

An extension theorem to rough paths

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Abstract

We show that any continuous path of finite p -variation can be lifted to a geometric q -rough path, where $q > p$.
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Résumé

Nous montrons que tout chemin continu de p -variation finie peut être relevé en un « geometric q -rough path », pour $q > p$.
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1. Introduction

Let

$$\begin{aligned} x : [0, 1] &\rightarrow \mathbb{R}^n, \\ t &\rightarrow (x_1(t), \dots, x_n(t)) \end{aligned}$$

be a continuous function of bounded variation, and V_1, \dots, V_n some smooth functions from \mathbb{R}^d into itself. Then there exists a (unique) solution to the control differential equation

$$\begin{cases} dy(t) = \sum_{i=1}^n V_i(y(t)) dx_i(t), \\ y(0) = y_0. \end{cases} \quad (1)$$

But without the smoothness assumption on x (which is for example almost surely not satisfied by Brownian motion), classical theory fails to give a meaning to the above equation. Rough paths theory [12,13,11] gives a meaning to Eq. (1), whenever x is a continuous path of finite p -variation lifted to a “geometric p -rough path”.

To understand what a geometric p -rough path is, consider a smooth path $x : [0, 1] \rightarrow \mathbb{R}^n$, and define

$$S(x)_{0,t} = 1 + \int_{0 < s_1 < t} dx_{s_1} + \int_{s < s_1 < \dots < s_{i[p]} < t} dx_{s_1} \otimes \dots \otimes dx_{s_{i[p]}}.$$

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$t \rightarrow S(x)_{0,t}$ takes its values in $G^{[p]}(\mathbb{R}^n)$, the free nilpotent group of step $[p]$ over \mathbb{R}^n , thought of a manifold immersed in the tensor algebra $\bigoplus_{k=0}^{[p]} (\mathbb{R}^n)^{\otimes k}$ [12,14]. For a given $p \geq 1$, the set of geometric rough paths is the closure under a given p -variation metric of the set $\{S(x), x \text{ smooth}\}$ (see definition (12)). A weak geometric p -rough path is a $G^{[p]}(\mathbb{R}^n)$ -valued path of finite p -variation, where the p -variation is computed using a homogeneous metric associated to the group. The distinction between these two spaces was glossed over in the paper [12] by the first name author. It is obvious that a geometric p -rough path is a weak geometric p -rough path and that a weak geometric p -rough path is a geometric q -rough path for any $q > p$. There are examples of weak geometric p -rough paths that are not geometric p -rough paths [6]. The difference between weak geometric rough paths and geometric rough paths is a bit like the difference between Lipschitz functions and C^1 functions.

If \mathbf{x} is a (weak) geometric p -rough path, \mathbf{x} projects onto a path x with values in \mathbb{R}^n . The solution to Eq. (1) is uniquely defined for any \mathbf{x} , and the solution is also a (weak) geometric p rough path \mathbf{y} . Moreover, the map $\mathbf{x} \rightarrow \mathbf{y}$ is continuous in an appropriate topology. In the classical setting where x is smooth, $p = 1$. There is a functional relationship between x and the solution of the differential equation (1). For $p \geq 2$, there will be infinitely many choices for \mathbf{x} projecting onto x . The corresponding solution \mathbf{y} and its projection y will in general depend on this choice.

There is often a “canonical” choice for the lift \mathbf{x} of x (for example, if x is smooth, $S(x)$ is a canonical lift of x to a geometric p -rough path, for any $p \geq 1$). “Canonical” lifts have been constructed for Brownian motion [13,10], fractional Brownian motion with Hurst parameter greater than $1/4$ [3,13], free Brownian motion [2,18], and a large class of random paths on fractals [1,8].

1.1. Our goal

Consider the following natural question: can every continuous path of finite p -variation in V (a Banach space) be lifted to a weak geometric p -rough path (a $G^{[p]}(V)$ -valued path of finite p -variation)? We will see, that provided that p is not an integer number greater than or equal to 2, the answer is affirmative. This is optimal, as a counter example for $p = 2$ was provided in [18]. In particular, any path of finite p -variation in V can be lifted to a geometric q -rough path, for any $q > p$. The theorem we prove is actually stronger.

Theorem 1. *We fix $p \in [1, +\infty)$. Let V be a Banach space and K a closed normal subgroup of $G^{([p])}(V)$. If x is a $(G^{([p])}(V)/K, \|\cdot\|_{G^{([p])}(V)/K})$ continuous path of finite p -variation, with $p \notin \mathbb{N} \setminus \{0, 1\}$, then one can lift x to a weak geometric p -rough path.*

Consider a path x of finite p -variation with values in $(G^{([p])}(V)/K, \|\cdot\|_{G^{([p])}(V)/K})$, where K is as above. x projects to a \mathbb{R}^n -valued path, and so one can consider again the differential equation (1). This one only makes sense once we lift x to a geometric q -rough path \mathbf{x} , for $q > p$. The solution depends in general on the choice of the lift \mathbf{x} . We will identify conditions on the Lie algebra generated by the vector fields $(V_i)_{1 \leq i \leq n}$ in (1) so that the projection y of the rough path solution \mathbf{y} of Eq. (1) does not depend on the lift of x . In general, \mathbf{y} and y will depend on the lift \mathbf{x} of x .

To help the comprehension of the paper, we start by presenting the main theorem for $p \in (2, 3)$ and $V = \mathbb{R}^2$, where no algebra is necessary and result are quite intuitive.

2. A simple case

We start with a non-surprising technical lemma, whose proof is inspired from the Kolmogorov–Centsov criteria [9].

Lemma 2. *Let y be a map from $\bigcup_{n \geq 0} \bigcup_{k=0}^{2^n} \{k2^{-n}\}$ into (E, d) , a metric space, such that for all $n, k \in \{0, \dots, 2^n\}$,*

$$d(y_{\frac{k}{2^n}}, y_{\frac{k+1}{2^n}}) \leq C2^{-n/p}. \quad (2)$$

Then, there exists a unique continuous path $\tilde{y} : [0, 1] \rightarrow (E, d)$ which coincides with y on $\bigcup_{n \geq 0} \bigcup_{k=0}^{2^n} \{k2^{-n}\}$. Moreover, \tilde{y} is $1/p$ -Hölder.

