case $F = A$ is the area integrand, Theorem 1.1 yields the pointwise curvature estimate of Schoen, Simon and Yau [19, Theorem 3], with $\mathbb{R}^{n+1}$ as the ambient manifold.

Moreover, we remark that curvature estimates and Bernstein type results for two-dimensional parametric functionals have been obtained by Jenkins [12], White [23,24], Lin [13], Sauvigny [17], Fröhlich [8,9], Räwer [16] and Clarenz [2]. All these results are strictly two-dimensional. Simon [22] has obtained a pointwise estimate for *minimizers* of $\mathcal{F}$ up to dimension $n \leqslant 6$ provided that $\|F - A\|_{C^3}$ is sufficiently small. It is unknown, whether our results can be improved to hold up to $n \leqslant 6$.

The paper is organized as follows. In Section 2 we recall some preliminary results taken from [25], including a generalized Simons inequality and an equivalent form of the integral curvature estimate. In Section 3 we then use these results to establish an L^p-estimate for the curvature. Here, we proceed similarly as in Dierkes [5,6], where a class of *singular* parametric functionals of the type

$$\mathcal{E}_\alpha(X) = \int_M |X_{n+1}|^\alpha\,\mathrm{d}\mu, \quad \alpha > 0,$$

is considered. Finally, in Section 4 we use the L^p-estimate together with a Moser type iteration argument to prove (4).

2. Preliminaries

In this section we recall some preliminary results from [25] concerning the geometry of parametric functionals.

Let $X : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of an n-dimensional oriented manifold without boundary into euclidean $\mathbb{R}^{n+1}$. Let $N : M \to S^n$ stand for the corresponding Gauß mapping and denote by μ the induced measure with respect to the pull back g of the euclidean metric. Consider the parametric functional

$$\mathcal{F}(X) = \int_M F(N)\,\mathrm{d}\mu$$

with an elliptic integrand $F \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ satisfying the homogeneity condition (1) and the ellipticity condition (2). We say that X is *F-stationary* if

$$\delta\mathcal{F}(X,\varphi) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(X+\varepsilon\varphi N)\Big|_{\varepsilon=0} = 0$$

for all $\varphi \in C_c^\infty(M)$ and we say that X is *F-stable*, if in addition

$$\delta^2\mathcal{F}(X,\varphi) := \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{F}(X+\varepsilon\varphi N)\Big|_{\varepsilon=0} \geqslant 0$$

for all $\varphi \in C_c^\infty(M)$.

In order to give a geometric description of F-stationary hypersurfaces, we need to recall the notion of *F-mean curvature* as introduced by Räwer [16] and Clarenz [2]: Let A_F be the symmetric, positive definite endomorphism-field given by

$$A_F := \mathrm{d}X^{-1} \circ F_{zz}(N) \circ \mathrm{d}X,$$

and let $S := -\mathrm{d}X^{-1} \circ \mathrm{d}N$ denote the Weingarten operator. Then

$$S_F := A_F \circ S$$

is called *F-Weingarten operator* and

$$H_F := \mathrm{tr}(S_F)$$

is the F-mean curvature of X. Now, according to [16] and [2] the first variation of $\mathcal{F}$ is given by

$$\delta\mathcal{F}(X,\varphi) = -\int_M H_F\varphi\,\mathrm{d}\mu.$$

Hence, X is F-stationary if and only if its F-mean curvature vanishes. Moreover, for an F-stationary hypersurface the second variation is given by

$$\delta^2\mathcal{F}(X,\varphi) = \int_M \big(g(A_F\nabla\varphi,\nabla\varphi) - \mathrm{tr}(A_F S^2)\varphi^2\big)\,\mathrm{d}\mu,$$

where $\nabla\varphi$ denotes the gradient of φ with respect to g. Consequently, X is F-stable if and only if $H_F = 0$ and

$$\int_M \mathrm{tr}(A_F S^2)\varphi^2\,\mathrm{d}\mu \leqslant \int_M g(A_F\nabla\varphi,\nabla\varphi)\,\mathrm{d}\mu \tag{5}$$

for all $\varphi \in C_c^\infty(M)$. For further details see also [3] and [4]. Also note that if F is the area integrand, then $A_F = \mathrm{id}$ and so the definitions coincide with their classical analogues.

