

# Singular limits for the bi-Laplacian operator with exponential nonlinearity in $\mathbb{R}^4$

Mónica Clapp<sup>a,\*</sup>, Claudio Muñoz<sup>b</sup>, Monica Musso<sup>c,d</sup>

<sup>a</sup> Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 México DF, Mexico

<sup>b</sup> Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170, Correo 3, Santiago, Chile

<sup>c</sup> Departamento de Matemática, Pontificia Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Macul, Santiago, Chile

<sup>d</sup> Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy

Received 29 December 2006; accepted 10 September 2007

Available online 3 December 2007

## Abstract

Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^4$  such that for some integer  $d \geq 1$  its  $d$ -th singular cohomology group with coefficients in some field is not zero, then problem

$$\begin{cases} \Delta^2 u - \rho^4 k(x) e^u = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution blowing-up, as  $\rho \rightarrow 0$ , at  $m$  points of  $\Omega$ , for any given number  $m$ .

© 2007 Elsevier Masson SAS. All rights reserved.

## 1. Introduction and statement of main results

Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^4$ . We are interested in studying existence and qualitative properties of positive solutions to the following boundary value problem

$$\begin{cases} \Delta^2 u - \rho^4 k(x) e^u = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $k \in C^2(\bar{\Omega})$  is a non-negative, not identically zero function, and  $\rho > 0$  is a small, positive parameter which tends to 0.

In a four-dimensional manifold, this type of equations and similar ones arise from the problem of prescribing the so-called  $Q$ -curvature, which was introduced in [7]. More precisely, given  $(M, g)$  a four-dimensional Riemannian manifold, the problem consists in finding a conformal metric  $\tilde{g}$  for which the corresponding  $Q$ -curvature  $Q_{\tilde{g}}$  is a priori prescribed. The  $Q$ -curvature for the metric  $g$  is defined as

$$Q_g = -\frac{1}{2}(\Delta_g R_g - R_g^2 + 3|\text{Ric}_g|^2),$$

\* Corresponding author.

E-mail addresses: [mclapp@matem.unam.mx](mailto:mclapp@matem.unam.mx) (M. Clapp), [cmunoz@dim.uchile.cl](mailto:cmunoz@dim.uchile.cl) (C. Muñoz), [mmusso@mat.puc.cl](mailto:mmusso@mat.puc.cl) (M. Musso).

where  $R_g$  is the scalar curvature and  $\text{Ric}_g$  is the Ricci tensor of  $(M, g)$ . Writing  $\tilde{g} = e^{2w}g$ , the problem reduces to finding a scalar function  $w$  which satisfies

$$P_g w + 2Q_g = 2Q_{\tilde{g}} e^{4w}, \tag{1.2}$$

where  $P_g$  is the Paneitz operator [32,10] defined as

$$P_g w = \Delta_g^2 w + \text{div} \left( \frac{2}{3} R_g g - 2 \text{Ric}_g \right) dw.$$

Problem (1.2) is thus an elliptic fourth-order partial differential equation with exponential non-linearity. Several results are already known for this problem [9,10] and related ones [1,18,30]. When the metric  $g$  is not Riemannian, the problem has been recently treated by Djadli and Malchiodi in [19] via variational methods.

In the special case where the manifold is the Euclidean space and  $g$  is the Euclidean metric, we recover the equation in (1.1), since (1.2) takes the simplified form

$$\Delta^2 w - 2Q e^{4w} = 0.$$

Problem (1.1) has a variational structure. Indeed, solutions of (1.1) correspond to critical points in  $H^2(\Omega) \cap H_0^1(\Omega)$  of the following energy functional

$$J_\rho(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 - \rho^4 \int_\Omega k(x) e^u.$$

For any  $\rho$  sufficiently small, the functional above has a local minimum which represents a solution to (1.1) close to 0. Furthermore, the Moser–Trudinger inequality assures the existence of a second solution, which can be obtained as a mountain pass critical point for  $J_\rho$ . Thus, as  $\rho \rightarrow 0$ , this second solution turns out not to be bounded. The aim of the present paper is to study multiplicity of solutions to (1.1), for  $\rho$  positive and small, under some topological assumption on  $\Omega$ , and to describe the asymptotic behavior of such solutions as the parameter  $\rho$  tends to zero. Indeed, we prove that, if some cohomology group of  $\Omega$  is not zero, then given any integer  $m$  we can construct solutions to (1.1) which concentrate and blow-up, as  $\rho \rightarrow 0$ , around some given  $m$  points of the domain. These are the singular limits.

Let us mention that concentration phenomena of this type, in domains with topology, appear also in other problems. As a first example, the two-dimensional version of problem (1.1) is the boundary value problem associated to Liouville’s equation [25]

$$\begin{cases} \Delta u + \rho^2 k(x) e^u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where  $k(x)$  is a non-negative function and now  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^2$ . In [14] it is proved that problem (1.3) admits solutions concentrating, as  $\rho \rightarrow 0$ , around some given set of  $m$  points of  $\Omega$ , for any given integer  $m$ , provided that  $\Omega$  is not simply connected. See also [5,6,21,20,11,8,24,29,31,35,38,36,37] for related results. A similar result holds true for another semilinear elliptic problem, still in dimension 2, namely

$$\begin{cases} \Delta u + u^p = 0, & u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where  $p$  now is a parameter converging to  $+\infty$ . Again in this situation, if  $\Omega$  is not simply connected, then for  $p$  large there exists a solution to (1.4) concentrating around some set of  $m$  points of  $\Omega$ , for any positive integer  $m$  [22].

In higher dimensions, the analogy is with the classical Bahri–Coron problem. In [2], Bahri and Coron show that, if  $N \geq 3$  and  $\Omega \subset \mathbb{R}^N$  is a bounded domain, then the presence of topology in the domain guarantees existence of solutions to

$$\begin{cases} \Delta u + u^{\frac{N+2}{N-2}} = 0, & u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Partial results in this direction are also known in the slightly super critical version of Bahri–Coron’s problem, namely

$$\begin{cases} \Delta u + u^{\frac{N+2}{N-2} + \varepsilon} = 0 & \text{in } \Omega, \\ u > 0, & u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with  $\varepsilon > 0$  small. In [12] it is proved that, under the assumption that  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^N$  with a sufficiently small hole, a solution to (1.6) exhibiting concentration in two points is present. See also [3,23,34,13,33].

The main point of this paper is to show that the presence of topology in the domain implies strongly existence of blowing-up solutions for problem (1.1).

We denote by  $H^d(\Omega)$  the  $d$ -th cohomology group of  $\Omega$  with coefficients in some field  $\mathbb{K}$ . We shall prove the following

**Theorem 1.** *Assume that there exists  $d \geq 1$  such that  $H^d(\Omega) \neq 0$  and that  $\inf_{\Omega} k > 0$ . Then, given any integer  $m \geq 1$ , there exists a family of solutions  $u_{\rho}$  to problem (1.1), for  $\rho$  small enough, with the property that*

$$\lim_{\rho \rightarrow 0} \rho^4 \int_{\Omega} k(x) e^{u_{\rho}(x)} dx = 64\pi^2 m.$$

Furthermore, there are  $m$  points  $\xi_1^{\rho}, \dots, \xi_m^{\rho}$  in  $\Omega$ , separated at uniform positive distance from each other and from the boundary as  $\rho \rightarrow 0$ , for which  $u_{\rho}$  remains uniformly bounded on  $\Omega \setminus \bigcup_{j=1}^m B_{\delta}(\xi_j^{\rho})$  and

$$\sup_{B_{\delta}(\xi_j^{\rho})} u_{\rho} \rightarrow +\infty,$$

for any  $\delta > 0$ .

The general behavior of arbitrary families of solutions to (1.1) has been studied by C.S. Lin and J.-C. Wei in [26], where they show that, when blow-up occurs for (1.1) as  $\rho \rightarrow 0$ , then it is located at a finite number of peaks, each peak being isolated and carrying the energy  $64\pi^2$  (at a peak,  $u \rightarrow +\infty$  and outside a peak,  $u$  is bounded). See [27] and [28] for related results.

We shall see that the sets of points where the solution found in Theorem 1 blows-up can be characterized in terms of Green’s function for the biharmonic operator in  $\Omega$  with the appropriate boundary conditions. Let  $G(x, \xi)$  be the Green function defined by

$$\begin{cases} \Delta_x^2 G(x, \xi) = 64\pi^2 \delta_{\xi}(x), & x \in \Omega, \\ G(x, \xi) = \Delta_x G(x, \xi) = 0, & x \in \partial\Omega \end{cases} \tag{1.7}$$

and let  $H(x, \xi)$  be its *regular part*, namely, the smooth function defined as

$$H(x, \xi) := G(x, \xi) + 8 \log |x - \xi|.$$

The location of the points of concentration is related to the set of critical points of the function

$$\varphi_m(\xi) = - \sum_{j=1}^m \{2 \log k(\xi_j) + H(\xi_j, \xi_j)\} - \sum_{i \neq j} G(\xi_i, \xi_j), \tag{1.8}$$

defined for points  $\xi = (\xi_1, \dots, \xi_m)$  such that  $\xi_i \in \Omega$  and  $\xi_i \neq \xi_j$  if  $i \neq j$ .

In [4] the authors prove that for each *non-degenerate* critical point of  $\varphi_m$  there exists a solution to (1.1), for any small  $\rho$ , which concentrates exactly around such critical point as  $\rho \rightarrow 0$ . We shall show the existence of a solution under a weaker assumption, namely, that  $\varphi_m$  has a *minimax value in an appropriate subset*.

More precisely, we consider the following situation. Let  $\Omega^m$  denote the Cartesian product of  $m$  copies of  $\Omega$ . Note that in any compact subset of  $\Omega^m$ , we may define, without ambiguity,

$$\varphi_m(\xi_1, \dots, \xi_m) = -\infty \quad \text{if } \xi_i = \xi_j \text{ for some } i \neq j.$$

We shall assume that there exists an open subset  $U$  of  $\Omega$  with smooth boundary, compactly contained in  $\Omega$ , and such that  $\inf_U k > 0$ , with the following properties:

(P1)  $U^m$  contains two closed subsets  $B_0 \subset B$  such that

$$\sup_{\xi \in B_0} \varphi_m(\xi) < \inf_{\gamma \in \Gamma} \sup_{\xi \in B} \varphi_m(\gamma(\xi)) =: c_0,$$

where  $\Gamma := \{\gamma \in \mathcal{C}(B, \bar{U}^m) : \gamma(\xi) = \xi \text{ for every } \xi \in B_0\}$ .

(P2) For every  $\xi = (\xi_1, \dots, \xi_m) \in \partial U^m$  with  $\varphi_m(\xi) = c_0$ , there exists an  $i \in \{1, \dots, m\}$  such that

$$\begin{aligned} \nabla_{\xi_i} \varphi_m(\xi) &\neq 0 && \text{if } \xi_i \in U, \\ \nabla_{\xi_i} \varphi_m(\xi) \cdot \tau &\neq 0 \text{ for some } \tau \in T_{\xi_i}(\partial U) && \text{if } \xi_i \in \partial U, \end{aligned}$$

where  $T_{\xi_i}(\partial U)$  denotes the tangent space to  $\partial U$  at the point  $\xi_i$ .

We will show that, under these assumptions,  $\varphi_m$  has a critical point  $\xi \in U^m$  with critical value  $c_0$ . Moreover, the same is true for any small enough  $C^1$ -perturbation of  $\varphi_m$ . Property (P1) is a common way of describing a change of topology of the sublevel sets of  $\varphi_m$  at the level  $c_0$ , and  $c_0$  is called a minimax value of  $\varphi_m$ . It is a critical value if  $U^m$  is invariant under the negative gradient flow of  $\varphi_m$ . If this is not the case, we use property (P2) to modify the gradient vector field of  $\varphi_m$  near  $\partial U^m$  at the level  $c_0$  and thus obtain a new vector field with the same stationary points, and such that  $\bar{U}^m$  is invariant and  $\varphi_m$  is a Lyapunov function for the associated negative flow near the level  $c_0$  (see Lemmas 6.3 and 6.4 below). This allows us to prove Theorem 1 and the following.

**Theorem 2.** *Let  $m \geq 1$  and assume that there exists an open subset  $U$  of  $\Omega$  with smooth boundary, compactly contained in  $\Omega$ , with  $\inf_U k > 0$ , which satisfies (P1) and (P2). Then, for  $\rho$  small enough, there exists a solution  $u_\rho$  to problem (1.1) with*

$$\lim_{\rho \rightarrow 0} \rho^4 \int_{\Omega} k(x) e^{u_\rho} = 64\pi^2 m.$$

Moreover, there is an  $m$ -tuple  $(x_1^\rho, \dots, x_m^\rho) \in U^m$ , such that as  $\rho \rightarrow 0$

$$\nabla \varphi_m(x_1^\rho, \dots, x_m^\rho) \rightarrow 0, \quad \varphi_m(x_1^\rho, \dots, x_m^\rho) \rightarrow c_0,$$

for which  $u_\rho$  remains uniformly bounded on  $\Omega \setminus \bigcup_{j=1}^m B_\delta(x_j^\rho)$ , and

$$\sup_{B_\delta(x_i^\rho)} u_\rho \rightarrow +\infty,$$

for any  $\delta > 0$ .

We will show that, for every  $m \geq 1$ , the set  $U := \{\xi \in \Omega : \text{dist}(\xi, \partial\Omega) > \delta\}$  has property (P2) at a given  $c_0$ , for  $\delta$  small enough (see Lemma 6.2). Thus, if  $\inf_\Omega k > 0$ , and if there exist closed subsets  $B_0 \subset B$  of  $\Omega^m$  with

$$\sup_{\xi \in B_0} \varphi_m(\xi) < \inf_{\gamma \in \Gamma} \sup_{\xi \in B} \varphi_m(\gamma(\xi)),$$

then both conditions (P1) and (P2) hold. Condition (P1) holds, for example, if  $\varphi_m$  has a (possibly degenerate) local minimum or local maximum. So a direct consequence of Theorem 2 is that in any bounded domain  $\Omega$  with  $\inf_\Omega k > 0$ , problem (1.1) has at least one solution concentrating exactly at one point, which corresponds to the minimum of the regular Green function  $H$ . Moreover if, for example,  $\Omega$  is a contractible domain obtained by joining together  $m$  disjoint bounded domains through thin enough tubes, then the function  $\varphi_m$  has a (possibly degenerate) local minimum, which gives rise to a solution exhibiting  $m$  points of concentration.

Finally, recall that problem (1.1) corresponds to a standard case of *uniform singular convergence*, in the sense that the associated non-linear coefficient in problem (1.1) –  $\rho^4 k(x)$  – goes to 0 uniformly in  $\bar{\Omega}$  as  $\rho \rightarrow 0$ , property that is also present in problem (1.3). Non-trivial topology strongly determines existence of solutions. However, we expect that this strong influence should decay under an inhomogeneous and *non-uniform* singular behavior, where critical points of an *external* function determine existence and multiplicity of solutions. See [16] for a recent two-dimensional case of this phenomenon.

