

Hardy inequalities with non-standard remainder terms

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Received 8 October 2006; accepted 7 May 2007

Available online 3 October 2007

Abstract

Improved Hardy inequalities, involving remainder terms, are established both in the classical and in the limiting case. The relevant remainders depend on a suitable distance from the families of the “virtual” extremals.

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Résumé

Nous établissons des inégalités de Hardy améliorées avec reste, à la fois dans le cas classique et dans le cas limite. Ce reste dépend d'une distance à la famille des extremas « virtuels ».

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Keywords: Quantitative Hardy inequalities; Quantitative Hardy–Littlewood inequalities; Symmetrization; Rearrangement invariant spaces

1. Introduction and main results

The plain Hardy inequality asserts that, if $n \geq 2$ and $1 < p < n$, then

$$\left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \leq \int_{\mathbb{R}^n} |\nabla u|^p dx \quad (1.1)$$

for every real-valued weakly differentiable function u in \mathbb{R}^n such that $|\nabla u| \in L^p(\mathbb{R}^n)$ and decaying to zero at infinity, in the sense that

$$|\{|u| > t\}| < \infty \quad \text{for every } t > 0. \quad (1.2)$$

Here, $|G|$ denotes the Lebesgue measure of a set $G \subset \mathbb{R}^n$.

The constant $(\frac{n-p}{p})^p$ is optimal in (1.1), as demonstrated by sequences obtained on truncating functions having the form

$$v_a(x) = a|x|^{\frac{p-n}{p}} \quad \text{for } x \in \mathbb{R}^n, \quad (1.3)$$

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with $a \in \mathbb{R} \setminus \{0\}$, at levels $1/k$ and k , and then letting $k \rightarrow \infty$. However, it is well known that equality is never achieved in (1.1), unless u is identically equal to 0. In fact, the natural candidates v_a to be extremals in (1.1) have a gradient which does not (even locally) belong to $L^p(\mathbb{R}^n)$.

The lack of extremals has recently inspired improved versions of (1.1) and of related inequalities, reminiscent of earlier results dealing with the Sobolev inequality [8,10], where \mathbb{R}^n is replaced by any open bounded subset Ω containing 0, and u is assumed to belong to the Sobolev space $W_0^{1,p}(\Omega)$ of those functions in $W^{1,p}(\Omega)$ vanishing, in the appropriate sense, on $\partial\Omega$. Typically, these improvements of (1.1) amount to extra terms on the left-hand side that either involve integrals of $|u|^p$ with weights depending on $|x|$ which are less singular than $|x|^{-p}$ at 0, or weighted integrals of $|\nabla u|^q$ with $q < p$ (see [2–5,9,18,19,22–24,27,31,33]).

In this paper we establish a strengthened version of (1.1) in the whole of \mathbb{R}^n , with a remainder term having a different nature. Such a remainder depends on a distance of u , in a suitable norm, from the family of those functions which have the form (1.3) and can be regarded as the virtual extremals in (1.1). In particular, our result entails that any extremizing sequence in inequality (1.1) must approach the family (1.3). Let us add that conclusions in a similar spirit are known for classical Sobolev inequalities – see e.g. [7,13,14,16,21]. The striking fact in connection with (1.1) is that a result of this kind can hold even though extremals do not exist.

In order to give a precise statement, we begin by noting that, via a symmetrization argument, inequality (1.1) is easily seen to be equivalent to the Lorentz-norm inequality

$$\omega_n^{1/n} \frac{n-p}{p} \|u\|_{L^{p^*,p}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^p(\mathbb{R}^n)}, \tag{1.4}$$

where $\omega_n = \pi^{n/2} / \Gamma(1 + \frac{n}{2})$, the measure of the unit ball, and $p^* = \frac{np}{n-p}$, the Sobolev conjugate of p . Recall that, given ρ and $\sigma \in [1, \infty]$, and a measurable set Ω , the Lorentz space $L^{\rho,\sigma}(\Omega)$ is the space of those measurable functions u in Ω for which the quantity

$$\|u\|_{L^{\rho,\sigma}(\Omega)} = \|s^{1/\rho-1/\sigma} u^*(s)\|_{L^\sigma(0,|\Omega|)}$$

is finite. Here, u^* denotes the decreasing rearrangement of u . Observe that, since $L^{\rho,\rho}(\Omega) = L^\rho(\Omega)$, and a constant $C = C(\rho, \sigma_1, \sigma_2)$ exists such that

$$\|u\|_{L^{\rho,\sigma_2}(\Omega)} \leq C \|u\|_{L^{\rho,\sigma_1}(\Omega)} \quad \text{if } 1 \leq \sigma_1 \leq \sigma_2 \leq \infty \text{ and } 1 < \rho < \infty, \tag{1.5}$$

inequality (1.4) improves the standard Sobolev inequality where $L^{p^*,p}(\mathbb{R}^n)$ is replaced by $L^{p^*}(\mathbb{R}^n)$ on the left-hand side (and $\omega_n^{1/n} \frac{n-p}{p}$ is replaced by a different constant).

