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Multi-bump type nodal solutions having a prescribed number of nodal domains: II

Solutions nodales de multi-bosses ayant un nombre de domaines nodales prescrites: II

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Abstract

This paper is a sequel to [Liu and Wang, preprint] in which we studied nodal property of multi-bump type sign-changing solutions constructed by Coti Zelati and Rabinowitz [Comm. Pure Appl. Math. 45 (1992) 1217]. In this paper we remove a technical condition that the nonlinearity is odd, which was used in [Comm. Pure Appl. Math. 45 (1992) 1217; Liu and Wang, Multi-bump type nodal solutions having a prescribed number of nodal domains: I, Ann. I. H. Poincaré – AN 22 (2005) 597–608] for constructing multi-bump type nodal solutions having a prescribed number of nodal domains.

Résumé

Cet article est la suite de [Liu and Wang, preprint] sur l'analyse de la propriété nodale des solutions des multi-bosses, construites par Coti Zelati et Rabinowitz dans [Comm. Pure Appl. Math. 45 (1992) 1217]. Nous supprimons la condition technique que le terme nonlinéaire impair comme elle est utilisée dans [Comm. Pure Appl. Math. 45 (1992) 1217; Liu and Wang, Multi-bump type nodal solutions having a prescribed number of nodal domains: I, Ann. I. H. Poincaré – AN 22 (2005) 597–608], pour construire des solutions nodales de multi-bosses ayant un nombre de domaines nodaux prescrits.

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1. Introduction

Building upon the work of Coti Zelati–Rabinowitz [3], in [5] we have given estimates on the number of nodal domains of multi-bump type nodal solutions and in some cases constructed multi-bump type nodal solutions which have exactly a prescribed number of nodal domains for nonlinear time-independent Schrödinger equations of the form

$$-\Delta u + V(x)u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

which satisfy $u(x) \to 0$ as $|x| \to \infty$, here Ω is a smooth cylindrical unbounded domain in $\mathbf{R}^N$ or the whole space $\mathbf{R}^N$, and the potential function is assumed to be periodic in the unbounded directions of Ω. In particular when the domain is a cylinder in $\mathbf{R}^N$, $\Omega = \omega \times \mathbf{R}$ with $\omega \in \mathbf{R}^{N-1}$ a bounded smooth domain, we have proved the existence of multi-bump type nodal solutions having exactly m nodal domains for any integer $m \geqslant 2$. The current paper is to remove one of the conditions imposed on the nonlinearity f, namely, f is odd in u. This condition plays a crucial role in the construction of *multi-bump nodal solutions* by Coti Zelati–Rabinowitz [3]. In order to remove this condition we shall combine the gluing procedure in [3] with some ideas in using invariant sets of descending flows which has been developed for unbounded domains recently in [1]. Following closely the framework of [3], this requires to use a more precise description of the basic one bump solutions and to modify the gluing procedure of [3] from the beginning, though most of the intermediate arguments of [3] can still be used. For reader's convenience we shall give a detailed construction for the setting studied in [3], namely,

$$-\Delta u + V(x)u = f(x,u), \quad \text{in } \mathbf{R}^N. \tag{2}$$

Let us make the following assumptions.

- (V_1) $V \in C(\mathbf{R}^N, \mathbf{R})$, $V_0 := \inf_{\mathbf{R}^N} V(x) > 0$, is periodic in each of $x_1, \dots, x_N$.
- (f_1) $f \in C^1(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ is periodic in each of $x_1, \dots, x_N$.
- (f_2) $f(x,0) = 0 = f_u(x,0)$.
- (f_3) There is $C > 0$ such that

$$\left|f_u(x,u)\right| \leqslant C\left(1 + |u|^{p-2}\right)$$

  for all $x \in \mathbf{R}^N$, $u \in \mathbf{R}$ where $2 < p < 2^*$.
- (f_4) There is $\mu > 2$ such that

$$0 < \mu F(x,u) := \mu \int_0^u f(x,t)\,\mathrm{d}t \leqslant u f(x,u)$$

  for all $x \in \mathbf{R}^N$, $u \in \mathbf{R} \setminus \{0\}$.

The periodicity conditions imply that Eq. (2) is $\mathbf{Z}^N$ invariant. The weak solutions of (2) correspond to critical points of

$$I(u) := \frac{1}{2} \int_{\mathbf{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x - \int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x,$$

in $E = W^{1,2}(\mathbf{R}^N)$. Define the mountain pass value c as

$$c = \inf_{g \in \Gamma} \sup_{t \in [0,1]} I\big(g(t)\big)$$

where

$$\Gamma = \big\{g \in C\big([0,1], E\big) \mid g(0) = 0,\ I\big(g(1)\big) < 0\big\}.$$

We shall follow [2,3] to use the notations: $I^b = \{u \in E \mid I(u) \leqslant b\}$, $I_a = \{u \in E \mid I(u) \geqslant a\}$, $I_a^b = \{u \in E \mid a \leqslant I(u) \leqslant b\}$, $\mathcal{K} = \{u \in E \mid I'(u) = 0\}$, $\mathcal{K}(c) = \{u \in E \mid I'(u) = 0, I(u) = c\}$, $\mathcal{K}^b = \mathcal{K} \cap I^b$, $\mathcal{K}_a^b = \mathcal{K} \cap I_a^b$.

In [3], it was proved that Eq. (2) has infinitely many k-bump solutions, and in particular that $\mathcal{K}_{kc-\alpha}^{kc+\alpha}/\mathbf{Z}^N$ is infinite, provided that (V_1) and (f_1)–(f_4) and the following condition are satisfied

$(*)$ there is $\alpha > 0$ such that $\mathcal{K}^{c+\alpha}/\mathbf{Z}^N$ is finite.

Under the additional condition that f is odd in u, it was proved that $\mathcal{K}_{kc-\alpha}^{kc+\alpha}/\mathbf{Z}^N$ also contains infinitely many nodal solutions. The condition f being odd in u allows the authors of [3] to use both positive and negative solutions at the same mountain pass level c as basic one-bump solutions which are glued into multi-bump nodal solutions. Without this condition the positive and negative mountain pass solutions may be at *different energy levels*, which makes the gluing procedure in [3] difficult to finish. The main purpose of this paper is to remove the condition that f is odd. We shall develop a modified version of the gluing procedure in [3] to glue the positive and negative mountain pass solutions of different energy levels. This will be done by building upon the main framework of [3] and by developing some new ideas of invariant sets of descending flows which have been very successful recently in dealing with nodal solutions.

Eq. (2) with V and f satisfying the assumptions (V_1) and (f_1)–(f_4) will be discussed in detail. As in [5], we will also discuss two other cases: Eq. (1) with V and f being periodic in x_N and Ω a cylindrical domain, and Eq. (2) with V and f being radially symmetric in $x_1, \ldots, x_n$ and periodic in $x_{n+1}, \ldots, x_N$ for some $1 < n < N$. Results for the latter two cases will only be stated in Sections 3 and 5 since the proofs are almost the same as for the first case.

The paper is organized as follows. Section 2 contains the constructions of basic one-bump positive and negative solutions which will be used as building blocks for constructing multi-bump nodal solutions. Section 3 is devoted to the statements of the main theorems on multi-bump nodal solutions, whose proofs will be given in Section 4. In Section 5 we will state results concerning number of nodal domains of multi-pump nodal solutions together with a few remarks.

2. Basic one-bump positive and negative solutions

In the following E denotes the Sobolev space $W^{1,2}(\mathbf{R}^N)$ with the norm

$$\|u\| = \left(\int_{\mathbf{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x \right)^{1/2}.$$

For two sets $\mathcal{A}, \mathcal{B} \subset E$, the distance between $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\|\mathcal{A} - \mathcal{B}\| = \inf_{u \in \mathcal{A}, v \in \mathcal{B}} \|u - v\|.$$

For $a > 0$, the a-neighborhood of a set $\mathcal{A} \subset E$ is defined by

$$N_a(\mathcal{A}) = \left\{u \in E \,\middle|\, \|u - \mathcal{A}\| < a\right\},$$

whose closure and boundary are denoted by $\overline{N}_a(\mathcal{A})$ and $\partial N_a(\mathcal{A})$, respectively. We will use $|\cdot|$ to represent the norm in $\mathbf{R}^N$. For two sets $A, B \subset \mathbf{R}^N$, the distance between A and B is given by

$$|A - B| = \inf_{x \in A, y \in B} |x - y|.$$

The ball in $\mathbf{R}^N$ centered at x and with radius R will be denoted by $B_R(x)$. The ball in E centered at u and with radius R will be denoted by $\mathcal{B}_R(u)$. Without loss of generality we assume the periods in all directions are equal to 1.

Let $j = (j_1, \dots, j_N) \in \mathbf{Z}^N$ and define translations on the $\mathbf{R}^N$ by

$$\tau_j u(x) = u(x_1 + j_1, \dots, x_N + j_N).$$

For a finite subset E_1 of E and an integer $l \geqslant 1$, we denote

$$\mathcal{T}_l(E_1) = \left\{ \sum_{i=1}^{j} \tau_{k_i} v_i \;\middle|\; 1 \leqslant j \leqslant l,\ v_i \in E_1,\ k_i \in \mathbf{Z}^N \right\}.$$

This set will be used later with a specifically constructed E_1. For any $u \in E$, denote

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

Consider the positive cone $\mathcal{P}^+$ and the negative cone $\mathcal{P}^-$ in E defined by

$$\mathcal{P}^\pm = \{u \in E \mid \pm u \geqslant 0\}.$$

Any $u \in \mathcal{K} \setminus (\mathcal{P}^+ \cup \mathcal{P}^-)$ will be a nodal solution of Eq. (2). In what follows, A_i will always stand for positive constants.