Proof. We fix $r \in \mathbb{N}$, and show by induction on m that for all $s, t \in D_m$ such that $0 < t - s < 2^{-r}$,

$$d(y_s, y_t) \leq 2C \sum_{k=r+1}^m 2^{-k/p}. \tag{3}$$

When $m = r + 1$, necessarily, (s, t) is of the form $(\frac{k}{2^m}, \frac{k+1}{2^m})$, $k \in \{0, \dots, 2^m - 1\}$, and so (3) is exactly formula (2). Suppose now that formula (3) is valid for $m = r + 1, \dots, M - 1$. Take $s, t \in D_M$ such that $0 < t - s < 2^{-r}$, and consider $t_1 = \max\{u \in D_{M-1}; u \leq t\}$ and $s_1 = \max\{u \in D_{M-1}; u \geq s\}$. Notice that $d(y_s, y_{s_1})$ and $d(y_{t_1}, y_t)$ are both bounded by $C2^{-M/p}$, and, by the induction assumption, that

$$d(y_{s_1}, y_{t_1}) \leq 2C \sum_{k=r+1}^{M-1} 2^{-k/p}.$$

Therefore,

$$\begin{aligned} d(y_s, y_t) &\leq 2C2^{-M/p} + 2C \sum_{k=r+1}^{M-1} 2^{-k/p} \\ &= 2C \sum_{k=r+1}^M 2^{-k/p}, \end{aligned}$$

which concludes the induction.

Now let us consider $(s, t) \in \bigcup_{m \geq 0} D_m$, and let r be the natural number such that $2^{-(r+1)} < t - s < 2^{-r}$. From the induction, we obtain

$$\begin{aligned} d(y_s, y_t) &\leq 2C \sum_{k=r+1}^{\infty} 2^{-k/p} \leq \tilde{C}_p 2^{-(r+1)/p} \\ &\leq \tilde{C}_p |t - s|^{1/p}. \end{aligned} \tag{4}$$

We finally define \tilde{y}_t for $0 \leq t \leq 1$ by

$$\tilde{y}_t = \lim_{r \rightarrow \infty} y_{\lfloor 2^r t \rfloor}.$$

From (4), the limit exists and \tilde{y} satisfies $d(\tilde{y}_s, \tilde{y}_t) \leq \tilde{C} |t - s|^{1/p}$. \square

Let x be a \mathbb{R}^2 -valued path, which is $1/p$ -Hölder with $p \in (2, 3)$. We want to prove that we can lift x to a $1/p$ -Hölder path with values in the Heisenberg group H^1 equipped with its Carnot–Carathéodory metric. Let us first recall a few facts about this group and its metric. The Heisenberg group H^1 is equal to \mathbb{R}^3 equipped with the product

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(y_1 x_2 - y_2 x_1) \right).$$

The Carnot–Carathéodory distance will be introduced later, the only property we need for this preliminary chapter is that there exists positive constants c, C such that

$$c \max \left\{ |x_1 - x_2|, |y_1 - y_2|, \left| z_1 - z_2 + \frac{1}{2}(y_1 z_2 - y_2 z_1) \right|^{1/2} \right\} \leq d((x_1, y_1, z_1), (x_2, y_2, z_2)),$$

and

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) \leq C \max \left\{ |x_1 - x_2|, |y_1 - y_2|, \left| z_1 - z_2 + \frac{1}{2}(y_1 z_2 - y_2 z_1) \right|^{1/2} \right\}.$$

It is easy to see that to lift x to a $1/p$ -Hölder H^1 -valued path, we need to construct the Levy area of x , i.e. a map $A : \{0 \leq s < t \leq 1\} \rightarrow \mathbb{R}$ such that

- for all $s < t < u \in [0, 1]$,

$$A_{s,u} = A_{s,t} + A_{t,u} + \frac{1}{2} (x_{s,t}^1 x_{t,u}^2 - x_{s,t}^2 x_{t,u}^1); \quad (5)$$

- for some constant C , for all $s, t \in [0, 1]$,

$$|A_{s,t}| \leq C |t - s|^{2/p}. \quad (6)$$

Of course, if x is of bounded variation, $A_{s,t} = \frac{1}{2} \iint_{s < v_1 < v_2 < t} (dx_{v_1}^1 dx_{v_2}^2 - dx_{v_2}^1 dx_{v_1}^2)$ satisfies the above condition. As we do not assume x smooth, we cannot use this area.

Proposition 3. *Let x be a \mathbb{R}^2 -valued path, which is $1/p$ -Hölder with $p \neq 2$. Then, one can lift x to a $1/p$ -Hölder path with values in the Heisenberg group H^1 equipped with its Carnot–Carathéodory metric.*

In rough paths language, using the fact that path of finite p -variation are $1/p$ -Hölder after a time change, this means that we can lift any \mathbb{R}^2 -valued path of finite p -variation to a geometric p -rough path, whenever $p \in (2, 3)$.

Proof. If $p < 2$, the result is just a easy consequence of Theorem 1 in [12], or just properties of Young integrals. We therefore assume $p > 2$.

Let C_x be the Hölder constant of x . We construct inductively the area of x between dyadic times, $A_{\frac{k}{2^n}, \frac{k+1}{2^n}}$ for $k = 0, \dots, 2^n$. We also define inductively

$$a_n = 2^{2n/p} \max_{0 \leq k \leq 2^n} |A_{\frac{k}{2^n}, \frac{k+1}{2^n}}|.$$

First, we set $A_{0,1} = 0$, and therefore we have $a_0 = 0$. Assume then, for a fixed n , that we have constructed $A_{\frac{k}{2^n}, \frac{k+1}{2^n}}$ for $k = 0, \dots, 2^n$. We define $A_{\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}}$ and $A_{\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}}$ so that they are both equal. Eq. (5) therefore forces them to be equal to

$$\frac{1}{2} A_{\frac{k}{2^n}, \frac{k+1}{2^n}} - \frac{1}{4} (x_{\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}}^1 x_{\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}}^2 - x_{\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}}^2 x_{\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}}^1).$$

In particular

$$a_{n+1} 2^{-2(n+1)/p} \leq 2^{-2n/p} \frac{a_n}{2} + \frac{C_x^2}{2} 2^{-2(n+1)/p},$$

i.e.

$$a_{n+1} \leq 2^{2/p-1} a_n + \frac{1}{2} C_x^2.$$

It is easy to see by induction that, if $p > 2$, the sequence a_n is bounded. Transferring this information in terms of the path $\mathbf{x} = (x, A)$, we see that we have constructed elements $\mathbf{x}_{k2^{-n}}$ of the metric space (H^1, d) such that for all $n, k \in \{0, \dots, 2^n\}$,

$$d(\mathbf{x}_{\frac{k}{2^n}}, \mathbf{x}_{\frac{k+1}{2^n}}) \leq M_p 2^{-n/p}. \quad (7)$$

We conclude the proof with Lemma 2. \square

The above construction and idea will be the main argument of the proof of the main theorem. To be able to explain it, we need to introduce a few algebraic and geometric notions.

