Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 23 (2006) 695–711
ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

A new definition of viscosity solutions for a class of second-order degenerate elliptic integro-differential equations

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Received 18 November 2004; received in revised form 5 July 2005; accepted 22 September 2005

Available online 2 February 2006

Abstract

We present a new definition of the viscosity solution for a class of integro-differential equations in a bounded open domain Ω in $\mathbf{R}^N$. We consider either the Dirichlet boundary condition or the Neumann boundary condition on the boundary. These equations correspond to the process which is a combination of the jumps and the Brownian motion. We give the comparison and the existence results for the viscosity solution in our new framework.

Résumé

Nous présentons une nouvelle définition de la solution de viscosité pour certaine classe des équations intégro-différentielles sur un ouvert borné dans $\mathbf{R}^N$. Sur le bord, on considère soit la condition de Dirichlet ou soit la condition de Neumann. Ces équations correspondent au processus de la combinaison des sauts et du mouvement brownien. Nous donnons des résultats de la comparaison et de l'existence de la solution de viscosité dans notre cadre.

Keywords: Integro-differential equations; Lévy operator; Viscosity solutions; Existence and uniqueness results

1. Introduction

We study the comparison and the uniqueness of viscosity solutions for the following classes of partial-integral differential equations.

(Stationary problem)

$$F\big(x,u,\nabla u,\nabla^2 u\big)-\sup_{\alpha\in\mathcal{A}}\left\{-\int_{\mathcal{P}} u\big(x+\beta(x,z,\alpha)\big)-u(x)-\big\langle\beta(x,z,\alpha),\nabla u(x)\big\rangle q(\mathrm{d}z)\right\}=0\quad\text{in }\Omega. \tag{1}$$

(Evolutionary problem)

$$\frac{\partial u}{\partial t}(x,t)+F\big(x,u,\nabla u,\nabla^2 u\big)$$

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2005.09.002

$$-\sup_{\alpha\in\mathcal{A}}\left\{-\int\limits_{\mathcal{P}} u\big(x+\beta(x,z,\alpha),t\big)-u(x,t)-\big\langle\beta(x,z,\alpha),\nabla u(x,t)\big\rangle q(\mathrm{d}z)\right\}=0\quad\text{in }\Omega\times(0,T),\tag{2}$$

$$u(x,0)=u_0(x)\quad\text{on }\Omega.\tag{3}$$

Here, Ω is a bounded smooth open subset of $\mathbf{R}^N$, u, ∇u, $\nabla^2 u$ denote respectively a scalar function on Ω, the gradient and the Hessian of u, and $\langle\,,\rangle$ stands for the scalar product in $\mathbf{R}^N$. The set $\mathcal{A}$ is a compact metric space, β is an $\mathbf{R}^N$-valued function defined on $\Omega\times\mathbf{R}^n\times\mathcal{A}$ such that

$$\begin{aligned}&\beta_1(x)|z|\leqslant\big|\beta(x,z,\alpha)\big|\leqslant\beta_2(x)|z|,\quad \big|\beta(x,z,\alpha)-\beta(y,z,\alpha)\big|\leqslant L|x-y||z|\\&\quad\forall(x,y)\in\Omega\times\Omega,\ \forall(z,\alpha)\in\mathbf{R}^n\times\mathcal{A},\end{aligned}\tag{4}$$

where $\beta_1(x)\geqslant 0$, $\beta_2(s)>0$ are continuous functions defined in Ω and $L>0$ is a constant, $\mathcal{P}=\mathcal{P}(x,\alpha)$ $((x,\alpha)\in\Omega\times\mathcal{A})$ is a subset of $\mathbf{R}^n$ defined by

$$\mathcal{P}(x,\alpha)=\big\{z\mid z\in\mathbf{R}^n,\ x+\theta\beta(x,z,\alpha)\in\Omega\ \forall 0\leqslant\theta\leqslant 1\big\},\tag{5}$$

where we remark that from (4) for some $\varepsilon>0$ small enough $U_\varepsilon(0)=\{|z|\leqslant\varepsilon\}\subset\mathcal{P}(x,\alpha)$. The so-called Lévy measure $q(\mathrm{d}z)$ is a positive Radon measure such that

$$\int\limits_{|z|\leqslant 1}|z|^2q(\mathrm{d}z)+\int\limits_{|z|>1}1q(\mathrm{d}z)<\infty\tag{6}$$

holds. A given continuous function $F(x,r,p,Q)$ defined on $\Omega\times\mathbf{R}\times\mathbf{R}^N\times\mathbf{S}^N$ satisfies the following conditions.

(Properness and degenerate ellipticity:)

$$F(x,r,p,X)\leqslant F(x,s,p,Y)\quad\text{if } r\leqslant s,\ Y\leqslant X.\tag{7}$$

On the boundary $\partial\Omega$, we consider Dirichlet and Neumann type boundary conditions.

(Dirichlet B.C.)

$$u=\phi\quad\text{on }\partial\Omega,\tag{8}$$

where ϕ is a given continuous function defined on $\partial\Omega$ in the case of (1), and on $\partial\Omega\times(0,T)$ in the case of (2).

(Neumann B.C.)

$$\langle\nabla u,\mathbf{n}\rangle=0\quad\text{on }\partial\Omega,\tag{9}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

The goal of this paper is to give the comparison and the existence of solutions for the above integro-differential equations. For this purpose, we propose the definitions of viscosity solutions, which is a little bit different from the ones in the preceding works. We refer the readers papers of Gimbert and Lions [11], Sayah [17], Alvarez and Tourin [2], Barles, Buckdahn and Pardoux [5], Jacobson and Karlsen [13], Imbert [12], etc. for the treatment of problems similar to (1) and (2) by the viscosity solutions theory. In this paper, differently from the former works we use the second-order superjet (resp. subjet) of an upper semi-continuous function $u\in\mathrm{USC}(\Omega)$ (resp. lower semi-continuous function $v\in\mathrm{LSC}(\Omega)$) in the non-local term of the equation, in particular around the origin ($\{|z|\leqslant\varepsilon\}$) to get rid of the singularity of the Lévy measure. From this reason, we need not only the superjet of u (resp. subjet of v) at a point $\hat{x}\in\Omega$, i.e. $(p,X)\in J^{2,+}_{\Omega}u(\hat{x})$ (resp. $J^{2,-}_{\Omega}v(\hat{x})$), but also the second-order remainders. If $(p,X)\in J^{2,+}_{\Omega}u(\hat{x})$ (resp. $J^{2,-}_{\Omega}v(\hat{x})$), then for any $\delta>0$ there exists $\varepsilon>0$ such that

$$u(x)\leqslant u(\hat{x})+\langle p,x-\hat{x}\rangle+\frac{1}{2}\big\langle X(x-\hat{x}),x-\hat{x}\big\rangle+\delta|x-\hat{x}|^2\quad\text{if }|x-\hat{x}|\leqslant\varepsilon$$

(resp.

$$v(x)\geqslant v(\hat{x})+\langle p,x-\hat{x}\rangle+\frac{1}{2}\big\langle X(x-\hat{x}),x-\hat{x}\big\rangle-\delta|x-\hat{x}|^2\quad\text{if }|x-\hat{x}|\leqslant\varepsilon).$$

To interpret this to the problem (1), from (4) for any $(\hat{x},\alpha)\in\Omega\times\mathcal{A}$, and any $\delta>0$ there exists $\varepsilon>0$ such that

$$U_\varepsilon=\big\{z\mid z\in\mathbf{R}^n,\ |z|\leqslant\varepsilon\big\}\subset\mathcal{P}(\hat{x},\alpha)\tag{10}$$

and for any $0 < \varepsilon' < \varepsilon$

$$u\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) \leqslant u(\hat{x}) + \big\langle p, \beta(\hat{x}, z, \alpha)\big\rangle + \frac{1}{2}\big\langle X\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle + \delta\big|\beta(\hat{x}, z, \alpha)\big|^2 \quad \text{if } |z| \leqslant \varepsilon' \tag{11}$$

(resp.

$$v\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) \geqslant v(\hat{x}) + \big\langle p, \beta(\hat{x}, z, \alpha)\big\rangle + \frac{1}{2}\big\langle X\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle - \delta\big|\beta(\hat{x}, z, \alpha)\big|^2 \quad \text{if } |z| \leqslant \varepsilon'). \tag{12}$$

We use this pair of numbers (ε, δ) satisfying (10) and (11) (resp. (12)) for any $(p, X) \in J^{2,+}_{\Omega} u(\hat{x})$ (resp. $(p, X) \in J^{2,-}_{\Omega} v(\hat{x})$) in the following definition. For simplicity, we shall denote

$$\mathcal{P}_\varepsilon(x, \alpha) = \big\{z \mid z \in \mathbf{R}^n,\ |z| \geqslant \varepsilon|\big\} \cap \mathcal{P}(x, \alpha).$$

Definition 1.1. Let $u \in \mathrm{USC}(\Omega)$ (resp. $v \in \mathrm{LSC}(\Omega)$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any $\hat{x} \in \Omega$, any $(p, X) \in J^{2,+}_{\Omega} u(\hat{x})$ (resp. $\in J^{2,-}_{\Omega} v(\hat{x})$), and any pair of numbers (ε, δ) satisfying (10) and (11) (resp. (12)), the following holds for any $0 < \varepsilon' \leqslant \varepsilon$

$$\begin{aligned} F\big(\hat{x}, u(\hat{x}), p, X\big) + \sup_{\alpha \in \mathcal{A}} \Bigg\{ &- \int\limits_{|z| < \varepsilon'} \frac{1}{2}\big\langle (X + 2\delta I)\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle q(\mathrm{d}z) \\ &- \int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x}, \alpha)} u\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) - u(\hat{x}) - \big\langle \beta(\hat{x}, z, \alpha), p\big\rangle q(\mathrm{d}z) \Bigg\} \leqslant 0 \end{aligned} \tag{13}$$