Lemma 2.1. *Let* (V) *and* (f_1)–(f_4) *be satisfied. Then*

(i) *there is $\nu > 0$ such that $\|u\| \geqslant \nu$ for all $u \in \mathcal{K} \setminus \{0\}$,*
(ii) *there is $\underline{c} > 0$ such that $I(u) \geqslant \underline{c}$ for all $u \in \mathcal{K} \setminus \{0\}$,*
(iii) *for all $u \in \mathcal{K} \setminus \{0\}$ with $I(u) \leqslant b$,*

$$\|u\| \leqslant \left(\frac{2\mu b}{\mu - 2} \right)^{1/2},$$

(iv) *for any $b > 0$, there is $\nu_1 > 0$ depending on b such that $\|u^\pm\|_{L^2(\mathbf{R}^N)} \geqslant \nu_1$ for all $u \in \mathcal{K} \setminus (\mathcal{P}^+ \cup \mathcal{P}^-)$ with $I(u) \leqslant b$.*

Proof. See [3, Remark 2.14] for (i) and [3, Lemma 2.17] for (ii), (iii). We will prove (iv) for the negative sign; it is the same for the positive sign. Let u be any nodal solution of Eq. (2). Multiplying (2) with u^- and taking integral we have

$$\|u^-\|^2 = \int_{\mathbf{R}^N} u^- f(x, u^-)\,\mathrm{d}x.$$

By (f_2)–(f_3), there exists $A_1 > 0$ such that

$$\left|f(x, u)\right| \leqslant \frac{V_0}{2}|u| + A_1|u|^{p-1}.$$

Then

$$\|u^-\|^2 \leqslant \frac{V_0}{2}\|u^-\|^2_{L^2(\mathbf{R}^N)} + A_1\|u^-\|^p_{L^p(\mathbf{R}^N)}.$$

Since

$$\|u^-\|_{L^p(\mathbf{R}^N)} \leqslant \|u^-\|^t_{L^2(\mathbf{R}^N)} \|u^-\|^{1-t}_{L^{2^*}(\mathbf{R}^N)}$$

where t satisfies

$$\frac{1}{p} = \frac{t}{2} + \frac{1-t}{2^*},$$

we have by the Sobolev inequality

$$\|u^-\|^2 \leqslant \frac{V_0}{2}\|u^-\|^2_{L^2(\mathbf{R}^N)} + A_2\|u^-\|^{pt}_{L^2(\mathbf{R}^N)}\|u^-\|^{p(1-t)}.$$

By the definition of V_0,

$$\|u^-\|^2 \geqslant V_0\|u^-\|^2_{L^2(\mathbf{R}^N)}.$$

Thus

$$\|u^-\|^2 \leqslant 2A_2\|u^-\|^{pt}_{L^2(\mathbf{R}^N)}\|u^-\|^{p(1-t)}, \tag{3}$$

which implies

$$\|u^-\|^2 \leqslant A_3\|u^-\|^p.$$

Since u is a nodal solution of Eq. (2), $u^- \neq 0$ and the last inequality yields

$$\|u^-\| \geqslant A_3^{-1/(p-2)}. \tag{4}$$

If $I(u) \leqslant b$ then the assertion (iii) and (3), (4) imply

$$A_3^{-2/(p-2)} \leqslant 2A_2\left(\frac{2\mu b}{\mu-2}\right)^{p(1-t)/2}\|u^-\|^{pt}_{L^2(\mathbf{R}^N)},$$

which yields the assertion (iv). □

Let $A: E \to E$ be given by $A(u) := (-\Delta + V)^{-1}[f(\cdot, u(\cdot))]$ for $u \in E$. Then the gradient of I has the form $I'(u) = u - A(u)$. Note that the set of fixed points of A is the same as the set of critical points of I, which is $\mathcal{K}$. By the proof of [3, Proposition 2.1], $I': E \to E$ is locally Lipschitz continuous. Indeed,

$$I(u) = \frac{1}{2}\|u\|^2 - J(u),$$

where

$$J(u) = \int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x,$$

and according to (2.11) in [3], we have for any $u, v \in E$,

$$\big\|J'(u) - J'(v)\big\| \leqslant \big(A_1 + A_2\big(\|u\|^{4/(N-2)} + \|v\|^{4/(N-2)}\big)\big)\|u-v\|.$$

Since nodal solutions are critical points of I outside of $\mathcal{P}^+$ and $\mathcal{P}^-$, our strategy to find nodal solutions is to construct subsets of E containing all the positive and negative solutions of Eq. (2) such that these subsets are strictly positively invariant for the descending flow of I; nodal solutions can then be found outside of these subsets.

The following lemma was proved in [1].

Lemma 2.2. *Let* (V) *and* (f_1)–(f_4) *be satisfied. There is an* $a_0 > 0$ *such that for* $0 < a \leqslant a_0$ *there holds*

(i) $A(\partial N_a(\mathcal{P}^-)) \subset N_a(\mathcal{P}^-)$, *and every nontrivial solution* $u \in N_a(\mathcal{P}^-)$ *of* (2) *is negative*;
(ii) $A(\partial N_a(\mathcal{P}^+)) \subset N_a(\mathcal{P}^+)$, *and every nontrivial solution* $u \in N_a(\mathcal{P}^+)$ *of* (2) *is positive.*

Remark 2.3. Furthermore, according to the proof of [1, Lemma 3.1], we have $A(\overline{N}_a(\mathcal{P}^\pm)) \subset N_a(\mathcal{P}^\pm)$. Lemma 2.2 implies that (cf. [4]) the sets $N_a(\mathcal{P}^\pm)$ are strictly positively invariant for the negative gradient flow φ defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,u) = -I'\big(\varphi(t,u)\big) \quad \text{for } t \geqslant 0 \quad \text{and} \quad \varphi(0,u) = u.$$

That is, $\varphi(t,u)\in N_a(\mathcal{P}^\pm)$ for any $0<t<T(u)$ and $u\in \overline{N}_a(\mathcal{P}^\pm)$, where $T(u)\in(0,\infty]$ is the maximal existence time for the trajectory $\varphi(t,u)$.

Using Lemma 2.2, we can study the behavior of (PS) sequences in the whole space E as well as in $\overline{N}_a(\mathcal{P}^\pm)$. The first part of the next lemma is [3, Proposition 2.31].

Lemma 2.4. *Let* (V) *and* (f_1)–(f_4) *be satisfied. Let* $(u_m)\subset E$ *be such that* $I(u_m)\to b>0$ *and* $I'(u_m)\to 0$. *Then there is an* $l\in\mathbf{N}$ (*depending on* b), $v_1,\dots,v_l\in\mathcal{K}\setminus\{0\}$, *a subsequence of* u_m *and corresponding* $(k_m^i)\subset\mathbf{Z}^N$ *such that*

$$\Bigl\|u_m-\sum_{i=1}^{l}\tau_{k_m^i}v_i\Bigr\|\to 0, \tag{5}$$

$$\sum_{i=1}^{l} I(v_i)=b, \tag{6}$$

and for $i\neq j$,

$$|k_m^i-k_m^j|\to\infty. \tag{7}$$

Moreover, there exists an $a_1\in(0,a_0]$ (*depending on* b) *such that if* $(u_m)\subset\overline{N}_{a_1}(\mathcal{P}^+)$ ($N_{a_1}(\mathcal{P}^-)$, *resp.*) *then* $v_1,\dots,v_l\in(\mathcal{K}\setminus\{0\})\cap\mathcal{P}^+$ ($(\mathcal{K}\setminus\{0\})\cap\mathcal{P}^-$, *resp.*).

Proof. We only need to prove the second part. This will be done for the positive sign $+$; the case for the negative sign $-$ is the same. Let ν_1 and a_0 be the two numbers from Lemmas 2.1 and 2.2, respectively. Define

$$a_1=\min\biggl(a_0,\frac{V_0\nu_1}{2}\biggr). \tag{8}$$

Suppose that $(u_m)\subset\overline{N}_{a_1}(\mathcal{P}^+)$ satisfies $I(u_m)\to b>0$ and $I'(u_m)\to 0$. Then according to the first part of the result, there is an $l\in\mathbf{N}$ (depending on b), $v_1,\dots,v_l\in\mathcal{K}\setminus\{0\}$, a subsequence of u_m and corresponding $(k_m^i)\subset\mathbf{Z}^N$ such that (5)–(7) hold. Choose $w_m\in\mathcal{P}^+$ such that

$$\|u_m-w_m\|\leqslant a_1. \tag{9}$$

By (5) and (9),

$$\limsup_{m\to\infty}\Bigl\|\sum_{i=1}^{l}\tau_{k_m^i}v_i-w_m\Bigr\|\leqslant a_1.$$

Arguing indirectly, we assume that $v_i\notin(\mathcal{K}\setminus\{0\})\cap\mathcal{P}^+$ for some $i\in\{1,\dots,l\}$. Rewrite the last inequality as

$$\limsup_{m\to\infty}\Bigl\|v_i+\sum_{j\neq i}\tau_{k_m^j-k_m^i}v_j-\tau_{-k_m^i}w_m\Bigr\|\leqslant a_1.$$

Denote

$$\Omega_i^-=\bigl\{x\in\mathbf{R}^N\mid v_i(x)<0\bigr\}.$$

For any $\epsilon>0$ and $R>0$, since v_j $(1\leqslant j\leqslant l)$ are solutions of (2) and $|k_m^j-k_m^i|\to\infty$ for $j\neq i$, if m is sufficiently large then for $x\in B_R(0)$,

$$\Bigl|\sum_{j\neq i}\tau_{k_m^j-k_m^i}v_j(x)\Bigr|\leqslant\epsilon_1:=\frac{\epsilon}{(\mathrm{meas}(B_R(0)))^{1/2}},$$

where meas$(B_R(0))$ is the measure of $B_R(0)$. For such m,

$$\begin{aligned}\Big\|v_i + \sum_{j\neq i}\tau_{k_m^j-k_m^i}v_j - \tau_{-k_m^i}w_m\Big\| &\geqslant V_0\Big\|v_i + \sum_{j\neq i}\tau_{k_m^j-k_m^i}v_j - \tau_{-k_m^i}w_m\Big\|_{L^2(\mathbf{R}^N)}\\ &\geqslant V_0\Big\|v_i + \sum_{j\neq i}\tau_{k_m^j-k_m^i}v_j - \tau_{-k_m^i}w_m\Big\|_{L^2(B_R(0))}\\ &\geqslant V_0\Big\|v_i + \sum_{j\neq i}\tau_{k_m^j-k_m^i}v_j - \epsilon_1 - \tau_{-k_m^i}w_m\Big\|_{L^2(B_R(0)\cap\Omega_i^-)} - V_0\epsilon.\end{aligned}$$

Since on $B_R(0)\cap\Omega_i^-$, v_i is negative,

$$-2\epsilon_1 \leqslant \sum_{j\neq i}\tau_{k_m^j-k_m^i}v_j - \epsilon_1 \leqslant 0,$$

and $\tau_{-k_m^i}w_m$ is positive, we have

$$\begin{aligned}\Big\|v_i + \sum_{j\neq i}\tau_{k_m^j-k_m^i}v_j - \epsilon_1 - \tau_{-k_m^i}w_m\Big\|_{L^2(B_R(0)\cap\Omega_i^-)} &\geqslant \Big\|v_i + \sum_{j\neq i}\tau_{k_m^j-k_m^i}v_j - \epsilon_1\Big\|_{L^2(B_R(0)\cap\Omega_i^-)}\\ &\geqslant \|v_i\|_{L^2(B_R(0)\cap\Omega_i^-)} - 2\epsilon.\end{aligned}$$

Thus

$$\limsup_{m\to\infty}\Big\|\sum_{i=1}^{l}\tau_{k_m^i}v_i - w_m\Big\| \geqslant V_0\|v_i\|_{L^2(B_R(0)\cap\Omega_i^-)} - 3V_0\epsilon,$$

which implies

$$a_1 \geqslant V_0\|v_i\|_{L^2(B_R(0)\cap\Omega_i^-)} - 3V_0\epsilon.$$

Letting $\epsilon\to 0$ and $R\to\infty$ yields

$$a_1 \geqslant V_0\|v_i^-\|_{L^2(\mathbf{R}^N)}.$$

By Lemma 2.1, we have $a_1 \geqslant V_0\nu_1$, contradicting (8). □

For $a\in[0,a_1]$, we define

$$\Gamma_a^{\pm} = \big\{g\in C\big([0,1],\overline{N}_a(\mathcal{P}^{\pm})\big)\mid g(0)=0 \text{ and } I\big(g(1)\big)<0\big\}$$

and

$$c_a^{\pm} = \inf_{g\in\Gamma_a^{\pm}}\max_{\theta\in[0,1]} I\big(g(\theta)\big).$$

For $a=0$, $\overline{N}_a(\mathcal{P}^{\pm}) = \mathcal{P}^{\pm}$. In this case, we denote $\Gamma^{\pm}=\Gamma_0^{\pm}$ and $c^{\pm}=c_0^{\pm}$.