The paper is organized as follows. Section 2 is devoted to describing a first approximation for the solution and to estimating the error. Furthermore, problem (1.1) is written as a fixed point problem, involving a linear operator. In Section 3 we study the invertibility of the linear problem. In Section 4 we solve a projected non-linear problem. In Section 5 we show that solving the entire non-linear problem reduces to finding critical points of a certain functional. Section 6 is devoted to the proofs of Theorems 1 and 2.

## 2. Preliminaries and ansatz for the solution

This section is devoted to construct a reasonably good approximation  $U$  for a solution of (1.1). The shape of this approximation will depend on some points  $\xi_i$ , which we leave as parameters yet to be adjusted, where the spikes are meant to take place. As we will see, a convenient set to select  $\xi = (\xi_1, \dots, \xi_m)$  is

$$\mathcal{O} := \left\{ \xi \in \Omega^m : \text{dist}(\xi_j, \partial\Omega) \geq 2\delta_0, \forall j = 1, \dots, m, \text{ and } \min_{i \neq j} |\xi_i - \xi_j| \geq 2\delta_0 \right\} \tag{2.1}$$

where  $\delta_0 > 0$  is a small fixed number. We thus fix  $\xi \in \mathcal{O}$ .

For numbers  $\mu_j > 0, j = 1, \dots, m$ , yet to be chosen,  $x \in \mathbb{R}^4$  and  $\varepsilon > 0$  we define

$$u_j(x) = 4 \log \frac{\mu_j(1 + \varepsilon^2)}{\mu_j^2 \varepsilon^2 + |x - \xi_j|^2} - \log k(\xi_j), \tag{2.2}$$

so that  $u_j$  solves

$$\Delta^2 u - \rho^4 k(\xi_j) e^u = 0 \quad \text{in } \mathbb{R}^4, \tag{2.3}$$

with

$$\rho^4 = \frac{384\varepsilon^4}{(1 + \varepsilon^2)^4}, \tag{2.4}$$

that is,  $\rho \sim \varepsilon$  as  $\varepsilon \rightarrow 0$ .

Since  $u_j$  and  $\Delta u_j$  are not zero on the boundary  $\partial\Omega$ , we will add to it a bi-harmonic correction so that the boundary conditions are satisfied. Let  $H_j(x)$  be the smooth solution of

$$\begin{cases} \Delta^2 H_j = 0 & \text{in } \Omega, \\ H_j = -u_j & \text{on } \partial\Omega, \\ \Delta H_j = -\Delta u_j & \text{on } \partial\Omega. \end{cases}$$

We define our first approximation  $U(\xi)$  as

$$U(\xi) \equiv \sum_{j=1}^m U_j, \quad U_j \equiv u_j + H_j. \tag{2.5}$$

As we will rigorously prove below,  $(u_j + H_j)(x) \sim G(x, \xi_j)$  where  $G(x, \xi)$  is the Green function defined in (1.7).

While  $u_j$  is a good approximation to a solution of (1.1) near  $\xi_j$ , it is not so much the case for  $U$ , unless the remainder  $U - u_j = (H_j + \sum_{k \neq j} u_k)$  vanishes at main order near  $\xi_j$ . This is achieved through the following precise choice of the parameters  $\mu_k$

$$\log \mu_j^4 = \log k(\xi_j) + H(\xi_j, \xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j). \tag{2.6}$$

We thus fix  $\mu_j$  *a priori* as a function of  $\xi$ . We write

$$\mu_j = \mu_j(\xi)$$

for all  $j = 1, \dots, m$ . Since  $\xi \in \mathcal{O}$ ,

$$\frac{1}{C} \leq \mu_j \leq C, \quad \text{for all } j = 1, \dots, m, \tag{2.7}$$

for some constant  $C > 0$ .

The following lemma expands  $U_j$  in  $\Omega$ .

**Lemma 2.1.** *Assume  $\xi \in \mathcal{O}$ . Then we have*

$$H_j(x) = H(x, \xi_j) - 4 \log \mu_j(1 + \varepsilon^2) + \log k(\xi_j) + O(\mu_j^2 \varepsilon^2), \tag{2.8}$$

uniformly in  $\Omega$ , and

$$u_j(x) = 4 \log \mu_j(1 + \varepsilon^2) - \log k(\xi_j) - 8 \log |x - \xi_j| + O(\mu_j^2 \varepsilon^2), \tag{2.9}$$

uniformly in the region  $|x - \xi_j| \geq \delta_0$ , so that in this region,

$$U_j(x) = G(x, \xi_j) + O(\mu_j^2 \varepsilon^2). \tag{2.10}$$

**Proof.** Let us prove (2.8). Define  $z(x) = H_j(x) + 4 \log \mu_j(1 + \varepsilon^2) - \log k(\xi_j) - H(x, \xi_j)$ . Then  $z$  is a bi-harmonic function which satisfies

$$\begin{cases} \Delta^2 z = 0 & \text{in } \Omega, \\ z = -u_j + 4 \log \mu_j(1 + \varepsilon^2) - \log k(\xi_j) - 8 \log |\cdot - \xi_j| & \text{on } \partial\Omega, \\ \Delta z = -\Delta u_j - \frac{16}{|\cdot - \xi_j|^2} & \text{on } \partial\Omega. \end{cases}$$

Let us define  $w \equiv -\Delta z$ . Thus  $w$  is harmonic in  $\Omega$  and

$$\sup_{\Omega} |w| \leq \sup_{\partial\Omega} |w| \leq C \mu_j^2 \varepsilon^2.$$

We also have  $\sup_{\partial\Omega} |z| \leq C \mu_j^2 \varepsilon^2$ . Standard elliptic regularity implies

$$\sup_{\Omega} |z| \leq C \left( \sup_{\Omega} |w| + \sup_{\partial\Omega} |z| \right) \leq C \mu_j^2 \varepsilon^2,$$

as desired. The second estimate is direct from the definition of  $u_j$ .  $\square$

Now, let us write

$$\Omega_\varepsilon = \varepsilon^{-1} \Omega, \quad \xi'_j = \varepsilon^{-1} \xi_j. \tag{2.11}$$

Then  $u$  solves (1.1) if and only if  $v(y) \equiv u(\varepsilon y) + 4 \log \rho \varepsilon$  satisfies

$$\begin{cases} \Delta^2 v - k(\varepsilon y) e^v = 0 & \text{in } \Omega_\varepsilon, \\ v = 4 \log \rho \varepsilon, \quad \Delta v = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.12}$$

Let us define  $V(y) = U(\varepsilon y) + 4 \log \rho \varepsilon$ , with  $U$  our approximate solution (2.5). We want to measure the size of the error of approximation

$$R \equiv \Delta^2 V - k(\varepsilon y) e^V. \tag{2.13}$$

It is convenient to do so in terms of the following norm

$$\|v\|_* = \sup_{y \in \Omega_\varepsilon} \left| \left[ \sum_{j=1}^m \frac{1}{(1 + |y - \xi'_j|^2)^{7/2}} + \varepsilon^4 \right]^{-1} v(y) \right|. \tag{2.14}$$

Here and in what follows,  $C$  denotes a generic constant independent of  $\varepsilon$  and of  $\xi \in \mathcal{O}$ .

**Lemma 2.2.** *The error  $R$  in (2.13) satisfies*

$$\|R\|_* \leq C \varepsilon \quad \text{as } \varepsilon \rightarrow 0.$$

**Proof.** We assume first  $|y - \xi'_k| < \delta_0/\varepsilon$ , for some index  $k$ . We have

$$\Delta^2 V(y) = \rho^4 \sum_{j=1}^m k(\xi_j) e^{u_j(\varepsilon y)} = \frac{384 \mu_k^4}{(\mu_k^2 + |y - \xi'_k|^2)^4} + O(\varepsilon^8).$$

Let us estimate  $k(\varepsilon y)e^{V(y)}$ . By (2.8) and the definition of  $\mu'_j$ 's,

$$\begin{aligned} H_k(x) &= H(\xi_k, \xi_k) - 4 \log \mu_k + \log k(\xi_j) + O(\mu_k^2 \varepsilon^2) + O(|x - \xi_k|) \\ &= - \sum_{j \neq k} G(\xi_j, \xi_k) + O(\mu_k^2 \varepsilon^2) + O(|x - \xi_k|), \end{aligned}$$

and if  $j \neq k$ , by (2.10)

$$U_j(x) = u_j(x) + H_j(x) = G(\xi_j, \xi_k) + O(|x - \xi_k|) + O(\mu_j^2 \varepsilon^2).$$

Then

$$H_k(x) + \sum_{j \neq k} U_j(x) = O(\varepsilon^2) + O(|x - \xi_k|). \tag{2.15}$$

Therefore,

$$\begin{aligned} k(\varepsilon y)e^{V(y)} &= k(\varepsilon y)\varepsilon^4 \rho^4 \exp \left\{ u_k(\varepsilon y) + H_k(\varepsilon y) + \sum_{j \neq k} U_j(\varepsilon y) \right\} \\ &= \frac{384 \mu_k^4 k(\varepsilon y)}{(\mu_k^2 + |y - \xi'_k|^2)^4 k(\xi_k)} \{ 1 + O(\varepsilon |y - \xi'_k|) + O(\varepsilon^2) \} \\ &= \frac{384 \mu_k^4}{(\mu_k^2 + |y - \xi'_k|^2)^4} \{ 1 + O(\varepsilon |y - \xi'_k|) \}. \end{aligned}$$

We can conclude that in this region

$$|R(y)| \leq C \frac{\varepsilon |y - \xi'_k|}{(1 + |y - \xi'_k|^2)^4} + O(\varepsilon^4).$$

If  $|y - \xi'_j| \geq \delta_0/\varepsilon$  for all  $j$ , using (2.8), (2.9) and (2.10) we obtain

$$\Delta^2 V = O(\varepsilon^4 \rho^4) \quad \text{and} \quad k(\varepsilon y)e^{V(y)} = O(\varepsilon^4 \rho^4).$$

Hence, in this region,

$$R(y) = O(\varepsilon^8)$$

so that finally

$$\|R\|_* = O(\varepsilon). \quad \square$$

Next we consider the energy functional associated with (1.1)

$$J_\rho[u] = \frac{1}{2} \int_\Omega (\Delta u)^2 - \rho^4 \int_\Omega k(x)e^u, \quad u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.16}$$

We will give an asymptotic estimate of  $J_\rho[U]$ , where  $U(\xi)$  is the approximation (2.5). Instead of  $\rho$ , we use the parameter  $\varepsilon$  (defined in (2.4)) to obtain the following expansion:

**Lemma 2.3.** *With the election of  $\mu_j$ 's given by (2.6),*

$$J_\rho[U] = -128\pi^2 m + 256\pi^2 m |\log \varepsilon| + 32\pi^2 \varphi_m(\xi) + \varepsilon \Theta_\varepsilon(\xi), \tag{2.17}$$

where  $\Theta_\varepsilon(\xi)$  is uniformly bounded together with its derivatives if  $\xi \in \mathcal{O}$ , and  $\varphi_m$  is the function defined in (1.8).

**Proof.** We have

$$\begin{aligned}
 J_\rho[U] &= \frac{1}{2} \sum_{j=1}^m \int_{\Omega} (\Delta U_j)^2 + \frac{1}{2} \sum_{j \neq i} \int_{\Omega} \Delta U_j \Delta U_i - \rho^4 \int_{\Omega} k(x) e^U \\
 &\equiv I_1 + I_2 + I_3;
 \end{aligned}$$

Note that  $\Delta^2 U_j = \Delta^2 u_j = \rho^4 k(\xi_j) e^{u_j}$  in  $\Omega$  and  $U_j = \Delta U_j = 0$  in  $\partial\Omega$ . Then

$$I_1 = \frac{1}{2} \rho^4 \sum_{j=1}^m k(\xi_j) \int_{\Omega} e^{u_j} U_j \quad \text{and} \quad I_2 = \frac{1}{2} \rho^4 \sum_{j \neq i} k(\xi_j) \int_{\Omega} e^{u_j} U_i.$$

Let us define the change of variables  $x = \xi_j + \mu_j \varepsilon y$ , where  $x \in \Omega$  and  $y \in \Omega_j \equiv (\mu_j \varepsilon)^{-1}(\Omega - \xi_j)$ . Using Lemma 2.1 and the definition of  $\rho$  in terms of  $\varepsilon$  in (2.4) we obtain

$$\begin{aligned}
 I_1 &= 192 \sum_{j=1}^m \int_{\Omega_j} \frac{1}{(1 + |y|^2)^4} \left\{ 4 \log \frac{1}{1 + |y|^2} - 8 \log \mu_j \varepsilon + H(\xi_j, \xi_j) + O(\mu_j \varepsilon |y|) \right\} \\
 &= 32\pi^2 \sum_{j=1}^m \{ H(\xi_j, \xi_j) - 8 \log \mu_j \varepsilon \} - 64\pi^2 m + O\left( \varepsilon \mu_j \int_{\Omega_j} \frac{|y|}{(1 + |y|^2)^4} \right) \\
 &= 32\pi^2 \sum_{j=1}^m \{ H(\xi_j, \xi_j) - 8 \log \mu_j \varepsilon \} - 64\pi^2 m + \varepsilon \Theta(\xi),
 \end{aligned}$$

where  $\Theta_\varepsilon(\xi)$  is bounded together with its derivatives if  $\xi \in \mathcal{O}$ . Besides we have used the explicit values

$$\int_{\mathbb{R}^4} \frac{1}{(1 + |y|^2)^4} = \frac{\pi^2}{6}, \quad \text{and} \quad \int_{\mathbb{R}^4} \frac{\log(1 + |y|^2)}{(1 + |y|^2)^4} = \frac{\pi^2}{12}.$$

We consider now  $I_2$ . As above,

$$\begin{aligned}
 \frac{1}{2} \rho^4 \int_{\Omega} e^{u_j} U_i &= \int_{\Omega_j} \frac{192}{(1 + |y|^2)^4} \{ u_i(\xi_j + \mu_j \varepsilon y) + H_i(\xi_j + \mu_j \varepsilon y) \} \\
 &= \int_{\Omega_j} \frac{192}{(1 + |y|^2)^4} \{ u_i(\xi_j + \mu_j \varepsilon y) - 4 \log \mu_i (1 + \varepsilon^2) + \log k(\xi_i) + 8 \log |\xi_j - \xi_i| \} \\
 &\quad + \int_{\Omega_j} \frac{192}{(1 + |y|^2)^4} \{ H_i(\xi_j + \mu_j \varepsilon y) - H_i(\xi_j) \} \\
 &\quad + \int_{\Omega_j} \frac{192}{(1 + |y|^2)^4} \{ H_i(\xi_j) - H(\xi_j, \xi_i) + 4 \log \mu_i (1 + \varepsilon^2) - \log k(\xi_i) \} \\
 &\quad + G(\xi_j, \xi_i) \int_{\Omega_j} \frac{192}{(1 + |y|^2)^4} \\
 &= 32\pi^2 G(\xi_i, \xi_j) + O\left( \varepsilon \mu_j \int_{\Omega_j} \frac{|y|}{(1 + |y|^2)^4} \right) + O(\mu_j^2 \varepsilon^2) \\
 &= 32\pi^2 G(\xi_i, \xi_j) + \varepsilon \Theta_\varepsilon(\xi).
 \end{aligned}$$