3. Algebraic preliminaries

3.1. Carnot groups

If G is a simply connected nilpotent Lie group with Lie algebra \mathcal{G} , then the Lie group exponential map $\exp: \mathcal{G} \rightarrow G$ is a diffeomorphism [15,17]. In this case we let $\ln: G \rightarrow \mathcal{G}$ denote the inverse of the exponential function. We start with a couple definitions.

Definition 4. A Carnot group¹ is a connected nilpotent Lie group G , such that its Lie algebra \mathcal{G} can be written as

$$\mathcal{G} = W_1 \oplus \cdots \oplus W_n,$$

where for all i , $W_{i+1} = [W_1, W_i]$. For an element $g = \exp(w_1 + \cdots + w_n) \in G$, with $w_i \in W_i$, we let, for $t \in \mathbb{R}$,

$$\delta_t g = \exp(tw_1 + \cdots + t^n w_n).$$

δ is called the dilation operator.

Definition 5. A (symmetric sub-additive) homogeneous norm [5] on a Carnot group G is a function $\|\cdot\|_G : G \rightarrow \mathbb{R}^+$ such that

- (i) $\|g\|_G = 0$ if and only if g is the neutral element of the group, $1 = \exp(0)$,
- (ii) $\|\delta_t g\|_G = |t| \|g\|_G$,
- (iii) for all $g, h \in G$, $\|g \otimes h\|_G \leq \|g\|_G + \|h\|_G$,
- (iv) for all g , $\|g\|_G = \|g^{-1}\|_G$.

Such a norm define a left invariant distance on the group by $d_G(g, h) = \|h^{-1} \otimes g\|_G$. We will say that $(G, \|\cdot\|_G)$ is a normed Carnot group.

If G is a fixed Carnot group with finite dimensional Lie algebra, all homogeneous norms on G are equivalent. The Carnot–Caratheodory norm is an example of a homogeneous norm on a Carnot group [7]. Any homogeneous norms $\|\cdot\|_G$ on G leads to a left invariant distance $d_G(x, y) = \|y^{-1}x\|$ (in particular, the Carnot–Caratheodory norm leads to the Carnot–Caratheodory distance). Let G be a normed Carnot group with Lie algebra \mathcal{G} , K a Lie subgroup of G , with Lie algebra \mathcal{K} . If K is a closed normal Lie subgroup of G , or equivalently if \mathcal{K} is closed ideal of \mathcal{G} , then G/K is then a Carnot group with Lie algebra \mathcal{G}/\mathcal{K} [15]. If G is equipped with a homogeneous norm $\|\cdot\|_G$, then we equip G/K with the quotient homogeneous norm on G/K

$$\begin{aligned} \|\cdot\|_{G/K} : G/K &\rightarrow \mathbb{R}, \\ gK &\rightarrow \inf_{k \in K} \|g \otimes k\|_G. \end{aligned}$$

We will denote by $\pi_{G,G/K}$ the canonical homomorphism from G onto G/K . Sometimes, it will be more convenient to write gK for $\pi_{G,G/K}(g)$.

Proposition 6. Let $(G, \|\cdot\|_G)$ be a normed Carnot group, K a closed normal Lie subgroup of G . There exists an injection $i_{G/K,G} : G/K \rightarrow G$ such that

- (i) $\pi_{G,G/K} \circ i_{G/K,G}$ is the identity map of G/K ,
- (ii) for all $t \in \mathbb{R}^+$, $gK \in G/K$, $\delta_t(i_{G/K,G}(gK)) = i_{G/K,G}(\delta_t(gK))$,
- (iii) for all $gK \in G/K$, $\|gK\|_{G/K} \leq \|i_{G/K,G}(gK)\|_G \leq 2\|gK\|_{G/K}$.

Proof. By definition of the homogeneous norm on G/K , for all $g \in G$ such that $\|gK\|_{G/K} = 1$, the set

$$M_g = \{g \otimes k \text{ such that } k \in K \text{ and } 1 \leq \|g \otimes k\|_G \leq 2\}$$

is non-empty. We define $i_{G/K,G}$ on the set of elements

$$\{gK, \text{ such that } \|gK\|_{G/K} = 1\}$$

to be any function which at gK associates an element of $\bigcup_{m \in \pi_{G,G/K}^{-1}(gK)} M_m$; such function exists by the axiom of choice. We then extend $i_{G/K,G}$ to G/K with the help of the formula $i_{G/K,G}(\delta_t gK) = \delta_t i_{G/K,G}(gK)$. \square

¹ In most definitions of a Carnot group, \mathcal{G} is assumed to be finite dimensional. We do not make such an assumption here.

3.2. Free nilpotent groups

We now introduce a fundamental example of a Carnot group.