(resp.

$$\begin{aligned} F\big(\hat{x}, v(\hat{x}), p, X\big) + \sup_{\alpha \in \mathcal{A}} \Bigg\{ &- \int\limits_{|z| < \varepsilon'} \frac{1}{2}\big\langle (X - 2\delta I)\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle q(\mathrm{d}z) \\ &- \int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x}, \alpha)} v\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) - v(\hat{x}) - \big\langle \beta(\hat{x}, z, \alpha), p\big\rangle q(\mathrm{d}z) \Bigg\} \geqslant 0). \end{aligned} \tag{14}$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

For the evolutionary problem, for a scalar function $u \in \mathrm{USC}(\Omega \times (0, T))$ (resp. $v \in \mathrm{LSC}(\Omega \times (0, T))$), we shall use any $(\hat{x}, \hat{t}) \in \Omega \times (0, T)$, any $(p, q, X) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{S}^N$ a superjet (resp. subjet) of u (resp. v) at $(\hat{x}, \hat{t})$, second-order with respect to x, first-order with respect to t, i.e. $(p, q, X) \in J^{2,1,+}_{\Omega \times (0,T)} u(\hat{x})$ (resp. $J^{2,1,-}_{\Omega \times (0,T)} v(\hat{x})$). Similar to (11) and (12), for any $\delta > 0$ there exists $\varepsilon > 0$ such that (10) holds, and for any $0 < \varepsilon' < \varepsilon$

$$\begin{aligned} u\big(\hat{x} + \beta(\hat{x}, z, \alpha), t\big) \leqslant u(\hat{x}, \hat{t}) + \big\langle p, \beta(\hat{x}, z, \alpha)\big\rangle + q(t - \hat{t}) + \frac{1}{2}\big\langle X\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle \\ + \delta\big|\beta(\hat{x}, z, \alpha)\big|^2 \quad \text{if } |z| \leqslant \varepsilon',\ |t - \hat{t}| < \varepsilon' \end{aligned} \tag{15}$$

(resp.

$$\begin{aligned} v\big(\hat{x} + \beta(\hat{x}, z, \alpha), t\big) \leqslant u(\hat{x}, \hat{t}) + \big\langle p, \beta(\hat{x}, z, \alpha)\big\rangle + q(t - \hat{t}) + \frac{1}{2}\big\langle X\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle \\ - \delta\big|\beta(\hat{x}, z, \alpha)\big|^2 \quad \text{if } |z| \leqslant \varepsilon',\ |t - \hat{t}| < \varepsilon'). \end{aligned} \tag{16}$$

Definition 1.2. Let $u \in \mathrm{USC}(\overline{\Omega} \times [0, T))$ (resp. $v \in \mathrm{LSC}(\overline{\Omega} \times [0, T))$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (2), if for any $(\hat{x}, \hat{t}) \in \Omega \times (0, T)$, any $(p, q, X) \in J^{2,1,+}_{\Omega \times (0,T)} u(\hat{x})$ (resp. $(p, q, X) \in J^{2,1,-}_{\Omega \times (0,T)} v(\hat{x})$), and any pair of numbers (ε, δ) satisfying (10) and (15) (resp. (16)), the following holds for any $0 < \varepsilon' < \varepsilon$

$$q + F\big(\hat{x}, u(\hat{x}), p, X\big) + \sup_{\alpha\in\mathcal{A}}\Bigg\{-\int\limits_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X+2\delta I)\beta(\hat{x},z,\alpha), \beta(\hat{x},z,\alpha)\big\rangle q(\mathrm{d}z)$$
$$-\int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} u\big(\hat{x}+\beta(\hat{x},z,\alpha)\big) - u(\hat{x}) - \big\langle \beta(\hat{x},z,\alpha), p\big\rangle q(\mathrm{d}z)\Bigg\} \leqslant 0 \tag{17}$$

(resp.

$$q + F\big(\hat{y}, v(\hat{y}), p, X\big) + \sup_{\alpha\in\mathcal{A}}\Bigg\{-\int\limits_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X-2\delta I)\beta(\hat{x},z,\alpha), \beta(\hat{x},z,\alpha)\big\rangle q(\mathrm{d}z)$$
$$-\int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} v\big(\hat{x}+\beta(\hat{x},z,\alpha)\big) - v(\hat{x}) - \big\langle \beta(\hat{x},z,\alpha), p\big\rangle q(\mathrm{d}z)\Bigg\} \geqslant 0). \tag{18}$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

As mentioned before, the difference between Definitions 1.1, 1.2 and the preceding definitions in [2,5], and [13] is the usage of the second-order subjet (resp. superjet) X in the non-local term. In view of (6), this avoids the singularity of the Lévy measure around $|z| < \varepsilon$. Moreover, (11) (resp. (12)) and (15) (resp. (16)) shows that we are only concerned with the local property of u (resp. v), while in previous definitions the global relationship of u and the so-called test function $\phi \in C^2(\Omega)$ was required. (We refer the readers Crandall, Ishii and Lions [8] for the meaning of the "test function" in the sense of viscosity solutions.) That is, $u - \phi$ (resp. $v - \phi$) should have taken the global maximum (resp. minimum) at $\hat{x}$. Since we only use the local property such as (11) and (12), the usual comparison argument in the PDE theory adapts well in our setting. We shall give a further remark on the relationship between our definitions and the preceding definitions in Remark 2.2 after the proof of Theorem 2.1. We would like to mention that in [12], the author uses the local argument similar to ours, but for parabolic integro-differential equations in the whole space with first-order Hamiltonian $F = F(x, u, p)$. However, even for the first-order Hamiltonian case, the definition of viscosity solutions in [12] and ours differs.

Now, we shall see how the classical comparison argument works with Definition 1.1.

Example 1.1. Let $\lambda > 0$, $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain, and consider

$$\begin{aligned} &\lambda u - \Delta u + |\nabla u| - \int\limits_{\mathcal{P}(x)} u(x+z) - u(x) - \langle z, \nabla u\rangle q(\mathrm{d}z) = 0 \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{19}$$

where $\mathcal{P} = \{z \mid z \in \mathbf{R}^N,\ x+z \in \Omega\}$, and

$$q(\mathrm{d}z) = \frac{1}{|z|^{N+1+\nu}}, \qquad \nu \in [0,1).$$

Let $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ be respectively a viscosity subsolution and a viscosity supersolution of (19). Then, we have $u \leqslant v$ in Ω.

We shall briefly see the proof of the above example. (The exact proof is included in the proof of more general Theorem 2.1 below.) We use the argument by contradiction, and thus assume that

$$M = (u-v)(\bar{x}) = \sup_{\overline{\Omega}} (u-v)(x) > 0.$$

Consider for each $\alpha > 0$

$$\Phi(x,y) = u(x) - v(y) - \alpha|x-y|^2, \qquad (x,y) \in \overline{\Omega}\times\overline{\Omega}$$

and let $(\hat{x}, \hat{y})$ be a maximum point of $\Phi(x,y)$. It is known (see M.G. Crandall, H. Ishii and P.-L. Lions [8]) that

$$\lim_{\alpha\to\infty} \alpha|\hat{x}-\hat{y}|^2 \to 0, \qquad \lim_{\alpha\to\infty} x_\alpha = \lim_{\alpha\to\infty} y_\alpha = \bar{x}.$$

We denote $p = \nabla_x \phi(\hat{x}, \hat{y}) = \alpha(\hat{x} - \hat{y})$. From the standard result in the viscosity solution theory there exist $\overline{X}, \overline{Y} \in \mathbf{S}^N$ such that $(p, \overline{X}) \in \bar{J}_\Omega^{2,+}(\hat{x})$ (resp. $(p, \overline{Y}) \in \bar{J}_\Omega^{2,-}(\hat{y})$),

$$\begin{pmatrix} \overline{X} & O \\ O & -\overline{Y} \end{pmatrix} \leqslant 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{20}$$

Take a sequence $(p, X_n) \in J_\Omega^{2,+}$ (resp. $(p, Y_n) \in J_\Omega^{2,-}$) such that $(p, X_n) \to (p, \overline{X})$ (resp. $(p, Y_n) \to (p, \overline{Y})$) as $n \to \infty$. Without any confusion, we shall denote $X = X_n$ (resp. $Y = Y_n$). Since u and v are respectively viscosity sub and super solutions in the sense of Definition 1.1, for (p, X) (resp. (p, Y)), for any $\delta > 0$ there exists $\varepsilon > 0$ such that (10) holds, that (11), (12) hold respectively, and that for any $0 < \varepsilon' < \varepsilon$

$$\lambda u(\hat{x}) - \operatorname{Tr} X + |p| - \int_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X + 2\delta I)z, z\big\rangle q(\mathrm{d}z) - \int_{\mathcal{P}_{\varepsilon'}(\hat{x})} u(\hat{x}+z) - u(\hat{x}) - \langle z, p\rangle q(\mathrm{d}z) \leqslant 0,$$

$$\lambda v(\hat{y}) - \operatorname{Tr} Y + |p| - \int_{|z|<\varepsilon'} \frac{1}{2}\big\langle (Y - 2\delta I)z, z\big\rangle q(\mathrm{d}z) - \int_{\mathcal{P}_{\varepsilon'}(\hat{y})} v(\hat{y}+z) - v(\hat{y}) - \langle z, p\rangle q(\mathrm{d}z) \geqslant 0.$$