Thus

$$I_2 = 32\pi^2 \sum_{j \neq i} G(\xi_i, \xi_j) + \varepsilon \Theta_\varepsilon(\xi). \tag{2.18}$$



Finally we consider  $I_3$ . Let us denote  $A_j \equiv B(\xi_j, \delta_0)$  and  $x = \xi_j + \mu_j \varepsilon y$ . Then using again Lemma 2.1

$$\begin{aligned} I_3 &= -\rho^4 \sum_{j=1}^m \int_{A_j} k(x)e^U + O(\varepsilon^4) \\ &= -\rho^4 \sum_{j=1}^m \int_{B(0, \frac{\delta_0}{\mu_j \varepsilon})} \frac{k(\xi_j + \mu_j \varepsilon y)}{k(\xi_j)(1 + |y|^2)^4} \frac{(1 + \varepsilon^2)^4}{\varepsilon^4} (1 + O(\varepsilon \mu_j |y|)) + O(\varepsilon^4) \\ &= -384m \int_{\mathbb{R}^4} \frac{1}{(1 + |y|^2)^4} + O\left(\varepsilon \mu_j \int_{\mathbb{R}^4} \frac{|y|}{(1 + |y|^2)^4}\right) \\ &= -64\pi^2 m + \varepsilon \Theta_\varepsilon(\xi), \end{aligned}$$

uniformly in  $\xi \in \mathcal{O}$ . Thus, we can conclude the following expansion of  $J_\rho[U]$ :

$$J_\rho[U] = -128m\pi^2 + 256m\pi^2 |\log \varepsilon| + 32\pi^2 \varphi_m(\xi) + \varepsilon \Theta_\varepsilon(\xi), \tag{2.19}$$

where  $\Theta_\varepsilon(\xi)$  is a bounded function together with its derivatives in the region  $\xi \in \mathcal{O}$ ,  $\varphi_m$  defined as in (1.8) and  $\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}$ .  $\square$

In the subsequent analysis we will stay in the expanded variable  $y \in \Omega_\varepsilon$  so that we will look for solutions of problem (2.12) in the form  $v = V + \psi$ , where  $\psi$  will represent a lower order correction. In terms of  $\psi$ , problem (2.12) now reads

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) \equiv \Delta^2 \psi - W\psi = -R + N(\psi) & \text{in } \Omega_\varepsilon, \\ \psi = \Delta \psi = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.20}$$

where

$$N(\psi) = W[e^\psi - \psi - 1] \quad \text{and} \quad W = k(\varepsilon y)e^V. \tag{2.21}$$

Note that

$$W(y) = \sum_{j=1}^m \frac{384\mu_j^4}{(\mu_j^2 + |y - \xi'_j|^2)^4} (1 + O(\varepsilon|y - \xi'_j|)) \quad \text{for } y \in \Omega_\varepsilon. \tag{2.22}$$

This fact, together with the definition of  $N(\psi)$  given in (2.21), give the validity of the following

**Lemma 2.4.** For  $\xi \in \mathcal{O}$ ,  $\|W\|_* = O(1)$  and  $\|N(\psi)\|_* = O(\|\psi\|_\infty^2)$  as  $\|\psi\|_\infty \rightarrow 0$ .

### 3. The linearized problem

In this section we develop a solvability theory for the fourth-order linear operator  $\mathcal{L}_\varepsilon$  defined in (2.20) under suitable orthogonality conditions. We consider

$$\mathcal{L}_\varepsilon(\psi) \equiv \Delta^2 \psi - W(y)\psi, \tag{3.1}$$

where  $W(y)$  was introduced in (2.20). By expression (2.22) and setting  $z = y - \xi'_j$ , one can easily see that formally the operator  $\mathcal{L}_\varepsilon$  approaches, as  $\varepsilon \rightarrow 0$ , the operator in  $\mathbb{R}^4$

$$\mathcal{L}_j(\psi) \equiv \Delta^2 \psi - \frac{384\mu_j^4}{(\mu_j^2 + |z|^2)^4} \psi, \tag{3.2}$$

namely, equation  $\Delta^2 v - e^v = 0$  linearized around the radial solution  $v_j(z) = \log \frac{384\mu_j^4}{(\mu_j^2 + |z|^2)^4}$ . Thus the key point to develop a satisfactory solvability theory for the operator  $\mathcal{L}_\varepsilon$  is the non-degeneracy of  $v_j$  up to the natural invariances of the equation under translations and dilations. In fact, if we set

$$Y_{0j}(z) = 4 \frac{|z|^2 - \mu_j^2}{|z|^2 + \mu_j^2}, \tag{3.3}$$

$$Y_{ij}(z) = \frac{8z_i}{\mu_j^2 + |z|^2}, \quad i = 1, \dots, 4, \tag{3.4}$$

the only bounded solutions of  $\mathcal{L}_j(\psi) = 0$  in  $\mathbb{R}^4$  are linear combinations of  $Y_{ij}, i = 0, \dots, 4$ ; see Lemma 3.1 in [4] for a proof.

We define for  $i = 0, \dots, 4$  and  $j = 1, \dots, m$ ,

$$Z_{ij}(y) \equiv Y_{ij}(y - \xi'_j), \quad i = 0, \dots, 4.$$

Additionally, let us consider  $R_0$  a large but fixed number and  $\chi$  a radial and smooth cut-off function with  $\chi \equiv 1$  in  $B(0, R_0)$  and  $\chi \equiv 0$  in  $\mathbb{R}^4 \setminus B(0, R_0 + 1)$ . Let

$$\chi_j(y) = \chi(|y - \xi'_j|), \quad j = 1, \dots, m.$$

Given  $h \in L^\infty(\Omega_\varepsilon)$ , we consider the problem of finding a function  $\psi$  such that for certain scalars  $c_{ij}$  one has

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = h + \sum_{i=1}^4 \sum_{j=1}^m c_{ij} \chi_j Z_{ij}, & \text{in } \Omega_\varepsilon, \\ \psi = \Delta \psi = 0, & \text{on } \partial \Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij} \psi = 0, & \text{for all } i = 1, \dots, 4, j = 1, \dots, m. \end{cases} \tag{3.5}$$

We will establish a priori estimates for this problem. To this end we shall introduce an adapted norm in  $\Omega_\varepsilon$ , which has been introduced previously in [15]. Given  $\psi : \Omega_\varepsilon \rightarrow \mathbb{R}$  and  $\alpha \in \mathbb{N}^m$  we define

$$\|\psi\|_{**} \equiv \sum_{j=1}^m \|\psi\|_{C^{4,\alpha}(r_j < 2)} + \sum_{j=1}^m \sum_{|\alpha| \leq 3} \|r_j^{|\alpha|} D^\alpha \psi\|_{L^\infty(r_j \geq 2)}, \tag{3.6}$$

with  $r_j = |y - \xi'_j|$ .

**Proposition 3.1.** *There exist positive constants  $\varepsilon_0 > 0$  and  $C > 0$  such that for any  $h \in L^\infty(\Omega_\varepsilon)$ , with  $\|h\|_* < \infty$ , and any  $\xi \in \mathcal{O}$ , there is a unique solution  $\psi = T(h)$  to problem (3.5) for all  $\varepsilon \leq \varepsilon_0$ , which defines a linear operator of  $h$ . Besides, we have the estimate*

$$\|T(h)\|_{**} \leq C |\log \varepsilon| \|h\|_*. \tag{3.7}$$

The proof will be split into a series of lemmas which we state and prove next. The first step is to obtain a priori estimates for the problem

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = h & \text{in } \Omega_\varepsilon, \\ \psi = \Delta \psi = 0 & \text{on } \partial \Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij} \psi = 0 & \text{for all } i = 0, \dots, 4, j = 1, \dots, m, \end{cases} \tag{3.8}$$

which involves more orthogonality conditions than those in (3.5). We have the following estimate.

**Lemma 3.1.** *There exist positive constants  $\varepsilon_0 > 0$  and  $C > 0$  such that for any  $\psi$  solution of problem (3.8) with  $h \in L^\infty(\Omega_\varepsilon)$ ,  $\|h\|_* < \infty$ , and  $\xi \in \mathcal{O}$ , then*

$$\|\psi\|_{**} \leq C \|h\|_* \tag{3.9}$$

for all  $\varepsilon \in (0, \varepsilon_0)$ .

**Proof.** We carry out the proof by a contradiction argument. If the above fact were false, then, there would exist a sequence  $\varepsilon_n \rightarrow 0$ , points  $\xi^n = (\xi_1^n, \dots, \xi_m^n) \in \mathcal{O}$ , functions  $h_n$  with  $\|h_n\|_* \rightarrow 0$  and associated solutions  $\psi_n$  with  $\|\psi_n\|_{**} = 1$  such that

$$\begin{cases} \mathcal{L}_{\varepsilon_n}(\psi_n) = h_n & \text{in } \Omega_{\varepsilon_n}, \\ \psi_n = \Delta \psi_n = 0 & \text{on } \partial \Omega_{\varepsilon_n}, \\ \int_{\Omega_{\varepsilon_n}} \chi_j Z_{ij} \psi_n = 0, & \text{for all } i = 0, \dots, 4, j = 1, \dots, m. \end{cases} \tag{3.10}$$

Let us set  $\tilde{\psi}_n(x) = \psi_n(x/\varepsilon_n)$ ,  $x \in \Omega$ . It is directly checked that for any  $\delta' > 0$  sufficiently small  $\tilde{\psi}_n$  solves the problem

$$\begin{cases} \Delta^2 \tilde{\psi}_n = O(\varepsilon_n^4) + \varepsilon_n^{-4} h_n = o(1), & \text{uniformly in } \Omega \setminus \bigcup_{k=1}^m B(\xi_j^n, \delta'), \\ \tilde{\psi}_n = \Delta \tilde{\psi}_n = 0 & \text{on } \partial\Omega, \end{cases}$$

together with  $\|\tilde{\psi}_n\|_\infty \leq 1$  and  $\|\Delta \tilde{\psi}_n\|_\infty \leq C\delta'$ , in the considered region. Passing to a subsequence, we then get that  $\xi^n \rightarrow \xi^* \in \mathcal{O}$  and  $\tilde{\psi}_n \rightarrow 0$  in the  $C^{3,\alpha}$  sense over compact subsets of  $\Omega \setminus \{\xi_1^*, \dots, \xi_m^*\}$ . In particular

$$\sum_{|\alpha| \leq 3} \frac{1}{\varepsilon_n^{|\alpha|}} |D^\alpha \psi_n(y)| \rightarrow 0, \quad \text{uniformly in } |y - (\xi_j^n)'| \geq \frac{\delta'}{2\varepsilon_n},$$

for any  $\delta' > 0$  and  $j \in \{1, \dots, m\}$ . We obtain thus that

$$\sum_{j=1}^m \sum_{|\alpha| \leq 3} \|r_j^{|\alpha|} D^\alpha \psi_n\|_{L^\infty(r_j \geq \delta'/\varepsilon_n)} \rightarrow 0, \tag{3.11}$$

for any  $\delta' > 0$ . In conclusion, the exterior portion of  $\|\psi_n\|_{**}$  goes to zero, see (3.6).

Let us consider now a smooth radial cut-off function  $\hat{\eta}$  with  $\hat{\eta}(s) = 1$  if  $s < \frac{1}{2}$ ,  $\hat{\eta}(s) = 0$  if  $s \geq 1$ , and define

$$\hat{\psi}_{n,j}(y) = \hat{\eta}_j(y) \psi_n(y) \equiv \hat{\eta}\left(\frac{\varepsilon_n}{\delta_0} |y - (\xi_j^n)'|\right) \psi_n(y),$$

such that

$$\text{supp } \hat{\psi}_{n,j} \subseteq B\left((\xi_j^n)', \frac{\delta_0}{\varepsilon_n}\right).$$

We observe that

$$\mathcal{L}_{\varepsilon_n}(\hat{\psi}_{n,j}) = \hat{\eta}_j h_n + F(\hat{\eta}_j, \psi_n),$$

where

$$F(f, g) = g \Delta^2 f + 2 \Delta f \Delta g + 4 \nabla(\Delta f) \cdot \nabla g + 4 \nabla f \cdot \nabla(\Delta g) + 4 \sum_{i,j=1}^4 \frac{\partial^2 f}{\partial y_i \partial y_j} \frac{\partial^2 g}{\partial y_i \partial y_j}. \tag{3.12}$$

Thus we get

$$\begin{cases} \Delta^2 \hat{\psi}_{n,j} = W_n(y) \hat{\psi}_{n,j} + \hat{\eta}_j h_n + F(\hat{\eta}_j, \psi_n) & \text{in } B((\xi_j^n)', \frac{\delta_0}{\varepsilon_n}), \\ \hat{\psi}_{n,j} = \Delta \hat{\psi}_{n,j} = 0 & \text{on } \partial B((\xi_j^n)', \frac{\delta_0}{\varepsilon_n}). \end{cases} \tag{3.13}$$

The following intermediate result provides an outer estimate. For notational simplicity we omit the subscript  $n$  in the quantities involved.

**Lemma 3.2.** *There exist constants  $C, R_0 > 0$  such that for large  $n$*

$$\sum_{|\alpha| \leq 3} \|r_j^{|\alpha|} D^\alpha \hat{\psi}_j\|_{L^\infty(r_j \geq R_0)} \leq C \{ \|\hat{\psi}_j\|_{L^\infty(r_j < 2R_0)} + o(1) \}. \tag{3.14}$$

**Proof.** We estimate the right-hand side of (3.13). If  $2 < r_j < \delta_0/\varepsilon$  we get

$$\Delta^2 \hat{\psi}_j = O\left(\frac{1}{r_j^8}\right) \hat{\psi}_j + \frac{1}{r_j^7} o(1) + O(\varepsilon^4) + O\left(\frac{\varepsilon^3}{r_j}\right) + O\left(\frac{\varepsilon^2}{r_j^2}\right) + O\left(\frac{\varepsilon}{r_j^3}\right).$$

From (3.13) and standard elliptic estimates we have

$$\sum_{|\alpha| \leq 3} |D^\alpha \hat{\psi}_j| \leq C \left\{ \frac{1}{r_j^8} \|\hat{\psi}_j\|_{L^\infty(r_j > 1)} + \frac{1}{r_j^7} o(1) + O\left(\frac{\varepsilon}{r_j^3}\right) \right\}, \quad \text{in } 2 \leq r_j \leq \frac{\delta_0}{\varepsilon}.$$

Now, if  $r_j \geq 2$

$$|r_j^{|\alpha|} D^\alpha \hat{\psi}_j| \leq C \left\{ \frac{1}{r_j^5} \|\hat{\psi}_j\|_{L^\infty(r_j > 1)} + o(1) \right\}, \quad |\alpha| \leq 3.$$

Finally

$$\frac{1}{r_j^5} \|\hat{\psi}_j\|_{L^\infty(r_j > 1)} \leq \|\hat{\psi}_j\|_{L^\infty(1 < r_j < R_0)} + \frac{1}{R_0^5} \|\hat{\psi}_j\|_{L^\infty(r_j > R_0)},$$

thus fixing  $R_0$  large enough we have

$$\sum_{|\alpha| \leq 3} \|r_j^{|\alpha|} D^\alpha \hat{\psi}_j\|_{L^\infty(r_j \geq R_0)} \leq C \{ \|\hat{\psi}_j\|_{L^\infty(1 < r_j < R_0)} + o(1) \}, \quad 2 < r_j < \frac{\delta_0}{\varepsilon},$$

and then (3.14).  $\square$

We continue with the proof of Lemma 3.1.