We fix (for the rest of the paper) a normed vector space $(V, \|\cdot\|_1)$. We let $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ be the tensor algebra over V . $T(V)$ equipped with standard addition $+$, tensor multiplication \otimes and scalar product is an associative algebra. $T^{(n)}(V)$, the quotient algebra of $T(V)$ by the ideal $\bigoplus_{m=n+1}^{\infty} V^{\otimes m}$, inherits this algebraic structure. One can define on $T^{(n)}(V)$ a Lie bracket by the formula

$$[a, b] = a \otimes b - b \otimes a,$$

which makes $T^{(n)}(V)$ into a Lie algebra. We let $\mathcal{G}^{(n)}(V)$ be the Lie subalgebra of $T^{(n)}(V)$ generated by elements in V . Note that

$$\mathcal{G}^{(n)}(V) \simeq \bigoplus_{i=1}^n V_i,$$

where

$$V_1 = V \quad \text{and} \quad V_{i+1} = [V, V_i]. \tag{8}$$

$\mathcal{G}^{(n)}(V)$ is the free nilpotent Lie algebra of step n [12–14]. The exponential, logarithm and inverse function are defined on $T^{(n)}(V)$ by means of their power series. We denote by $G^{(n)}(V) = \exp(\mathcal{G}^{(n)}(V))$. By the Baker–Campbell–Hausdorff formula, $(G^{(n)}(V), \otimes)$ is a connected nilpotent Lie group, called the free nilpotent Lie group of step n over V . By construction, $(G^{(n)}(V), \otimes)$ is a Carnot group, with Lie algebra $\mathcal{G}^{(n)}(V)$.

We are now going to equip $G^{(n)}(V)$ with a homogeneous norm. We first let $\|\cdot\|_i$ be some norms on $V^{\otimes i}$ such that for all $(a_i, a_j) \in V^{\otimes i} \times V^{\otimes j}$, $\|a_i \otimes a_j\|_{i+j} \leq \|a_i\|_i + \|a_j\|_j$. To simplify notations, we will write $\|\cdot\|$ for all these norms. Now define

$$\|g\|_{G^{(n)}(V)} = \max_{i=1, \dots, n} (i! \|g_i\|)^{1/i} + \max(i! \|(g^{-1})_i\|)^{1/i},$$

where $g = 1 + g_1 + \dots + g_n$, $g_i \in V^{\otimes i}$ is an element of the group $G^{(n)}(V)$, and g^{-1} is its inverse. The binomial equality quickly shows that $g \rightarrow \sum_{i=1, \dots, n} (i! \|g_i\|)^{1/i}$ is a sub-additive homogeneous norm (but a priori not symmetric). That implies that $\|\cdot\|_{G^{(n)}(V)}$ defines a homogeneous norm on $G^{(n)}(V)$. We also let

$$d_{G^{(n)}(V)}(g, h) = \|h^{-1} \otimes g\|_{G^{(n)}(V)}.$$

Proposition 7. *Let $g = \exp(l_1 + \dots + l_n)$, with $l_i \in V_i$. Then,*

$$c_n \max_{i=1, \dots, n} \|l_i\|^{1/i} \leq \|g\|_{G^{(n)}(V)} \leq C_n \max_{i=1, \dots, n} \|l_i\|^{1/i},$$

for some constants c_n and C_n which depends only on n .

Proof. Let us fix $i \in \{1, \dots, n\}$ and write $g = 1 + g_1 + \dots + g_n$, with $g_i \in V^{\otimes i}$. By definition of the exponential function,

$$g^k = \sum_{i=1}^k \frac{1}{i!} \sum_{\substack{j_1, \dots, j_i \\ j_1 + \dots + j_i = k}} \otimes \dots \otimes l_{j_i}.$$

Hence,

$$\begin{aligned} (k! \|g^k\|)^{1/k} &\leq \left(\sum_{i=1}^k \frac{k!}{i!} \sum_{\substack{j_1, \dots, j_i \\ j_1 + \dots + j_i = k}} \|l_{j_1}\| \dots \|l_{j_i}\| \right)^{1/k} \\ &\leq (k!(\exp k - 1))^{1/k} \max_{i=1, \dots, n} \|l_i\|^{1/i}. \end{aligned}$$

Applying this to g^{-1} , we obtain

$$(k! \|(g^{-1})_k\|_k)^{1/k} \leq (k!(\exp k - 1))^{1/k} \max_{i=1, \dots, n} \| -l_i \|^{1/i} = (k!(\exp k - 1))^{1/k} \max_{i=1, \dots, n} \| -l_i \|^{1/i}.$$

That gives us the upper bound. For the lower bound, observe that by definition of the logarithm function,

$$l_k = \sum_{i=1}^k \frac{(-1)^i}{i} \sum_{\substack{j_1, \dots, j_i \\ j_1 + \dots + j_i = k}} g_{j_1} \otimes \dots \otimes g_{j_i},$$

which, when applied to both g and its inverse, gives that for all $1 \leq k \leq n$,

$$\|l_k\|^{1/k} \leq c_n^{-1} \|g\|_{G^{(n)}(V)}$$

for a constant $c_n > 0$. \square

Corollary 8. Let $K = \exp(K)$ be a closed normal subgroup of $G^{(n)}(V)$. Then, if $g = \exp(l_1 + \dots + l_n)$ with $l_i \in V_i$,

$$c_n \leq \frac{\|gK\|_{G^{(n)}(V)/K}}{\max_{i=1, \dots, n} (\inf_{k_i \in K \cap V_i} \|l_i + k_i\|)^{1/i}} \leq C_n.$$

Corollary 9. Let $C(G^{(n)}(V))$ be the centre of $G^{(n)}(V)$ and θ the canonical isomorphism between $G^{(n-1)}(V)$ and $G^{(n)}(V)/C(G^{(n)}(V))$. Then the homogeneous norm $\|\cdot\|_{G^{(n-1)}(V)}$ and $\|\theta(\cdot)\|_{G^{(n)}(V)/C(G^{(n)}(V))}$ are equivalent. We will therefore not distinguish between them.

4. Rough paths

In this paper, by E -valued path, we mean a function from $[0, 1]$ into E .

4.1. On p -variation

Definition 10. Let (E, d) be a metric space. A (E, d) -valued path x is said to have finite p -variation if

$$\sup_D \sum_{i=1}^{\#D-1} d(x_{t_i}, x_{t_{i+1}})^p < \infty,$$

where the supremum runs over all subdivisions $D = (0 \leq t_1 \leq \dots \leq t_{\#D} \leq 1)$ of the interval $[0, 1]$.

Note that x is continuous and of finite regular p -variation if and only if for all $s \leq t$, $d(x_s, x_t) \leq \omega(s, t)$, where

- (i) $\omega : \{(s, t), 0 \leq s \leq t \leq 1\} \rightarrow \mathbb{R}^+$ is continuous.
 - (ii) ω is super-additive, i.e. $\forall s < t < u$, $\omega(s, t) + \omega(t, u) \leq \omega(s, u)$.
 - (iii) $\omega(t, t) = 0$ for all $t \in [0, 1]$.
- (9)

We will say in such case that x has finite p -variation controlled by ω .