Lemma 2.5. *Let* (V) *and* (f_1)–(f_4) *be satisfied. Then there exists* $a_2\in(0,a_1)$ *such that* $c_a^{\pm}=c^{\pm}$ *for all* $a\in(0,a_2]$.

Proof. We only prove $c_a^+=c^+$. It is similar to prove $c_a^-=c^-$. By (f_2)–(f_3), for any $\epsilon>0$ there exists $A_\epsilon>0$ such that for $u\in E$

$$\int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x \leqslant \epsilon\|u\|_{L^2(\mathbf{R}^N)}^2 + A_\epsilon\|u\|_{L^p(\mathbf{R}^N)}^p.$$

For $r \in [2, 2^*]$ there exists $K_r > 0$ such that for $u \in E$,

$$\|u^-\|^r_{L^r(\mathbf{R}^N)} \leqslant \inf_{v\in\mathcal{P}^+} \|u - v\|^r_{L^r(\mathbf{R}^N)} \leqslant K_r \inf_{v\in\mathcal{P}^+} \|u - v\|^r \leqslant K_r \|u - \mathcal{P}^+\|^r.$$

For $u \in E$, since $\|u^-\| \geqslant \|u - \mathcal{P}^+\|$, we have

$$\begin{aligned} I(u^-) &= \frac{1}{2}\|u^-\|^2 - \int_{\mathbf{R}^N} F(x, u^-)\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\|u - \mathcal{P}^+\|^2 - \epsilon K_2 \|u - \mathcal{P}^+\|^2 - A_\epsilon K_p \|u - \mathcal{P}^+\|^p. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, there exists $a_2 \in (0, a_1)$ such that $I(u^-) > 0$ if $0 < \|u - \mathcal{P}^+\| \leqslant a_2$. Let $0 < a \leqslant a_2$. The definition of c_a^+ implies $c_a^+ \leqslant c_0^+$. Now for any $\epsilon > 0$ there exists $g \in \Gamma_a^+$ such that

$$\max_{\theta\in[0,1]} I\big(g(\theta)\big) \leqslant c_a^+ + \epsilon.$$

Since $\|g(\theta) - \mathcal{P}^+\| \leqslant a \leqslant a_2$, $I((g(\theta))^-) \geqslant 0$. But $I(g(\theta)) = I((g(\theta))^-) + I((g(\theta))^+)$. Therefore

$$\max_{\theta\in[0,1]} I\big(\big(g(\theta)\big)^+\big) \leqslant c_a^+ + \epsilon.$$

Since the map $\varphi^+ : E \to E$ defined by $\varphi^+(u) = u^+$ is continuous [3, Proposition 7.2], $(g(\cdot))^+$ is continuous from $[0, 1]$ to $\mathcal{P}^+$, which yields $c_0^+ \leqslant c_a^+ + \epsilon$. Letting $\epsilon \to 0$, we have $c_0^+ \leqslant c_a^+$ for $0 < a \leqslant a_2$, finishing the proof. □

Denote $\mathcal{K}^i = \mathcal{K} \cap \mathcal{P}^i$ for $i \in \{+, -\}$. We will also use the notations: $(\mathcal{K}^i)^b = \mathcal{K}^i \cap I^b$, $(\mathcal{K}^i)^b_a = \mathcal{K}^i \cap I^b_a$, and $\mathcal{K}^i(c^i) = \mathcal{K}(c^i) \cap \mathcal{P}^i$ for $i \in \{+, -\}$. Instead of $(*)$, we need the following conditions.

$(*)_\pm$ There is $\alpha > 0$ such that $(\mathcal{K}^\pm)^{c^\pm+\alpha}/\mathbf{Z}^N$ is finite.

Choose a representative in E from each equivalent class in $(\mathcal{K}^i)^{c^i+\alpha}/\mathbf{Z}^N$ and denote the resulting set by $\mathcal{F}^i$, $i \in \{+, -\}$. Let $\underline{c} > 0$ be the number from Lemma 2.1 which satisfies $I(u) \geqslant \underline{c}$ for all $u \in \mathcal{K} \setminus \{0\}$. Denote $l^\pm = [(c^\pm + \alpha)/\underline{c}]$. According to [3, Proposition 2.57] or [2, Proposition 1.55], we have

Lemma 2.6. $\mu(\mathcal{T}_{l^\pm}(\mathcal{F}^\pm)) = \inf\{\|u - w\| \mid u \neq w \in \mathcal{T}_{l^\pm}(\mathcal{F}^\pm)\} > 0$.

Now we have a deformation lemma in $\overline{N}_a(\mathcal{P}^\pm)$, which is an analogue of [3, Proposition 2.60].

Lemma 2.7. *Let $i \in \{+, -\}$ and $a \in [0, a_2]$. Assume* (V), (f$_1$)–(f$_4$), *and $(*)_i$. If $b \in (0, c^i + \alpha)$, $\bar\epsilon$ satisfies $0 < b - \bar\epsilon < b + \bar\epsilon < c^i + \alpha$, and $r < \frac{1}{3}\mu(\mathcal{T}_{l^i}(\mathcal{F}^i))$, then there exist $\epsilon \in (0, \bar\epsilon)$, $\eta \in C([0, 1] \times \overline{N}_a(\mathcal{P}^i), \overline{N}_a(\mathcal{P}^i))$, and $\sigma \in C(I^{b+\epsilon} \cap \overline{N}_a(\mathcal{P}^i), [0, 1])$ such that*

1° $\eta(0, u) = u$ *for all* $\overline{N}_a(\mathcal{P}^i)$,
2° $\eta(s, u) = u$ *if* $u \notin I^{b+\bar\epsilon}_{b-\bar\epsilon} \cap \overline{N}_a(\mathcal{P}^i)$,
3° $I(\eta(s, u))$ *is nonincreasing in* s,
4° $\eta(1, I^{b+\epsilon} \cap \overline{N}_a(\mathcal{P}^i) \setminus N_r((\mathcal{K}^i)^{b+\bar\epsilon}_{b-\bar\epsilon})) \subset I^{b-\epsilon} \cap \overline{N}_a(\mathcal{P}^i)$,
5° $\sigma(u) = 0$ *if* $u \in I^{b-\epsilon} \cap \overline{N}_a(\mathcal{P}^i) \setminus N_r((\mathcal{K}^i)^{b+\bar\epsilon}_{b-\bar\epsilon})$ *and* $I(\eta(\sigma(u), u)) = b - \epsilon$ *for all* $u \in I^{b+\epsilon}_{b-\epsilon} \cap \overline{N}_a(\mathcal{P}^i) \setminus N_r((\mathcal{K}^i)^{b+\bar\epsilon}_{b-\bar\epsilon})$,
6° $\|\eta(\sigma(u), u) - u\| \leqslant r$ *for all* $u \in \overline{N}_a(\mathcal{P}^i)$,
7° $\eta(s, \tau_j u) = \tau_j \eta(s, u)$ *for all* $j \in \mathbf{Z}^N$, $s \in [0, 1]$, $u \in \overline{N}_a(\mathcal{P}^i)$.

Proof. This is similar to the proof of [2, Proposition 2.3]. However, we should construct a descending flow of I which makes $\overline{N}_a(\mathcal{P}^i)$ invariant so that the deformation is from $\overline{N}_a(\mathcal{P}^i)$ to itself. First of all, there exists $\delta > 0$ such that

$$\|I'(u)\| \geqslant \delta \quad \text{for } u \in I_{b-\bar\epsilon}^{b+\bar\epsilon} \cap \overline{N}_a(\mathcal{P}^i) \setminus N_{r/50}\big(\mathcal{T}_{l^i}(\mathcal{F}^i)\big). \tag{10}$$

Indeed, if not, there is a sequence $(u_m) \subset I_{b-\bar\epsilon}^{b+\bar\epsilon} \cap \overline{N}_a(\mathcal{P}^i) \setminus N_{r/50}(\mathcal{T}_{l^i}(\mathcal{F}^i))$ such that $I'(u_m) \to 0$ and $I(u_m) \to \gamma \in [b-\bar\epsilon, b+\bar\epsilon]$. By Lemma 2.4, along a subsequence, $u_m \to \mathcal{T}_{l^i}(\mathcal{F}^i)$, contrary to $u_m \notin N_{r/50}(\mathcal{T}_{l^i}(\mathcal{F}^i))$. Now, choose ϵ and $\hat\epsilon$ such that

$$0 < \epsilon < \hat\epsilon < \min\left(\bar\epsilon, \frac{r\delta}{100}\right). \tag{11}$$

Similar to [2], for $u \in E$ let

$$\phi(u) = \frac{\|u - N_{r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})\|}{\|u - N_{r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})\| + \|u - \overline{N}_a(\mathcal{P}^i) \setminus N_{r/4}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})\|}$$

and

$$\psi(u) = \frac{\|u - (I^{b-\hat\epsilon} \cup I_{b+\hat\epsilon}) \cap \overline{N}_a(\mathcal{P}^i)\|}{\|u - (I^{b-\hat\epsilon} \cup I_{b+\hat\epsilon}) \cap \overline{N}_a(\mathcal{P}^i)\| + \|u - I_{b-\epsilon}^{b+\epsilon} \cap \overline{N}_a(\mathcal{P}^i)\|}.$$

Define $\mathcal{V}(u) = 3\hat\epsilon I'(u)/\|I'(u)\|^2$ for $u \in E \setminus \mathcal{K}$. Then $\mathcal{V}$ satisfies

(a) $\|\mathcal{V}(u)\| \leqslant \frac{4\hat\epsilon}{\|I'(u)\|}$,
(b) $I'(u)\mathcal{V}(u) \geqslant 2\hat\epsilon$,
(c) $\mathcal{V}(\tau_k u) = \mathcal{V}(u)$ for all $k \in \mathbf{Z}^N$, $u \in E \setminus \mathcal{K}$.