Now, as in [25], let us introduce as an additional geometric quantity the abstract metric

$$g_F(v,w) := g\big(A_F^{-1}(v),w\big), \quad (v,w \in TM)$$

which can be seen as an n-dimensional analogue of Sauvigny's [17] weighted first fundamental form. In [25, Propositions 2.2 and 2.3] we have shown the estimates

$$c(F)|S| \leqslant |S_F|_F \leqslant C(F)|S|, \tag{6}$$

$$c(F)|\nabla h| \leqslant |\overline{\nabla} h|_F \leqslant C(F)|\nabla h| \tag{7}$$

and

$$c(F)\,\mathrm{d}\mu \leqslant \mathrm{d}\mu_F \leqslant C(F)\,\mathrm{d}\mu \tag{8}$$

for suitable positive constants $c(F)$ and $C(F)$, where $\overline{\nabla}\varphi$, $|T|_F$ and μ_F denote the gradient of φ, the length of the tensor T and the measure, all taken with respect to g_F. In particular, these estimates imply the equivalence of the integral curvature estimate (3) to an estimate of the form

$$\int_M |S_F|_F^p \varphi^p \,\mathrm{d}\mu_F \leqslant C(n,p,F) \int_M |\overline{\nabla}\varphi|_F^p \,\mathrm{d}\mu_F \tag{9}$$

for all nonnegative functions $\varphi \in C_c^\infty(M)$.

Now, it was shown in [25, Proposition 2.5] that the stability inequality (5) implies

$$\int_M |S_F|_F^2 \varphi^2 \,\mathrm{d}\mu_F \leqslant C(F) \int_M |\overline{\nabla}\varphi|_F^2 \,\mathrm{d}\mu_F \tag{10}$$

for all $\varphi \in C_c^\infty(M)$, where $C(F)$ is a positive constant that tends to 1 as $\|F - A\|_{C^2} \to 0$. Moreover, we have shown that any F-stationary hypersurface X satisfies the generalized Simons inequality

$$\frac{1}{2}\Delta_F |S_F|_F^2 \geqslant \left(\frac{1-\eta}{1+\theta}\right)\left(1+\frac{2}{n}\right)|\overline{\nabla}|S_F|_F|_F^2 - \left(\frac{1}{\lambda(F)} + C(\eta,\theta)\varepsilon(F)\right)|S_F|_F^4 \tag{11}$$

for all $\theta > 0$ and $\eta \in (0,1]$ with a nonnegative constant $\varepsilon(F)$ that tends to 0 as $\|F - A\|_{C^4} \to 0$, cp. [25, Theorem 4.2]. Here, of course, Δ_F denotes the Laplace-Beltrami operator with respect to g_F.

Using (10) and (11), we have shown that if $p \in (4, 4+\sqrt{8/n})$ and if F is an elliptic integrand satisfying $\|F - A\|_{C^4} < \delta(n,p)$, then (9), and hence (3), holds for any F-stable hypersurface X.

In the next section, we do use the generalized Simons inequality (11) and the integral curvature estimate (9) to establish an L^p-estimate for the curvature $|S_F|_F$. To accomplish this, we will need the following version of the Michael–Simon Sobolev inequality [14]:

Lemma 2.1. *Let F be an elliptic integrand and let $X : M \to \mathbb{R}^{n+1}$ be an immersed hypersurface. If $1 \leqslant p < n$, then we have*

$$\left(\int_M |h|^{\frac{np}{n-p}} \,\mathrm{d}\mu_F\right)^{\frac{n-p}{np}} \leqslant C(n,p,F)\left(\int_M \left(|h|^p |S_F|_F^p + |\overline{\nabla} h|_F^p\right)\mathrm{d}\mu_F\right)^{\frac{1}{p}} \tag{12}$$

for all $h \in C_c^\infty(M)$.

Proof. From [14] we infer

$$\left(\int_M |h|^{\frac{np}{n-p}} \,\mathrm{d}\mu\right)^{\frac{n-p}{np}} \leqslant C(n,p)\left(\int_M \left(|h|^p |H|^p + |\nabla h|^p\right)\mathrm{d}\mu\right)^{\frac{1}{p}},$$

so the obvious inequality $|H|^2 \leqslant n|S|^2$ together with (6), (7) and (8) gives the desired result. □

3. L^p-estimate

Let $2 \leqslant n \leqslant 5$, let F be an elliptic integrand satisfying $\|F - A\|_{C^4} < \delta_\star(n)$, and let $X : M \to \mathbb{R}^{n+1}$ be an F-stable hypersurface. Suppose that $\mathcal{B}_1(x_0) \Subset M$ with $\mu(\mathcal{B}_1(x_0)) \leqslant K$. The main result of this section is the following L^p-estimate:

Proposition 3.1. *Let $f := |S_F|_F^p$, $p > 1$, and let $\eta \in C_c^\infty(M)$ be a nonnegative testfunction with* $\operatorname{supp}(\eta) \subset \mathcal{B}_\tau(x_0)$, $0 < \tau < 1$. *Define*

$$q = \begin{cases} \frac{n}{n-2}, & \text{if } n \geqslant 3,\\ 2, & \text{if } n = 2. \end{cases}$$

Then we have

$$\left(\int_M (f\eta)^{2q}\, \mathrm{d}\mu_F\right)^{1/q} \leqslant Cp^\alpha \int_M f^2\eta^2\, \mathrm{d}\mu_F + C\int_M f^2|\overline{\nabla}\eta|_F^2\, \mathrm{d}\mu_F, \tag{13}$$

where $C = C(n, F, K, \tau)$ and $\alpha = \alpha(n, F) > 1$.

Proof. Put $h := f\eta$ in the Sobolev inequality (12). By approximation, this choice of h is valid and if $n \geqslant 3$, then we obtain

$$\left(\int_M (f\eta)^{\frac{2n}{n-2}}\, \mathrm{d}\mu_F\right)^{\frac{n-2}{n}} \leqslant C(n, F)\int_M \left(|\overline{\nabla} f|_F^2\eta^2 + f^2|\overline{\nabla}\eta|_F^2 + f^2\eta^2|S_F|_F^2\right) \mathrm{d}\mu_F.$$

On the other hand, if $n = 2$, then we have

$$\left(\int_M (f\eta)^{\frac{2r}{2-r}}\, \mathrm{d}\mu_F\right)^{\frac{2-r}{2r}} \leqslant C(n, r, F)\left(\int_M \left(\left|\overline{\nabla}(f\eta)\right|_F^r + f^r\eta^r|S_F|_F^r\right) \mathrm{d}\mu_F\right)^{\frac{1}{r}}$$

for all $r \in (1, 2)$, and by virtue of Hölder's inequality together with $\operatorname{supp}(\eta) \subset \mathcal{B}_1(x_0)$ and $\mu_F(\mathcal{B}_1(x_0)) \leqslant C(F)K$, see (8), we infer

$$\int_M \left(\left|\overline{\nabla}(f\eta)\right|_F^r + f^r\eta^r|S_F|_F^r\right) \mathrm{d}\mu_F \leqslant C(r, F, K)\left(\int_M \left|\overline{\nabla}(f\eta)\right|_F^2 \mathrm{d}\mu_F\right)^{\frac{r}{2}} + C(r, F, K)\left(\int_M f^2\eta^2|S_F|_F^2\, \mathrm{d}\mu_F\right)^{\frac{r}{2}},$$

hence

$$\left(\int_M (f\eta)^{\frac{2r}{2-r}}\, \mathrm{d}\mu_F\right)^{\frac{2-r}{r}} \leqslant C(n, r, F, K)\int_M \left(|\overline{\nabla} f|_F^2\eta^2 + f^2|\overline{\nabla}\eta|_F^2 + f^2\eta^2|S_F|_F^2\right) \mathrm{d}\mu_F.$$

Choose $r = \frac{4}{3}$, so that $\frac{r}{2-r} = 2$. Then we have shown

$$\left(\int_M (f\eta)^{2q}\, \mathrm{d}\mu_F\right)^{\frac{1}{q}} \leqslant C_1(n, F, K)\int_M \left(|\overline{\nabla} f|_F^2\eta^2 + f^2|\overline{\nabla}\eta|_F^2 + f^2\eta^2|S_F|_F^2\right) \mathrm{d}\mu_F \tag{14}$$

with

$$q=\begin{cases}\frac{n}{n-2}, & \text{if } n\geqslant 3,\\ 2, & \text{if } n=2.\end{cases}$$

We now use the generalized Simons inequality to estimate the first integral on the right hand side as follows:

Lemma 3.2. *We have*

$$\int\limits_M |\overline{\nabla} f|_F^2\eta^2\,\mathrm{d}\mu_F \leqslant C_2(n,F)p\int\limits_M f^2\eta^2|S_F|_F^2\,\mathrm{d}\mu_F+4\int\limits_M f^2|\overline{\nabla}\eta|_F^2\,\mathrm{d}\mu_F. \tag{15}$$