Let $m(\cdot)$ be the Lebesgue measure in $\mathbf{R}^N$. Remark that there exists a constant $C > 0$ such that

$$m\big(\mathcal{P}_{\varepsilon'}(x) - \big(\mathcal{P}_{\varepsilon'}(x) \cap \mathcal{P}_{\varepsilon'}(y)\big)\big) \leqslant C|x - y|$$

(see Lemma 2.2 in below, for the proof). We also remark that since $(\hat{x}, \hat{y})$ is the maximum point of $\Phi(x, y)$,

$$u(\hat{x}+z) - v(\hat{x}+z) - \alpha|\hat{x}-\hat{y}|^2 \leqslant u(\hat{x}) - v(\hat{y}) - \alpha|\hat{x}-\hat{y}|^2$$

holds. We take the difference of two preceding inequalities, by taking account of the above remarks, and get

$$\begin{aligned} 0 < \frac{\lambda M}{2} < \lambda\big(u(\hat{x}) - v(\hat{y})\big) &\leqslant \operatorname{Tr}(X - Y) + \int_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X - Y + 4\delta I)z, z\big\rangle q(\mathrm{d}z) \\ &\quad + \int_{\mathcal{P}_{\varepsilon'}(\hat{x}) - (\mathcal{P}_{\varepsilon'}(\hat{x}) \cap \mathcal{P}_{\varepsilon'}(\hat{y}))} u(\hat{x}+z) - u(\hat{x}) - \langle z, p\rangle q(\mathrm{d}z) \\ &\quad - \int_{\mathcal{P}_{\varepsilon'}(\hat{y}) - (\mathcal{P}_{\varepsilon'}(\hat{x}) \cap \mathcal{P}_{\varepsilon'}(\hat{y}))} v(\hat{y}+z) - v(\hat{y}) - \langle z, p\rangle q(\mathrm{d}z) \\ &\leqslant \operatorname{Tr}(X - Y) + \int_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X - Y + 4\delta I)z, z\big\rangle q(\mathrm{d}z) + \mathrm{o}(1), \end{aligned}$$

where we have used the fact that $\lim_{\alpha\to\infty} \alpha|\hat{x}-\hat{y}|^2 = 0$. From (20), for any small number $r > 0$, there exists $n \in \mathbf{N}$ such that

$$X - Y = X_n - Y_n \leqslant rI.$$

Thus by taking $\delta > 0$ small enough, from (6) we get a contradiction. Therefore, $u \leqslant v$ in Ω is proved.

If the Hamiltonian F is in the form of the Hamilton–Jacobi–Bellman operator, then (1), (2) correspond respectively to the optimal control problems of jump-diffusion processes. We shall see in the following example some Lévy measures used in practical models in mathematical finances (mostly in the geometrically transformed form).

Example 1.2. Let $N = n = 1$, $\Omega = \mathbf{R}$, and consider

$$\frac{\partial u}{\partial t} + \sup_{\alpha\in\mathcal{A}} \left\{ -a\frac{\partial^2 u}{\partial x^2} - b\frac{\partial u}{\partial x} - f(x,\alpha) - \int_{\mathbf{R}} u(x+z,t) - u(x,t) - \big\langle z, \nabla u(x,t)\big\rangle q(\mathrm{d}z) \right\} = 0$$
$$\text{in } \mathbf{R} \times (0, T). \tag{21}$$

According to the intensity of jump-frequencies and lengths of jumps, some models exist.

(i) The compound Poisson process: for $C > 0$

$$q(\mathrm{d}z) = C\sum_{j=1}^{d} p_j \delta_{a_j}\,\mathrm{d}z, \quad p_j \geqslant 0, \quad \sum_{j=1}^{d} p_j = 1.$$

(ii) The stable process: for $C_1, C_2 \geqslant 0$ ($C_1C_2 \neq 0$)

$$q(\mathrm{d}z) = \begin{cases} C_1 \dfrac{1}{|z|^{N+1+\alpha}}\,\mathrm{d}z, & z < 0, \\ C_2 \dfrac{1}{|z|^{N+1+\alpha}}\,\mathrm{d}z, & z > 0, \end{cases}$$

where $\alpha \in [0, 1)$.

(iii) The *CGMY* process: for $C > 0$, $G \geqslant 0$, $M \geqslant 0$, $0 \leqslant Y < 2$

$$q(\mathrm{d}z) = C\big(I_{z<0}\,\mathrm{e}^{-G|z|} + I_{z>0}\,\mathrm{e}^{-M|z|}\big)|z|^{-(1+Y)}\,\mathrm{d}z.$$

We refer the readers to Cont [7], Miyahara [14], Framstad, Oksendal and Sulem [9], Achdou and Pironneau [1], and Sato [16] for more details.

Now, we give the plan of this paper. In Section 2, we study the comparison of viscosity solutions of the Dirichlet boundary value problem (1), (8). In Section 3, we study the comparison of viscosity solutions of the Neumann boundary value problem. In Section 4, the existence of viscosity solutions of (1) with Dirichlet and Neumann boundary conditions are established.

To conclude this introduction, we shall mention further perspective in future researches. Since, we treat the simplest cases in this paper, such as Dirichlet and Neumann boundary conditions under elementary assumptions on the Hamiltonian F (see Section 2 in below), many extensions will be possible. In particular, first it is important to establish the comparison and the existence of viscosity solutions in unbounded domains. In fact, in Bensoussan and Lions [6], Garroni and Menaldi [10], Pham [15], similar problems to (1) and (2) are studied in unbounded domains (the first two were framework of weak solutions in distributions sense). Next, it is interesting to study the regularizing effect of the non-local operator according to the singularity of Lévy measures. We intend to visit this in our future work. Thirdly, we would naturally expect that the present treatment of non-local operators by Definitions 1.1, 1.2 would be applicable to larger classes of general integro-differential operators.

2. Comparisons for the Dirichlet boundary value problem

In this section, we study the comparison of viscosity solutions of (1), (8) in the sense of Definition 1.1. The Hamiltonian F is assumed to satisfy the following conditions.

There exists $\gamma > 0$ such that

$$F(x, r, p, X) - F(x, s, p, X) \geqslant \gamma(r - s) \quad \text{if } r \geqslant s, \quad \text{in } \overline{\Omega} \times \mathbf{R}^N \times \mathbf{S}^N. \tag{22}$$

There exists a function $w(\cdot) : [0, \infty) \to [0, \infty)$, $w(0+) = 0$ such that

$$F(y, r, p, Y) - F(x, r, p, X) \leqslant w\big(\alpha|x - y|^2 + |x - y|\big(|p| + 1\big)\big) \quad \text{for } x, y \in \overline{\Omega},\ r \in \mathbf{R},\ p \in \mathbf{R}^N \tag{23}$$

for any $\alpha > 0$, and for any $X, Y \in \mathbf{S}^N$ such that

$$-3\alpha \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqslant \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leqslant 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{24}$$

The N-dimensional vector valued function $\beta(x, z, \alpha)$ is assumed to satisfy the following condition. For the Lebesgue measure $m(\cdot)$ in $\mathbf{R}^n$, and the Lebesgue measure $m'(\cdot)$ in $\mathbf{R}^N$, for any measurable subset $V \in \mathbf{R}^N$

$$m\big(\{z \mid z \in \mathbf{R}^n,\ \beta(x, z, \alpha) \in V\}\big) \leqslant Mm'(V) \quad \text{in } \Omega \times \mathcal{A} \tag{25}$$

where $M > 0$ is a constant independent on V.

Our main comparison result is the following.

Theorem 2.1. *Assume that* (4), (6), (7), (22), (23) *and* (25) *hold. Let* $u \in \mathrm{USC}(\Omega)$ *and* $v \in \mathrm{LSC}(\Omega)$ *be respectively a viscosity subsolution and a supersolution of* (1) *in* Ω, *which satisfy* $u \leqslant v$ *on* $\partial\Omega$. *Then,*

$$u \leqslant v \quad \text{in } \Omega.$$

The proof of Theorem 2.1 uses the following lemma.

Lemma 2.2. *For* $(x,\alpha) \in \Omega \times \mathcal{A}$, *let*

$$\mathcal{P}(x,\alpha) = \{z \mid z \in \mathbf{R}^n,\ x + \beta(x,z,\alpha) \in \Omega\}.$$

Then, there exists a constant $C > 0$ *such that*

$$m\big(\mathcal{P}(x,\alpha) - \big(\mathcal{P}(x,\alpha) \cap \mathcal{P}(x',\alpha)\big)\big) \leqslant C|x - x'| \quad \forall \alpha \in \mathcal{A},$$

where $m(\cdot)$ *is the Lebesgue measure in* $\mathbf{R}^n$.