Set $W(u) = \phi(u)\psi(u)\mathcal{V}(u)$ and let $\eta(s,u)$ with maximal existence interval $[0, S(u))$ be the solution of

$$\frac{d\eta}{ds} = -W(\eta) \quad \text{for } s \geqslant 0 \quad \text{and} \quad \eta(0,u) = u.$$

Then Remark 2.3 shows that $\eta(s,u) \in N_a(\mathcal{P}^i)$ for any $s \in (0, S(u))$ and $u \in \overline{N}_a(\mathcal{P}^i)$, since $\eta(s,u)$ is just a reparameterization of $\varphi(t,u)$ defined there. Indeed,

$$\eta(s,u) = \varphi(t,u)$$

with

$$t = \int_0^s \frac{3\hat\epsilon\phi(\eta(\alpha,u))\psi(\eta(\alpha,u))}{\|I'(\eta(\alpha,u))\|^2}\, d\alpha.$$

In view of this fact, we can get the assertions 1°–3° and 7° immediately. By Lemma 2.4, we can prove that $\eta(s,u)$ exists for all $s > 0$ and $u \in \overline{N}_a(\mathcal{P}^i)$ in the same way as in [2], distinguishing the two cases $u \in Y := (I^{b-\hat\epsilon} \cup I_{b+\hat\epsilon} \cup N_{r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})) \cap \overline{N}_a(\mathcal{P}^i)$ and $u \in \overline{N}_a(\mathcal{P}^i) \setminus Y$. Next we define the required $\sigma \in C(I^{b+\epsilon} \cap \overline{N}_a(\mathcal{P}^i), [0,1])$. For $u \in I_{b-\epsilon}^{b+\epsilon} \cap \overline{N}_a(\mathcal{P}^i) \setminus N_{3r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})$ and $s \in [0,1]$, at least one of the three cases must occur:

(i) $\eta(s,u)$ reaches neither $\partial\mathcal{B}_{r/8}(u)$ nor $\partial I^{b-\epsilon}$,
(ii) $\eta(s,u)$ reaches $\partial\mathcal{B}_{r/8}(u)$ before it reaches $\partial I^{b-\epsilon}$,
(iii) $\eta(s,u)$ reaches $\partial I^{b-\epsilon}$ before it reaches $\partial\mathcal{B}_{r/8}(u)$.

Since $u \notin N_{3r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})$, $\mathcal{B}_{r/8}(u) \cap N_{r/4}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon}) = \emptyset$. In case (i), the definitions of ϕ and ψ yield

$$\phi\big(\eta(s,u)\big) = \psi\big(\eta(s,u)\big) = 1 \quad \text{for all } 0 \leqslant s \leqslant 1.$$

But then we obtain a contradiction

$$2\epsilon \geqslant I(u) - I\big(\eta(1,u)\big) \geqslant \int_0^1 I'\big(\eta(s,u)\big)\mathcal{V}\big(\eta(s,u)\big)\,\mathrm{d}s \geqslant 2\hat\epsilon,$$

which rules out (i). In case (ii), we have either

$$\mathcal{B}_{r/24}(u) \cap N_{r/50}\big(\mathcal{T}_{l^i}(\mathcal{F}^i)\big) = \emptyset \tag{12}$$

or

$$\big(\mathcal{B}_{r/8}(u) \setminus \mathcal{B}_{r/12}(u)\big) \cap N_{r/50}\big(\mathcal{T}_{l^i}(\mathcal{F}^i)\big) = \emptyset. \tag{13}$$

Otherwise, there exist $v \in \mathcal{B}_{r/24}(u) \cap N_{r/50}(\mathcal{T}_{l^i}(\mathcal{F}^i))$ and $w \in (\mathcal{B}_{r/8}(u) \setminus \mathcal{B}_{r/12}(u)) \cap N_{r/50}(\mathcal{T}_{l^i}(\mathcal{F}^i))$. Choose $v_1, w_1 \in \mathcal{T}_{l^i}(\mathcal{F}^i)$ such that $\|v_1 - v\| < r/50$ and $\|w_1 - w\| < r/50$. Then a direct computation shows that $0 < \|v_1 - w_1\| < r$. This contradicts the assumption $r < \frac{1}{3}\mu(\mathcal{T}_{l^i}(\mathcal{F}^i))$ and the definition of $\mu(\mathcal{T}_{l^i}(\mathcal{F}^i))$. No matter (12) or (13), as a consequence of (10) there exist $0 \leqslant s_1 < s_2 \leqslant 1$ such that

$$\big\|\eta(s_1,u) - \eta(s_2,u)\big\| \geqslant \frac{r}{24},$$
$$\big\|I'\big(\eta(s,u)\big)\big\| \geqslant \delta \quad \text{for } s_1 \leqslant s \leqslant s_2,$$

and

$$b - \epsilon \leqslant I\big(\eta(s,u)\big) \leqslant b + \epsilon \quad \text{for } s_1 \leqslant s \leqslant s_2.$$

Then we have

$$\frac{r}{24} \leqslant \big\|\eta(s_1,u) - \eta(s_2,u)\big\| \leqslant \int_{s_1}^{s_2} \phi\psi\|\mathcal{V}\|\,\mathrm{d}s \leqslant \frac{4\hat\epsilon}{\delta}\int_{s_1}^{s_2} \phi\psi\,\mathrm{d}s$$

and

$$2\epsilon \geqslant I\big(\eta(s_1,u)\big) - I\big(\eta(s_2,u)\big) = \int_{s_1}^{s_2} \phi\psi I'\mathcal{V}\,\mathrm{d}s \geqslant 2\hat\epsilon \int_{s_1}^{s_2} \phi\psi\,\mathrm{d}s.$$

The last two inequalities imply $\frac{r}{24} \leqslant \frac{4\epsilon}{\delta}$, which contradicts (11). Thus (ii) is also impossible and (iii) occurs. Now define $\sigma(u)$ to be the time s at which $\eta(s,u)$ reaches $\partial I^{b-\epsilon}$ for $u \in I_{b-\epsilon}^{b+\epsilon} \cap \overline{N}_a(\mathcal{P}^i) \setminus N_{3r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})$; $\sigma(u) = 0$ for $u \in I^{b-\epsilon} \cap \overline{N}_a(\mathcal{P}^i)$; and

$$\sigma(u) = \sup\big\{s\colon\ 0 \leqslant s \leqslant 1,\ I\big(\eta(s,u)\big) \geqslant b - \epsilon\big\}$$

for $u \in I_{b-\epsilon}^{b+\epsilon} \cap \overline{N}_a(\mathcal{P}^i) \cap N_{3r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})$. Then 4° and 5° are satisfied. Obviously, 6° is satisfied for $u \in I_{b-\epsilon}^{b+\epsilon} \cap \overline{N}_a(\mathcal{P}^i) \setminus N_{3r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})$ and $u \in I^{b-\epsilon} \cap \overline{N}_a(\mathcal{P}^i)$. For $u \in I_{b-\bar\epsilon}^{b+\epsilon} \cap \overline{N}_a(\mathcal{P}^i) \cap N_{3r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})$, if $\eta(s,u)$ stays inside $N_{3r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})$ for $0 \leqslant s \leqslant \sigma(u)$ then the fact that $(\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon} \subset \mathcal{T}_{l^i}(\mathcal{F}^i)$ and $r < \frac{1}{3}\mu(\mathcal{T}_{l^i}(\mathcal{F}^i))$ implies that there is a $v \in (\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon}$ such that $\eta(s,u)$ stays inside $\mathcal{B}_{3r/8}(v)$ for $0 \leqslant s \leqslant \sigma(u)$ and 6° is satisfied; if not, there is $\sigma_1(u) \in (0, \sigma(u))$ which is the first time for $\eta(s,u)$ to reach $\partial N_{3r/8}((\mathcal{K}^i)_{b-\bar\epsilon}^{b+\bar\epsilon})$ and the case (iii) above must occur with $\eta(\sigma_1(u),u)$ in place of u and again we have

$$\big\|\eta\big(\sigma(u),u\big) - u\big\| \leqslant \big\|\eta\big(\sigma(u),u\big) - \eta\big(\sigma_1(u),u\big)\big\| + \big\|\eta\big(\sigma_1(u),u\big) - u\big\| \leqslant \frac{r}{8} + \frac{6r}{8} < r. \qquad \square$$

The following theorem asserts existence of one-bump positive and negative solutions at the mountain pass level. These one-bump solutions will be used later to construct multi-bump nodal solutions.

Lemma 2.8. *Let* (V), (f_1)–(f_4) *and* $(*)_\pm$ *be satisfied. Then* $c^\pm$ *are critical values of* I *and there is a critical point* $u^\pm \in \mathcal{K}^\pm$ *such that* $I(u^\pm) = c^\pm$.

Proof. We follow the same way as in the proof of [3, Theorem 2.61]. Let $i \in \{+, -\}$. If the result was not true for c^i then $(*)_i$ would imply $(\mathcal{K}^i)_{c^i-\bar\epsilon}^{c^i+\bar\epsilon} = \emptyset$ for all small $\bar\epsilon > 0$. Choosing any such $\bar\epsilon$, $r < \frac{1}{3}\mu(\mathcal{T}_{l^i}(\mathcal{F}^i))$, and ϵ as given by Lemma 2.7, select $g \in \Gamma^i$ such that

$$\max_{\theta\in[0,1]} I\big(g(\theta)\big) \leqslant c^i + \epsilon.$$

Then by 4° of Lemma 2.7,

$$\max_{\theta\in[0,1]} I\big(\eta\big(1, g(\theta)\big)\big) \leqslant c^i - \epsilon.$$

But 2° of Lemma 2.7 implies $\eta(1, g) \in \Gamma^i$, a contradiction to the definition of c^i. □

By $(*)_\pm$, there is an $\alpha_1 \in (0, \alpha)$ such that

$$(\mathcal{K}^i)_{c^i-\alpha_1}^{c^i+\alpha_1} = \mathcal{K}^i(c^i).$$

Lemma 2.9. *Let* (V), (f_1)–(f_4) *and* $(*)_\pm$ *be satisfied. Then there exist finite sets* $A^+ \subset \mathcal{K}^+(c^+)$ *and* $A^- \subset \mathcal{K}^-(c^-)$ *having the property that for any* $\bar\epsilon_1 \leqslant \frac{\alpha_1}{2}$, $r_1 \leqslant \frac{1}{12}\mu(\mathcal{T}_{l^\pm}(\mathcal{F}^\pm))$, *and* $p \in \mathbf{N}$, *there is an* $\epsilon_1 \in (0, \bar\epsilon_1)$ *and* $g_1^\pm \in \Gamma^\pm$ *such that*

1° $\max_{\theta\in[0,1]} I(g_1^\pm(\theta)) \leqslant c^\pm + \frac{\epsilon_1}{p}$,
2° *if* $I(g_1^\pm(\theta)) > c^\pm - \epsilon_1$ *then* $g_1^\pm(\theta) \in N_{r_1}(A^\pm)$.