We are going to show that a continuous (E, d) -valued path of finite p -variation is, up to reparametrisation of time, $1/p$ -Hölder continuous. If ω satisfies (9), then

$$(s, t) \rightarrow \omega(0, 1) \left(\frac{\omega(0, t)}{\omega(0, 1)} - \frac{\omega(0, s)}{\omega(0, 1)} \right)$$

is a continuous additive map, equal to zero on the diagonal, and $\omega(0, t) - \omega(0, s) \geq \omega(s, t)$ (by the super-additivity of ω). Therefore, a path is of finite p -variation if and only if there exists a non-decreasing and continuous surjection γ from $[0, 1]$ onto $[0, 1]$ and a positive constant C such that

$$\text{for all } s \leq t, \quad d(x_s, x_t)^p \leq C |\gamma(t) - \gamma(s)|.$$

For such a γ , we define

$$\begin{aligned} \gamma^{-1} : [0, 1] &\rightarrow [0, 1], \\ t &\rightarrow \inf\{u, \gamma(u) = t\}. \end{aligned}$$

The following is straightforward to check.

Lemma 11. *Let x be a continuous (E, d) -valued path of finite p -variation controlled by $(s, t) \rightarrow C|\gamma(t) - \gamma(s)|$, where γ is a continuous increasing surjection from $[0, 1]$ onto $[0, 1]$. Define*

$$\begin{aligned} y : [0, 1] &\rightarrow E, \\ t &\rightarrow x_{\gamma^{-1}(t)}. \end{aligned}$$

Then, y is a $1/p$ -Hölder (E, d) -valued path.

Reciprocally, if y is a $1/p$ -Hölder (E, d) -valued path then, it is a continuous path of finite p -variation controlled by $(s, t) \rightarrow C|t - s|$.

4.2. Geometric p -rough paths

Definition 12. A weak geometric p -rough path is a $(G^{(lp)}(V), \|\cdot\|_{G^{(lp)}(V)})$ -valued path which has finite p -variation.²

When x is a path with values in a group (G, \otimes) , we will write $x_{s,t} = x_s^{-1} \otimes x_t$.

5. The extension theorem

We first need an important lemma.

Lemma 13. *Let $(G, \|\cdot\|_G)$ be a normed Carnot group with graded Lie algebra*

$$\mathcal{G} = W_1 \oplus W_2 \oplus \dots \oplus W_n.$$

Define K to be a closed subgroup of $\exp(W_n)$, which gives us a normed Carnot group $(G/K, \|\cdot\|_{G/K})$. Let x be a $1/p$ -Hölder $(G/K, \|\cdot\|_{G/K})$ -valued path. Then, if $p > n$, there exists a $1/p$ -Hölder $(G, \|\cdot\|_G)$ -valued path \tilde{x} such that $\pi_{G,G/K}(\tilde{x}) = x$.

Proof. As $\exp(W_n)$ is in the center of G , K is a subgroup of the center of G . In particular, K is a closed normal subgroup of G .

To construct our path \tilde{x} , we are first going to construct its increments $\tilde{x}_{s,t}$ when $s, t \in D_m = \{\frac{k}{2^m}, k \in \{0, \dots, 2^m - 1\}\}$ with $t - s = 2^{-m}$, doing this for all m . $\tilde{x}_{s,t}$ will be constructed in such a way that $\|\tilde{x}_{s,t}\|_G \leq C|t - s|^{1/p}$ for a given $C < \infty$. Multiplying the increments, we will then have defined x on all dyadics, and the proof will be finished thanks to Lemma 2.

So we define recursively on m some elements $y_{\frac{k}{2^m}, \frac{k+1}{2^m}} \in K, k \in \{0, \dots, 2^m - 1\}, m \in \mathbb{N}$, in the aim of defining the elements $\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}}$ with the formula

$$\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}} = i_{G/K,G}(x_{\frac{k}{2^m}, \frac{k+1}{2^m}}) \otimes y_{\frac{k}{2^m}, \frac{k+1}{2^m}}$$

where $i_{G/K,G}$ is the injection of Proposition 6. This will ensure that $\pi_{G,G/K}(\tilde{x}) = x$. First, we let, $y_{0,1} = \exp(0)$. Then, we assume that $y_{\frac{k}{2^m}, \frac{k+1}{2^m}}$ (and hence $\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}}$) has been constructed for all $0 \leq k \leq 2^m - 1$ and a fixed m , and we define the two elements $y_{\frac{2k}{2^{m+1}}, \frac{2k+1}{2^{m+1}}}$ and $y_{\frac{2k+1}{2^{m+1}}, \frac{2k+2}{2^{m+1}}}$ to be both equal, and equal to the inverse of

$$\delta_{2^{-1/n}}(i_{G/K,G}(x_{\frac{2k}{2^{m+1}}, \frac{2k+1}{2^{m+1}}}) \otimes i_{G/K,G}(x_{\frac{2k+1}{2^{m+1}}, \frac{2k+2}{2^{m+1}}}) \otimes \tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}}^{-1}).$$

² The definition of a geometric p -rough path is presented quite differently in [12], as the notion of a homogeneous norm was not mentioned there. Nonetheless, the difference is easily seen to be only notational.