Since  $\|\psi_n\|_{**} = 1$  and using (3.11) and Lemma 3.2 we have that there exists an index  $j \in \{1, \dots, m\}$  such that

$$\liminf_{n \rightarrow \infty} \|\psi_n\|_{L^\infty(r_j < R_0)} \geq \alpha > 0. \tag{3.15}$$

Let us set  $\tilde{\psi}_n(z) = \psi_n((\xi_j^n)' + z)$ . We notice that  $\tilde{\psi}_n$  satisfies

$$\Delta^2 \tilde{\psi}_n - W((\xi_j^n)' + z) \tilde{\psi}_n = h_n((\xi_j^n)' + z), \quad \text{in } \Omega_n \equiv \Omega_\varepsilon - (\xi_j^n)'.$$

Since  $\psi_n, \Delta \psi_n$  are bounded uniformly, standard elliptic estimates allow us to assume that  $\tilde{\psi}_n$  converges uniformly over compact subsets of  $\mathbb{R}^4$  to a bounded, non-zero solution  $\tilde{\psi}$  of

$$\Delta^2 \psi - \frac{384\mu_j^4}{(\mu_j^2 + |z|^2)^4} \psi = 0.$$

This implies that  $\tilde{\psi}$  is a linear combination of the functions  $Y_{ij}, i = 0, \dots, 4$ . But orthogonality conditions over  $\tilde{\psi}_n$  pass to the limit thanks to  $\|\tilde{\psi}_n\|_\infty \leq 1$  and dominated convergence. Thus  $\tilde{\psi} \equiv 0$ , a contradiction with (3.15). This concludes the proof.  $\square$

Now we will deal with problem (3.8) lifting the orthogonality constraints  $\int_{\Omega_\varepsilon} \chi_j Z_{0j} \psi = 0, j = 1, \dots, m$ , namely

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = h & \text{in } \Omega_\varepsilon, \\ \psi = \Delta \psi = 0 & \text{on } \partial \Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij} \psi = 0, & \text{for all } i = 1, \dots, 4, j = 1, \dots, m. \end{cases} \tag{3.16}$$

We have the following a priori estimates for this problem.

**Lemma 3.3.** *There exist positive constants  $\varepsilon_0$  and  $C$  such that, if  $\psi$  is a solution of (3.16), with  $h \in L^\infty(\Omega_\varepsilon)$ ,  $\|h\|_* < \infty$  and with  $\xi \in \mathcal{O}$ , then*

$$\|\psi\|_{**} \leq C |\log \varepsilon| \|h\|_* \tag{3.17}$$

for all  $\varepsilon \in (0, \varepsilon_0)$ .

**Proof.** Let  $R > R_0 + 1$  be a large and fixed number. Let us consider  $\hat{Z}_{0j}$  be the following function

$$\hat{Z}_{0j}(y) = Z_{0j}(y) - 1 + a_{0j} G(\varepsilon y, \xi_j), \tag{3.18}$$

where  $a_{0j} = (H(\xi_j, \xi_j) - 8 \log(\varepsilon R))^{-1}$ . It is clear that if  $\varepsilon$  is small enough

$$\begin{aligned} \hat{Z}_{0j}(y) &= Z_{0j}(y) + a_{0j}(G(\varepsilon y, \xi_j) - H(\xi_j, \xi_j) + 8 \log(\varepsilon R)) \\ &= Z_{0j}(y) + \frac{1}{|\log \varepsilon|} \left( O(\varepsilon r_j) + 8 \log \frac{R}{r_j} \right) \end{aligned} \tag{3.19}$$

and  $Z_{0j}(y) = O(1)$ . Next we consider radial smooth cut-off functions  $\eta_1$  and  $\eta_2$  with the following properties:

$$\begin{aligned} 0 \leq \eta_1 \leq 1, \quad \eta_1 \equiv 1 \text{ in } B(0, R), \quad \eta_1 \equiv 0 \text{ in } \mathbb{R}^4 \setminus B(0, R + 1), \quad \text{and} \\ 0 \leq \eta_2 \leq 1, \quad \eta_2 \equiv 1 \text{ in } B\left(0, \frac{\delta_0}{3\varepsilon}\right), \quad \eta_2 \equiv 0 \text{ in } \mathbb{R}^4 \setminus B\left(0, \frac{\delta_0}{2\varepsilon}\right). \end{aligned}$$

Then we set

$$\eta_{1j}(y) = \eta_1(r_j), \quad \eta_{2j}(y) = \eta_2(r_j), \tag{3.20}$$

and define the test function

$$\tilde{Z}_{0j} = \eta_{1j} Z_{0j} + (1 - \eta_{1j}) \eta_{2j} \hat{Z}_{0j}.$$

Note the  $\tilde{Z}_{0j}$ 's behavior through  $\Omega_\varepsilon$

$$\tilde{Z}_{0j} = \begin{cases} Z_{0j}, & r_j \leq R, \\ \eta_{1j}(Z_{0j} - \hat{Z}_{0j}) + \hat{Z}_{0j}, & R < r_j \leq R + 1, \\ \hat{Z}_{0j}, & R + 1 < r_j \leq \frac{\delta_0}{3\varepsilon}, \\ \eta_{2j} \hat{Z}_{0j}, & \frac{\delta_0}{3\varepsilon} < r_j \leq \frac{\delta_0}{2\varepsilon}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

In the subsequent, we will label these four regions as

$$\Omega_0 \equiv \{r_j \leq R\}, \quad \Omega_1 \equiv \{R < r_j \leq R + 1\}, \quad \Omega_2 \equiv \left\{R + 1 < r_j \leq \frac{\delta_0}{3\varepsilon}\right\}, \quad \text{and} \quad \Omega_3 \equiv \left\{\frac{\delta_0}{3\varepsilon} < r_j \leq \frac{\delta_0}{2\varepsilon}\right\}.$$

Let  $\psi$  be a solution to problem (3.16). We will modify  $\psi$  so that the extra orthogonality conditions with respect to  $Z_{0j}$ 's hold. We set

$$\tilde{\psi} = \psi + \sum_{j=1}^m d_j \tilde{Z}_{0j}. \tag{3.22}$$

We adjust the constants  $d_j$  so that

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij} \tilde{\psi} = 0, \quad \text{for all } i = 0, \dots, 4; j = 1, \dots, m. \tag{3.23}$$

Then,

$$\mathcal{L}_\varepsilon(\tilde{\psi}) = h + \sum_{j=1}^m d_j \mathcal{L}_\varepsilon(\tilde{Z}_{0j}). \tag{3.24}$$

If (3.23) holds, the previous lemma allows us to conclude

$$\|\tilde{\psi}\|_{**} \leq C \left\{ \|h\|_* + \sum_{j=1}^m |d_j| \|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* \right\}. \tag{3.25}$$

Estimate (3.17) is a direct consequence of the following claim:

**Claim 1.** *The constants  $d_j$  are well defined,*

$$|d_j| \leq C |\log \varepsilon| \|h\|_* \quad \text{and} \quad \|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* \leq \frac{C}{|\log \varepsilon|}, \quad \text{for all } j = 1, \dots, m. \tag{3.26}$$

After these facts have been established, using the fact that

$$\|\tilde{Z}_{0j}\|_{**} \leq C,$$

we obtain (3.17), as desired.

Let us prove now Claim 1. First we find  $d_j$ . From definition (3.22), orthogonality conditions (3.23) and the fact that  $\text{supp } \chi_j \eta_{1k} = \emptyset$  and  $\text{supp } \chi_j \eta_{2k} = \emptyset$  if  $j \neq k$ , we can write

$$d_j \int_{\Omega_\varepsilon} \chi_j Z_{0j}^2 = - \int_{\Omega_\varepsilon} \chi_j Z_{0j} \psi, \quad \forall j = 1, \dots, m. \tag{3.27}$$

Thus  $d_j$  is well defined. Note that the orthogonality conditions in (3.23) for  $i = 1, \dots, 4$  are also satisfied for  $\tilde{\psi}$  thanks to the fact that  $R > R_0 + 1$ .

We prove now the second inequality in (3.26). From (3.21), (3.18) and estimate (2.22) we obtain,

$$\mathcal{L}_\varepsilon(\tilde{Z}_{0j}) = \begin{cases} O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right) & \text{in } \Omega_0, \\ \eta_{1j} \mathcal{L}_\varepsilon(Z_{0j} - \hat{Z}_{0j}) + \mathcal{L}_\varepsilon(\hat{Z}_{0j}) + F(\eta_{1j}, Z_{0j} - \hat{Z}_{0j}) & \text{in } \Omega_1, \\ \mathcal{L}_\varepsilon(\hat{Z}_{0j}) & \text{in } \Omega_2, \\ \eta_{2j} \mathcal{L}_\varepsilon(\hat{Z}_{0j}) + F(\eta_{2j}, \hat{Z}_{0j}) & \text{in } \Omega_3, \end{cases} \tag{3.28}$$

and where  $F$  was defined in (3.12). We compute now  $\mathcal{L}_\varepsilon(\tilde{Z}_{0j})$  in  $\Omega_i$ ,  $i = 1, 2, 3$ . In  $\Omega_1$ , thanks to (3.19) (we consider  $R$  here because we will need this dependence below to prove estimate (3.38))

$$|Z_{0j} - \hat{Z}_{0j}|, |R \nabla(Z_{0j} - \hat{Z}_{0j})| \text{ and } |R^2 \Delta(Z_{0j} - \hat{Z}_{0j})| = O\left(\frac{1}{|\log \varepsilon|}\right); \tag{3.29}$$

moreover

$$|R \nabla(\Delta(Z_{0j} - \hat{Z}_{0j}))| \text{ and } |\Delta^2(Z_{0j} - \hat{Z}_{0j})| = O\left(\frac{1}{R^2 |\log \varepsilon|}\right). \tag{3.30}$$

Thus, using (3.12) and the fact that, in  $\Omega_1$ ,  $|D^\alpha \eta_{1j}| \leq C R^{-|\alpha|}$ , for any multi-index  $|\alpha| \leq 4$ ,

$$F(\eta_{1j}, Z_{0j} - \hat{Z}_{0j}) = O\left(\frac{1}{R^4 |\log \varepsilon|}\right).$$

On the other hand,

$$\mathcal{L}_\varepsilon(Z_{0j} - \hat{Z}_{0j}) = O\left(\frac{1}{R^4 |\log \varepsilon|}\right), \tag{3.31}$$

and

$$\mathcal{L}_\varepsilon(\hat{Z}_{0j}) = O(\varepsilon R) + O\left(\frac{1}{R^4 |\log \varepsilon|}\right). \tag{3.32}$$

In conclusion, if  $y \in \Omega_1$ ,

$$\mathcal{L}_\varepsilon(\tilde{Z}_{0j})(y) = O\left(\frac{1}{R^4 |\log \varepsilon|}\right). \tag{3.33}$$

In  $\Omega_2$ ,

$$\begin{aligned} W(1 - a_{0j} G(\varepsilon y, \xi_j)) &= O\left(\frac{\mu_j^4 a_{0j}}{(\mu_j^2 + r_j^2)^4} \left\{ H(\xi_j, \xi_j) - H(\varepsilon y, \xi_j) + 8 \log \frac{r_j}{R} \right\}\right) \\ &= O\left(\frac{\mu_j^4 a_{0j}}{(\mu_j^2 + r_j^2)^{7/2}} \frac{\log r_j}{(\mu_j^2 + r_j^2)^{1/2}}\right) \\ &= O\left(\frac{1}{|\log \varepsilon|} \frac{\mu_j^4}{(\mu_j^2 + r_j^2)^{7/2}}\right), \end{aligned}$$

and

$$\mathcal{L}_\varepsilon(\hat{Z}_{0j}) = O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right).$$

Thus, in this region

$$\mathcal{L}(\tilde{Z}_{0j}) = O\left(\frac{\mu_j^4 |\log \varepsilon|^{-1}}{(\mu_j^2 + r_j^2)^{7/2}}\right). \tag{3.34}$$

In  $\Omega_3$ , thanks to (3.18),  $|\hat{Z}_{0j}| = O(\frac{1}{|\log \varepsilon|})$ ,  $|\nabla \hat{Z}_{0j}| = O(\frac{\varepsilon}{|\log \varepsilon|})$ ,  $|\Delta \hat{Z}_{0j}| = O(\frac{\varepsilon^2}{|\log \varepsilon|})$ ,  $|\nabla(\Delta \hat{Z}_{0j})| = O(\frac{\varepsilon^3}{|\log \varepsilon|})$  and  $|\Delta^2 \hat{Z}_{0j}| = O(\frac{\varepsilon^4}{|\log \varepsilon|})$ . Thus,  $F(\eta_{2j}, \hat{Z}_{0j}) = O(\frac{\varepsilon^4}{|\log \varepsilon|})$ .