Proof. We just need to modify the proof of [2, Proposition 2.22] with the help of Lemma 2.7. For the present case, c, $\mathcal{T}_{\bar l}(\mathcal{F})$, Γ, and $\mathcal{K}(c)$ in the proof of [2, Proposition 2.22] should be replaced with $c^\pm$, $\mathcal{T}_{l^\pm}(\mathcal{F}^\pm)$, $\Gamma^\pm$, and $\mathcal{K}^\pm(c^\pm)$ respectively. Then as in the proof of [2, Proposition 2.22], there exists a finite set $A^\pm \subset \mathcal{K}^\pm(c^\pm)$ such that for $\bar\epsilon_0 = \alpha_1/2$, $r_0 = \frac{1}{12}\mu(\mathcal{T}_{l^\pm}(\mathcal{F}^\pm))$, and $p \in \mathbf{N}$, there exist $\epsilon_0 \in (0, \bar\epsilon_0)$ and $g_0^\pm \in \Gamma^\pm$ such that

$$\max_{\theta\in[0,1]} I\big(g_0^\pm(\theta)\big) \leqslant c^\pm + \frac{\epsilon_0}{p}$$

and

$$I\big(g_0^\pm(\theta)\big) > c^\pm - \epsilon_0 \quad \text{implies} \quad g_0^\pm(\theta) \in N_{r_0}(A^\pm).$$

To prove this $A^\pm$ is valid for any $\bar\epsilon_1 \leqslant \bar\epsilon_0$, $r_1 \leqslant r_0$, and $p \in \mathbf{N}$, we can proceed as in the proof of [2, Proposition 2.22]. Instead of (2.28) in [2], we choose a $\rho > 0$ such that

$$\max_{u\in N_\rho(\mathcal{K}^\pm(c^\pm))} I(u) < c^\pm + \frac{\epsilon_1}{p}.$$

The function $\hat\phi$ in [2] should be replaced with

$$\hat\phi(u) = \frac{\|u - N_{\rho/8}(\mathcal{K}^\pm(c^\pm))\|}{\|u - N_{\rho/8}(\mathcal{K}^\pm(c^\pm))\| + \|u - \mathcal{P}^\pm \setminus N_{\rho/4}(\mathcal{K}^\pm(c^\pm))\|},$$

while setting $\hat\epsilon = \max\{\bar\epsilon_1, \epsilon_0\} < \bar\epsilon_0$, instead of $\hat f$ we define

$$\hat\psi(u) = \frac{\|u - (I^{b-\bar\epsilon} \cup I_{b+\bar\epsilon}) \cap \mathcal{P}^{\pm}\|}{\|u - (I^{b-\bar\epsilon} \cup I_{b+\bar\epsilon}) \cap \mathcal{P}^{\pm}\| + \|u - I_{b-\hat\epsilon}^{b+\hat\epsilon} \cap \mathcal{P}^{\pm}\|}.$$

Note that $\mathcal{K}$ on page [2, p. 710] should also be replaced with $\mathcal{K}^{\pm}(c^{\pm})$. Then one can follow the same line of the proof of [2, Proposition 2.22] to complete the present proof. □

3. Existence of multi-bump type nodal solutions

Depending on whether the domain Ω is the whole space $\mathbf{R}^N$ or a cylindrical unbounded domain and on whether V and f are periodic in all $x_1, \dots, x_N$ or only partially, the results will be stated in distinguished three cases in the following three subsections. In Section 3.1, we will state a result for Eq. (2) in the case where V and f satisfy (V_1) and (f_1)–(f_4). Similar results in two other cases will be stated in Sections 3.2 and 3.3. In Section 3.2, a result for Eq. (1) will be given provided that V and f are periodic in x_N and Ω is a cylindrical domain. A result also for Eq. (2) will be stated in Section 3.3 where it is assumed that V and f are radially symmetric in $x_1, \dots, x_n$ and periodic in $x_{n+1}, \dots, x_N$ for some $1 < n < N$.

3.1. Eq. (2) with V and f satisfying (V_1) and (f_1)–(f_4)

Let $A = A^+ \cup A^-$ with $A^{\pm}$ given in Lemma 2.9. For any fixed integer $k \geqslant 2$ we fix two positive integers k^+ and k^- such that $k = k^+ + k^-$. Denote $\Lambda^+ = \{1, \dots, k^+\}$, $\Lambda^- = \{k^+ + 1, \dots, k\}$. Let $j_i \in \mathbf{Z}^N$ for $i = 1, \dots, k$ be fixed such that $j_i \neq j_m$ for $i \neq m$ and if $v_i \in A^+$ for $i \in \Lambda^+$ and $v_i \in A^-$ for $i \in \Lambda^-$ then

$$\left\| \sum_{i=1}^{k} \tau_{j_i} v_i \right\| \geqslant \frac{k\nu}{2}$$

and

$$\left| I\left(\sum_{i=1}^{k} \tau_{j_i} v_i\right) - (k^+c^+ + k^-c^-) \right| < \frac{\alpha}{2}.$$

Define

$$\mathcal{M}(j_1, \dots, j_k, A, k^+, k^-) = \left\{ \sum_{i=1}^{k} \tau_{j_i} v_i \;\middle|\; v_i \in A^+ \text{ for } i \in \Lambda^+,\ v_i \in A^- \text{ for } i \in \Lambda^- \right\}$$

and

$$b_k = k^+c^+ + k^-c^-.$$

Our main theorem in this paper reads as

Theorem 3.1. *Let* (V_1), (f_1)–(f_4), *and* $(*)_{\pm}$ *be satisfied. Then there is an* $r_0 > 0$ *such that for any* $r \in (0, r_0)$,

$$N_r\big(\mathcal{M}(lj_1, \dots, lj_k, A, k^+, k^-)\big) \cap (\mathcal{K}_{b_k-\alpha}^{b_k+\alpha}/\mathbf{Z}^N) \neq \emptyset$$

for all but finitely many $l \in \mathbf{N}$.

3.2. Eq. (1) with Ω being an unbounded cylindrical domain

In this subsection, we state a result for Eq. (1) in the case where Ω is a cylinder type domain such that the set $\{x' \in \mathbf{R}^{N-1} \mid (x', x_N) \in \Omega$ for some $x_N \in \mathbf{R}\}$ is bounded and $(x', x_N + j) \in \Omega$ for any $(x', x_N) \in \Omega$ and $j \in \mathbf{Z}$. We assume that

($V_{1'}$) $V \in C(\Omega, \mathbf{R})$, $\inf_\Omega V(x) > 0$, is 1-periodic in x_N.
($f_{1'}$) $f \in C^1(\Omega \times \mathbf{R}, \mathbf{R})$ is 1-periodic in x_N.

We understand the assumptions (f_2)–(f_4) are now satisfied for $x \in \Omega$. In this case Eq. (1) is $\mathbf{Z}$ invariant. We define $E = W_0^{1,2}(\Omega)$ with the norm

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x\right)^{1/2}.$$

For $j \in \mathbf{Z}$ and $u \in E$, we define

$$\tau_j u(x', x_N) = u(x', x_N + j)$$

for $(x', x_N) \in \Omega$. Define the same notations as in Sections 2 and 3.1 accordingly. We need to assume

$(*')_\pm$ There is $\alpha > 0$ such that $(\mathcal{K}^\pm)^{c^\pm + \alpha}/\mathbf{Z}$ is finite.

Then all the results in Section 2 have analogues valid in the present case. In particular, we also have two finite sets $A^+ \subset \mathcal{K}^+(c^+)$ and $A^- \subset \mathcal{K}^-(c^-)$ having the property in Lemma 2.9.

Using the same notations before Theorem 3.1 with an understanding of $j_i \in \mathbf{Z}$, we can state the following theorem for Eq. (1).

Theorem 3.2. *Let* ($V_{1'}$), ($f_{1'}$), (f_2)–(f_4), *and* $(*')_\pm$ *be satisfied. Then there is an* $r_0 > 0$ *such that for any* $r \in (0, r_0)$,

$$N_r\left(\mathcal{M}(lj_1, \ldots, lj_k, A, k^+, k^-)\right) \cap (\mathcal{K}_{b_k - \alpha}^{b_k + \alpha}/\mathbf{Z}) \neq \emptyset$$

for all but finitely many $l \in \mathbf{N}$.

3.3. Eq. (2) with V and f being partially radially symmetric and partially periodic

In this subsection, we state a result for Eq. (2). We assume that there is $1 < n < N$ such that

($V_{1''}$) $V \in C(\mathbf{R}^N, \mathbf{R})$, $\inf_{\mathbf{R}^N} V(x) > 0$, is radially symmetric in $x_1, \ldots, x_n$ and 1-periodic in $x_{n+1}, \ldots, x_N$.
($f_{1''}$) $f \in C^1(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ is radially symmetric in $x_1, \ldots, x_n$ and 1-periodic in $x_{n+1}, \ldots, x_N$.

In this case Eq. (2) is $\mathbf{Z}^{N-n}$ invariant. We define

$$E = \left\{u \in W^{1,2}(\mathbf{R}^N) \;\Big|\; \int_{\mathbf{R}^N} V(x)u^2\,\mathrm{d}x < \infty,\ u \text{ is radially symmetric in } x_1, \ldots, x_n\right\}$$

with the norm

$$\|u\| = \left(\int_{\mathbf{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x\right)^{1/2}.$$

Let $j \in \mathbf{Z}^{N-n}$ and $u \in E$ and we define

$$\tau_j u(x_1, \dots, x_n, x_{n+1}, \dots, x_N) = u(x_1, \dots, x_n, x_{n+1} + j_{n+1}, x_N + j_N)$$

for $(x_1, \dots, x_N) \in \mathbf{R}^N$. Define the same notations as in Sections 2 and 3.1 accordingly. Since everything can be confined in E, critical points in $\mathcal{K}$ are radially symmetric in $x_1, \dots, x_n$. We need to assume

$(*'')_\pm$ There is $\alpha > 0$ such that $(\mathcal{K}^\pm)^{c^\pm+\alpha}/\mathbf{Z}^{N-n}$ is finite.

Then all the results in Section 2 are also valid in the present case. With $j_i \in \mathbf{Z}^{N-n}$ being understood, we can state the following theorem for Eq. (2).

Theorem 3.3. *Let* $(\mathrm{V}_{1''})$, $(\mathrm{f}_{1''})$, (f_2)–(f_4), *and* $(*'')_\pm$ *be satisfied. Then there is an* $r_0 > 0$ *such that for any* $r \in (0, r_0)$,

$$N_r\big(\mathcal{M}(lj_1, \dots, lj_k, A, k^+, k^-)\big) \cap (\mathcal{K}_{b_k-\alpha}^{b_k+\alpha}/\mathbf{Z}^{N-n}) \neq \emptyset$$

for all but finitely many $l \in \mathbf{N}$.