Proof of Lemma 2.2. Let $z \in \mathcal{P}(x,\alpha) \cap \mathcal{P}(x',\alpha)^c$. Since $x + \beta(x,z,\alpha) \in \Omega$, and since Ω is a bounded set, from (4) there exists a constant $C_0 > 0$ (independent on z) such that $|z| < C_0$. Thus, by putting $w = x - x' + \beta(x,z,\alpha) - \beta(x',z,\alpha)$, we have from (4)

$$x' + \beta(x',z,\alpha) + w \in \Omega, \quad |w| \leqslant C'|x - x'|,$$

where $C' > 0$ is a constant. This means

$$x' + \beta(x',z,\alpha) \in \{y \mid y \in \mathbf{R}^N \cap \Omega^c,\ \mathrm{dist}(y,\Omega) \leqslant C'|x - x'|\} = V.$$

By denoting m' the N-dimensional Lebesgue measure, and by taking account that Ω is bounded, we have thus

$$m'\big(V - \{x'\}\big) = m'(V) \leqslant C''|x - x'|,$$

where $C'' > 0$ is a constant. Therefore, from (25) we proved the claim. □

Proof of Theorem 2.1. We use the argument by contradiction, and thus assume that there exists $\bar{x} \in \Omega$ and a constant $d > 0$ such that

$$(u - v)(\bar{x}) = \sup_{x \in \Omega}(u - v)(x) = d > 0.$$

Consider

$$\Phi(x,y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2,$$

and let $(\hat{x},\hat{y}) = (\hat{x}_\alpha,\hat{y}_\alpha)$ be a maximum point of $\Phi(x,y)$. Denote $\phi(x,y) = \frac{\alpha}{2}|x - y|^2$, $p = \nabla_x\phi(\hat{x}_\alpha,\hat{y}_\alpha) = \alpha(\hat{x}_\alpha - \hat{y}_\alpha)$. The standard argument in the viscosity solutions theory (see [8]) leads

$$\lim_{\alpha\to\infty} \frac{\alpha}{2}|\hat{x}_\alpha - \hat{y}_\alpha|^2 = 0, \qquad \lim_{\alpha\to\infty} \hat{x}_\alpha = \lim_{\alpha\to\infty} \hat{y}_\alpha = \bar{x},$$
$$\lim_{\alpha\to\infty} u(\hat{x}_\alpha) = u(\bar{x}), \qquad \lim_{\alpha\to\infty} v(\hat{y}_\alpha) = v(\bar{x}).$$

For simplicity, we shall abridge the dependence on α without any confusion. There also exist $\overline{X}, \overline{Y} \in \mathbf{S}^N$ which satisfy the following (see [8])

$$-3\alpha\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqslant \begin{pmatrix} \overline{X} & O \\ O & -\overline{Y} \end{pmatrix} \leqslant 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{26}$$

$(p,\overline{X}) \in \bar{J}_\Omega^{2,+}u(\hat{x})$, $(p,\overline{Y}) \in \bar{J}_\Omega^{2,-}v(\hat{y})$. We take a sequence $(p,X_n) \in J_\Omega^{2,+}u(\hat{x})$ (resp. $(p,Y_n) \in J_\Omega^{2,-}v(\hat{y})$) such that $(p,X_n) \to (p,\overline{X})$ (resp. $(p,Y_n) \to (p,\overline{Y})$), and denote $X = X_n$, $Y = Y_n$ without any confusion. Then, from Definition 1.1 we have the following inequalities. For (p,X) and for any $\delta > 0$, there exists $\varepsilon > 0$ such that (10), (11) hold, and that for any $0 < \varepsilon' < \varepsilon$

$$F\big(\hat{x}, u(\hat{x}), p, X\big) + \sup_{\alpha\in\mathcal{A}}\bigg\{-\int_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X+2\delta I)\beta(\hat{x},z,\alpha), \beta(\hat{x},z,\alpha)\big\rangle q(\mathrm{d}z)$$
$$-\int_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} u\big(\hat{x}+\beta(\hat{x},z,\alpha)\big) - u(\hat{x}) - \big\langle \beta(\hat{x},z,\alpha), p\big\rangle q(\mathrm{d}z)\bigg\} \leqslant 0. \tag{27}$$

For (p, X) and for any $\delta > 0$, there exists $\varepsilon > 0$ such that (10), (12) hold, and that for any $0 < \varepsilon' < \varepsilon$

$$F\big(\hat{y}, v(\hat{y}), p, Y\big) + \sup_{\alpha\in\mathcal{A}}\bigg\{-\int_{|z|<\varepsilon'} \frac{1}{2}\big\langle (Y-2\delta I)\beta(\hat{y},z,\alpha), \beta(\hat{y},z,\alpha)\big\rangle q(\mathrm{d}z)$$
$$-\int_{\mathcal{P}_{\varepsilon'}(\hat{y},\alpha)} v\big(\hat{y}+\beta(\hat{y},z,\alpha)\big) - v(\hat{y}) - \big\langle \beta(\hat{y},z,\alpha), p\big\rangle q(\mathrm{d}z)\bigg\} \geqslant 0. \tag{28}$$

For any $\rho > 0$, there exists a control $\bar{\alpha} \in \mathcal{A}$ such that

$$F\big(\hat{y}, v(\hat{y}), p, Y\big) - \int_{|z|<\varepsilon'} \frac{1}{2}\big\langle (Y-2\delta I)\beta(\hat{y},z,\bar{\alpha}), \beta(\hat{y},z,\bar{\alpha})\big\rangle q(\mathrm{d}z)$$
$$-\int_{\mathcal{P}_{\varepsilon'}(\hat{y},\bar{\alpha})} v\big(\hat{y}+\beta(\hat{y},z,\bar{\alpha})\big) - v(\hat{y}) - \big\langle \beta(\hat{y},z,\bar{\alpha}), p\big\rangle q(\mathrm{d}z) \geqslant -\rho. \tag{29}$$

Therefore, from (27)–(29) we have

$$F\big(\hat{x}, u(\hat{x}), p, X\big) - \int_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X+2\delta I)\beta(\hat{x},z,\bar{\alpha}), \beta(\hat{x},z,\bar{\alpha})\big\rangle q(\mathrm{d}z) - F\big(\hat{y}, v(\hat{y}), p, Y\big)$$
$$+\int_{|z|<\varepsilon'} \frac{1}{2}\big\langle (Y-2\delta I)\beta(\hat{y},z,\bar{\alpha}), \beta(\hat{y},z,\bar{\alpha})\big\rangle q(\mathrm{d}z) - \int_{\mathcal{P}_{\varepsilon'}(\hat{x},\bar{\alpha})} u\big(\hat{x}+\beta(\hat{x},z,\bar{\alpha})\big) - u(\hat{x}) - \big\langle \beta(\hat{x},z,\bar{\alpha}), p\big\rangle q(\mathrm{d}z)$$
$$+\int_{\mathcal{P}_{\varepsilon'}(\hat{y},\bar{\alpha})} v\big(\hat{y}+\beta(\hat{y},z,\bar{\alpha})\big) - v(\hat{y}) - \big\langle \beta(\hat{y},z,\bar{\alpha}), p\big\rangle q(\mathrm{d}z) \leqslant \rho. \tag{30}$$

Since $(\hat{x}, \hat{y})$ is the maximum point of $\Phi(x, y)$, from (4)

$$\begin{aligned} u\big(\hat{x}+\beta(\hat{x},z,\bar{\alpha})\big) - u(\hat{x}) &\leqslant v\big(\hat{y}+\beta(\hat{y},z,\bar{\alpha})\big) - v(\hat{y}) + \alpha\big|\hat{x}-\hat{y}+\beta(\hat{x},z,\bar{\alpha})-\beta(\hat{y},z,\bar{\alpha})\big|^2 - \alpha|\hat{x}-\hat{y}|^2 \\ &\leqslant v\big(\hat{y}+\beta(\hat{y},z,\bar{\alpha})\big) - v(\hat{y}) + \alpha L^2|z|^2\,|\hat{x}-\hat{y}|^2 \quad \forall z \in \mathcal{P}_{\varepsilon'}(\hat{x},\bar{\alpha}) \cap \mathcal{P}_{\varepsilon'}(\hat{y},\bar{\alpha}). \end{aligned}$$

We have also

$$\big|\big\langle \beta(\hat{x},z,\bar{\alpha}), p\big\rangle - \big\langle \beta(\hat{y},z,\bar{\alpha}), p\big\rangle\big| \leqslant L|z|\,\alpha|\hat{x}-\hat{y}|^2.$$

By introducing above inequalities into (30), together with (4), (6), from Lemma 2.2, we get

$$\begin{aligned} &F\big(\hat{x}, u(\hat{x}), p, X\big) - F\big(\hat{y}, v(\hat{y}), p, Y\big) \\ &\quad\leqslant \frac{1}{2}\int_{|z|<\varepsilon'} \big\langle (X-Y)\beta(\hat{x},z,\bar{\alpha}), \beta(\hat{x},z,\bar{\alpha}\big\rangle q(\mathrm{d}z) + C'\delta + \rho \\ &\qquad+ \int_{\mathcal{P}_{\varepsilon'}(\hat{x},\bar{\alpha})\cap\mathcal{P}_{\varepsilon'}(\hat{y},\bar{\alpha})^c} u\big(\hat{x}+\beta(\hat{x},z,\bar{\alpha})\big) - u(\hat{x}) - \big\langle \beta(\hat{x},z,\bar{\alpha}), p\big\rangle q(\mathrm{d}z) \\ &\qquad- \int_{\mathcal{P}_{\varepsilon'}(\hat{y},\bar{\alpha})\cap\mathcal{P}_{\varepsilon'}(\hat{x},\bar{\alpha})^c} v\big(\hat{y}+\beta(\hat{y},z,\bar{\alpha})\big) - v(\hat{y}) - \big\langle \beta(\hat{y},z,\bar{\alpha}), p\big\rangle q(\mathrm{d}z) \end{aligned}$$

$$\leqslant \frac{1}{2} \int\limits_{|z|<\varepsilon'} \big\langle (X-Y)\beta(\hat{x}, z, \bar{\alpha}), \beta(\hat{x}, z, \bar{\alpha})\big\rangle q(\mathrm{d}z) + C'\big(\delta + \alpha|\hat{x}-\hat{y}|^2\big) + \rho, \tag{31}$$

where $C' > 0$ is a constant. From (7), (22), (23), and from the continuity of F, we can take $n \in \mathbf{N}$ large enough so that for $X = X_n$ and for $Y = Y_n$

$$F\big(\hat{x}, u(\hat{x}), p, X\big) - F\big(\hat{y}, v(\hat{y}), p, Y\big) \geqslant \frac{1}{2}\gamma\big(u(\hat{x}) - v(\hat{y})\big) - w\big(\alpha|\hat{x}-\hat{y}|^2 + |\hat{x}-\hat{y}|\big(|p|+1\big)\big).$$

By introducing the above inequality into (31), and by letting α to ∞, since $\delta > 0$, $\rho > 0$ are arbitrary we get a contradiction. Thus we proved $u \leqslant v$ in Ω.