We easily check that $\pi_{G,G/K}(y_{\frac{2k}{2m+1}, \frac{2k+1}{2m+1}}) = \exp(0)$, i.e. that $y_{\frac{2k}{2m+1}, \frac{2k+1}{2m+1}} = y_{\frac{2k+1}{2m+1}, \frac{2k+2}{2m+1}} \in K$. As elements of K commute with elements of G , and with the help of the formula $\delta_{2^{1/n}}(y) = y^{\otimes 2}$ for $y \in K$, we check that this choice for $y_{\frac{2k}{2m+1}, \frac{2k+1}{2m+1}}$ and $y_{\frac{2k+1}{2m+1}, \frac{2k+2}{2m+1}}$ gives

$$\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}} = \tilde{x}_{\frac{2k}{2m+1}, \frac{2k+1}{2m+1}} \otimes \tilde{x}_{\frac{2k+1}{2m+1}, \frac{2k+2}{2m+1}}.$$

We then define $a_m = 2^{m/p} \sup_{k \in \{0, \dots, 2^m-1\}} \|y_{\frac{k}{2^m}, \frac{k+1}{2^m}}\|_G$. By the assumption that x is $1/p$ -Hölder and by the definition of $i_{G/K,G}$,

$$\|i_{G/K,G}(x_{\frac{k}{2^m}, \frac{k+1}{2^m}})\|_G \leq 2 \|x_{\frac{k}{2^m}, \frac{k+1}{2^m}}\|_{G/K} \leq 2C2^{-m/p}.$$

Hence, from the previous inequality, we obtain that

$$2^{1/n}2^{-(m+1)/p}a_{m+1} \leq a_m2^{-m/p} + 2^{-(m+1)/p}2^{2+1/p}C,$$

i.e.

$$a_{m+1} \leq 2^{1/p-1/n}a_m + 2^{2+1/p-1/n}C.$$

As $n < p$, we have $2^{1/p-1/n} < 1$ which forces the sequence a_m to be bounded. So we have constructed for every $m \geq 0, k \in \{0, \dots, 2^m - 1\}$ some elements $\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}} = i_{G/K,G}(x_{\frac{k}{2^m}, \frac{k+1}{2^m}}) \otimes y_{\frac{k}{2^m}, \frac{k+1}{2^m}} \in G$, such that

$$\begin{aligned} \|\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}}\|_G &\leq 2 \|x_{\frac{k}{2^m}, \frac{k+1}{2^m}}\|_{G/K} + \|y_{\frac{k}{2^m}, \frac{k+1}{2^m}}\|_G \\ &\leq \left(2C + \sup_m a_m\right)2^{-m/p} = C_{p,n}2^{-m/p}. \end{aligned}$$

Remember also that for all dyadic $\frac{k}{2^m}$,

$$\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}} = \tilde{x}_{\frac{2k}{2m+1}, \frac{2k+1}{2m+1}} \otimes \tilde{x}_{\frac{2k+1}{2m+1}, \frac{2k+2}{2m+1}}.$$

That allows us to define

$$\tilde{x}_{\frac{k}{2^m}, \frac{l}{2^m}} = \bigotimes_{j=k}^{l-1} \tilde{x}_{\frac{j}{2^m}, \frac{j+1}{2^m}}.$$

We have proved that for all $n, k \in \{0, \dots, 2^n\}$,

$$\|\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}}\|_G \leq C'2^{-m/p}. \tag{10}$$

The proof is therefore finished using Lemma 2. \square

We are now ready for our main theorem.

Theorem 14. *We fix $p \in [1, +\infty)$. Let K be a closed normal subgroup of $G^{(l,p)}(V)$. If x is a $(G^{(l,p)}(V)/K, \|\cdot\|_{G^{(l,p)}(V)/K})$ continuous path of finite p -variation, with $p \notin \mathbb{N} \setminus \{0, 1\}$, then there exists a continuous $(G^{(l,p)}(V), \|\cdot\|_{G^{(l,p)}(V)})$ -valued path \tilde{x} of finite p -variation such that*

$$\pi_{G^{(l,p)}(V), G^{(l,p)}(V)/K}(\tilde{x}) = x.$$

Proof. As noticed in Section 4.1, we assume without loss of generalities that x is $1/p$ -Hölder. We denote by $\mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_{[p]}$ the Lie algebra of K , with $\mathcal{K}_i \subset V_i$. We define for $k = 1, \dots, n$,

$$H^{(k)} = G^{(k)}(V)/\exp(\mathcal{K}_k)$$

and

$$\begin{aligned} M^{(k)} &= G^{(k)}(V)/\exp(\mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_k) \\ &\simeq H^{(k)}/\exp(\mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_{k-1}). \end{aligned}$$

We are going to construct recursively some $H^{(k)}$ -valued paths $y^{(k)}$ and $G^{(k)}(V)$ -valued paths $x^{(k)}$ which are $1/p$ -Hölder and such that the projections of x , $x^{(k)}$, and $y^{(k)}$ onto $M^{(k)}$ are equal, i.e. such that

$$\pi_{H^{(k)}, M^{(k)}}(y^{(k)}) = \pi_{M^{(l(p))}, M^{(k)}}(x), \tag{11}$$

$$\pi_{G^{(k)}(V), M^{(k)}}(x^{(k)}) = \pi_{M^{(l(p))}, M^{(k)}}(x). \tag{12}$$

Using Lemma 13, $x^{(k)}$ is easily constructed from $y^{(k)}$, so we only need to construct the paths $y^{(k)}$. Constructing those paths are not very difficult intuitively, as to construct $y^{(k)}$, we just need to “paste” together $x^{(k-1)}$ and x .

For $k = 1$ $y^{(1)} = \pi_{M^{(l(p))}, M^{(1)}}(x)$ is a $H^{(1)} = G^{(1)}(V)/\exp(\mathcal{K}_1)$ valued path, which is $1/p$ -Hölder.

We now assume that we have a $G^{(k)}(V)$ -valued path $x^{(k)}$ which is $1/p$ -Hölder and which satisfies equality (12), and we aim to construct a $1/p$ -Hölder $H^{(k+1)}$ -valued path $y^{(k+1)}$.

The set $Z^{(k+1)}$ defined by

$$\{(g, m) \in G^{(k)}(V) \times M^{(k+1)} \text{ such that } \pi_{G^{(k)}(V), M^{(k)}}(g) = \pi_{M^{(k+1)}, M^{(k)}}(m)\}$$

equipped with the product

$$(g_1, m_1) \otimes (g_2, m_2) = (g_1 \otimes g_2, m_1 \otimes m_2)$$

is a group, and the application $\Psi : Z^{(k+1)} \rightarrow H^{(k+1)}$ defined by the formula: if $\ell_k \in V_1 \oplus \dots \oplus V_k$ and $l^{k+1} \in V_{k+1}$,

$$\Psi(\exp(\ell_k), \exp(\ell_k + l^{k+1}) \exp(\mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_{k+1})) = \exp(\ell_k) \exp(l^{k+1}) \exp(\mathcal{K}_{k+1}),$$

is easy seen to be an isomorphism by Baker–Campbell–Hausdorff formula.