Finally,

$$\begin{aligned} \mathcal{L}_\varepsilon(\hat{Z}_{0j}) &= \mathcal{L}_\varepsilon(Z_{0j}) + Wa_{0j} \left( H(\xi_j, \xi_j) - H(\varepsilon y, \xi_j) + 8 \log \frac{r_j}{R} \right) \\ &= O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right) + O\left(\frac{\mu_j^4}{(\mu_j^2 + r_j^2)^4}\right) \\ &= O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right) \end{aligned}$$

and then, combining (3.33), (3.34) and the previous estimate, we can again write the estimate (3.28):

$$\mathcal{L}_\varepsilon(\tilde{Z}_{0j}) = \begin{cases} O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right) & \text{in } \Omega_0, \\ O\left(\frac{1}{|\log \varepsilon|}\right) & \text{in } \Omega_1, \\ O\left(\frac{\mu_j^4 |\log \varepsilon|^{-1}}{(\mu_j^2 + r_j^2)^{7/2}}\right) & \text{in } \Omega_2, \\ O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right) & \text{in } \Omega_3. \end{cases} \tag{3.35}$$

In conclusion,

$$\|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* = O\left(\frac{1}{|\log \varepsilon|}\right). \tag{3.36}$$

Finally, we prove the bounds of  $d_j$ . Testing equation (3.24) against  $\tilde{Z}_{0j}$  and using relations (3.25) and the above estimate, we get

$$\begin{aligned} |d_j| \left| \int_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} \right| &= \left| \int_{\Omega_\varepsilon} h \tilde{Z}_{0j} + \int_{\Omega_\varepsilon} \tilde{\psi} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \right| \\ &\leq C \|h\|_* + C \|\tilde{\psi}\|_\infty \|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* \\ &\leq C \|h\|_* \{1 + \|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_*\} + C \sum_{k=1}^m |d_k| \|\mathcal{L}_\varepsilon(\tilde{Z}_{0k})\|_* \|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* \end{aligned}$$

where we have used that

$$\int_{\Omega_\varepsilon} \frac{\mu_j^4}{(\mu_j^2 + r_j^2)^{7/2}} \leq C \quad \text{for all } j.$$

But estimate (3.36) imply

$$|d_j| \left| \int_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} \right| \leq C \|h\|_* + C \sum_{k=1}^m \frac{|d_k|}{|\log \varepsilon|^2}. \tag{3.37}$$

It only remains to estimate the integral term of the left side. For this purpose, we have the following

**Claim 2.** *If  $R$  is sufficiently large,*

$$\left| \int_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} \right| = \frac{E}{|\log \varepsilon|} (1 + o(1)), \tag{3.38}$$

where  $E$  is a positive constant independent of  $\varepsilon$  and  $R$ .

Assume for the moment the validity of this claim. We replace (3.38) in (3.37), we get

$$|d_j| \leq C |\log \varepsilon| \|h\|_* + C \sum_{k=1}^m \frac{|d_k|}{|\log \varepsilon|}, \tag{3.39}$$

and then,

$$|d_j| \leq C |\log \varepsilon| \|h\|_*.$$

Claim 1 is thus proven. Let us proof Claim 2. We decompose

$$\begin{aligned} \int_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} &= O(\varepsilon) + \int_{\Omega_1} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} + \int_{\Omega_2} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} + \int_{\Omega_3} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} \\ &\equiv O(\varepsilon) + I_1 + I_2 + I_3. \end{aligned}$$

First we estimate  $I_2$ . From (3.35),

$$\begin{aligned} I_2 &= O\left(\frac{1}{|\log \varepsilon|} \int_{\Omega_2} \frac{\mu_j^4 \hat{Z}_{0j}}{(\mu_j^2 + r_j^2)^{7/2}}\right) \\ &= O\left(\frac{1}{R^3 |\log \varepsilon|}\right). \end{aligned}$$

Now we estimate  $I_3$ . From the estimates in  $\Omega_3$ ,  $|I_3| = O(\frac{\varepsilon^4}{|\log \varepsilon|})$ . On the other hand, since (3.33) holds true and  $\hat{Z}_{0j} = Z_{0j}(1 + O(\frac{1}{R|\log \varepsilon|}))$ , we conclude

$$\begin{aligned} |I_1| &= \frac{1}{R^4 |\log \varepsilon|} \int_{R < r_j \leq R+1} \tilde{Z}_{0j}(y) dy \\ &= \frac{1}{R^4 |\log \varepsilon|} \int_{R < r_j \leq R+1} \left\{ O\left(\frac{1}{R|\log \varepsilon|}\right) + \hat{Z}_{0j}(y) \right\} dy \\ &= \frac{1}{R^5 |\log \varepsilon|^2} + \frac{|S^3|}{R^4 |\log \varepsilon|} \int_R^{R+1} r^3 \left(\frac{r^2 - \mu_j^2}{\mu_j^2 + r^2}\right) (1 + o(1)) dr \\ &= \frac{E}{|\log \varepsilon|} (1 + o(1)), \end{aligned}$$

where  $E$  is a positive constant independent of  $\varepsilon$  and  $R$ . Thus, for fixed  $R$  large and  $\varepsilon$  small, we obtain (3.38).  $\square$

Now we can try with the original linear problem (3.5).

**Proof of Proposition 3.1.** We first establish the validity of the a priori estimate (3.7) for solutions  $\psi$  of problem (3.5), with  $h \in L^\infty(\Omega_\varepsilon)$  and  $\|h\|_* < \infty$ . Lemma 3.3 implies

$$\|\psi\|_{**} \leq C |\log \varepsilon| \left\{ \|h\|_* + \sum_{i=1}^2 \sum_{j=1}^m |c_{ij}| \|\chi_j Z_{ij}\|_* \right\}. \tag{3.40}$$



On the other hand,

$$\|\chi_j Z_{ij}\|_* \leq C,$$

then, it is sufficient to estimate the values of the constants  $c_{ij}$ . To this end, we multiply the first equation in (3.5) by  $Z_{ij}\eta_{2j}$ , with  $\eta_{2j}$  the cut-off function introduced in (3.20), and integrate by parts to find

$$\int_{\Omega_\varepsilon} \psi \mathcal{L}_\varepsilon(Z_{ij}\eta_{2j}) = \int_{\Omega_\varepsilon} h Z_{ij}\eta_{2j} + c_{ij} \int_{\Omega_\varepsilon} \eta_{2j} Z_{ij}^2. \tag{3.41}$$

It is easy to see that  $\int_{\Omega_\varepsilon} \eta_{2j} Z_{ij} h = O(\|h\|_*)$  and  $\int_{\Omega_\varepsilon} \eta_{2j} Z_{ij}^2 = C > 0$ . On the other hand we have

$$\begin{aligned} \mathcal{L}_\varepsilon(\eta_{2j} Z_{ij}) &= \eta_{2j} \mathcal{L}_\varepsilon(Z_{ij}) + F(\eta_{2j}, Z_{ij}) \\ &= O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right) \eta_{2j} |Z_{ij}| + F(\eta_{2j}, Z_{ij}). \end{aligned}$$

Directly from (3.12) we get

$$F(\eta_{2j}, Z_{ij}) = O\left(\frac{\varepsilon^4}{(\mu_j^2 + r_j^2)^{1/2}}\right) + O\left(\frac{\varepsilon^3}{\mu_j^2 + r_j^2}\right) + O\left(\frac{\varepsilon^2}{(\mu_j^2 + r_j^2)^{3/2}}\right) + O\left(\frac{\varepsilon}{(\mu_j^2 + r_j^2)^2}\right),$$

in the region  $\frac{\delta_0}{3\varepsilon} \leq r_j \leq \frac{\delta_0}{2\varepsilon}$ . Thus

$$\begin{aligned} \|\mathcal{L}_\varepsilon(\eta_{2j} Z_{ij})\|_* &= O(\varepsilon) \quad \text{and} \\ \left| \int_{\Omega_\varepsilon} \psi \mathcal{L}_\varepsilon(\eta_{2j} Z_{ij}) \right| &\leq C\varepsilon |\log \varepsilon| \|\psi\|_\infty \leq C\varepsilon |\log \varepsilon| \|\psi\|_{**}. \end{aligned} \tag{3.42}$$

Using the above estimates in (3.41), we obtain

$$|c_{ij}| \leq C\{\varepsilon |\log \varepsilon| \|\psi\|_{**} + \|h\|_*\}, \tag{3.43}$$

and then

$$|c_{ij}| \leq C \left\{ (1 + \varepsilon |\log \varepsilon|^2) \|h\|_* + \varepsilon |\log \varepsilon|^2 \sum_{l,k} |c_{lk}| \right\}.$$

Then  $|c_{ij}| \leq C\|h\|_*$  and putting this estimate in (3.40), we conclude the validity of (3.17).

We now prove the solvability assertion. To this purpose we consider the space

$$\mathcal{H} = \left\{ \psi \in H^3(\Omega_\varepsilon): \psi = \Delta \psi = 0 \text{ on } \partial\Omega_\varepsilon, \text{ and such that } \int_{\Omega_\varepsilon} \chi_j Z_{ij} \psi = 0, \text{ for all } i = 1, \dots, 4; j = 1, \dots, m \right\},$$

endowed with the usual inner product  $(\psi, \varphi) = \int_{\Omega_\varepsilon} \Delta \psi \Delta \varphi$ . Problem (3.16) expressed in a weak form is equivalent to that of finding a  $\psi \in \mathcal{H}$ , such that

$$(\psi, \varphi) = \int_{\Omega_s} \{h + W\psi\} \varphi, \quad \text{for all } \varphi \in \mathcal{H}.$$

With the aid of Riesz’s representation theorem, this equation can be rewritten in  $\mathcal{H}$  in the operator form  $\psi = K(W\psi + h)$ , where  $K$  is a compact operator in  $\mathcal{H}$ . Fredholm’s alternative guarantees unique solvability of this problem for any  $h$  provided that the homogeneous equation  $\psi = K(W\psi)$  has only the zero solution in  $\mathcal{H}$ . This last equation is equivalent to (3.16) with  $h \equiv 0$ . Thus existence of a unique solution follows from the a priori estimate (3.17). This concludes the proof.  $\square$

The result of Proposition 3.1 implies that the unique solution  $\psi = T(h)$  of (3.5) defines a continuous linear map from the Banach space  $C_*$  of all functions  $h \in L^\infty(\Omega_\varepsilon)$  with  $\|h\|_* < +\infty$ , into  $W^{3,\infty}(\Omega_\varepsilon)$ , with norm bounded uniformly in  $\varepsilon$ .

**Remark 3.1.** The operator  $T$  is differentiable with respect to the variables  $\xi'$ . In fact, computations similar to those used in [14] yield the estimate

$$\|\partial_{\xi'} T(h)\|_{**} \leq C |\log \varepsilon|^2 \|h\|_*, \quad \text{for all } l = 1, 2; k = 1, \dots, m. \tag{3.44}$$

**4. The intermediate non-linear problem**

In order to solve problem (2.20) we consider first the intermediate non-linear problem.

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = -R + N(\psi) + \sum_{i=1}^4 \sum_{j=1}^m c_{ij} \chi_j Z_{ij} & \text{in } \Omega_\varepsilon, \\ \psi = \Delta \psi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij} \psi = 0, & \text{for all } i = 1, \dots, 4, j = 1, \dots, m. \end{cases} \tag{4.1}$$

For this problem we will prove

**Proposition 4.1.** *Let  $\xi \in \mathcal{O}$ . Then, there exists  $\varepsilon_0 > 0$  and  $C > 0$  such that for all  $\varepsilon \leq \varepsilon_0$  the non-linear problem (4.1) has a unique solution  $\psi \in$  which satisfies*

$$\|\psi\|_{**} \leq C \varepsilon |\log \varepsilon|. \tag{4.2}$$

Moreover, if we consider the map  $\xi' \in \mathcal{O} \rightarrow \psi \in C^{4,\alpha}(\bar{\Omega}_\varepsilon)$ , the derivative  $D_{\xi'} \psi$  exists and defines a continuous map of  $\xi'$ . Besides

$$\|D_{\xi'} \psi\|_{**} \leq C \varepsilon |\log \varepsilon|^2. \tag{4.3}$$

**Proof.** In terms of the operator  $T$  defined in Proposition 3.1, problem (4.1) becomes

$$\psi = \mathcal{B}(\psi) \equiv T(N(\psi) - R).$$

Let us consider the region

$$\mathcal{F} \equiv \{ \psi \in C^{4,\alpha}(\bar{\Omega}_\varepsilon) : \|\psi\|_{**} \leq \varepsilon |\log \varepsilon| \}.$$

From Proposition 3.1,

$$\|\mathcal{B}(\psi)\|_{**} \leq C |\log \varepsilon| \{ \|N(\psi)\|_* + \|R\|_* \},$$

and Lemma 2.2 implies

$$\|R\|_* \leq C \varepsilon.$$

Also, from Lemma 2.4

$$\|N(\psi)\|_* \leq C \|\psi\|_\infty^2 \leq C \|\psi\|_{**}^2.$$

Hence, if  $\psi \in \mathcal{F}$ ,  $\|\mathcal{B}(\psi)\|_{**} \leq C \varepsilon |\log \varepsilon|$ . Along the same way we obtain

$$\|N(\psi_1) - N(\psi_2)\|_* \leq C \max_{i=1,2} \|\psi_i\|_\infty \|\psi_1 - \psi_2\|_\infty \leq C \max_{i=1,2} \|\psi_i\|_{**} \|\psi_1 - \psi_2\|_{**},$$

for any  $\psi_1, \psi_2 \in \mathcal{F}$ . Then, we conclude

$$\|\mathcal{B}(\psi_1) - \mathcal{B}(\psi_2)\|_{**} \leq C |\log \varepsilon| \|N(\psi_1) - N(\psi_2)\|_* \leq C \varepsilon |\log \varepsilon|^2 \|\psi_1 - \psi_2\|_{**}.$$

It follows that for all  $\varepsilon$  small enough  $\mathcal{B}$  is a contraction mapping of  $\mathcal{F}$ , and therefore a unique fixed point of  $\mathcal{B}$  exists in this region. The proof of (4.3) is similar to one included in [14] and we thus omit it.  $\square$

### 5. Variational reduction

We have solved the non-linear problem (4.1). In order to find a solution to the original problem (2.20) we need to find  $\xi$  such that

$$c_{ij} = c_{ij}(\xi') = 0, \quad \text{for all } i, j, \tag{5.1}$$

where  $c_{ij}(\xi')$  are the constants in (4.1). problem (5.1) is indeed variational: it is equivalent to finding critical points of a function of  $\xi'$ . In fact, we define the function for  $\xi \in \mathcal{O}$

$$\mathcal{F}_\varepsilon(\xi) \equiv J_\rho[U(\xi) + \hat{\psi}_\xi] \tag{5.2}$$

where  $J_\rho$  is defined in (2.16),  $\rho$  is given by (2.4),  $U = U(\xi)$  is our approximate solution from (2.5) and  $\hat{\psi}_\xi = \psi(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon})$ ,  $x \in \Omega$ , with  $\psi = \psi_{\xi'}$  the unique solution to problem (4.1) given by Proposition 4.1. Then we obtain that critical points of  $\mathcal{F}$  correspond to solutions of (5.1) for small  $\varepsilon$ . That is,

**Lemma 5.1.**  $\mathcal{F}_\varepsilon : \mathcal{O} \rightarrow \mathbb{R}$  is of class  $C^1$ . Moreover, for all  $\varepsilon$  small enough, if  $D_\xi \mathcal{F}_\varepsilon(\xi) = 0$  then  $\xi$  satisfies (5.1).