Remark 2.1. Example 1.1 is a special case of Theorem 2.1. We can treat more general class of (1), (8) by weaken conditions (22), (23), and also can get the comparison result for the evolutionary problem (2), (3).

Remark 2.2. As we have mentioned in the introduction, in the preceding works of [5,15], etc. a different definition was used for the following problem, in which the domain of the integral is the whole space.

$$\frac{\partial u}{\partial t}(x,t) + F\big(x, u, \nabla u, \nabla^2 u\big) - \sup_{\alpha \in \mathcal{A}} \left\{ -\int\limits_{\mathbf{R}^n} u(x+\beta(x,z,\alpha),t) - u(x,t) - \big\langle \beta(x,z,\alpha), \nabla u(x,t)\big\rangle q(\mathrm{d}z) \right\} = 0$$
$$\text{in } \mathbf{R}^N \times (0,T). \tag{32}$$

For simplicity, we shall focus on the case when $n = N$ and $\beta(x,z,\alpha) = z$, and shall compare two definitions. In the preceding definition, a function $u \in \mathrm{USC}(\mathbf{R}^N \times (0,T])$ is called a viscosity subsolution of (32) if for any $(\hat{x}, \hat{t})$, and for any $\phi(x,t) \in C^{2,1}$ such that $u - \phi$ takes a global maximum at $(\hat{x}, \hat{t})$, the following holds for any $\varepsilon > 0$:

$$\frac{\partial u}{\partial t}(\hat{x},\hat{t}) + F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x},\hat{t}), \nabla^2\phi(\hat{x},\hat{t})\big) - \int\limits_{|z|\leqslant\varepsilon} \phi(\hat{x}+z,t) - \phi(x,t) - \big\langle z, \nabla\phi(\hat{x},\hat{t})\big\rangle q(\mathrm{d}z)$$
$$- \int\limits_{|z|\geqslant\varepsilon} u(\hat{x}+z,\hat{t}) - u(\hat{x},\hat{t}) - \big\langle z, \nabla\phi(\hat{x},\hat{t})\big\rangle q(\mathrm{d}z) \leqslant 0. \tag{33}$$

(The definition of the supersolution is given in parallel.) The difference between ours in (17) (Definition 1.2) and (33) is the following. In (17), we localize the integral around the singularity (if the Lévy measure is singular), and replace the integrand to the superjet, just in the domain such that the one-sided Taylor expansion (15) holds. While, in (33), the test function ϕ is chosen so that $u - \phi$ takes a global maximum, and (33) must hold for any $\varepsilon > 0$. To resume, as for the number of test functions our definition (ex. (17)) uses more than the preceding definition (ex. (33)), however as for the localizations around the singularity ours need less than the preceding, that is (33) is not required to hold for all $\varepsilon > 0$. Therefore, we cannot conclude which definition is stronger or not. By the way, we can present some regularity results for (1) and (2), by using Definitions 1.1 and 1.2. (M. Arisawa [4]), and in the proof of which the usage of the second-order superjet (resp. subjet) in the integral is essential.

Remark 2.3. In the case that F is a Hamilton–Jacobi–Bellman operator, it is expected that the unique viscosity solutions of (1), (8), or (9) and (2), (8), or (9) correspond to the value function of the infinite or finite horizon problems. As far as we know, there is no work which establishes these relationships for the types of integral operators we are concerned with (jumps are restricted inside the domain), and we would like to revisit this problem in our future work.

3. Comparison result for the Neumann boundary value problem

We consider the problem (1) with the Neumann boundary condition (9)

$$\big\langle \nabla u(x), \mathbf{n}(x)\big\rangle = 0 \quad \text{on } \partial\Omega.$$

Throughout this section, we assume that Ω is a compact C^1 domain in $\mathbf{R}^N$ which satisfies the uniform exterior sphere condition: there exists $r > 0$ such that

$$U\big(x + r\mathbf{n}(x), r\big) \cap \Omega = \emptyset \quad \text{on } \partial\Omega, \tag{34}$$

where $\mathbf{n}(\mathbf{x})$ is the outward unit normal to $\partial\Omega$ at x.

We modify Definition 1.1 to treat a class of general boundary value problems including (9). Let B be a scalar valued function defined on $\partial\Omega \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ and consider

$$B\big(x, u, \nabla u, \nabla^2 u\big) = 0 \quad \text{on } \partial\Omega. \tag{35}$$

Definition 3.1. Let $u \in \mathrm{USC}(\overline{\Omega})$ (resp. $v \in \mathrm{LSC}(\overline{\Omega})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), (35) if the following two conditions hold.

(i) If $\hat{x} \in \Omega$, then for any $(p, X) \in J^{2,+}_{\overline{\Omega}} u(\hat{x})$ (resp. $J^{2,-}_{\overline{\Omega}} v(\hat{x})$) and for any $\delta > 0$ there exists $\varepsilon > 0$ such that (10), (11) (resp. (12)) hold, and that for any $0 < \varepsilon' < \varepsilon$

$$F\big(\hat{x}, u(\hat{x}), p, X\big) + \sup_{\alpha\in\mathcal{A}}\bigg\{-\int\limits_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X + 2\delta I)\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle q(\mathrm{d}z)$$
$$- \int\limits_{\mathcal{P}_\varepsilon(\hat{x},\alpha)} u\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) - u(\hat{x}) - \big\langle \beta(\hat{x}, z, \alpha), p\big\rangle q(\mathrm{d}z)\bigg\} \leqslant 0 \tag{36}$$

(resp.

$$F\big(\hat{x}, v(\hat{x}), p, X\big) + \sup_{\alpha\in\mathcal{A}}\bigg\{-\int\limits_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X - 2\delta I)\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle q(\mathrm{d}z)$$
$$- \int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} v\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) - v(\hat{x}) - \big\langle \beta(\hat{x}, z, \alpha), p\big\rangle q(\mathrm{d}z)\bigg\} \geqslant 0). \tag{37}$$

(ii) If $\hat{x} \in \partial\Omega$, then for any $(p, X) \in J^{2,+}_{\overline{\Omega}} u(\hat{x})$ (resp. $J^{2,-}_{\overline{\Omega}} v(\hat{x})$) and for any $\delta > 0$, there exists $\varepsilon > 0$ such that (10), (11) (resp. (12)) hold, and that for any $0 < \varepsilon' < \varepsilon$

$$\min\bigg\{F\big(\hat{x}, u(\hat{x}), p, X\big) + \sup_{\alpha\in\mathcal{A}}\bigg\{-\int\limits_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X + 2\delta I)\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle q(\mathrm{d}z)$$
$$- \int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} u\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) - u(\hat{x}) - \big\langle \beta(\hat{x}, z, \alpha), p\big\rangle q(\mathrm{d}z)\bigg\}, B\big(\hat{x}, u(\hat{x}), p, X\big)\bigg\} \leqslant 0 \tag{38}$$

(resp.

$$\max\bigg\{F\big(\hat{x}, v(\hat{x}), p, X\big) + \sup_{\alpha\in\mathcal{A}}\bigg\{-\int\limits_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X - 2\delta I)\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle q(\mathrm{d}z)$$
$$- \int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} v\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) - v(\hat{x}) - \big\langle \beta(\hat{x}, z, \alpha), p\big\rangle q(\mathrm{d}z)\bigg\}, B\big(\hat{x}, v(\hat{x}), p, X\big)\bigg\} \geqslant 0). \tag{39}$$

Our result is the following.

Theorem 3.1. *Assume that* (4), (6), (7), (22), (23) *and* (25) *hold. Let* $u \in \mathrm{USC}(\overline{\Omega})$ *and* $v \in \mathrm{LSC}(\overline{\Omega})$ *be respectively a viscosity subsolution and a supersolution of* (1), (9). *Then,* $u \leqslant v$ *in* $\overline{\Omega}$.

Remark 3.1. It is possible to generalize Theorem 3.1 for more general boundary condition (35) by modifying $\Phi(x, y)$ in the proof in below.