Using Proposition 7 and Corollary 8, we see that there exists a constant C_{k+1} such that for all $(g, m) \in Z^{(k+1)}$,

$$\|\Psi(g, m)\|_{H^{(k+1)}} \leq C_{k+1} (\|g\|_{G^{(k)}(V)} + \|m\|_{M^{(k+1)}}).$$

For all $t \in [0, 1]$, $(x_t^{(k)}, \pi_{M^{(l(p))}, M^{(k+1)}}(x_t)) \in Z^{(k+1)}$, hence we can define

$$y_t^{(k+1)} = \Psi(x_t^{(k)}, \pi_{M^{(l(p))}, M^{(k+1)}}(x_t)).$$

Note first that $y^{(k+1)}$ satisfies the equality (11). Because Ψ is an isomorphism, $y_{s,t}^{(k+1)} = \Psi(x_{s,t}^{(k)}, \pi_{M^{(l(p))}, M^{(k+1)}}(x_{s,t}))$ and hence

$$\begin{aligned} \|y_{s,t}^{(k+1)}\|_{H^{(k+1)}} &\leq C_{k+1} (\|x_{s,t}^{(k)}\|_{G^{(k)}(V)} + \|\pi_{M^{(l(p))}, M^{(k+1)}}(x_{s,t})\|_{M^{(k+1)}}) \\ &\leq C_{k+1} (\|x_{s,t}^{(k)}\|_{G^{(k)}(V)} + \|x_{s,t}\|_{M^{(l(p))}}). \end{aligned}$$

By hypothesis and induction hypothesis,

$$\|x_{s,t}^{(k)}\|_{G^{(k)}(V)} + \|x_{s,t}\|_{M^{(l(p))}} \leq (C + C'_k) |t - s|^{1/p},$$

hence $\|y_{s,t}^{(k+1)}\|_{H^{(k+1)}} \leq C'_{k+1} |t - s|^{1/p}$.

Using the induction step until we reach the level $[p]$, we obtain a $G^{(l(p))}(V)$ -valued path $x^{(l(p))}$ which is $1/p$ -Hölder and such that

$$\pi_{G^{(l(p))}(V), M^{(l(p))}}(x^{(l(p))}) = \pi_{M^{(l(p))}, M^{(l(p))}}(x) = x. \quad \square$$

We ought to make a couple comments on our main theorem.

Remark 15. Note that we could have considered a continuous path of finite p -variation with values in a quotient space of $G^{(n)}(V)$, with $n > [p]$. If $K^{(n)}$ is a closed normal subgroup of $G^{(n)}(V)$ and x is a $1/p$ -Hölder path with values in $(G^{(n)}(V)/K^{(n)}, \|\cdot\|_{G^{(n)}(V)/K^{(n)}})$, with $p \notin \mathbb{N} \setminus \{0, 1\}$, then there exists a $1/p$ -Hölder $G^{(n)}(V)$ -valued path \tilde{x} such that

$$\pi_{G^{(n)}(V), G^{(n)}(V)/K^{(n)}}(\tilde{x}) = x.$$

To prove this, first let $K^{(l(p))} = \pi_{G^{(n)}, G^{(l(p))}}(K^{(n)})$. The canonical projection of x into $G^{(l(p))}(V)/K^{(l(p))}$ is a $1/p$ -Hölder path. Hence, by the previous theorem, there exists a $1/p$ -Hölder $G^{(l(p))}(V)$ -valued path $x^{(l(p))}$ such that

$$\pi_{G^{(l(p))}(V), G^{(l(p))}(V)/K^{(l(p))}}(x^{(l(p))}) = \pi_{G^{(n)}(V)/K^{(n)}, G^{(l(p))}(V)/K^{(l(p))}}(x).$$

Then, by Theorem 1 in [12], $x^{(l p)}$ can be (uniquely) extended to a $1/p$ -Hölder $(G^{(n)}(V), \|\cdot\|_{G^{(n)}(V)})$ -valued path $x^{(n)}$. By the uniqueness statement, $x^{(n)}$ must satisfy

$$\pi_{G^{(n)}(V), G^{(n)}(V)/K^{(n)}}(x^{(n)}) = x.$$

Remark 16. As already pointed out in [12,13], if $p \geq 2$ and if there exists one $(G^{(lp)}(V), \|\cdot\|_{G^{(lp)}(V)})$ -valued path \tilde{x} of finite p variation such that

$$\pi_{G^{(lp)}(V), G^{(lp)}(V)/K}(\tilde{x}) = x,$$

then there exists infinitely many such paths.

Remark 17. The condition $p \notin \mathbb{N} \setminus \{0, 1\}$ is necessary. In [18], it was proven that, for a particular choice of tensor norm, there does not exist a 2-rough path lying above the free Brownian motion (which is a path of finite 2-variation).

Remark 18. If p is a natural number greater than or equal to 2, keeping the notation of the previous theorem, we can find, for any fixed $\varepsilon > 0$, a continuous $G^{(lp)}(V)$ -valued path \tilde{x} of finite $p + \varepsilon$ variation such that

$$\pi_{G^{(p)}(V), G^{(p)}(V)/K}(\tilde{x}) = x.$$

This is obtained just by noticing that a path of finite p -variation has finite $(p + \varepsilon)$ -variation.

We end up with a corollary, which was the original motivation of this paper.

Corollary 19. *If $p \in [1, \infty) \setminus \{2, 3, \dots\}$, a continuous V -valued path of finite p -variation can be lifted to a geometric p -rough path. For any p , a continuous path of finite p -variation can be lifted to a geometric $(p + \varepsilon)$ -rough path.*

Proof. Apply Theorem 14 to $K = \exp(\bigoplus_{i=2}^{[p]} V_i)$ and use the previous remark. \square

That means, in particular, that one can always define a notion of solution to differential equations controlled by a continuous path of finite p -variation, whatever the p is.