Proof of Lemma 3.2. We have

$$\begin{aligned}\frac{1}{2}\Delta_F|S_F|_F^{2p}&=p(2p-1)|S_F|_F^{2p-2}|\overline{\nabla}|S_F|_F|_F^2+p|S_F|_F^{2p-1}\Delta_F|S_F|_F\\&=2p(p-1)|S_F|_F^{2p-2}|\overline{\nabla}|S_F|_F|_F^2+\frac{1}{2}p|S_F|_F^{2p-2}\Delta_F|S_F|_F^2\end{aligned}$$

in a weak sense, and applying the generalized Simons inequality (11) we obtain

$$\begin{aligned}\frac{1}{2}\Delta_F|S_F|_F^{2p}\geqslant{}& p(p-1)|S_F|_F^{2p-2}|\overline{\nabla}|S_F|_F|_F^2+p\left\{p-1+\left(\frac{1-\eta}{1+\theta}\right)\left(1+\frac{2}{n}\right)\right\}|S_F|_F^{2p-2}|\overline{\nabla}|S_F|_F|_F^2\\&-p\left(\frac{1}{\lambda(F)}+C(\eta,\theta)\varepsilon(F)\right)|S_F|_F^{2p+2}\end{aligned}$$

for all $\theta>0$ and $\eta\in(0,1]$. Choose $\eta(n)$ and $\theta(n)$ so small that

$$\left(\frac{1-\eta}{1+\theta}\right)\left(1+\frac{2}{n}\right)\geqslant 1.$$

Then we infer

$$\frac{1}{2}\Delta_F|S_F|_F^{2p}\geqslant p^2|S_F|_F^{2p-2}|\overline{\nabla}|S_F|_F|_F^2-p\left(\frac{1}{\lambda(F)}+C(\eta,\theta)\varepsilon(F)\right)|S_F|_F^{2p+2},$$

whence

$$\frac{1}{2}\Delta_F f^2\geqslant|\overline{\nabla} f|_F^2-C_3(n,F)p|S_F|_F^2f^2.$$

We now multiply this inequality by η^2 and integrate by parts. This gives

$$\int\limits_M |\overline{\nabla} f|_F^2\eta^2\,\mathrm{d}\mu_F\leqslant-2\int\limits_M f\eta\overline{\nabla} f\overline{\nabla}\eta\,\mathrm{d}\mu_F+C_3p\int\limits_M f^2\eta^2|S_F|_F^2\,\mathrm{d}\mu_F$$

and in view of Young's inequality

$$2f\eta|\overline{\nabla} f|_F|\overline{\nabla}\eta|_F\leqslant\frac{1}{2}|\overline{\nabla} f|_F^2\eta^2+2f^2|\overline{\nabla}\eta|_F^2$$

the desired estimate follows. □

Combining (14) and (15) we obtain

$$\left(\int\limits_M (f\eta)^{2q}\,\mathrm{d}\mu_F\right)^{\frac{1}{q}}\leqslant C_4(n,F,K)p\int\limits_M f^2\eta^2|S_F|_F^2\,\mathrm{d}\mu_F+C_5(n,F,K)\int\limits_M f^2|\overline{\nabla}\eta|_F^2\,\mathrm{d}\mu_F. \tag{16}$$

We now employ the integral curvature estimate to estimate the first integral on the right-hand side.

Lemma 3.3. *We have*

$$\int_M f^2\eta^2|S_F|_F^2\,\mathrm{d}\mu_F \leqslant C_6\gamma\left(\int_M (f\eta)^{2q}\,\mathrm{d}\mu_F\right)^{\frac{1}{q}} + \gamma^{-\frac{1}{s-1}}\int_M f^2\eta^2\,\mathrm{d}\mu_F \tag{17}$$

for all $\gamma > 0$, where $C_6 = C_6(n, F, K, \tau)$ and $s = s(n, F) > 1$.