Proof. Under the assumption that Ω is C^1 and compact, there exists a function $\phi \in C^2(\overline{\Omega})$ such that $\phi \geqslant 0$ in $\overline{\Omega}$ and

$$\langle \mathbf{n}(x), \nabla\phi(x)\rangle \geqslant 1 \quad \text{on } \partial\Omega. \tag{40}$$

(We refer the readers to [3] for this result.) For constants $\rho > 0$ and $C > 0$ which will be determined later, let

$$u_\rho(x) = u(x) - \rho\phi(x) - C, \qquad v_\rho(x) = v(x) + \rho\phi(x) + C. \tag{41}$$

For appropriate choices of ρ and C, we shall show that u_ρ is a viscosity subsolution of (1) in Ω and

$$\langle \mathbf{n}(x), \nabla u_\rho(x)\rangle \leqslant -\rho \quad \text{on } \partial\Omega \tag{42}$$

and that v_ρ is a viscosity supersolution of (1) in Ω and

$$\langle \mathbf{n}(x), \nabla v_\rho(x)\rangle \geqslant \rho \quad \text{on } \partial\Omega. \tag{43}$$

In order to see that u_ρ satisfies (1), (42), for $x \in \overline{\Omega}$, for $(p, X) \in J^{2,+}_{\overline{\Omega}} u_\rho(x)$, and for $\delta > 0$, let $\varepsilon > 0$ be such that (10), (11) hold. We may assume that

$$\phi(y) \leqslant \phi(x) + \langle \nabla\phi(x), y - x\rangle + \frac{1}{2}\langle \nabla^2\phi(x)(y - x), y - x\rangle + \delta|y - x|^2 \quad \text{if } |y - x| \leqslant \varepsilon$$

holds. Then, $(p + \rho\nabla\phi(x), X + \rho\nabla^2\phi) \in J^{2,+}_{\overline{\Omega}} u(x)$ and

$$u(y) \leqslant u(x) + \langle p + \rho\nabla\phi(x), y - x\rangle + \frac{1}{2}\langle \big(X + \rho\nabla^2\phi(x)\big)(y - x), y - x\rangle + 2\delta|y - x|^2 \quad \text{if } |y - x| \leqslant \varepsilon.$$

Hence, by taking $\varepsilon > 0$ small enough for any $0 < \varepsilon' < \varepsilon$

$$\begin{aligned}
&F\big(x, u_\rho(x), p, X\big) + \sup_{\alpha\in\mathcal{A}}\Big\{-\int_{|z|<\varepsilon'} \frac{1}{2}\langle (X + 2\delta I)\beta(x,z,\alpha), \beta(x,z,\alpha)\rangle q(\mathrm{d}z) \\
&\quad - \int_{\mathcal{P}_{\varepsilon'}(x,\alpha)} u_\rho\big(x + \beta(x,z,\alpha)\big) - u_\rho(x) - \langle \beta(x,z,\alpha), p\rangle q(\mathrm{d}z)\Big\} \\
&\leqslant F\big(x, u(x), p + \rho\nabla\phi(x), X + \rho\nabla^2\phi(x)\big) \\
&\quad + \sup_{\alpha\in\mathcal{A}}\Big\{-\int_{|z|<\varepsilon'} \frac{1}{2}\langle \big(X + \rho\nabla^2\phi(x) + 4\delta I\big)\beta(x,z,\alpha), \beta(x,z,\alpha)\rangle q(\mathrm{d}z) \\
&\quad - \int_{\mathcal{P}_{\varepsilon'}(x,\alpha)} u\big(x + \beta(x,z,\alpha)\big) - u(x) - \langle \beta(x,z,\alpha), \big(p + \rho\nabla\phi(x)\big)\rangle q(\mathrm{d}z) \\
&\quad + \int_{|z|<\varepsilon'} \langle \delta\beta(x,z,\alpha), \beta(x,z,\alpha)\rangle q(\mathrm{d}z)\Big\} - \gamma C + w(\rho M),
\end{aligned}$$

where

$$\begin{aligned}
M = \max_{\overline{\Omega}}\Big\{&\big|\nabla\phi(x)\big| + \big\|\nabla^2\phi(x)\big\| + \int_{|z|<\varepsilon'} \frac{1}{2}\langle \nabla^2\phi\beta(x,z,\alpha), \beta(x,z,\alpha)\rangle q(\mathrm{d}z) \\
&+ \int_{\mathcal{P}_{\varepsilon'}(x,\alpha)} \phi\big(x + \beta(x,z,\alpha)\big) - \phi(x) - \langle \beta(x,z,\alpha), \nabla\phi(x)\rangle q(\mathrm{d}z)\Big\}.
\end{aligned}$$

Thus, if we take $C > 0$ so that

$$\gamma C = \sup_{\overline{\Omega}\times\mathcal{A}}\Big\{\int_{|z|<1} \langle \beta(x,z,\alpha), \beta(x,z,\alpha)\rangle q(\mathrm{d}z)\Big\} + w(\rho M),$$

we have

$$F(x,u_\rho,p,X)+\sup_{\alpha\in A}\bigg\{-\int_{|z|<\varepsilon'}\frac{1}{2}\big\langle (X+2\delta I)\beta(x,z,\alpha),\beta(x,z,\alpha)\big\rangle q(\mathrm{d}z)$$
$$-\int_{\mathcal{P}_{\varepsilon'}(x,\alpha)}u_\rho\big(x+\beta(x,z,\alpha)\big)-u_\rho(x)-\big\langle\beta(x,z,\alpha),p\big\rangle q(\mathrm{d}z)\bigg\}\leqslant 0.$$

If $x\in\partial\Omega$ from (40)

$$\big\langle \mathbf{n}(x),p\big\rangle\leqslant\big\langle \mathbf{n}(x),p+\rho\phi(x)\big\rangle-\rho\leqslant-\rho.$$

Thus, we have shown (1), (42). The similar argument for v_ρ leads (1), (43), which we do not write here. Now, it is enough to prove for any $\rho>0$ small enough

$$u_\rho\leqslant v_\rho\quad\text{in }\overline{\Omega}$$

since by tending ρ to 0, we shall have $u\leqslant v$ in $\overline{\Omega}$. Denote $u_\rho=u$, $v_\rho=v$ for simplicity which are respectively solutions of (1), (42) and (1), (43).

We use the argument by contradiction, and assume that there exists $z\in\overline{\Omega}$ such that

$$(u-v)(z)=\max_{x\in\overline{\Omega}}(u-v)(x)>0.$$

We may assume that $z\in\partial\Omega$ (for if not, we can use the same argument as in Theorem 2.1). Consider for arbitrary $\alpha>0$

$$\Phi(x,y)=u(x)-v(y)-\frac{\alpha}{2}|x-y|^2$$

and let (x_α,y_α) be a maximum point of $\Phi(x,y)$. We denote $\phi(x,y)=\frac{\alpha}{2}|x-y|^2$, $p=\nabla_x\phi(x_\alpha,y_\alpha)=\alpha(x_\alpha-y_\alpha)$. As in the proof of Theorem 2.1

$$\lim_{\alpha\to\infty}\frac{\alpha}{2}|x_\alpha-y_\alpha|^2=0,\qquad \lim_{\alpha\to\infty}x_\alpha=\lim_{\alpha\to\infty}y_\alpha=z,$$
$$\lim_{\alpha\to\infty}u(x_\alpha)=u(z),\qquad \lim_{\alpha\to\infty}v(y_\alpha)=v(z)$$

and for any $\alpha>0$ there exist $\overline{X},\overline{Y}\in\mathbf{S}^N$ such that

$$-3\alpha\begin{pmatrix} I & O\\ O & I\end{pmatrix}\leqslant\begin{pmatrix}\overline{X} & O\\ O & -\overline{Y}\end{pmatrix}\leqslant 3\alpha\begin{pmatrix} I & -I\\ -I & I\end{pmatrix},$$

where $(p,\overline{X})\in\bar{J}^{2,+}_{\overline{\Omega}}u(x_\alpha)$, $(p,\overline{Y})\in\bar{J}^{2,-}_{\overline{\Omega}}v(y_\alpha)$. As before, we take a sequence $(p,X_n)\in J^{2,+}_{\overline{\Omega}}u(x_\alpha)$ (resp. $(p,Y_n)\in J^{2,-}_{\overline{\Omega}}v(y_\alpha)$), and shall denote $X=X_n$ (resp. $Y=Y_n$) without any confusion. From now on, we shall denote $(x_\alpha,y_\alpha)=(\hat{x},\hat{y})$ for simplicity. If $\hat{x}\in\partial\Omega$, the exterior sphere condition (34) leads

$$\big\langle\mathbf{n}(\hat{x}),p\big\rangle\geqslant \mathrm{o}(1)\quad\text{as }\alpha\to\infty$$

and this leads

$$\big\langle\mathbf{n}(\hat{x}),p\big\rangle+\rho>\frac{\rho}{2}>0. \tag{44}$$

Since u is a viscosity subsolution of (1), (42), from Definition 3.1 for (p,X), for any $\rho>0$, there exists $\varepsilon>0$ such that (10), (12) hold, and that for any $0<\varepsilon'<\varepsilon$

$$F\big(\hat{x},u(\hat{x}),p,X\big)+\sup_{\alpha\in\mathcal{A}}\bigg\{-\int_{|z|<\varepsilon'}\frac{1}{2}\big\langle (X+2\delta I)\beta(\hat{x},z,\alpha),\beta(\hat{x},z,\alpha)\big\rangle q(\mathrm{d}z)$$
$$-\int_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)}u\big(\hat{x}+\beta(\hat{x},z,\alpha)\big)-u(\hat{x})-\big\langle\beta(\hat{x},z,\alpha),p\big\rangle q(\mathrm{d}z)\bigg\}\leqslant 0. \tag{45}$$

Similarly, for (p, Y), for any $\delta > 0$, there exists $\varepsilon > 0$ such that (10), (12) hold, and that for any $0 < \varepsilon' < \varepsilon$ we also have

$$F\big(\hat{y}, v(\hat{y}), p, Y\big) + \sup_{\alpha\in\mathcal{A}}\bigg\{-\int_{|z|<\varepsilon'} \frac{1}{2}\big\langle (Y - 2\delta I)\beta(\hat{y}, z, \alpha), \beta(\hat{y}, z, \alpha)\big\rangle q(\mathrm{d}z)$$
$$- \int_{\mathcal{P}_{\varepsilon'}(\hat{y},\alpha)} v\big(\hat{y} + \beta(\hat{y}, z, \alpha)\big) - v(\hat{y}) - \big\langle \beta(\hat{y}, z, \alpha), p\big\rangle q(\mathrm{d}z)\bigg\} \geqslant 0. \tag{46}$$

Since (45) and (46) hold for all $(\hat{x}, \hat{y}) = (x_\alpha, y_\alpha)$, by using the same argument as in the proof of Theorem 2.1, we get a contradiction. Thus, we have proved the claim.