**Proof.** We define

$$I_\varepsilon[v] \equiv \frac{1}{2} \int_{\Omega_\varepsilon} (\Delta v)^2 - \int_{\Omega_\varepsilon} k(\varepsilon y) e^v.$$

Let us differentiate the function  $\mathcal{F}_\varepsilon$  with respect to  $\xi$ . Since  $J_\rho[U(\xi) + \hat{\psi}_\xi] = I_\varepsilon[V(\xi') + \psi_{\xi'}]$ , we can differentiate directly under the integral sign, so that

$$\begin{aligned} \partial_{(\xi_k)_l} \mathcal{F}_\varepsilon(\xi) &= \varepsilon^{-1} DI_\varepsilon[V + \psi](\partial_{(\xi_k)_l} V + \partial_{(\xi_k)_l} \psi) \\ &= \varepsilon^{-1} \sum_{i=1}^4 \sum_{j=1}^m \int_{\Omega_\varepsilon} c_{ij} \chi_j Z_{ij} (\partial_{(\xi_k)_l} V + \partial_{(\xi_k)_l} \psi). \end{aligned}$$

From the results of the previous section, this expression defines a continuous function of  $\xi'$ , and hence of  $\xi$ . Let us assume that  $D_\xi \mathcal{F}_\varepsilon(\xi) = 0$ . Then

$$\sum_{i=1}^4 \sum_{j=1}^m \int_{\Omega_\varepsilon} c_{ij} \chi_j Z_{ij} (\partial_{(\xi_k)_l} V + \partial_{(\xi_k)_l} \psi) = 0, \quad \text{for } k = 1, 2, 3, 4; \quad l = 1, \dots, m.$$

Since  $\|D_{\xi'} \psi_{\xi'}\| \leq C\varepsilon |\log \varepsilon|^2$ , we have

$$\partial_{(\xi_k)_l} V + \partial_{(\xi_k)_l} \psi = Z_{kl} + o(1),$$

where  $o(1)$  is uniformly small as  $\varepsilon \rightarrow 0$ . Thus, we have the following linear system of equation

$$\sum_{i=1}^4 \sum_{j=1}^m c_{ij} \int_{\Omega_\varepsilon} \chi_j Z_{ij} (Z_{kl} + o(1)) = 0, \quad \text{for } k = 1, 2, 3, 4; \quad l = 1, \dots, m.$$

This system is dominant diagonal, thus  $c_{ij} = 0$  for all  $i, j$ . This concludes the proof.  $\square$

We also have the validity of the following lemma

**Lemma 5.2.** Let  $\rho$  be given by (2.4). For points  $\xi \in \mathcal{O}$  the following expansion holds

$$\mathcal{F}_\varepsilon(\xi) = J_\rho[U(\xi)] + \theta_\varepsilon(\xi), \tag{5.3}$$

where  $|\theta_\varepsilon| + |\nabla \theta_\varepsilon| = o(1)$ , uniformly on  $\xi \in \mathcal{O}$  as  $\varepsilon \rightarrow 0$ .

**Proof.** The proof follows directly from an application of Taylor expansion for  $\mathcal{F}_\varepsilon$  in the expanded domain  $\Omega_\varepsilon$  and from the estimates for the solution  $\psi_{\xi'}$  to problem (4.1) obtained in Proposition 4.1.  $\square$

### 6. Proof of the theorems

In this section we carry out the proofs of our main results.

#### 6.1. Proof of Theorem 1

Taking into account the result of Lemma 5.1, a solution to problem (1.1) exists if we prove the existence of a critical point of  $\mathcal{F}_\varepsilon$ , which automatically implies that  $c_{ij} = 0$  in (2.20) for all  $i, j$ . The qualitative properties of the solution found follow from the ansatz.

Finding critical points of  $\mathcal{F}_\varepsilon(\xi)$  is equivalent to finding critical points of

$$\tilde{\mathcal{F}}_\varepsilon(\xi) = \mathcal{F}_\varepsilon(\xi) - 256\pi^2 m |\log \varepsilon|. \tag{6.1}$$

On the other hand, if  $\xi \in \mathcal{O}$ , from Lemmas 2.3 and 5.2 we get the existence of constants  $\alpha > 0$  and  $\beta$  such that

$$\alpha \tilde{\mathcal{F}}_\varepsilon(\xi) + \beta = \varphi_m(\xi) + \varepsilon \Theta_\varepsilon(\xi), \tag{6.2}$$

with  $\Theta_\varepsilon$  and  $\nabla_\xi \Theta_\varepsilon$  uniformly bounded in the considered region as  $\varepsilon \rightarrow 0$ .

We shall prove that, under the assumptions of Theorems 1 and 2,  $\tilde{\mathcal{F}}_\varepsilon$  has a critical point in  $\mathcal{O}$  for  $\varepsilon$  small enough. We start with a topological lemma. We denote by  $D$  the diagonal

$$D := \{\xi \in \Omega^m : \xi_i = \xi_j \text{ for some } i \neq j\},$$

and we write  $H^* := H^*(\cdot; \mathbb{K})$  for singular cohomology with coefficients in a field  $\mathbb{K}$ .

**Lemma 6.1.** *If  $H^d(\Omega) \neq 0$  for some  $d \geq 1$ , and  $H^j(\Omega) = 0$  for  $j > d$ , then the homomorphism*

$$H^{md}(\Omega^m, D) \longrightarrow H^{md}(\Omega^m),$$

*induced by the inclusion of pairs  $(\Omega^m, \emptyset) \hookrightarrow (\Omega^m, D)$ , is an epimorphism. In particular,  $H^{md}(\Omega^m, D) \neq 0$ .*

**Proof.** Let us prove first that  $H^j(D) = 0$  if  $j > (m - 1)d$ . For this purpose we write

$$D = \bigcup_{1 \leq i < j \leq m} X_{i,j}, \quad \text{where } X_{i,j} := \{(x_1, \dots, x_m) \in \Omega^m : x_i = x_j\},$$

and consider the sets  $\mathcal{F}_0 := \{\Omega^m\}$ ,  $\mathcal{F}_1 := \{X_{i,j} : 1 \leq i < j \leq m\}$ , and

$$\mathcal{F}_{k+1} := \{Z \cap Z' : Z, Z' \in \mathcal{F}_k \text{ and } Z \neq Z'\}, \quad k = 1, \dots, m - 2.$$

Note that

$$Z \cong \Omega^{m-k'} \quad \text{for some } k \leq k' \leq m - 1 \text{ if } Z \in \mathcal{F}_k, \quad k = 0, \dots, m - 1,$$

where  $\cong$  means that the sets are homeomorphic. Künneth's formula

$$H^j(\Omega^{m-k}) = \bigoplus_{p+q=j} (H^p(\Omega) \otimes H^q(\Omega^{m-k-1})) \tag{6.3}$$

(see, for example, [17, Proposition 8.18]) yields inductively that, for  $0 \leq k \leq m - 1$ ,

$$H^j(Z) = 0 \quad \text{if } Z \in \mathcal{F}_k \text{ and } j > (m - k)d. \tag{6.4}$$

We claim that, for each  $0 \leq k \leq m - 1$ , one has that

$$H^j(Z_1 \cup \dots \cup Z_\ell) = 0 \quad \text{if } Z_1, \dots, Z_\ell \in \mathcal{F}_k \text{ and } j > (m - k)d. \tag{6.5}$$

Let us prove this claim. Since  $\mathcal{F}_{m-1}$  has only one element and (6.4) holds, we have that the claim is true for  $k = m - 1$ . Assume that the claim is true for  $k + 1$  with  $k + 1 \leq m - 1$  and let us then prove it for  $k$ . We do this by induction on  $\ell$ . If  $\ell = 1$  the assertion reduces to (6.4). Now assume that the assertion is true for every union of at most  $\ell - 1$  sets in  $\mathcal{F}_k$ , and let  $Z_1, \dots, Z_\ell \in \mathcal{F}_k$  be pairwise distinct sets. Consider the Mayer–Vietoris sequence

$$\dots \rightarrow H^{j-1} \left( \bigcup_{i=1}^{\ell-1} (Z_i \cap Z_\ell) \right) \rightarrow H^j(Z_1 \cup \dots \cup Z_\ell) \rightarrow H^j(Z_1 \cup \dots \cup Z_{\ell-1}) \oplus H^j(Z_\ell) \rightarrow \dots \tag{6.6}$$

Our induction hypothesis on  $\ell$  yields that  $H^j(Z_1 \cup \dots \cup Z_{\ell-1}) = 0$  and  $H^j(Z_\ell) = 0$  if  $j > (m - k)d$ . Since  $Z_1, \dots, Z_\ell$  are pairwise distinct, we have that  $Z_i \cap Z_\ell \in \mathcal{F}_{k+1}$  for each  $i = 1, \dots, \ell - 1$  and, since we are assuming that the claim is true for  $k + 1$  we have that

$$H^{j-1} \left( \bigcup_{i=1}^{\ell-1} (Z_i \cap Z_\ell) \right) = 0 \quad \text{if } j - 1 > (m - (k + 1))d.$$

Note that  $j > (m - k)d$  implies  $j - 1 > (m - (k + 1))d$ . This proves that both ends of the exact sequence (6.6) are zero if  $j > (m - k)d$ , hence the middle term is also zero in this case. This concludes the proof of claim (6.5).

Now, since  $D = \bigcup_{Y \in \mathcal{F}_1} Y$ , assertion (6.5) with  $k = 1$  yields that  $H^j(D) = 0$  if  $j > (m - 1)d$ . So the exact cohomology sequence

$$H^{md}(\Omega^m, D) \longrightarrow H^{md}(\Omega^m) \longrightarrow H^{md}(D) = 0$$

gives that  $H^{md}(\Omega^m, D) \rightarrow H^{md}(\Omega^m)$  is an epimorphism. But (6.3) implies that  $H^{md}(\Omega^m) \neq 0$ . Therefore,  $H^{md}(\Omega^m, D) \neq 0$ , as claimed.  $\square$

For each positive number  $\delta$  define

$$\begin{aligned} \Omega_\delta &:= \{ \xi \in \Omega : \text{dist}(\xi, \partial\Omega) > \delta \}, \\ \mathcal{D}_\delta &:= \{ \xi = (\xi_1, \dots, \xi_m) \in \Omega^m : \xi_j \in \Omega_\delta \}. \end{aligned}$$

**Lemma 6.2.** *Given  $K > 0$  there exists  $\delta_0 > 0$  such that, for each  $\delta \in (0, \delta_0)$ , the following holds: For every  $\xi = (\xi_1, \dots, \xi_m) \in \partial\mathcal{D}_\delta$  with  $|\varphi_m(\xi)| \leq K$  there exists an  $i \in \{1, \dots, m\}$  such that*

$$\begin{aligned} \nabla_{\xi_i} \varphi_m(\xi) &\neq 0 && \text{if } \xi_i \in \Omega_\delta, \\ \nabla_{\xi_i} \varphi_m(\xi) \cdot \tau &\neq 0 \text{ for some } \tau \in T_{\xi_i}(\partial\Omega_\delta) && \text{if } \xi_i \in \partial\Omega_\delta \end{aligned}$$

where  $T_{\xi_i}(\partial\Omega_\delta)$  denotes the tangent space to  $\partial\Omega_\delta$  at the point  $\xi_i$ .

**Proof.** We first need to establish some facts related to the regular part of the Green function on the half hyperplane

$$\mathcal{H} := \{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 \geq 0 \}.$$

It is well known that the regular part of the Green function on  $\mathcal{H}$  is given by

$$H(x, y) = 8 \log |x - \bar{y}|, \quad \bar{y} = (y_1, y_2, y_3, -y_4),$$

for  $x, y \in \mathcal{H}$  and the Green function is

$$G(x, y) = -8 \log |x - y| + 8 \log |x - \bar{y}|.$$

Consider the function of  $k \geq 2$  distinct points of  $\mathcal{H}$

$$\Psi_k(x_1, \dots, x_k) := -8 \sum_{i \neq j} \log |x_i - x_j|,$$

and denote by  $I_+$  and  $I_0$  the set of indices  $i$  for which  $(x_i)_4 > 0$  and  $(x_i)_4 = 0$ , respectively. Define also

$$\varphi_{k, \mathcal{H}}(x_1, \dots, x_k) = -8 \sum_{j=1}^k \log |x_j - \bar{x}_j| + 8 \sum_{i \neq j} \log \frac{|x_i - x_j|}{|x_i - \bar{x}_j|}.$$

**Claim 3.** *We have the following alternative: Either*

$$\nabla_{x_i} \Psi_k(x_1, \dots, x_k) \neq 0 \quad \text{for some } i \in I_+,$$

or

$$\partial_{(x_i)_j} \Psi_k(x_1, \dots, x_k) \neq 0 \quad \text{for some } i \in I_0 \text{ and } j \in \{1, 2, 3\},$$

where  $\partial_{(x_i)_j} \equiv \frac{\partial}{\partial (x_i)_j}$ .

**Proof.** We have that

$$\frac{\partial}{\partial \lambda} \Psi_k(\lambda x_1, \dots, \lambda x_k)|_{\lambda=1} = \sum_{i \in I_+} \nabla_{x_i} \Psi_k(x_1, \dots, x_k) \cdot x_i + \sum_{i \in I_0} \nabla_{x_i} \Psi_k(x_1, \dots, x_k) \cdot x_i.$$

On the other hand

$$\frac{\partial}{\partial \lambda} \Psi_k(\lambda x_1, \dots, \lambda x_k)|_{\lambda=1} = -8k(k - 1) \neq 0,$$

and Claim 3 follows.  $\square$

**Claim 4.** For any  $k$  distinct points  $x_i \in \text{Int } \mathcal{H}$  we have  $\nabla \varphi_{k, \mathcal{H}}(x_1, \dots, x_k) \neq 0$ .

**Proof.** We have that

$$\frac{\partial}{\partial \lambda} \varphi_{k, \mathcal{H}}(\lambda x_1, \dots, \lambda x_k)|_{\lambda=1} = \sum_{i=1}^k \nabla_{x_i} \varphi_{k, \mathcal{H}}(x_1, \dots, x_k) \cdot x_i.$$

On the other hand

$$\frac{\partial}{\partial \lambda} \varphi_{k, \mathcal{H}}(\lambda x_1, \dots, \lambda x_k)|_{\lambda=1} = -8k(k - 1) \neq 0,$$

and Claim 4 follows.  $\square$

Now we will need an estimate for the regular part  $H(x, y)$  of the Green’s function for points  $x, y$  close to  $\partial\Omega$ .