Proof of Lemma 3.3. We need the following interpolation inequality from [10, p. 145]

$$ab \leqslant \gamma a^s + \gamma^{-\frac{1}{s-1}} b^{\frac{s}{s-1}}$$

for $a, b \geqslant 0$, $\gamma > 0$ and $s > 1$. Letting $a = |S_F|_F^2$ and $b = 1$, we obtain

$$\int_M f^2\eta^2|S_F|_F^2\,\mathrm{d}\mu_F \leqslant \gamma\int_M f^2\eta^2|S_F|_F^{2s}\,\mathrm{d}\mu_F + \gamma^{-\frac{1}{s-1}}\int_M f^2\eta^2\,\mathrm{d}\mu_F$$

for arbitrary $\gamma > 0$ and $s > 1$. Applying the Hölder inequality with $\frac{1}{q} + \frac{1}{q'} = 1$ and noticing that $\operatorname{supp}(\eta) \subset \mathcal{B}_\tau(x_0)$, it follows that

$$\int_M f^2\eta^2|S_F|_F^2\,\mathrm{d}\mu_F \leqslant \gamma\left(\int_M (f\eta)^{2q}\,\mathrm{d}\mu_F\right)^{\frac{1}{q}}\left(\int_{\mathcal{B}_\tau(x_0)} |S_F|_F^{2sq'}\,\mathrm{d}\mu_F\right)^{\frac{1}{q'}} + \gamma^{-\frac{1}{s-1}}\int_M f^2\eta^2\,\mathrm{d}\mu_F. \tag{18}$$

We now choose s in the following way: According to the definition of $\delta_\star(n)$ there exists a number $t = t(F) \in (4, 4+\sqrt{8/n})$ with $t > n$, such that $\|F - A\|_{C^4} < \delta(n, t)$. Choose

$$s = \begin{cases} \frac{t}{n}, & \text{if } n \geqslant 3,\\ \frac{t}{4}, & \text{if } n = 2.\end{cases}$$

Then, clearly, $s = s(n, F) > 1$. Moreover, on account of

$$q' = \begin{cases} \frac{n}{2}, & \text{if } n \geqslant 3,\\ 2, & \text{if } n = 2\end{cases}$$

we have

$$2sq' = t. \tag{19}$$

Now, let φ be the cut-off function defined by

$$\varphi(x) := \Phi\left(\frac{r(x)-\tau}{1-\tau}\right), \quad x \in M,$$

where $\Phi \in C^1(\mathbb{R})$ is a nonincreasing function with $\Phi(y) = 1$ for $y \leqslant 0$, $\Phi(y) = 0$ for $y \geqslant 1$ and $|\Phi'(y)| \leqslant 2$ for all $y \in \mathbb{R}$. Then $\varphi = 1$ in $\mathcal{B}_\tau(x_0)$, $\varphi = 0$ in $M \setminus \mathcal{B}_1(x_0)$ and using (7) we see that

$$|\overline{\nabla}\varphi|_F \leqslant C(F)|\nabla\varphi| \leqslant \frac{2C(F)}{1-\tau}$$

μ_F-a.e.. Also note that φ is compactly supported in M subject to $\mathcal{B}_1(x_0) \Subset M$. Hence, we can apply φ in the integral curvature estimate (9) with exponent t and together with (19) and $\mu_F(\mathcal{B}_1(x_0)) \leqslant C(F)K$ this leads to

$$\int_{\mathcal{B}_\tau(x_0)} |S_F|_F^{2sq'}\,\mathrm{d}\mu_F \leqslant \int_M |S_F|_F^t\varphi^t\,\mathrm{d}\mu_F \leqslant C(n, F, t)\int_{\mathcal{B}_1(x_0)} |\overline{\nabla}\varphi|_F^t\,\mathrm{d}\mu_F \leqslant C(n, F, K)\left(\frac{2C(F)}{1-\tau}\right)^t.$$

Thus, we have

$$\left(\int\limits_{\mathcal{B}_\tau(x_0)} |S_F|_F^{2sq'}\,\mathrm{d}\mu_F\right)^{\frac{1}{q'}} \leqslant C_6(n,F,K,\tau)$$

and inserting this into (18) gives the desired result. □

According to (16) and (17) we now have

$$\left(\int\limits_M (f\eta)^{2q}\,\mathrm{d}\mu_F\right)^{\frac{1}{q}} \leqslant C_7(n,F,K,\tau)p\gamma\left(\int\limits_M (f\eta)^{2q}\,\mathrm{d}\mu_F\right)^{\frac{1}{q}} + C_4 p\gamma^{-\frac{1}{s-1}}\int\limits_M f^2\eta^2\,\mathrm{d}\mu_F + C_5\int\limits_M f^2|\overline{\nabla}\eta|_F^2\,\mathrm{d}\mu_F$$

for all $\gamma > 0$, and choosing $\gamma := \frac{1}{2C_7 p}$ we finally arrive at