4. Proofs of the main theorems

Theorem 3.1 will be proved in detail. Theorems 3.2 and 3.3 can be proved similarly and their proofs will be omitted. As in [3], for $\theta = (\theta_1, \dots, \theta_k) \in [0,1]^k$, let $0_i = (\theta_1, \dots, \theta_{i-1}, 0, \theta_{i+1}, \dots, \theta_k)$ and $1_i = (\theta_1, \dots, \theta_{i-1}, 1, \theta_{i+1}, \dots, \theta_k)$, $1 \leqslant i \leqslant k$. Let a_2 be as in Lemma 2.5 and $a \in [0, a_2]$ and define

$$\Gamma_k(a) = \big\{G = g_1 + \dots + g_k \mid g_i \text{ satisfies } (g_1) - (g_3),\ 1 \leqslant i \leqslant k\big\},$$

where

- (g_1) $g_i \in C([0,1]^k, \overline{N}_a(\mathcal{P}^\pm))$ for $i \in \Lambda^\pm$,
- (g_2) $g_i(0_i) = 0$ and $I(g_i(1_i)) < 0$, $1 \leqslant i \leqslant k$,
- (g_3) There are bounded open sets $\mathcal{O}_i$, $1 \leqslant i \leqslant k$, such that $\overline{\mathcal{O}}_i \cap \overline{\mathcal{O}}_j = \emptyset$ if $i \neq j$ and $\operatorname{supp} g_i(\theta) \subset \mathcal{O}_i$ for all $\theta \in [0,1]^k$.

Lemma 4.1. *Let* (V_1), (f_1)–(f_4), *and* $(*)_\pm$ *be satisfied. Define*

$$b_k(a) = \inf_{G \in \Gamma_k(a)} \max_{\theta \in [0,1]^k} I\big(G(\theta)\big).$$

Then $b_k(a) = b_k = k^+c^+ + k^-c^-$ *for* $a \in (0, a_2]$.

Proof. For each $G \in \Gamma_k(a)$, by the proof of [2, Proposition 3.4], there exists a $\bar\theta \in [0,1]^k$ such that $I(g_i(\bar\theta)) \geqslant c_a^\pm$ for $i \in \Lambda^\pm$. By Lemma 2.5, $I(g_i(\bar\theta)) \geqslant c^\pm$ for $i \in \Lambda^\pm$. Thus

$$\max_{\theta \in [0,1]^k} I\big(G(\theta)\big) \geqslant I\big(G(\bar\theta)\big) = \sum_{i=1}^k I\big(g_i(\bar\theta)\big) \geqslant k^+c^+ + k^-c^- = b_k,$$

and $b_k(a) \geqslant b_k$. Let $\epsilon > 0$. To prove the reversed inequality, choose $g^\pm \in \Gamma^\pm$ such that

$$\max_{t \in [0,1]} I\big(g^\pm(t)\big) \leqslant c^\pm + \frac{\epsilon}{2k}.$$

Let $R > 0$ and $\chi_R \in C^\infty(\mathbf{R}^+, \mathbf{R}^+)$ such that $\chi_R(z) = 1$ if $z \leqslant R$, $-1 \leqslant \chi_R'(z) \leqslant 0$, and $\chi_R(z) = 0$ if $z \geqslant R + 2$. Define

$$\hat{g}^{\pm}(t)(x) = \chi_R\big(|x|\big) g^{\pm}(t)(x).$$

As in the proof of [3, Proposition 3.4], if R is sufficiently large then $\hat{g}^{\pm} \in \Gamma^{\pm}$ and

$$\max_{t\in[0,1]} I\big(\hat{g}^{\pm}(t)\big) \leqslant c^{\pm} + \frac{\epsilon}{k}.$$

Then for $j \in \mathbf{Z}^N$ such that $j_i \neq j_m$ for $i \neq m$ and $l \in \mathbf{N}$ sufficiently large,

$$G(\theta)(x) := \sum_{i\in\Gamma^+} \hat{g}^+(\theta_i)(x + lj_i) + \sum_{i\in\Gamma^-} \hat{g}^-(\theta_i)(x + lj_i) \in \Gamma_k(a)$$

and

$$\max_{\theta\in[0,1]^k} I\big(G(\theta)\big) \leqslant k^+c^+ + k^-c^- + \epsilon.$$

Letting $\epsilon \to 0$ yields $b_k(a) \leqslant k^+c^+ + k^-c^- = b_k$. This completes the proof. □

Define

$$\mathcal{M}^* = \mathcal{M}^*(j_1, \ldots, j_k, A, k^+, k^-) = \bigcup_{l\in\mathbf{N}} \mathcal{M}(lj_1, \ldots, lj_k, A, k^+, k^-).$$

As [2, Proposition 3.12] and [3, Proposition 3.22], we have the following lemma.

Lemma 4.2. *Let* (V_1), (f_1)–(f_4), *and* $(*)_\pm$ *be satisfied. There is an* $r_k = r_k(A, \alpha)$ *such that if* $r \leqslant r_k$ *and* $w \in \overline{N}_r(\mathcal{M}^*(j_1, \ldots, j_k, A, k^+, k^-)) \cap \mathcal{K}$, *then* $w \in \mathcal{K}_{b_k-\alpha}^{b_k+\alpha}$.

As in [2, Remark 3.19], we also assume that $r_k < r_{k-1} < \cdots < r_1$.

Lemma 4.3. *Let* (V_1), (f_1)–(f_4), *and* $(*)_\pm$ *be satisfied and*

$$r < \min\left(\frac{1}{12}\mu\big(\mathcal{T}_{l^\pm}(\mathcal{F}^\pm)\big), \frac{\nu}{2}, r_k\right). \tag{14}$$

Then either

(i) *there is a* $\delta_l = \delta_l(j_1, \ldots, j_k, A, k^+, k^-, r)$ *such that* $\|I'(w)\| \geqslant \delta_l$ *for all* $w \in N_r(\mathcal{M}(lj_1, \ldots, lj_k, A, k^+, k^-))$, *or*
(ii) *there is a* $w \in \overline{N}_r(\mathcal{M}(lj_1, \ldots, lj_k, A, k^+, k^-)) \cap \mathcal{K}$.

Moreover, if

$$\mathcal{L} = \big\{l \in \mathbf{N} \mid \text{(i) holds for } N_r\big(\mathcal{M}(lj_1, \ldots, lj_k, A, k^+, k^-)\big)\big\}$$

and

$$\mathcal{W} = \bigcup_{l\in\mathcal{L}} \mathcal{M}(lj_1, \cdots, lj_k, A, k^+, k^-),$$

then there is a $\delta = \delta(j_1, \ldots, j_k, A, k^+, k^-, r)$ *independent of* l *such that* $\|I'(w)\| \geqslant \delta$ *for all* $w \in N_r(\mathcal{W}) \setminus N_{r/8}(\mathcal{W})$.

This lemma is the same as [3, Proposition 3.23] and can be proved as [2, Proposition 3.20].
Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We will follow the five steps in the proof of [3, Theorem 3.27] and indicate only the differences. Arguing indirectly, we assume that $\mathcal{L}$ is an infinite set.

Step 1: The construction of G. Let r and δ be as in Lemma 4.3 and α_1 be defined before Lemma 2.9. We further require that

$$r < \min\left(\frac{1}{8}, \frac{a_2}{16}\right), \tag{15}$$

where a_2 is the number from Lemma 2.5. Choose

$$\bar{\epsilon}_1 < \min\left(\frac{r\delta}{40}, \frac{\alpha_1}{2}, c^+, c^-\right). \tag{16}$$

With this choice of $\bar{\epsilon}_1$, $r_1 = \frac{r}{16k}$, and $p = 6k$, by Lemma 2.9, there is an $\epsilon = \frac{\epsilon_1}{2} \in (0, \frac{\bar{\epsilon}_1}{2})$ and $g_1^\pm \in \Gamma^\pm$ such that

$$\max_{t\in[0,1]} I\left(g_1^\pm(t)\right) \leqslant c^\pm + \frac{\epsilon}{3k}$$

and

$$I\left(g_1^\pm(t)\right) > c^\pm - 2\epsilon \quad \text{implies} \quad g_1^\pm(t) \in N_{r/(16k)}(A^\pm).$$

By an approximation argument as in Lemma 4.1, there is $g^\pm \in \Gamma^\pm$ and $R > 0$ such that

$$\left\|g^\pm(t) - g_1^\pm(t)\right\| \leqslant \frac{r}{16k},$$
$$\left|I\left(g^\pm(t)\right) - I\left(g_1^\pm(t)\right)\right| \leqslant \frac{\epsilon}{6k},$$

and

$$\operatorname{supp} g^\pm(t) \subset B_{R/2}(0) \quad \text{for all } t \in [0,1]. \tag{17}$$

Then we have

$$\max_{t\in[0,1]} I\left(g^\pm(t)\right) \leqslant c^\pm + \frac{\epsilon}{2k}$$

and

$$I\left(g^\pm(t)\right) > c^\pm - \frac{3\epsilon}{2} \quad \text{implies} \quad g^\pm(t) \in N_{r/(8k)}(A^\pm).$$

For $\theta \in [0,1]^k$ and $l \in \mathcal{L}$, set

$$G(\theta) = \sum_{i\in\Lambda^+} \tau_{lj_i} g^+(\theta_i) + \sum_{i\in\Lambda^-} \tau_{lj_i} g^-(\theta_i). \tag{18}$$

Then

$$\operatorname{supp} G(\theta) \subset \bigcup_{i=1}^{k} B_{R/2}(lj_i). \tag{19}$$

For any $\beta > 0$, since $\mathcal{L}$ is an infinite set, there is an $l \in \mathcal{L}$ such that

$$\left|B_R(lj_i) - B_R(lj_m)\right| \geqslant 2\beta + 4 \quad \text{for } i \neq m. \tag{20}$$

Fix such an $l = l(\beta)$. Then $G \in \Gamma_k(0)$ and G satisfies

$$I\big(G(\theta)\big) = \sum_{i\in\Lambda^+} I\big(g^+(\theta_i)\big) + \sum_{i\in\Lambda^-} I\big(g^-(\theta_i)\big) < k^+c^+ + k^-c^- + \epsilon = b_k + \epsilon. \tag{21}$$

Now if $I(G(\theta)) > b_k - \epsilon$ then for $i \in \Lambda^+$,

$$I\big(g^+(\theta_i)\big) > b_k - \epsilon - (k^+ - 1)\left(c^+ + \frac{\epsilon}{2k}\right) - k^-\left(c^- + \frac{\epsilon}{2k}\right) > c^+ - \frac{3\epsilon}{2},$$

which implies $g^+(\theta_i) \in N_{r/8k}(A^+)$. Similarly, if $I(G(\theta)) > b_k - \epsilon$ then for $i \in \Lambda^-$, $g^-(\theta_i) \in N_{r/8k}(A^-)$. For θ satisfying $I(G(\theta)) > b_k - \epsilon$, choosing $v_i \in A^\pm$ for $i \in \Lambda^\pm$ such that

$$\big\|g^\pm(\theta_i) - v_i\big\| < \frac{r}{8k},$$

we have

$$\left\|G(\theta) - \sum_{i=1}^{k} \tau_{l j_i} v_i\right\| \leqslant \sum_{i\in\Lambda^+} \big\|g^+(\theta_i) - v_i\big\| + \sum_{i\in\Lambda^-} \big\|g^-(\theta_i) - v_i\big\| < \frac{r}{8}.$$

Thus

$$I\big(G(\theta)\big) > b_k - \epsilon \quad \text{implies} \quad G(\theta) \in N_{r/8}(\mathcal{W}). \tag{22}$$