4. Existence of viscosity solutions

In this section, we show the existence of viscosity solutions for Dirichlet and Neumann boundary value problems by the Perron's method. The main results are the following.

Theorem 4.1. *Assume that* (4), (6), (7), (22), (23) *and* (25) *hold. Let* $u \in \mathrm{USC}(\overline{\Omega})$ *and* $v \in \mathrm{LSC}(\overline{\Omega})$ *be respectively a viscosity subsolution and a viscosity supersolution of* (1) *in* Ω *such that* $u = v = \phi$ *on* $\partial\Omega$. *Then*

$$W(x) = \sup\big\{w(x) \mid u \leqslant w \leqslant v,\ w \text{ is a subsolution of } (1) \text{ in } \Omega\big\}$$

is a viscosity solution of (1), (8).

Proposition 4.2. *Assume that* (4), (6), (7), (22), (23) *and* (25) *hold. Let* $u \in \mathrm{USC}(\overline{\Omega})$ *and* $v \in \mathrm{LSC}(\overline{\Omega})$ *be respectively a viscosity subsolution and a viscosity supersolution of* (1) *in* Ω *such that* $B(x, u, \nabla u, \nabla^2 u) = B(x, v, \nabla v, \nabla^2 v) = 0$ *on* $\partial\Omega$. *Then*

$$W(x) = \sup\big\{w(x) \mid u \leqslant w \leqslant v,\ w \text{ is a subsolution of } (1) \text{ in } \overline{\Omega}\big\}$$

is a viscosity solution of (1), (35).

Proof of Theorem 4.1. *Step 1*. First, we shall show that $W(x)$ is a viscosity subsolution of (1). Since Definition 1.1 uses not only sub and super jets but also second-order remainders satisfying (10), and (11) (resp. (12)), we re-examine the following standard stability result in the PDE theory.

Lemma 4.3. *Let* $\mathcal{S}$ *be a family of viscosity subsolutions of* (1) *in* Ω. *Put* $U(x) = \sup\{u(x) \mid u \in \mathcal{S}\}$. *If* $U^*(x) = \limsup_{r\downarrow 0}\{U(y) \mid y \in \Omega,\ |y - x| \leqslant r\} < \infty$, *then* U^* *also belongs to* S.

Proof of Lemma 4.3. For any $\bar{x} \in \Omega$, any $(p, X) \in J^{2,+}_{\Omega} U^*(\bar{x})$, and any $\delta > 0$, take $\varepsilon > 0$ such that (10) and the following hold

$$U^*(x) \leqslant U^*(\bar{x}) + \langle p, x - \bar{x}\rangle + \frac{1}{2}\big\langle X(x - \bar{x}), x - \bar{x}\big\rangle + \delta|x - \bar{x}|^2 \quad \text{if } |x - \bar{x}| < \varepsilon. \tag{47}$$

Then, we are to show that for any $0 < \varepsilon' < \varepsilon$

$$F\big(\bar{x}, U^*(\bar{x}), p, X\big) + \sup_{\alpha\in\mathcal{A}}\bigg\{-\frac{1}{2}\int_{|z|<\varepsilon'} \big\langle (X + 2\delta I)\beta(\bar{x}, z, \alpha), \beta(\bar{x}, z, \alpha)\big\rangle q(\mathrm{d}z)$$
$$- \int_{\mathcal{P}_{\varepsilon'}(\bar{x},\alpha)} U^*\big(\bar{x} + \beta(\bar{x}, z, \alpha)\big) - U^*(\bar{x}) - \big\langle \beta(\bar{x}, z, \alpha), p\big\rangle q(\mathrm{d}z)\bigg\} \leqslant 0. \tag{48}$$

To simplify the argument, from now on we assume that $\bar{x} = 0$. (The other case is immediate from the present argument.) From the definition of U, there exists a sequence $\{u_n\} \subset \mathcal{S}$ such that $U(\bar{x}) = \lim_{n\to\infty} u_n(\bar{x})$. Consider

$$u_n(x) - \bigg(\langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle + (\delta + \delta_0)|x|^2\bigg) \quad \text{for } |x| \leqslant \varepsilon$$

where $\delta_0 > 0$ is an arbitrary small number, and let $\hat{x}_n$ be a maximum point of the above in $\overline{U_\varepsilon(0)} = \{|x| \leqslant \varepsilon\}$. That is

$$u_n(x) \leqslant u_n(\hat{x}_n) + \langle p, x - \hat{x}_n\rangle + \frac{1}{2}\big(\langle Xx, x\rangle - \langle X\hat{x}_n, \hat{x}_n\rangle\big) + (\delta + \delta_0)\big(|x|^2 - |\hat{x}_n|^2\big) \quad \text{in } |x| \leqslant \varepsilon. \tag{49}$$

Thus, by denoting $p_n = p + 2(\delta + \delta_0)\hat{x}_n + X\hat{x}_n$, $X_n = X + 2(\delta + \delta_0)I$, we get

$$u_n(x) \leqslant u_n(\hat{x}_n) + \langle p_n, x - \hat{x}_n\rangle + \frac{1}{2}\big\langle X_n(x, -\hat{x}_n), x - \hat{x}_n\big\rangle \quad \text{in } |x| \leqslant \varepsilon. \tag{50}$$

Since $\overline{U_\varepsilon(0)}$ is compact, there exists a subsequence of $\hat{x}_n$ denoting again $\hat{x}_n$ which converges to $y \in \overline{U_\varepsilon(0)}$ as $n \to \infty$. On the other hand, from the definition of U^* there exists $x_n \in \Omega$ such that $u_n(x_n) \to U^*(0)$, $x_n \to 0$. By putting $x = x_n$ in (49), and then by passing $n \to \infty$ we get

$$\begin{aligned}U^*(0) &\leqslant \lim\inf_{n\to\infty} u_n(\hat{x}_n) - \langle p, y\rangle - \frac{1}{2}\langle Xy, y\rangle - (\delta + \delta_0)|y|^2 \\ &\leqslant U^*(y) - \langle p, y\rangle - \frac{1}{2}\langle Xy, y\rangle - (\delta + \delta_0)|y|^2 \leqslant U^*(0) - \delta_0|y|^2\end{aligned}$$

and thus $y = 0 = \bar{x}$. From (50), $(p_n, X_n) \in J_\Omega^{2,+}u_n(\hat{x}_n)$ and the fact that u_n is a subsolution leads the existence of $\varepsilon > 0$ uniformly in n such that (10) holds (because of (4)), and that for any $0 < \varepsilon' < \varepsilon$

$$\begin{aligned}&F\big(\hat{x}_n, u_n(\hat{x}_n), p_n, X_n\big) + \sup_{\alpha\in\mathcal{A}}\Bigg\{-\frac{1}{2}\int_{|z|<\varepsilon'} \big\langle X_n\beta(\hat{x}_n, z, \alpha), \beta(\hat{x}_n, z, \alpha)\big\rangle q(\mathrm{d}z) \\ &\quad - \int_{\mathcal{P}_{\varepsilon'}(\hat{x}_n,\alpha)} u_n\big(\hat{x}_n + \beta(\hat{x}_n, z, \alpha)\big) - u_n(\hat{x}_n) - \big\langle\beta(\hat{x}_n, z, \alpha), p_n\big\rangle q(\mathrm{d}z)\Bigg\} \leqslant 0.\end{aligned}$$

From the Lebesgue's convergence theorem and from Lemma 2.2, by tending first $n \to \infty$, then $\delta_0 \to 0$, the above inequality leads

$$\begin{aligned}&F\big(\bar{x}, u(\bar{x}), p, X\big) + \sup_{\alpha\in A}\Bigg\{-\frac{1}{2}\int_{|z|<\varepsilon'} \big\langle(X + 2\delta I)\beta(\bar{x}, z, \alpha), \beta(\bar{x}, z, \alpha)\big\rangle q(\mathrm{d}z) \\ &\quad - \int_{\mathcal{P}_{\varepsilon'}(\bar{x},\alpha)} U^*\big(\bar{x} + \beta(\bar{x}, z, \alpha)\big) - U^*(\bar{x}) - \big\langle\beta(\bar{x}, z, \alpha), p\big\rangle q(\mathrm{d}z)\Bigg\} \leqslant 0.\end{aligned}$$

Therefore, we proved (48). □

To complete the proof of Theorem 4.1, we show that U_* is a viscosity supersolution of (1) in Ω. We use the argument by contradiction. Thus, we assume that there exist $\bar{x} \in \Omega$, $(p, X) \in J_\Omega^{2,-}U_*(\bar{x})$, $\delta > 0$ and a sequence of numbers ε' tending to 0 ($\varepsilon' \to 0$), such that (10) and the following hold

$$\begin{aligned}&U_*(x) \geqslant U_*(\bar{x}) + \langle p, x - \bar{x}\rangle + \frac{1}{2}\big\langle X(x - \bar{x}), x - \bar{x}\big\rangle - \delta|x - \bar{x}|^2 \quad \text{if } |x - \bar{x}| \leqslant \varepsilon', \\ &F\big(\bar{x}, U_*(\bar{x}), p, X\big) + \sup_{\alpha\in\mathcal{A}}\Bigg\{-\frac{1}{2}\int_{|z|<\varepsilon'} \big\langle(X - 2\delta I)\beta(\bar{x}, z, \alpha), \beta(\bar{x}, z, \alpha)\big\rangle q(\mathrm{d}z) \\ &\quad - \int_{\mathcal{P}_{\varepsilon'}(\bar{x},\alpha)} U_*\big(\bar{x} + \beta(\bar{x}, z, \alpha)\big) - U_*(\bar{x}) - \big\langle\beta(\bar{x}, z, \alpha), p\big\rangle q(\mathrm{d}z)\Bigg\} \leqslant -\theta < 0,\end{aligned} \tag{51}$$

where $\theta > 0$ is a constant. Now, define

$$w(x) = U_*(\bar{x}) + \langle p, x - \bar{x}\rangle + \frac{1}{2}\big\langle X(x - \bar{x}), x - \bar{x}\big\rangle + \mu - \gamma|x - \bar{x}|^2,$$

where $\mu > 0$, $\gamma > 0$ are constants. Then in view of (51), $w(x)$ is a viscosity subsolution of (1) in $\{x \mid |x - \bar{x}| < r\}$ for r, μ, γ small enough. In particular, we choose $\mu = \gamma r^2/8$. Then

$$U(x) > w(x) \quad \text{for } \frac{r}{2} < |x - \bar{x}| < r$$

and $w(\bar{x}) > U(\bar{x})$. Define a "bump" function:

$$\overline{U}(x) = \begin{cases} \max\{U(x), w(x)\} & \text{if } |x| \leqslant r, \\ U(x) & \text{if } |x| > r. \end{cases} \tag{52}$$