$$\begin{aligned}\left(\int\limits_M (f\eta)^{2q}\,\mathrm{d}\mu_F\right)^{\frac{1}{q}} &\leqslant 2C_4 p^{1+\frac{1}{s-1}}(2C_7)^{\frac{1}{s-1}}\int\limits_M f^2\eta^2\,\mathrm{d}\mu_F + 2C_5\int\limits_M f^2|\overline{\nabla}\eta|_F^2\,\mathrm{d}\mu_F\\ &\leqslant Cp^\alpha\int\limits_M f^2\eta^2\,\mathrm{d}\mu_F + C\int\limits_M f^2|\overline{\nabla}\eta|_F^2\,\mathrm{d}\mu_F,\end{aligned}$$

where $\alpha = \alpha(n,F) := \frac{s}{s-1} > 1$ and $C = C(n,F,K,\tau)$. This completes the proof of the L^p-estimate. □

4. Proof of the curvature estimate

Before we turn to the proof of Theorem 1.1, let us make sure that it suffices to establish the curvature estimate (4) for $R = 1$.

Indeed, suppose that X satisfies the assumptions of Theorem 1.1. We then consider the hypersurface

$$\widetilde{X}: M \to \mathbb{R}^{n+1}, \quad \widetilde{X} := \frac{1}{R}X.$$

On account of

$$\delta\mathcal{F}(\widetilde{X},\tilde\varphi) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(\widetilde{X}+\varepsilon\tilde\varphi\widetilde{N})\Big|_{\varepsilon=0} = \frac{1}{R^n}\delta\mathcal{F}(X,R\tilde\varphi) = 0$$

and

$$\delta^2\mathcal{F}(\widetilde{X},\tilde\varphi) = \frac{1}{R^n}\delta^2\mathcal{F}(X,R\tilde\varphi) \geqslant 0$$

for all $\tilde\varphi \in C_c^\infty(M)$, we see that $\widetilde{X}$ is F-stable as well. Moreover, if r denotes the *abstract distance function* of X, i.e. the Lipschitz function with $r(x_0) = 0$ and $|\nabla r| \leqslant 1$ μ-a.e., then

$$\tilde r: M \to \mathbb{R}, \quad \tilde r := \frac{1}{R}r$$

defines an abstract distance function for $\widetilde{X}$ and for the corresponding balls we have

$$\widetilde{\mathcal{B}}_\theta(x_0) := \{x \in M:\ \tilde r(x) < \theta\} = \mathcal{B}_{\theta R}(x_0)$$

for all $\theta > 0$. In particular, we see that $\widetilde{\mathcal{B}}_1(x_0) \Subset M$ and since the measure scales with $1/R^n$ we obtain

$$\tilde{\mu}\big(\widetilde{\mathcal{B}}_1(x_0)\big) = \frac{1}{R^n}\mu\big(\mathcal{B}_R(x_0)\big) \leqslant K.$$

Assume now, that the curvature estimate (4) holds for $R = 1$. Then we can apply it to $\widetilde{X}$ and obtain

$$\sup_{\widetilde{\mathcal{B}}_\theta(x_0)} |\widetilde{S}|^2_{\tilde{g}} \leqslant C(n, F, K, \theta)$$

for all $\theta \in (0, 1)$. Moreover, since the principal curvatures scale with R, we have

$$|\widetilde{S}|^2_{\tilde{g}} = R^2|S|^2_g$$

and therefore

$$\sup_{\mathcal{B}_{\theta R}(x_0)} |S|^2_g = \sup_{\widetilde{\mathcal{B}}_\theta(x_0)} \frac{|\widetilde{S}|^2_{\tilde{g}}}{R^2} \leqslant \frac{C(n, F, K, \theta)}{R^2}$$

with the same constant C. Hence, the curvature estimate (4) holds for every $R > 0$.

Proof of Theorem 1.1. According to the discussion above we only have to consider the case $R = 1$, so let us assume that $X : M \to \mathbb{R}^{n+1}$ is F-stable, $\mathcal{B}_1(x_0) \Subset M$ and $\mu(\mathcal{B}_1(x_0)) \leqslant K$. It is our goal to prove the following estimate

$$\sup_{\mathcal{B}_\theta(x_0)} |S_F|^2_F \leqslant C(n, F, K, \theta) \tag{20}$$

for all $\theta \in (0, 1)$ since this will imply the desired curvature estimate

$$\sup_{\mathcal{B}_\theta(x_0)} |S|^2 \leqslant C(n, F, K, \theta)$$

in view of (6).