6. Rough differential equations for which the extension does not matter

We fix a real $p \geq 1$. $\mathcal{X}^{k+\varepsilon}(\mathbb{R}^d)$ denotes the class of k -times differentiable vector fields with the k th-derivatives being ε -Hölder and with all the first k -derivatives being bounded. We consider A_1, \dots, A_m some elements of $\mathcal{X}^\gamma(\mathbb{R}^d)$, with $\gamma > p$. We fix a basis e_1, \dots, e_m of \mathbb{R}^m , and extend the linear application

$$\begin{aligned} \mathbb{R}^m &\rightarrow \mathcal{X}^\gamma(\mathbb{R}^d), \\ e_i &\rightarrow A_i \end{aligned}$$

to an algebra homomorphism $F_{[p]}^A$ from $T^{[p]}(\mathbb{R}^m)$ into the space of continuous differential operators, in other words, for a smooth function g , we have

$$F_{[p]}^A\left(\sum \alpha_{i_1, \dots, i_n} e_{i_1} \otimes \dots \otimes e_{i_n}\right)g = \sum \alpha_{i_1, \dots, i_n} A_{i_1} \dots A_{i_n} g.$$

Note that $F_{[p]}^A$ restricted to the free Lie algebra $\mathcal{G}^{(lp)}(\mathbb{R}^m)$, i.e. $(F_{[p]}^A)|_{\mathcal{G}^{(lp)}(\mathbb{R}^m)}$ is a Lie homomorphism into $\mathcal{X}^0(\mathbb{R}^d)$.

Recall that if \mathbf{x} is a p -geometric rough path, a solution of the differential equation

$$\begin{aligned} d\mathbf{y}_t &= A(\mathbf{y}_t) d\mathbf{x}_t, \\ \mathbf{y}_0 &= a \end{aligned}$$

is an extension of \mathbf{x} to $\mathbf{z} \in G\Omega(\mathbb{R}^{m+d})$ that projects onto (\mathbf{x}, \mathbf{y}) , $(x_0, y_0) = (0, a)$, and such that

$$\mathbf{z}_{s,t} = \int_s^t h(z_u) dz_u,$$

with

$$h: \mathbb{R}^d \oplus \mathbb{R}^m \rightarrow \text{Hom}(\mathbb{R}^d \oplus \mathbb{R}^m, \mathbb{R}^d \oplus \mathbb{R}^m),$$

$$(x, y) \rightarrow ((dX, dY) \rightarrow (dX, V(y) dX)).$$

The map $\mathbf{x} \rightarrow \mathbf{z}$ is called the Itô map, denoted $I_V: G\Omega(\mathbb{R}^n) \rightarrow G\Omega(\mathbb{R}^{n+d})$.

Theorem 20. *Let x be a $1/p$ -Hölder path in $G^{(1/p)}(\mathbb{R}^m)/K$, where K is a normal subgroup of $G^{(1/p)}(\mathbb{R}^m)$ with Lie algebra \mathcal{K} and $\tilde{\mathbf{x}}$ an extension of \mathbf{x} to a $1/p$ -Hölder path in $G^{(1/p)}(\mathbb{R}^m)$ ($1/(p + \varepsilon)$ if p is an integer). Assume that the kernel of the Lie algebra homomorphism $(F_{[p]}^A)|_{\mathcal{G}^{(1/p)}(\mathbb{R}^m)}$ contains \mathcal{K} . Then $I_A(\tilde{\mathbf{x}})$ is a $1/p$ -Hölder path in $G^{(1/p)}(\mathbb{R}^{m+d})$ which, in general depends on the extension of x to \mathbf{x} . Nevertheless, the projection of $I_A(\mathbf{x})$ onto \mathbb{R}^d depends only on x .*

Proof. To see that $I_A(\mathbf{x})$ depends on general of the extension of x to \mathbf{x} , just consider the Itô map which is the identity. Now let y be the projection of $I_A(\mathbf{x})$ onto \mathbb{R}^d . From [12], we know that

$$|y_t - F_{[p]}^A(\mathbf{x}_{s,t})(y_s)| \leq C|t - s|^\theta, \quad (13)$$

where $\theta > 1$ and $C \geq 0$. Define the path y^n by the inductive formula

$$y_0^n = a,$$

$$y_{\frac{k+1}{2^n}}^n = F_{[p]}^A(\mathbf{x}_{\frac{k}{2^n}, \frac{k+1}{2^n}})(y_{\frac{k}{2^n}}^n), \quad k = 0, \dots, 2^n - 1,$$

$$y_t^n = (k + 1 - 2^n t)y_{\frac{k}{2^n}}^n + (2^n t - k)y_{\frac{k+1}{2^n}}^n, \quad t \in \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right].$$

By Eq. (13) and an argument similar to Euler construction of a solution to an ordinary differential equation, we see that y^n converges to y in uniform topology. Due to our assumption on the vector fields A_1, \dots, A_m , $F_{[p]}^A(\mathbf{x}_{s,t})(y)$ only depends on $x_{s,t}$ (and not on the choice of the lift). In particular, y^n does not depend on the choice of the lift. Letting n tends to infinity, we obtain our theorem. \square

A simple case of the above is the following:

Example 21. Let A_1, \dots, A_d be d vector fields which commute, i.e. such that $[A_i, A_j] = 0$ for all i, j . Let $x: [0, 1] \rightarrow \mathbb{R}^d$ be a continuous path of finite p -variation, lifted to a geometric $(p + \varepsilon)$ -rough path \mathbf{x} . Then, the projection of $I_A(\mathbf{x})$ into \mathbb{R}^d depends only on x , and not on the choice of the lift. This could be seen more directly from Doss–Sussman’s theorem [4,16]

The following less trivial example should illustrate a bit more the interest of Theorem 20.

Example 22. Let A_1, A_2, A_3 be 3 vector fields, such that $[A_1, A_2] = [A_1, A_3] = 0$, but we do not assume that $[A_2, A_3]$ is equal to 0. Let $x = (x^1, x^2, x^3): [0, 1] \rightarrow \mathbb{R}^3$ be a $1/p$ -Hölder path ($p > 2$), equipped with a Levy area $A^{2,3}$ between x^2 and x^3 , such that $|A_{s,t}^{2,3}| \leq C|t - s|^{2/p}$. We lift (x, A) to a geometric $(p + \varepsilon)$ -rough path \mathbf{x} . Then, the projection of $I_A(\mathbf{x})$ into \mathbb{R}^d depends only on x and $A^{2,3}$, and not on the choice of the lift.

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