Step 2: The deformation of G. Let r and ϵ be as in Step 1. Set $\bar\epsilon = \alpha$ and choose $\hat\epsilon \in (\epsilon, \bar\epsilon)$. Define for $u \in E$,

$$\phi(u) = \frac{\|u - N_{r/8}(\mathcal{K}_{b_k-\bar\epsilon}^{b_k+\bar\epsilon})\|}{\|u - N_{r/8}(\mathcal{K}_{b_k-\bar\epsilon}^{b_k+\bar\epsilon})\| + \|u - E \setminus N_{r/4}(\mathcal{K}_{b_k-\bar\epsilon}^{b_k+\bar\epsilon})\|}$$

and

$$\psi(u) = \frac{\|u - (I^{b_k-\hat\epsilon} \cup I_{b_k+\hat\epsilon})\|}{\|u - (I^{b_k-\hat\epsilon} \cup I_{b_k+\hat\epsilon})\| + \|u - I_{b_k-\epsilon}^{b_k+\epsilon}\|}.$$

As before, set $\mathcal{V}(u) = 3\hat\epsilon I'(u)/\|I'(u)\|^2$ and $W(u) = \phi(u)\psi(u)\mathcal{V}(u)$ for $u \in E \setminus \mathcal{K}$ and let $\eta(s,u)$ be the solution of

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -W(\eta) \quad \text{for } s \geqslant 0 \quad \text{and} \quad \eta(0,u) = u.$$

Set $v = G(\theta)$. Then by (21), $I(v) < b_k + \epsilon$. If $I(v) \leqslant b_k - \epsilon$, set $\sigma(v) = 0$ so that $\eta(\sigma(v), v) \in I^{b_k-\epsilon}$. If $I(v) > b_k - \epsilon$ then (22) shows that $v \in N_{r/8}(\mathcal{W})$; we will show in this case there is a unique $\sigma(v) \in (0,1)$ such that $I(\eta(\sigma(v), v)) = b_k - \epsilon$ and $\|\eta(\sigma(v), v) - v\| < r$. Choose $u \in \mathcal{W}$ such that $v \in \mathcal{B}_{r/8}(u)$. For $s \in [0,1]$, one of the three cases must occur:

(i) $\eta(s,v)$ reaches neither $\partial\mathcal{B}_{r/2}(u)$ nor $\partial I^{b_k-\epsilon}$,
(ii) $\eta(s,v)$ reaches $\partial\mathcal{B}_{r/2}(u)$ before it reaches $\partial I^{b_k-\epsilon}$,
(iii) $\eta(s,v)$ reaches $\partial I^{b_k-\epsilon}$ before it reaches $\partial\mathcal{B}_{r/2}(u)$.

In case (i), since $u \in \mathcal{W}$ implies $B_r(u) \cap \mathcal{K} = \emptyset$, the definition of ϕ and ψ yields

$$\phi\big(\eta(s,v)\big) = \psi\big(\eta(s,v)\big) = 1 \quad \text{for all } 0 \leqslant s \leqslant 1,$$

which implies

$$2\epsilon \geqslant I(v) - I\big(\eta(1,v)\big) \geqslant \int_0^1 I'\big(\eta(s,v)\big)\mathcal{V}\big(\eta(s,v)\big)\,\mathrm{d}s \geqslant 2\hat{\epsilon},$$

a contradiction. In case (ii), by Lemma 4.3, there exist $0 \leqslant s_1 < s_2 \leqslant 1$ such that

$$\big\|\eta(s_1,v) - \eta(s_2,v)\big\| \geqslant \frac{3r}{8},$$
$$\big\|I'\big(\eta(s,v)\big)\big\| \geqslant \delta \quad \text{for } s_1 \leqslant s \leqslant s_2,$$

and

$$b_k - \epsilon \leqslant I\big(\eta(s,v)\big) \leqslant b_k + \epsilon \quad \text{for } s_1 \leqslant s \leqslant s_2.$$

These inequalities imply

$$\frac{3r}{8} \leqslant \int_{s_1}^{s_2} \left\|\frac{\mathrm{d}\eta}{\mathrm{d}s}\right\| \mathrm{d}s \leqslant \int_{s_1}^{s_2} \phi\psi\|\mathcal{V}\|\,\mathrm{d}s \leqslant \frac{4\hat{\epsilon}}{\delta}\int_{s_1}^{s_2} \phi\psi\,\mathrm{d}s$$

and

$$2\epsilon \geqslant I\big(\eta(s_1,u)\big) - I\big(\eta(s_2,u)\big) = \int_{s_1}^{s_2} \phi\psi I'\mathcal{V}\,\mathrm{d}s \geqslant 2\hat{\epsilon}\int_{s_1}^{s_2} \phi\psi\,\mathrm{d}s.$$

Then, $\frac{3r}{8} \leqslant \frac{4\epsilon}{\delta}$, which contradicts (16). Thus case (iii) occurs. Then there is a unique $\sigma(v) \in (0,1)$ such that $I(\eta(\sigma(v),v)) = b_k - \epsilon$. Since $\eta(\sigma(v),v) \in \mathcal{B}_{r/2}(u)$ and $v \in \mathcal{B}_{r/8}(u)$, $\|\eta(\sigma(v),v) - v\| < r$. As in [3], we define $\overline{G}(\theta) = \eta(\sigma(G(\theta)), G(\theta))$ so that for all $\theta \in [0,1]^k$,

$$I\big(\overline{G}(\theta)\big) \leqslant b_k - \epsilon \tag{23}$$

and

$$\big\|\overline{G}(\theta) - G(\theta)\big\| \leqslant r. \tag{24}$$

In addition, for $i \in \Lambda^+$,

$$G(0_i) = \sum_{m\in\Lambda^+,\ m\neq i} \tau_{l_{j_m}} g^+(\theta_m) + \sum_{m\in\Lambda^-} \tau_{l_{j_m}} g^-(\theta_m),$$

which implies

$$I(G(0_i)) \leqslant (k^+ - 1)\left(c^+ + \frac{\epsilon}{2k}\right) + k^-\left(c^- + \frac{\epsilon}{2k}\right) < b_k - c^+ + \frac{\epsilon}{2} < b_k - \epsilon.$$

Here, we have used $\epsilon < \frac{1}{2}c^+$ which was deduced from $\epsilon \in (0, \frac{\bar{\epsilon}}{2})$ and (16). In the same way, for $i \in \Lambda^-$,

$$I\big(G(0_i)\big) < b_k - \epsilon.$$

Thus, for $1 \leqslant i \leqslant k$,

$$\overline{G}(0_i) = G(0_i). \tag{25}$$

Similarly, for $1 \leqslant i \leqslant k$,

$$\overline{G}(1_i) = G(1_i). \tag{26}$$

Step 3: Modifying $\overline{G}$. Using a convolution operator J_{ϵ^*} with a smooth peaking kernel to mollify $\overline{G}$ to get $G^* = J_{\epsilon^*}(\overline{G})$ and then cutting down G^* (see [3] for more details), we get a $\widehat{G} \in C([0,1]^k, E)$ such that $\widehat{G}(\theta) \in C^\infty(\mathbf{R}^N, \mathbf{R})$ for each $\theta \in [0,1]^k$ and for some $\widehat{R} > 0$,

$$I(\widehat{G}(\theta)) \leqslant b_k - \frac{\epsilon}{4}, \tag{27}$$

$$\|\widehat{G}(\theta) - G(\theta)\| \leqslant 2r, \tag{28}$$

$$\operatorname{supp} \widehat{G}(\theta) \subset \bigcup_{i=1}^{k} B_R(lj_i) \quad \text{for } \theta = 0_i \text{ and } 1_i,\ 1 \leqslant i \leqslant k, \tag{29}$$

and

$$\operatorname{supp} \widehat{G}(\theta) \subset B_{\hat{R}+2}(0) \quad \text{for all } \theta \in [0,1]^k. \tag{30}$$

Here, (27) is obtained from (23); (28) is from (24); (29) comes from (19), (25), and (26); and (30) is a result of cutting down. Also by (25) and (26), we have

$$G^*(\theta) = J_{\epsilon^*}(\overline{G}(\theta)) = J_{\epsilon^*}(G(\theta)) \quad \text{for } \theta = 0_i \text{ and } 1_i,\ 1 \leqslant i \leqslant k,$$

which together with (19) imply

$$\widehat{G}(\theta) = G^*(\theta) = J_{\epsilon^*}(G(\theta)) \quad \text{for } \theta = 0_i \text{ and } 1_i,\ 1 \leqslant i \leqslant k. \tag{31}$$

Step 4: Modifying $\widehat{G}$. Let

$$S = \left\{ x \in \mathbf{R}^N \mid |x| < \widehat{R} + 2 \text{ and } x \notin \bigcup_{i=1}^{k} B_R(lj_i) \right\}.$$

It can be assumed that for $1 \leqslant i \leqslant k$,

$$|\partial B_{\hat{R}+2}(0) - B_R(lj_i)| \geqslant \min_{i \neq m} |B_R(lj_i) - B_R(lj_m)|. \tag{32}$$

Let

$$\widehat{E}(\theta) = \{ v \in W^{1,2}(S) \mid v = \widehat{G}(\theta) \text{ on } \partial S \text{ and } \|v\|_{W^{1,2}(S)} < 8r \}$$

and

$$\Psi(v) = \int_S \left(\frac{1}{2}(|\nabla v|^2 + v^2) - F(x,v) \right) \mathrm{d}x.$$

Consider the minimization problem

$$\text{minimize}_{v \in \hat{E}(\theta)} \Psi(v).$$

We further restrict r such that

$$A_8 K_1^{2^*} (8r)^{2^*-2} < \frac{1}{8} \quad \text{and} \quad \overline{A}_8 K_1^{2^*} (8r)^{2^*-2} < \frac{7}{8}, \tag{33}$$

where A_8, $\bar{A}_8$, and K_1 are positive constants satisfying

$$F(x,z) \leqslant \frac{V_0}{8}|z|^2 + A_8|z|^{2^*} \quad \text{for } x \in \mathbf{R}^N,\ z \in \mathbf{R},$$

$$|f_u(x,z)| \leqslant \frac{V_0}{8} + \bar{A}_8|z|^{2^*-2} \quad \text{for } x \in \mathbf{R}^N,\ z \in \mathbf{R},$$

and

$$\|w\|_{L^{2^*}(S)} \leqslant K_1 \|w\|_{W^{1,2}(S)} \quad \text{for } w \in W^{1,2}(S),$$

respectively. Here K_1 depends only on N but not S. Then according to [3, Proposition 5.7] and its proof, there is a unique $v = v(\theta) \in \widehat{E}(\theta)$ minimizing Ψ, $v(\theta) \in C^{2,\gamma}(S)$ for all $\gamma \in (0,1)$ and $\theta \in [0,1]^k$, v depends continuously on $\theta \in [0,1]^k$ (in $\|\cdot\|_{W^{1,2}(S)}$), and $v(\theta)$ satisfies

$$\big\|v(\theta)\big\|_{W^{1,2}(S)} \leqslant 4r \tag{34}$$

and

$$-\Delta v + V(x)v = f(x,v) \quad \text{in } S, \qquad v = \widehat{G}(\theta) \quad \text{on } \partial S. \tag{35}$$