To establish (20), let $\tilde{\theta} := \frac{1+\theta}{2}$, $\tau := \frac{1+\tilde{\theta}}{2}$ and let ρ', ρ be radii satisfying

$$\theta < \rho' < \rho \leqslant \tilde{\theta} < \tau < 1 \tag{21}$$

and

$$\rho' \geqslant \rho - \frac{1}{2}(\rho - \theta). \tag{22}$$

Define a cut-off function η by

$$\eta(x) := \Phi\left(\frac{r(x) - \rho'}{\rho - \rho'}\right), \quad x \in M,$$

where Φ is to be chosen as in the proof of Lemma 3.3. Then we have $\eta = 1$ in $\mathcal{B}_{\rho'}(x_0)$, $\eta = 0$ in $M \setminus \mathcal{B}_\rho(x_0)$ and

$$|\overline{\nabla}\eta|_F \leqslant \frac{2C(F)}{\rho - \rho'}$$

μ_F-a.e.. Inserting η into the L^p-estimate (13) we infer

$$\left(\int_{\mathcal{B}_{\rho'}(x_0)} f^{2q}\,\mathrm{d}\mu_F\right)^{\frac{1}{q}} \leqslant C_1 p^\alpha \int_{\mathcal{B}_\rho(x_0)} f^2\,\mathrm{d}\mu_F + C_1\left(\frac{2C(F)}{\rho - \rho'}\right)^2 \int_{\mathcal{B}_\rho(x_0)} f^2\,\mathrm{d}\mu_F,$$

where $C_1 = C_1(n, F, K, \tau)$ and $\alpha = \alpha(n, F) > 1$. Since $p > 1$ and $\frac{\tilde{\theta}-\theta}{2(\rho-\rho')} \geqslant 1$ it follows that

$$\left(\int\limits_{\mathcal{B}_{\rho'}(x_0)} f^{2q}\,\mathrm{d}\mu_F\right)^{\frac{1}{q}} \leqslant \frac{C_2(n,F,K,\theta)}{(\rho-\rho')^2} p^\alpha \int\limits_{\mathcal{B}_\rho(x_0)} f^2\,\mathrm{d}\mu_F$$

with $C_2 = \frac{C_1(\tilde{\theta}-\theta)^2}{4} + 4C_1C(F)^2$. Now, let $u := |S_F|_F^2$. Then we have $f^2 = |S_F|_F^{2p} = u^p$ and $f^{2q} = u^{qp}$, and therefore

$$\left(\int\limits_{\mathcal{B}_{\rho'}(x_0)} u^{qp}\,\mathrm{d}\mu_F\right)^{\frac{1}{qp}} \leqslant C_2^{\frac{1}{p}} p^{\frac{\alpha}{p}} (\rho-\rho')^{-\frac{2}{p}} \left(\int\limits_{\mathcal{B}_\rho(x_0)} u^p\,\mathrm{d}\mu_F\right)^{\frac{1}{p}}.$$

Hence, abbreviating

$$I(\sigma,t) := \left(\int\limits_{\mathcal{B}_\sigma(x_0)} u^t\,\mathrm{d}\mu_F\right)^{\frac{1}{t}}$$

we obtain

$$I(\rho',qp) \leqslant C_2^{\frac{1}{p}} p^{\frac{\alpha}{p}} (\rho-\rho')^{-\frac{2}{p}} I(\rho,p) \tag{23}$$

for all $p > 1$ and ρ', ρ satisfying (21) and (22).

From here we intend to employ a well-known iteration scheme originally due to Moser [15]. Let

$$\rho_k := \theta + 2^{-k}(\tilde{\theta}-\theta),$$
$$\rho'_k := \rho_{k+1} = \rho_k - \frac{1}{2}(\rho_k - \theta)$$

and

$$p_k := q^k p_0 \quad \text{with } p_0 > 1$$

for $k = 0, 1, 2, \ldots$. Then we infer from (23)

$$\begin{aligned} I(\rho_{k+1},p_{k+1}) &= I(\rho'_k, qp_k) \leqslant C_2^{\frac{1}{p_k}} p_k^{\frac{\alpha}{p_k}} (\rho_k-\rho'_k)^{-\frac{2}{p_k}} I(\rho_k,p_k) \\ &= C_2^{\frac{1}{p_k}} p_0^{\frac{\alpha}{p_k}} q^{\frac{\alpha k}{p_k}} 4^{\frac{k+1}{p_k}} (\tilde{\theta}-\theta)^{-\frac{2}{p_k}} I(\rho_k,p_k) \leqslant C_3^{\frac{k+1}{p_k}} p_0^{\frac{\alpha}{p_k}} I(\rho_k,p_k), \end{aligned}$$

where $C_3 = C_3(n,F,K,\theta) := \max(4C_2(\tilde{\theta}-\theta)^{-2}, 4q^\alpha)$, and iterating this inequality yields

$$I(\rho_{k+1},p_{k+1}) \leqslant C_3^{\sum_{j=0}^k \frac{j+1}{p_j}} p_0^{\sum_{j=0}^k \frac{\alpha}{p_j}} I(\rho_0,p_0).$$