**Claim 5.** There exists  $C_1, C_2 > 0$  constants such that for any  $x, y \in \Omega$

$$|\nabla_x H(x, y)| + |\nabla_y H(x, y)| \leq C_1 \min \left\{ \frac{1}{|x - y|}, \frac{1}{\text{dist}(y, \partial\Omega)} \right\} + C_2.$$

**Proof.** For  $y \in \Omega$  a point close to  $\partial\Omega$  we denote by  $\bar{y}$  its uniquely determined reflection with respect to  $\partial\Omega$ . Define  $\psi(x, y) = H(x, y) + 8 \log \frac{1}{|x - \bar{y}|}$ . It is straightforward to see that  $\psi$  is bounded in  $\bar{\Omega} \times \bar{\Omega}$  and that  $|\nabla_x \psi(x, y)| + |\nabla_y \psi(x, y)| \leq C$  for some positive constant  $C$ . Claim 5 follows.  $\square$

We have now all elements to prove Lemma 6.2. Assume, by contradiction, that for some sequence  $\delta_n \rightarrow 0$  there are points  $\xi^n \in \partial\mathcal{D}_{\delta_n}$ , such that  $|\varphi_m(\xi^n)| \leq K$  and, for every  $i \in \{1, \dots, m\}$ ,

$$\nabla_{\xi_i^n} \varphi_m(\xi^n) = 0 \quad \text{if } \xi_i^n \in \Omega_{\delta_n}, \tag{6.7}$$

and

$$\nabla_{\xi_i^n} \varphi_m(\xi^n) \cdot \tau = 0 \quad \text{if } \xi_i^n \in \partial\Omega_{\delta_n}, \tag{6.8}$$

for any vector  $\tau$  tangent to  $\partial\Omega_{\delta_n}$  at  $\xi_i^n$ . It follows that there exists a point  $\xi_l^n \in \partial\Omega_{\delta_n}$  such that  $H(\xi_l^n, \xi^n) \rightarrow -\infty$  as  $n \rightarrow \infty$ . Since  $|\varphi_m(\xi^n)| \leq K$ , there are necessarily two distinct points  $\xi_i^n$  and  $\xi_j^n$  coming closer to each other, that is,

$$\rho_n := \inf_{i \neq j} |\xi_i^n - \xi_j^n| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Without loss of generality we can assume  $\rho_n = |\xi_1^n - \xi_2^n|$ . We define  $x_j^n := (\xi_j^n - \xi_1^n) / \rho_n$ . Thus, up to a subsequence, there exists a  $k, 2 \leq k \leq m$ , such that

$$\lim_{n \rightarrow \infty} |x_j^n| < +\infty, \quad j = 1, \dots, k, \quad \text{and} \quad \lim_{n \rightarrow \infty} |x_j^n| = +\infty, \quad j > k.$$

For  $j \leq k$  we set

$$\bar{x}_j = \lim_{n \rightarrow \infty} x_j^n.$$

We consider two cases:

(1) Either

$$\frac{\text{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} \rightarrow +\infty,$$

(2) or there exists  $C_0 < +\infty$  such that for almost all  $n$  we have

$$\frac{\text{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} < C_0.$$

Case 1. It is easy to see that in this case we actually have

$$\frac{\text{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} \rightarrow +\infty, \quad j = 1, \dots, k.$$

Furthermore, the points  $\xi_1^n, \dots, \xi_k^n$  are all in the interior of  $\Omega_{\delta_n}$ , hence (6.7) is satisfied for all partial derivatives  $\nabla_{\xi_j}$ ,  $j \leq k$ . Define  $\tilde{\varphi}_m(x_1, \dots, x_m) := \varphi_m(\xi_1^n + \rho_n x_1, \xi_1^n + \rho_n x_2, \dots, \xi_1^n + \rho_n x_k, \xi_{k+1}^n + \rho_n x_{k+1}, \dots, \xi_m^n + \rho_n x_m)$ , and  $x = (x_1, \dots, x_m)$ . We have that, for all  $l = 1, 2, j = 1, \dots, k, \partial_{(x_j)_l} \tilde{\varphi}_m(x) = \rho_n \partial_{(\xi_j)_l} \varphi_m(\xi_1^n + \rho_n x_1, \dots, \xi_1^n + \rho_n x_k, \xi_{k+1}^n + \rho_n x_{k+1}, \dots, \xi_m^n + \rho_n x_m)$ . Then at  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_k, 0, \dots, 0)$  we have

$$\partial_{(x_j)_l} \tilde{\varphi}_m(\bar{x}) = 0.$$

On the other hand, using Claim 5 and letting  $n \rightarrow \infty$ , we obtain

$$\lim_{n \rightarrow \infty} \rho_n \partial_{(\xi_j)_l} \varphi_m(\xi_1^n + \rho_n \bar{x}_1, \dots, \xi_m^n + \rho_n \bar{x}_m) = 8 \sum_{i \neq j, i \leq k} \partial_{(x_j)_i} \log |\bar{x}_i - \bar{x}_j| = 0,$$

a contradiction with Claim 3.

Case 2. In this case we actually have

$$\frac{\text{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} < C_1, \quad j = 1, \dots, m,$$

for some constant  $C_1 > 0$  and for almost all  $n$ . If the points  $\xi_j^n$  are all interior to  $\Omega_{\delta_n}$ , we argue as in Case 1 above to reach a contradiction to Claim 4.

Therefore, we assume that for some  $j^*$  we have  $\xi_{j^*}^n \in \partial\Omega_{\delta_n}$ . Assume first that there exists a constant  $C$  such that  $\delta_n \leq C\rho_n$ . Consider the following sum

$$s_n := \sum_{i \neq j} G(\xi_j^n, \xi_i^n).$$

In this case it is not difficult to see that  $s_n = O(1)$  as  $n \rightarrow +\infty$ . On the other hand

$$\sum_j H(\xi_j^n, \xi_j^n) \leq H(\xi_{j^*}^n, \xi_{j^*}^n) + C \leq 8 \log |\xi_{j^*}^n - \bar{\xi}_{j^*}^n| + C,$$

where  $\bar{\xi}_{j^*}^n$  is the reflection of the point  $\xi_{j^*}^n$  with respect to  $\partial\Omega$ . Since  $|\xi_{j^*}^n - \bar{\xi}_{j^*}^n| \leq 2\delta_n$  we have that

$$\sum_j H(\xi_j^n, \xi_j^n) \rightarrow -\infty, \quad \text{as } n \rightarrow \infty.$$

But  $|\varphi_m(\xi^n)| \leq K$ , a contradiction.

Finally assume that  $\rho_n = o(\delta_n)$ . In this case after scaling with  $\rho_n$  around  $\xi_{j^*}^n$ , and arguing similarly as in Case 1 we get a contradiction with Claim 3 since those points  $\xi_j^n$  which lie on  $\partial\Omega_{\delta_n}$ , after passing to the limit, give rise to points that lie on the same straight line. Thus this case cannot occur.  $\square$

We shall now show that we can perturb the gradient vector field of  $\varphi_m$  near  $\partial\mathcal{D}_\delta$  to obtain a new vector field with the same stationary points, such that  $\varphi_m$  is a Lyapunov function for the associated flow and  $\mathcal{D}_\delta \cap \varphi_m^{-1}[-K, K]$  is positively invariant.

We consider the following more general situation. Let  $U$  be a bounded open subset of  $\mathbb{R}^N$  with smooth boundary, and let  $m \in \mathbb{N}$ . We consider a decomposition of  $\bar{U}^m$  as follows. Let  $S$  be the set of all functions  $\sigma : \{1, \dots, m\} \rightarrow \{U, \partial U\}$ , and define

$$\mathcal{Y}_\sigma := \sigma(1) \times \dots \times \sigma(m) \subset \mathbb{R}^{mN}.$$

Then

$$\bar{U}^m = \bigcup_{\sigma \in S} \mathcal{Y}_\sigma, \quad \partial(U^m) = \bigcup_{\sigma \in S \setminus \sigma_U} \mathcal{Y}_\sigma, \quad \text{and} \quad \mathcal{Y}_\sigma \cap \mathcal{Y}_\zeta = \emptyset \quad \text{if } \sigma \neq \zeta,$$

where  $\sigma_U$  stands for the constant function  $\sigma_U(i) = U$ . Note that  $\mathcal{Y}_\sigma$  is a manifold of dimension  $\leq mN$ . We denote by  $T_\xi(\mathcal{Y}_\sigma)$  the tangent space to  $\mathcal{Y}_\sigma$  at the point  $\xi \in \mathcal{Y}_\sigma$ . The following holds.

**Lemma 6.3.** *Let  $\mathcal{F}$  be a function of class  $\mathcal{C}^1$  in a neighborhood of  $\bar{U}^m \cap \mathcal{F}^{-1}[b, c]$ . Assume that*

$$\nabla_\sigma \mathcal{F}(\xi) \neq 0 \quad \text{for every } \xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c] \text{ with } \sigma \neq \sigma_U, \tag{6.9}$$

where  $\nabla_\sigma \mathcal{F}(\xi)$  is the projection of  $\nabla \mathcal{F}(\xi)$  onto the tangent space  $T_\xi(\mathcal{Y}_\sigma)$ . Then there exists a locally Lipschitz continuous vector field  $\chi : \mathcal{U} \rightarrow \mathbb{R}^N$ , defined in an open neighborhood  $\mathcal{U}$  of  $\bar{U}^m \cap \mathcal{F}^{-1}[b, c]$ , with the following properties: For  $\xi \in \mathcal{U}$ ,

- (i)  $\chi(\xi) = 0$  if and only if  $\nabla \mathcal{F}(\xi) = 0$ ,
- (ii)  $\chi(\xi) \cdot \nabla \mathcal{F}(\xi) > 0$  if  $\nabla \mathcal{F}(\xi) \neq 0$ ,
- (iii)  $\chi(\xi) \in T_\xi(\mathcal{Y}_\sigma)$  if  $\xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c]$ .

**Proof.** Let  $\mathcal{N}_\alpha := \{x \in \mathbb{R}^N : \text{dist}(x, \partial U) < \alpha\}$ . Fix  $\alpha > 0$  small enough so that there exists a smooth retraction  $r : \mathcal{N}_\alpha \rightarrow \partial U$ . For every  $\sigma \in S$ , let  $\hat{\sigma} : \{1, \dots, m\} \rightarrow \{U, \partial \mathcal{N}_\alpha\}$  be the function  $\hat{\sigma}(i) = \sigma(i)$  if  $\sigma(i) = U$  and  $\hat{\sigma}(i) = \mathcal{N}_\alpha$  if  $\sigma(i) = \partial U$ . Set

$$\mathcal{U}_\sigma := \hat{\sigma}(1) \times \dots \times \hat{\sigma}(m).$$

Then  $\mathcal{U}_\sigma$  is an open neighborhood of  $\mathcal{Y}_\sigma$ . Let  $r_\sigma : \mathcal{U}_\sigma \rightarrow \mathcal{Y}_\sigma$  be the obvious retraction. Assumption (6.9) implies that  $\mathcal{F}$  has no critical points on  $\partial(U^m) \cap \mathcal{F}^{-1}[b, c]$  and, moreover, that

$$\nabla_\sigma \mathcal{F}(\xi) \cdot \nabla \mathcal{F}(\xi) > 0 \quad \text{if } \xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c] \text{ and } \nabla \mathcal{F}(\xi) \neq 0.$$

So taking  $\alpha$  even smaller if necessary, we may assume that  $\mathcal{F}$  has no critical points in  $\mathcal{U}_\sigma \cap \mathcal{F}^{-1}[b, c]$  if  $\sigma \neq \sigma_U$ , and that

$$\nabla_\sigma \mathcal{F}(r_\sigma(\xi)) \cdot \nabla \mathcal{F}(\xi) > 0 \quad \text{if } \xi \in \mathcal{U}_\sigma \cap \mathcal{F}^{-1}(b - \alpha, c + \alpha) \text{ and } \nabla \mathcal{F}(\xi) \neq 0.$$

Let  $\{\pi_\sigma : \sigma \in S\}$  be a locally Lipschitz partition of unity subordinated to the open cover  $\{\mathcal{U}_\sigma : \sigma \in S\}$ . Define

$$\chi(\xi) := \sum_{\sigma \in S} \pi_\sigma(\xi) \nabla_\sigma \mathcal{F}(r_\sigma(\xi)), \quad \xi \in \mathcal{U} := \bigcup_{\sigma \in S} \mathcal{U}_\sigma \cap \mathcal{F}^{-1}(b - \alpha, c + \alpha).$$

One can easily verify that  $\chi$  has the desired properties.  $\square$

As usual, set  $\mathcal{F}^c := \{\xi \in \text{dom } \mathcal{F} : \mathcal{F}(\xi) \leq c\}$ .

**Lemma 6.4 (Deformation lemma).** *Let  $\mathcal{F}$  be a function of class  $\mathcal{C}^1$  in a neighborhood of  $\bar{U}^m \cap \mathcal{F}^{-1}[b, c]$ . Assume that*

$$\nabla_\sigma \mathcal{F}(\xi) \neq 0 \quad \text{for every } \xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c] \text{ with } \sigma \neq \sigma_U.$$

*If  $\mathcal{F}$  has no critical points in  $U^m \cap \mathcal{F}^{-1}[b, c]$ , then there exists a continuous deformation  $\tilde{\eta} : [0, 1] \times (\bar{U}^m \cap \mathcal{F}^c) \rightarrow \bar{U}^m \cap \mathcal{F}^c$  such that*

- $\tilde{\eta}(0, \xi) = \xi \quad \text{for all } \xi \in \bar{U}^m \cap \mathcal{F}^c,$
- $\tilde{\eta}(s, \xi) = \xi \quad \text{for all } (s, \xi) \in [0, 1] \times (\bar{U}^m \cap \mathcal{F}^b),$
- $\tilde{\eta}(1, \xi) \in \bar{U}^m \cap \mathcal{F}^b \quad \text{for all } \xi \in \bar{U}^m \cap \mathcal{F}^c.$



**Proof.** Let  $\chi : \mathcal{U} \rightarrow \mathbb{R}^N$  be as in Lemma 6.3 and consider the flow  $\eta$  defined by

$$\begin{cases} \frac{\partial}{\partial t} \eta(t, \xi) = -\chi(\eta(t, \xi)), \\ \eta(0, \xi) = \xi, \end{cases} \tag{6.10}$$

for  $\xi \in \mathcal{U}$  and  $t \in [0, t^+(\xi))$ , where  $t^+(\xi)$  is the maximal existence time of the trajectory  $t \mapsto \eta(t, \xi)$  in  $\mathcal{U}$ . For each  $\xi \in \mathcal{U}$ , let

$$t_b(\xi) := \inf\{t \geq 0 : \mathcal{F}(\eta(t, \xi)) \leq b\} \in [0, \infty]$$

be the entrance time into the sublevel set  $\mathcal{F}^b$ . Property (ii) in Lemma 6.3 implies that

$$\frac{d}{dt} \mathcal{F}(\eta(t, \xi)) = -\nabla \mathcal{F}(\eta(t, \xi)) \cdot \chi(\eta(t, \xi)) \leq 0,$$

therefore  $\mathcal{F}(\eta(t, \xi))$  is non-increasing in  $t$ . This, together with (iii) in Lemma 6.3 yields

$$\eta(t, \xi) \in \bar{U}^m \cap \mathcal{F}^{-1}[b, c] \quad \text{if } \xi \in \bar{U}^m \cap \mathcal{F}^{-1}[b, c] \text{ and } t \in [0, t_b(\xi)].$$