We claim that $\overline{U}(x)$ is a subsolution of (1). To see this, first assume that $\overline{U}(\hat{x}) = U(\hat{x})$ ($\hat{x} \in \Omega$). For $(p, X) \in J_{\Omega}^{2,+}\overline{U}(\hat{x})$, for $\delta > 0$, there exists $\varepsilon_0 > 0$ which satisfies (10) and (11), i.e.

$$\overline{U}(x) \leqslant \overline{U}(\hat{x}) + \langle p, x - \hat{x}\rangle + \frac{1}{2}\big\langle (X + 2\delta I)(x - \hat{x}), x - \hat{x}\big\rangle, \quad |x - \hat{x}| < \varepsilon_0 \tag{53}$$

and the fact that $U \leqslant \overline{U}$ leads

$$U(x) \leqslant U(\hat{x}) + \langle p, x - \hat{x}\rangle + \frac{1}{2}\big\langle (X + 2\delta I)(x - \hat{x}), x - \hat{x}\big\rangle', \quad |x - \hat{x}| < \varepsilon_0.$$

Moreover, since $U(x)$ is the viscosity subsolution of (1) there exists $\varepsilon > 0$ ($\varepsilon < \varepsilon_0$) such that (10) holds and for any $0 < \varepsilon' < \varepsilon$

$$F\big(\hat{x}, U(\hat{x}), p, X\big) + \sup_{\alpha \in \mathcal{A}}\Bigg\{-\frac{1}{2}\int\limits_{|z|<\varepsilon'} \big\langle (X + 2\delta I)\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle q(\mathrm{d}z)$$
$$- \int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} U\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) - U(\hat{x}) - \big\langle \beta(\hat{x}, z, \alpha), p\big\rangle q(\mathrm{d}z)\Bigg\} \leqslant 0.$$

Since $\overline{U}(\hat{x} + \beta) > U(\hat{x} + \beta)$, $\overline{U}(\hat{x}) = U(\hat{x})$

$$F\big(\hat{x}, \overline{U}(\hat{x}), p, X\big) + \sup_{\alpha \in \mathcal{A}}\Bigg\{-\frac{1}{2}\int\limits_{|z|<\varepsilon'} \big\langle (X + 2\delta I)\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle q(\mathrm{d}z)$$
$$- \int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} \overline{U}\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) - \overline{U}(\hat{x}) - \big\langle \beta(\hat{x}, z, \alpha), q\big\rangle q(\mathrm{d}z)\Bigg\} \leqslant 0.$$

Next, assume that there exists $\hat{x} \in \Omega$ such that $\overline{U}(\hat{x}) = w(\hat{x}) > U(\hat{x})$ and $|x - \hat{x}| \leqslant r$. Let $\varepsilon_0 > 0$ be such that $w(x) > U(x)$ for $|x - \hat{x}| < \varepsilon_0$. Then, for any $(q, Y) \in J_{\Omega}^{2,+}\overline{U}(\hat{x})$, for $\delta > 0$, we take $0 < \varepsilon < \varepsilon_0$ such that $\beta_1(\hat{x})\varepsilon < \varepsilon_0$, that (10) holds, and

$$w(x) \leqslant w(\hat{x}) + \langle q, x - \hat{x}\rangle + \frac{1}{2}\big\langle (Y + 2\delta I)(x - \hat{x}), x - \hat{x}\big\rangle \quad \text{if } |x - \hat{x}| \leqslant \varepsilon.$$

From the definition of w

$$q = p - 2\gamma(\hat{x} - \bar{x}) + (X - 2\delta I)(\hat{x} - \bar{x}), \qquad Y \geqslant X - 2\gamma I.$$

We remark that for $|z| < \varepsilon$, from (4)

$$\begin{aligned}\overline{U}\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) - \overline{U}(\hat{x}) - \big\langle \beta(\hat{x}, z, \alpha), q\big\rangle &= w\big(\hat{x} + \beta(\hat{x}, z, \alpha)\big) - w(\hat{x}) - \big\langle \beta(\hat{x}, z, \alpha), \nabla w(\hat{x})\big\rangle \\ &= \frac{1}{2}\big\langle \nabla^2 w(\hat{x})\beta(\hat{x}, z, \alpha), \beta(\hat{x}, z, \alpha)\big\rangle.\end{aligned}$$

And thus, for any $0 < \varepsilon' < \varepsilon$

$$
\begin{aligned}
&F\big(\hat{x}, \overline{U}(\hat{x}), q, Y\big) + \sup_{\alpha\in\mathcal{A}}\bigg\{-\int\limits_{|z|<\varepsilon'} \frac{1}{2}\big\langle (Y+2\delta I)\beta(\hat{x},z,\alpha), \beta(\hat{x},z,\alpha)\big\rangle q(\mathrm{d}z) \\
&\quad - \int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} \overline{U}\big(\hat{x}+\beta(\hat{x},z,\alpha)\big) - \overline{U}(\hat{x}) - \big\langle \beta(\hat{x},z,\alpha), q\big\rangle q(\mathrm{d}z)\bigg\} \\
&\quad \leqslant F\big(\hat{x}, w(\hat{x}), q, Y\big) + \sup_{\alpha\in\mathcal{A}}\bigg\{-\int\limits_{|z|<\varepsilon'} \frac{1}{2}\big\langle \big(X+2(\delta-\gamma)I\big)\beta(\hat{x},z,\alpha), \beta(\hat{x},z,\alpha)\big\rangle q(\mathrm{d}z) \\
&\quad - \int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} \overline{U}\big(\hat{x}+\beta(\hat{x},z,\alpha)\big) - \overline{U}(\hat{x}) - \big\langle \beta(\hat{x},z,\alpha), q\big\rangle q(\mathrm{d}z)\bigg\}
\end{aligned}
$$

then from (51) the right-hand side is majorated by

$$
\begin{aligned}
&\leqslant -\theta + F\big(\hat{x}, w(\hat{x}), q, Y\big) - F\big(\bar{x}, U_*(\bar{x}), p, X\big) \\
&\quad + \sup_{\alpha\in\mathcal{A}}\bigg\{-\int\limits_{|z|<\varepsilon'} \frac{1}{2}\big\langle \big(X+2(\delta-\gamma)I\big)\beta(\hat{x},z,\alpha), \beta(\hat{x},z,\alpha)\big\rangle q(\mathrm{d}z) \\
&\quad - \int\limits_{\mathcal{P}_{\varepsilon'}(\hat{x},\alpha)} \overline{U}\big(\hat{x}+\beta(\hat{x},z,\alpha)\big) - \overline{U}(\hat{x}) - \big\langle \beta(\hat{x},z,\alpha), q\big\rangle q(\mathrm{d}z)\bigg\} \\
&\quad - \sup_{\alpha\in\mathcal{A}}\bigg\{-\int\limits_{|z|<\varepsilon'} \frac{1}{2}\big\langle (X+2\delta I)\beta(\bar{x},z,\alpha), \beta(\bar{x},z,\alpha)\big\rangle q(\mathrm{d}z) \\
&\quad - \int\limits_{\mathcal{P}_{\varepsilon'}(\bar{x},\alpha)} U_*\big(\bar{x}+\beta(\bar{x},z,\alpha)\big) - U_*(\bar{x}) - \big\langle \beta(\bar{x},z,\alpha), p\big\rangle q(\mathrm{d}z)\bigg\} \leqslant -\frac{\theta}{2}
\end{aligned}
$$

provided that we take $r>0$, $\gamma>0$ small enough, where we have used (4), (6) and Lemma 2.2. Thus, $\overline{U}$ defined in (52) is a subsolution of (1), and we get a contradiction. Therefore, $U_*(x)$ is a supersolution of (1). From the comparison (Theorem 2.1), $U_* \geqslant U^*$, and we proved that $U(x)$ is the unique viscosity solution of (1), (8). □

Proof of Proposition 4.2. The similar argument in the proof of Theorem 4.1 and the comparison result in Theorem 3.1 leads the claim. □

Remark 4.1. Some examples of the way to construct the sub and super solutions of Dirichlet problem (1), (8) are given in [8]. For the Neumann boundary problem (9), we can take $\underline{u} = M$, $\bar{v} = -M$, where

$$
M = \sup_{\overline{\Omega}\times\mathbf{R}\times\mathbf{R}^N\times\mathbf{S}^N} F(x,r,p,Q)
$$

are respectively a subsolution and a supersolution of (1) satisfying (9). Therefore, there exists a unique viscosity solution of (1), (9).

Acknowledgements

Finally, the author would like to express her gratitudes to Professors P.-L. Lions, O. Pironneau and R. Cont for the interesting discussions and the encouragements.

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