Now we have $\sum_{j=0}^\infty \frac{\alpha}{p_j} = \frac{\alpha q}{p_0(q-1)}$ and $\sum_{j=0}^\infty \frac{j+1}{p_j} = \frac{q^2}{p_0(q-1)^2}$. Consequently,

$$I(\rho_{k+1},p_{k+1}) \leqslant C_4^{\frac{q^2}{p_0(q-1)^2}} p_0^{\frac{\alpha q}{p_0(q-1)}} I(\rho_0,p_0) \tag{24}$$

for $k = 0, 1, 2, \ldots$ with $C_4 = C_4(n,F,K,\theta) := \max(C_3, 1)$. Now we need the following simple lemma:

Lemma 4.1. *We have*

$$\sup_{\mathcal{B}_\theta(x_0)} u \leqslant \liminf_{k\to\infty} I(\rho_k,p_k).$$

Proof of Lemma 4.1. Given $\varepsilon > 0$, there exists a set $A_\varepsilon \subset \mathcal{B}_\theta(x_0)$ with $\mu_F(A_\varepsilon) > 0$ and

$$u \geqslant \sup_{\mathcal{B}_\theta(x_0)} u - \varepsilon$$

in A_ε. Hence,

$$I(\rho_k, p_k) = \left(\int_{\mathcal{B}_{\rho_k}(x_0)} u^{p_k} \, \mathrm{d}\mu_F \right)^{\frac{1}{p_k}} \geqslant \left(\int_{A_\varepsilon} u^{p_k} \, \mathrm{d}\mu_F \right)^{\frac{1}{p_k}} \geqslant \Big(\sup_{\mathcal{B}_\theta(x_0)} u - \varepsilon \Big) \big(\mu_F(A_\varepsilon)\big)^{\frac{1}{p_k}},$$

and letting $k \to \infty$ we obtain

$$\liminf_{k\to\infty} I(\rho_k, p_k) \geqslant \sup_{\mathcal{B}_\theta(x_0)} u - \varepsilon.$$

The assertion now follows as $\varepsilon \to 0$. □

We now apply Lemma 4.1 to (24) and find

$$\sup_{\mathcal{B}_\theta(x_0)} u \leqslant C_4^{\frac{q^2}{p_0(q-1)^2}} p_0^{\frac{\alpha q}{p_0(q-1)}} I(\rho_0, p_0) = C_4^{\frac{q^2}{p_0(q-1)^2}} p_0^{\frac{\alpha q}{p_0(q-1)}} \left(\int_{\mathcal{B}_{\tilde\theta}(x_0)} u^{p_0} \, \mathrm{d}\mu_F \right)^{\frac{1}{p_0}}.$$

Here, $p_0 > 1$ can be arbitrarily chosen. In particular, letting $p_0 \to 1$ we obtain

$$\sup_{\mathcal{B}_\theta(x_0)} |S_F|_F^2 \leqslant C_5(n, F, K, \theta) \int_{\mathcal{B}_{\tilde\theta}(x_0)} |S_F|_F^2 \, \mathrm{d}\mu_F,$$

where $C_5 := C_4^{\frac{q^2}{(q-1)^2}}$.

Finally, using a suitable cut-off function in a similar fashion as we did before, we infer

$$\int_{\mathcal{B}_{\tilde\theta}(x_0)} |S_F|_F^2 \, \mathrm{d}\mu_F \leqslant C(F, K, \tilde\theta)$$

from our stability inequality (10). Hence, we have

$$\sup_{\mathcal{B}_\theta(x_0)} |S_F|_F^2 \leqslant C_6(n, F, K, \theta)$$

which is the desired estimate (20), and this completes the proof of Theorem 1.1. □

Acknowledgement

The author wishes to thank Prof. Dr. U. Dierkes for many interesting and helpful discussions on this work.

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