Since  $\mathcal{F}$  has no critical points in  $U^m \cap \mathcal{F}^{-1}[b, c]$ , we have that  $t_b(\xi) < \infty$  for every  $\xi \in \bar{U}^m \cap \mathcal{F}^{-1}[b, c]$ , and the entrance time map  $t_b : \bar{U}^m \cap \mathcal{F}^c \cap \mathcal{U} \rightarrow [0, \infty)$  is continuous. It follows that the map

$$\tilde{\eta} : [0, 1] \times (\bar{U}^m \cap \mathcal{F}^c) \rightarrow \bar{U}^m \cap \mathcal{F}^c$$

given by

$$\tilde{\eta}(s, \xi) := \begin{cases} \eta(st_b(\xi), \xi) & \text{if } \xi \in (\bar{U}^m \cap \mathcal{F}^c) \cap \mathcal{U}, \\ \xi & \text{if } \xi \in \bar{U}^m \cap \mathcal{F}^b \end{cases}$$

is a continuous deformation of  $\bar{U}^m \cap \mathcal{F}^c$  into  $\bar{U}^m \cap \mathcal{F}^b$  which leaves  $\bar{U}^m \cap \mathcal{F}^b$  fixed, as claimed.  $\square$

**Proof of Theorem 1.** Fix  $\delta_1$  small enough so that the inclusions

$$\mathcal{D}_{\delta_1} \hookrightarrow \Omega^m \quad \text{and} \quad \mathcal{D}_{\delta_1} \cap D \hookrightarrow B_{\delta_1}(D) := \{x \in \Omega^m : \text{dist}(x, D) \leq \delta_1\} \tag{6.11}$$

are homotopy equivalences, where  $D := \{\xi \in \Omega^m : \xi_i = \xi_j \text{ for some } i \neq j\}$ . Since  $\varphi_m$  is bounded above on  $\mathcal{D}_{\delta_1}$  and bounded below on  $\Omega^m \setminus B_{\delta_1}(D)$ , we may choose  $b_0, c_0 > 0$  such that

$$\mathcal{D}_{\delta_1} \subset \varphi_m^{c_0} \quad \text{and} \quad \varphi_m^{b_0} \subset B_{\delta_1}(D).$$

Fix  $K > \max\{-b_0, c_0\}$  and, for this  $K$ , fix  $\delta \in (0, \delta_1)$  as in Lemma 6.2. By property (6.2), for each  $\varepsilon$  small enough, there exist  $b < c$  such that

$$\varphi_m^{c_0} \subset \tilde{\mathcal{F}}_\varepsilon^c \subset \varphi_m^K, \quad \varphi_m^{-K} \subset \tilde{\mathcal{F}}_\varepsilon^b \subset \varphi_m^{b_0},$$

and such that, for every  $\xi = (\xi_1, \dots, \xi_m) \in \partial \mathcal{D}_\delta$  with  $\tilde{\mathcal{F}}_\varepsilon(\xi) \in [b, c]$  there is an  $i \in \{1, \dots, m\}$  with

$$\nabla_{\xi_i} \tilde{\mathcal{F}}_\varepsilon(\xi) \neq 0 \quad \text{if } \xi_i \in \Omega_\delta,$$

$$\nabla_{\xi_i} \tilde{\mathcal{F}}_\varepsilon(\xi) \cdot \tau \neq 0 \quad \text{for some } \tau \in T_{\xi_i}(\partial \Omega_\delta) \quad \text{if } \xi_i \in \partial \Omega_\delta.$$

We wish to prove that  $\tilde{\mathcal{F}}_\varepsilon$  has a critical point in  $\mathcal{D}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[b, c]$ . We argue by contradiction: Assume that  $\tilde{\mathcal{F}}_\varepsilon$  has no critical points in  $\mathcal{D}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[b, c]$ . Then Lemma 6.4 gives a continuous deformation

$$\tilde{\eta} : [0, 1] \times (\bar{\mathcal{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c) \rightarrow \bar{\mathcal{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c$$

of  $\bar{\mathcal{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c$  into  $\bar{\mathcal{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^b$  which keeps  $\bar{\mathcal{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^b$  fixed. Our choices of  $b$  and  $c$  imply that  $\mathcal{D}_{\delta_1} \subset \bar{\mathcal{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c$  and  $\tilde{\eta}$  induces a deformation of  $\mathcal{D}_{\delta_1}$  into  $\bar{\mathcal{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^b \subset B_{\delta_1}(D)$ , which keeps the diagonal  $D$  fixed. Consequently, the homomorphism

$$\iota^* : H^*(\Omega^m, B_{\delta_1}(D)) \rightarrow H^*(\mathcal{D}_{\delta_1}, \mathcal{D}_{\delta_1} \cap D),$$

induced by the inclusion map  $\iota : \mathcal{D}_{\delta_1} \hookrightarrow \Omega^m$ , factors through  $H^*(B_{\delta_1}(D), B_{\delta_1}(D)) = 0$ . Hence,  $\iota^*$  is the zero homomorphism. On the other hand, our choice (6.11) of  $\delta_1$  implies that  $\iota^*$  is an isomorphism. Therefore,  $H^*(\Omega^m, B_{\delta_1}(D)) = H^*(\Omega^m, D) = 0$ . But, by assumption,  $H^d(\Omega) \neq 0$  for some  $d \geq 1$ . If we choose  $d$  so that  $H^j(\Omega) = 0$  for  $j > d$ , then Lemma 6.1 asserts that  $H^{md}(\Omega^m, D) \neq 0$ . This is a contradiction. Consequently,  $\tilde{\mathcal{F}}_\varepsilon$  must have critical point in  $\mathcal{D}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[b, c]$ , as claimed.  $\square$

## 6.2. Proof of Theorem 2

Assume that there exist an open subset  $U$  of  $\Omega$  with smooth boundary, compactly contained in  $\Omega$ , and two closed subsets  $B_0 \subset B$  of  $U^m$ , which satisfy conditions (P1) and (P2) stated in Section 1. By property (6.2), for  $\varepsilon$  small enough,  $\tilde{\mathcal{F}}_\varepsilon$  satisfies those conditions too, that is,

$$b_\varepsilon := \sup_{\xi \in B_0} \tilde{\mathcal{F}}_\varepsilon(\xi) < \inf_{\gamma \in \Gamma} \sup_{\xi \in B} \tilde{\mathcal{F}}_\varepsilon(\gamma(\xi)) =: c_\varepsilon,$$

where  $\Gamma := \{\gamma \in \mathcal{C}(B, \bar{U}^m) : \gamma(\xi) = \xi \text{ for every } \xi \in B_0\}$  and, for every  $\xi = (\xi_1, \dots, \xi_m) \in \partial U^m$  with  $\tilde{\mathcal{F}}_\varepsilon(\xi) \in [c_\varepsilon - \alpha, c_\varepsilon + \alpha]$ ,  $\alpha \in (0, c_\varepsilon - b_\varepsilon)$  small enough, one has that

$$\begin{aligned} \nabla_{\xi_i} \tilde{\mathcal{F}}_\varepsilon(\xi) &\neq 0 && \text{if } \xi_i \in U, \\ \nabla_{\xi_i} \tilde{\mathcal{F}}_\varepsilon(\xi) \cdot \tau &\neq 0 && \text{for some } \tau \in T_{\xi_i}(\partial U) \text{ if } \xi_i \in \partial U, \end{aligned}$$

for some  $i \in \{1, \dots, m\}$ . If  $\tilde{\mathcal{F}}_\varepsilon$  has no critical points in  $U^m \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[c_\varepsilon - \alpha, c_\varepsilon + \alpha]$ , then Lemma 6.4 gives a continuous deformation

$$\tilde{\eta} : [0, 1] \times (\bar{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon + \alpha}) \rightarrow \bar{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon + \alpha}$$

of  $\bar{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon + \alpha}$  into  $\bar{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon - \alpha}$  which keeps  $\bar{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon - \alpha}$  fixed. Let  $\gamma \in \Gamma$  be such that  $\tilde{\mathcal{F}}_\varepsilon(\gamma(\xi)) \leq c_\varepsilon + \alpha$  for every  $\xi \in B$ . Since  $b_\varepsilon < c_\varepsilon - \alpha$ , the map  $\tilde{\gamma}(\xi) := \tilde{\eta}(1, \gamma(\xi))$  belongs to  $\Gamma$ . But  $\tilde{\mathcal{F}}_\varepsilon(\tilde{\gamma}(\xi)) \leq c_\varepsilon - \alpha$  for every  $\xi \in B$ , contradicting the definition of  $c_\varepsilon$ . Therefore,  $c_\varepsilon$  is a critical value of  $\tilde{\mathcal{F}}_\varepsilon$ .  $\square$

## Acknowledgement

This work has been partly supported by grants Fondecyt 1040936, Chile, and by grants CONACYT 43724 and PAPIIT IN105106, Mexico.

## References

- [1] Adimurthi, F. Robert, M. Struwe, Concentration phenomena for Liouville's equation in dimension four, *J. Eur. Math. Soc. (JEMS)* 8 (2) (2006) 171–180.
- [2] A. Bahri, J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, *Comm. Pure Appl. Math.* 41 (3) (1988) 253–294.
- [3] A. Bahri, Y.-Y. Li, O. Rey, On a variational problem with lack of compactness: the topological effect of the critical points at infinity, *Calc. Var.* 3 (1) (1995) 67–93.
- [4] S. Baraket, M. Dammak, F. Pacard, T. Ouni, Singular limits for 4-dimensional semilinear elliptic problems with exponential nonlinearity, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (2007) 875–895.
- [5] S. Baraket, F. Pacard, Construction of singular limits for a semilinear elliptic equation in dimension 2, *Calc. Var. Partial Differential Equations* 6 (1) (1998) 1–38.
- [6] D. Bartolucci, G. Tarantello, The Liouville equation with singular data: a concentration-compactness principle via a local representation formula, *J. Differential Equations* 185 (1) (2002) 161–180.
- [7] T.P. Branson, Differential operators canonically associated to a conformal structure, *Math. Scand.* 57 (1985) 293–345.
- [8] H. Brezis, F. Merle, Uniform estimates and blow-up behavior for solutions of  $-\Delta u = V(x)e^u$  in two dimensions, *Comm. Partial Differential Equations* 16 (8–9) (1991) 1223–1253.
- [9] S.Y.A. Chang, On a fourth order operator – the Paneitz operator – in conformal geometry, in: *Proceedings of the Conference for the 70th of A.P. Calderon*, in press.
- [10] S.Y.A. Chang, P.C. Yang, On a fourth order curvature invariant, in: T. Branson (Ed.), *Spectral Problems in Geometry and Arithmetic*, in: *Contemporary Math.*, vol. 237, Amer. Math. Soc., 1999, pp. 9–28.
- [11] C.-C. Chen, C.-S. Lin, Topological degree for a mean field equation on Riemann surfaces, *Comm. Pure Appl. Math.* 56 (12) (2003) 1667–1727.
- [12] M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri–Coron's problem, *Calc. Var. Partial Differential Equations* 16 (2) (2003) 113–145.
- [13] M. del Pino, J. Dolbeault, M. Musso, “Bubble-tower” radial solutions in the slightly supercritical Brezis–Nirenberg problem, *J. Differential Equations* 193 (2) (2003) 280–306.
- [14] M. del Pino, M. Kowalczyk, M. Musso, Singular limits in Liouville-type equations, *Calc. Var. Partial Differential Equations* 24 (2005) 47–81.
- [15] M. del Pino, M. Kowalczyk, M. Musso, Variational reduction for Ginzburg–Landau vortices, *J. Funct. Anal.* 239 (2) (2006) 497–541.
- [16] M. del Pino, C. Muñoz, The two-dimensional Lazer–McKenna conjecture for an exponential nonlinearity, *J. Differential Equations* 231 (2006) 108–134.

- [17] A. Dold, Lectures on Algebraic Topology, Springer-Verlag, Berlin, 1972.
- [18] O. Druet, F. Robert, Bubbling phenomena for fourth-order four-dimensional PDEs with exponential growth, Proc. Amer. Math. Soc. 134 (3) (2006) 897–908.
- [19] Z. Djadli, A. Malchiodi, Existence of conformal metrics with constant  $Q$ -curvature, Ann. Math., in press.
- [20] P. Esposito, A class of Liouville-type equations arising in Chern–Simons vortex theory: asymptotics and construction of blowing-up solutions, Ph.D. Thesis, Università di Roma “Tor Vergata”, 2004.
- [21] P. Esposito, M. Grossi, A. Pistoia, On the existence of blowing-up solutions for a mean field equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2) (2005) 227–257.
- [22] P. Esposito, M. Musso, A. Pistoia, Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent, J. Differential Equations 227 (1) (2006) 29–68.
- [23] J. Kazdan, F. Warner, Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures, Ann. Math. 101 (1975) 317–331.
- [24] Y.-Y. Li, I. Shafrir, Blow-up analysis for solutions of  $-\Delta u = Ve^u$  in dimension two, Indiana Univ. Math. J. 43 (4) (1994) 1255–1270.
- [25] J. Liouville, Sur l’équation aux différences partielles  $\frac{d^2 \log \lambda}{dudv} \pm \frac{\lambda}{2a^2} = 0$ , C. R. Acad. Sci. Paris 36 (1853) 71–72.
- [26] C.S. Lin, J. Wei, Locating the peaks of solutions via the maximum principle. II. A local version of the method of moving planes, Comm. Pure Appl. Math. 56 (6) (2003) 784–809.
- [27] C.S. Lin, J. Wei, Sharp estimates for bubbling solutions of a fourth order mean field equation, Preprint, 2007.
- [28] C.S. Lin, L.P. Wang, J. Wei, Topological degree for solutions of a fourth order mean field equation, Preprint, 2007.
- [29] L. Ma, J. Wei, Convergence for a Liouville equation, Comment. Math. Helv. 76 (3) (2001) 506–514.
- [30] A. Malchiodi, M. Struwe,  $Q$ -curvature flow on  $S^4$ , J. Differential Geom. 73 (1) (2006) 1–44.
- [31] K. Nagasaki, T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities, Asymptotic Anal. 3 (2) (1990) 173–188.
- [32] S. Paneitz, Essential unitarization of symplectics and applications to field quantization, J. Funct. Anal. 48 (1982) 310–359.
- [33] A. Pistoia, O. Rey, Multiplicity of solutions to the supercritical Bahri–Coron’s problem in pierced domains, Adv. Differential Equations 11 (6) (2006) 647–666.
- [34] O. Rey, The role of the Green’s function in a nonlinear elliptic equation involving the critical Sobolev exponent, J. Funct. Anal. 89 (1) (1990) 1–52.
- [35] T. Suzuki, Two-dimensional Emden–Fowler equation with exponential nonlinearity, in: Nonlinear Diffusion Equations and their Equilibrium States, 3, Gregynog, 1989, in: Progr. Nonlinear Differential Equations Appl., vol. 7, Birkhäuser Boston, Boston, MA, 1992.
- [36] G. Tarantello, A quantization property for blow-up solutions of singular Liouville-type equations, J. Funct. Anal. 219 (2) (2005) 368–399.
- [37] G. Tarantello, Analytical aspects of Liouville-type equations with singular sources, in: Stationary Partial Differential Equations, vol. I, in: Handbook of Differential Equations, North-Holland, Amsterdam, 2004, pp. 491–592.
- [38] V.H. Weston, On the asymptotic solution of a partial differential equation with an exponential nonlinearity, SIAM J. Math. Anal. 9 (6) (1978) 1030–1053.