For $\theta \in [0,1]^k$, define

$$U(\theta)(x) = \begin{cases} \widehat{G}(\theta)(x) & \text{for } x \notin S, \\ v(\theta)(x) & \text{for } x \in S. \end{cases}$$

By (19) and (28),

$$\big\|\widehat{G}(\theta)\big\|_{W^{1,2}(S)} = \big\|\widehat{G}(\theta) - G(\theta)\big\|_{W^{1,2}(S)} \leqslant 2r.$$

Then (34) implies

$$\big\|U(\theta) - \widehat{G}(\theta)\big\| \leqslant \|v\|_{W^{1,2}(S)} + \big\|\widehat{G}(\theta)\big\|_{W^{1,2}(S)} \leqslant 4r + 2r = 6r.$$

Thus, for all $\theta \in [0,1]^k$,

$$\big\|U(\theta) - G(\theta)\big\| \leqslant \big\|U(\theta) - \widehat{G}(\theta)\big\| + \big\|\widehat{G}(\theta) - G(\theta)\big\| \leqslant 8r. \tag{36}$$

Also, for all $\theta \in [0,1]^k$, by (27) and the definition of v,

$$I\big(U(\theta)\big) \leqslant I\big(\widehat{G}(\theta)\big) \leqslant b_k - \frac{\epsilon}{4}. \tag{37}$$

For $\theta = 0_i$ and $\theta = 1_i$, $1 \leqslant i \leqslant k$, by (29)

$$\widehat{G}(\theta)(x) = 0 \quad \text{for } x \in S,$$

which implies by the definition of v

$$v(\theta)(x) = 0 \quad \text{for } x \in S.$$

Thus for $\theta = 0_i$ and $\theta = 1_i$, $1 \leqslant i \leqslant k$ and $x \in \mathbf{R}^N$,

$$U(\theta)(x) = \widehat{G}(\theta)(x) \tag{38}$$

and by (29) again

$$\operatorname{supp} U(\theta) \subset \bigcup_{i=1}^{k} B_R(lj_i). \tag{39}$$

For $\rho > 0$, let $\mathcal{D}_\rho = \{x \in S \mid |x - \partial S| \geqslant \rho\}$. Since v satisfies (35), by [3, Proposition 5.24] where the requirement $r < \frac{1}{8}$ from (15) was needed, there is a $K_2 > 0$ depending only on ρ, p, and N such that

$$\|v\|_{L^\infty(\mathcal{D}_\rho)} \leqslant K_2 \|v\|_{W^{1,2}(S)}. \tag{40}$$

According to [3], (40) implies that if

$$r \leqslant (8K_2)^{-1}\bar{z}, \tag{41}$$

where $\bar z$ is a number such that $|z| \leqslant \bar z$ implies $|f(x,z)| \leqslant |z|/2$, then

$$v^2(x) \leqslant 2\bar z^2 \, \mathrm{e}^{-\beta/2} \cosh 1 \tag{42}$$

for all $x \in \bigcup_{1\leqslant i\leqslant k} \mathcal{A}_i$ where

$$\mathcal{A}_i = \bigl\{x \in \mathbf{R}^N \mid R+\beta-2 < |x-lj_i| < R+\beta+2\bigr\}.$$

Step 5: The construction of H. In this last step we will construct an $H \in \Gamma_k(a)$ with $a \in (0, a_2]$ such that

$$\max_{\theta\in[0,1]^k} I\bigl(H(\theta)\bigr) \leqslant b_k - \frac{\epsilon}{8}, \tag{43}$$

which is a contradiction to Lemma 4.1. As in [3], we define for $1 \leqslant i \leqslant k$,

$$h_i(\theta)(x) = \begin{cases} U(\theta)(x), & |x-lj_i| \leqslant R+\beta, \\ \bigl||x-lj_i| - (R+\beta+1)\bigr| U(\theta)(x), & R+\beta < |x-lj_i| < R+\beta+1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$H(\theta) = \sum_{i=1}^{k} h_i(\theta).$$

Then as a consequence of (20), h_i satisfies (g_3). For $\theta = 0_i$ and $\theta = 1_i$, $i = 1, \ldots, k$, by (39) we have

$$\operatorname{supp} h_i(\theta) \subset B_R(lj_i).$$

By (17), (18), (31), and (38) we see that, for $x \in B_R(lj_i)$ with $i \in \Lambda^{\pm}$,

$$h_i(0_i)(x) = U(0_i)(x) = \widehat{G}(0_i)(x) = J_{\epsilon^*}\bigl(G(0_i)\bigr)(x) = J_{\epsilon^*}\bigl(g^{\pm}(0)\bigr)(x) = 0 \tag{44}$$

and

$$h_i(1_i)(x) = U(1_i)(x) = \widehat{G}(1_i)(x) = J_{\epsilon^*}\bigl(G(1_i)\bigr)(x) = J_{\epsilon^*}\bigl(g^{\pm}(1)\bigr)(x). \tag{45}$$

By (45), for ϵ^* small enough

$$I\bigl(h_i(1_i)\bigr) < 0 \quad \text{for } i = 1, \ldots, k. \tag{46}$$

That h_i satisfy (g_2) follows from (44) and (46). Define $\underline{S} = \bigcup_{i=1}^{k} B_{R+\beta}(lj_i)$ and $\mathcal{D} = S \setminus \underline{S}$. Since

$$F(x,z) \leqslant \frac{V_0}{4}|z|^2 + A_4|z|^{2^*} \quad \text{for } x \in \mathbf{R}^N,\ z \in \mathbf{R},$$

we see that for $v = v(\theta)$,

$$\int_{\mathcal{D}} F(x,v)\,\mathrm{d}x \leqslant \left(\frac{1}{4} + A_5\|v\|_{W^{1,2}(S)}^{2^*-2}\right)\|v\|_{W^{1,2}(\mathcal{D})}^2.$$

By further requiring

$$A_5(4r)^{2^*-2} \leqslant \frac{1}{4}, \tag{47}$$

it can be deduced (see [3]) from (42) that for β (or equivalently $l \in \mathcal{L}$) large enough,

$$\bigl|I\bigl(H(\theta)\bigr) - I\bigl(U(\theta)\bigr)\bigr| \leqslant \frac{\epsilon}{8}. \tag{48}$$

Now (43) follows from (37) and (48). To verify that h_i satisfies (g_1), using (36) and the definition of $h_i(\theta)$ we see that

$$\begin{aligned}
&\left\| h_i(\theta) - G(\theta) \right\|_{W^{1,2}(B_{R+\beta+1}(lj_i))} \\
&\quad \leqslant \left\| h_i(\theta) - U(\theta) \right\|_{W^{1,2}(B_{R+\beta+1}(lj_i))} + \left\| U(\theta) - G(\theta) \right\|_{W^{1,2}(B_{R+\beta+1}(lj_i))} \\
&\quad \leqslant \left\| h_i(\theta) - U(\theta) \right\|_{W^{1,2}(B_{R+\beta+1}(lj_i) \setminus B_{R+\beta}(lj_i))} + 8r.
\end{aligned}$$

By (20) and (32), $B_{R+\beta+1}(lj_i) \setminus B_{R+\beta}(lj_i) \subset S$. Then (34) and the definition of $U(\theta)$ and $h_i(\theta)$ imply

$$\left\| h_i(\theta) - U(\theta) \right\|_{W^{1,2}(B_{R+\beta+1}(lj_i) \setminus B_{R+\beta}(lj_i))} \leqslant 2 \left\| v(\theta) \right\|_{W^{1,2}(S)} \leqslant 2 \cdot 4r = 8r.$$

Therefore

$$\left\| h_i(\theta) - G(\theta) \right\|_{W^{1,2}(B_{R+\beta+1}(lj_i))} \leqslant 16r.$$

By (17), (18), and (20), $G(\theta)|_{B_{R+\beta+1}(lj_i)} \in \mathcal{P}^{\pm}$ and $h_i \in C([0,1], \overline{N}_{16r}(\mathcal{P}^{\pm}))$ for $i \in \Lambda^{\pm}$. Thus, as a consequence of (15), h_i satisfies (g_1). Let $r = r_0$ be a number satisfying (14), (15), (33), (41), and (47). Then r_0 is a valid number for the theorem. □

5. Further remarks

Combining the theorems in Section 3 and the argument from [5], we can obtain information on the number of nodal domains of non-symmetric multi-bump nodal solutions for Eq. (1) and Eq. (2), extending the results in [3] and improving the results in [5].

Theorem 5.1. *Assume* (V_1) *and* (f_1)–(f_4). *Suppose* $(*)_\pm$ *holds. For multi-bump nodal solutions of Eq.* (2), *the number of nodal domains is bounded by the number of bumps. In particular, the two-bump nodal solutions have exactly two nodal domains. Moreover, there are infinitely many, geometrically different, two-bump, nodal solutions which have exactly two nodal domains.*

Theorem 5.2. *Assume* $(\mathrm{V}_{1'})$, $(\mathrm{f}_{1'})$, *and* (f_2)–(f_4). *Suppose* $(*')_\pm$ *holds. Then for any integers* $k \geqslant m \geqslant 2$, *Eq.* (1) *has infinitely many, geometrically different, k-bump, nodal solutions in* $I_{kc-\alpha}^{kc+\alpha}$ *which have exactly m nodal domains. More precisely, given any positive integers* $k_1, k_2, \ldots, k_m$ *such that* $\sum_{i=1}^m k_i = k \geqslant 2$, *there are infinitely many, geometrically different, k-bump, nodal solutions in* $I_{kc-\alpha}^{kc+\alpha}$ *which have exactly m nodal domains* D_i, $i = 1, \ldots, m$ *such that* $u|_{D_i}$ *is a* k_i*-bump positive or negative solution.*

Theorem 5.3. *Assume* $(\mathrm{V}_{1''})$, $(\mathrm{f}_{1''})$, *and* (f_2)–(f_4). *Suppose* $(*'')_\pm$ *holds. For any integer* $k \geqslant 2$, *Eq.* (2) *has infinitely many, geometrically different, k-bump, nodal solutions in* $I_{kc-\alpha}^{kc+\alpha}$ *such that the numbers of their nodal domains are bounded between* $[\frac{k}{2}] + 1$ *and k. In particular, there are nodal solutions such that the numbers of their nodal domains tend to infinity.*

Looking back at the proof, we see that if we take $k_- = 0$, we will end up obtaining k-bump solutions with only positive bumps. Together with Theorem 1.1 of [5] we get k-bump positive solutions. This is an alternative way of obtaining positive multi-bump solutions (see Theorem 7.22 in [3]).

Recently, the construction of multi-bump solutions [3] has been extended to the case that the nonlinearity is asymptotically linear instead of superlinear. This was done by van Heerden in [6]. Obviously, our results on multi-bump nodal solutions can be carried to this case and we refer to [6] for precise conditions.

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