In view of (1.4), the norm $\|\cdot\|_{L^{p^*,p}(\mathbb{R}^n)}$ could be considered the natural one to measure the distance of any u from the family (1.3) in terms of the gap between the two sides of (1.1). Unfortunately, this is not possible, since the functions v_a , whose gradient is not in $L^p(\mathbb{R}^n)$, neither belong to $L^{p^*,p}(\mathbb{R}^n)$. They do not even belong to the larger space $L^{p^*}(\mathbb{R}^n)$, appearing in the usual Sobolev inequality. In fact, the smallest rearrangement invariant space containing v_a is the Marcinkiewicz space $L^{p^*,\infty}(\mathbb{R}^n)$, also called the weak- L^{p^*} space (see e.g. Proposition 2.3, Section 2). Recall that a rearrangement invariant (briefly, r.i.) space $X(\Omega)$ on a measurable set Ω is a Banach function space – in the sense of Luxemburg – endowed with a norm $\|u\|_{X(\Omega)}$ such that

$$\|u\|_{X(\Omega)} = \|v\|_{X(\Omega)} \quad \text{whenever } u^* = v^*$$

(we refer the reader to [6] for more details on r.i. spaces). Thus, the $L^{p^*,\infty}(\mathbb{R}^n)$ norm appearing in the (normalized) distance

$$d_p(u) = \inf_{a \in \mathbb{R}} \frac{\|u - v_a\|_{L^{p^*,\infty}(\mathbb{R}^n)}}{\|u\|_{L^{p^*,p}(\mathbb{R}^n)}}, \quad 1 < p < n, \tag{1.6}$$

which will be employed in our first result, is actually the strongest possible in this setting.

Theorem 1.1. *Let $n \geq 2$ and let $1 < p < n$. Then a constant $C = C(p, n)$ exists such that*

$$\left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx [1 + Cd_p(u)^{2p^*}] \leq \int_{\mathbb{R}^n} |\nabla u|^p dx \tag{1.7}$$

for every real-valued weakly differentiable function u in \mathbb{R}^n decaying to zero at infinity and such that $|\nabla u| \in L^p(\mathbb{R}^n)$.

Of course, Theorem 1.1 continues to hold if \mathbb{R}^n is replaced by any open set Ω , provided that functions $u \in W_0^{1,p}(\Omega)$ are taken into account. Moreover, if $|\Omega| < \infty$ and $1 \leq q < p^*$, the space $L^{p^*,\infty}(\Omega)$ is continuously embedded into $L^q(\Omega)$, and hence an inequality analogous to (1.7), with $\|u - v_a\|_{L^{p^*,\infty}(\mathbb{R}^n)}$ replaced by $\|u - v_a\|_{L^q(\Omega)}$ in the definition of $d_p(u)$, follows from Theorem 1.1. In fact, if $0 \in \Omega$, minor changes in the proof yield a version of inequality (1.7) where the functions v_a , which do not vanish on $\partial\Omega$, are replaced by the functions $\bar{v}_a : \Omega \rightarrow [0, \infty)$ given by

$$\bar{v}_a(x) = a(|x|^{\frac{p-n}{p}} - Q)_+ \quad \text{for } x \in \Omega.$$

Here, subscript “+” stands for positive part, and Q is any positive number such that the support of \bar{v}_a is contained in Ω ; namely $Q > r_{\Omega}^{\frac{p-n}{n}}$, where

$$r_{\Omega} = \sup\{r > 0: B_r(0) \subset \Omega\},$$

and $B_r(x)$ denotes the ball centered at x and having radius r . Precisely, if $|\Omega| < \infty$ and $1 \leq q < p^*$, then on setting

$$d_{p,q}(u) = \inf_{a \in \mathbb{R}} \frac{\|u - \bar{v}_a\|_{L^q(\Omega)}}{\|u\|_{L^{p^*,p}(\Omega)}},$$

a constant $C = C(p, q, n, Q, |\Omega|)$ exists such that

$$\left(\frac{n-p}{p}\right)^p \int_{\Omega} \frac{|u(x)|^p}{|x|^p} dx [1 + Cd_{p,q}(u)^{2p^*}] \leq \int_{\Omega} |\nabla u|^p dx$$

for every $u \in W_0^{1,p}(\Omega)$.

Inequality (1.1) breaks down when $p = n$. In fact, no estimate like (1.1) (with $(\frac{n-p}{p})^p$ replaced by any constant) can hold in this case, since the weight $|x|^{-n}$ is not (even locally) integrable in \mathbb{R}^n . However, an inequality in the same spirit can be restored, provided that $|x|^{-n}$ is replaced by a suitable less singular weight at 0, and \mathbb{R}^n is replaced by any open bounded subset Ω . On defining

$$R_{\Omega} = \sup_{x \in \Omega} |x|, \tag{1.8}$$

the relevant inequality tells us that

$$\left(\frac{n-1}{n}\right)^n \int_{\Omega} \frac{|u(x)|^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \leq \int_{\Omega} |\nabla u|^n dx \tag{1.9}$$

for every $D \geq R_{\Omega}$ and for every function $u \in W_0^{1,n}(\Omega)$. A similar phenomenon as in (1.1) occurs in (1.9), in the sense that the constant $(\frac{n-1}{n})^n$ is the best possible for any bounded Ω containing 0, but it is not attained. Again, the optimality is witnessed by sequences of truncated (at levels k , with $k \rightarrow \infty$) of a suitable family of functions, which in this case have the form

$$w_a(x) = a \left[\left(1 + \log \frac{D}{|x|}\right)^{1/n'} - Q \right]_+ \quad \text{for } x \in \Omega, \tag{1.10}$$

for some $a \in \mathbb{R} \setminus \{0\}$. Here, Q is any positive number fulfilling $Q > (1 + \log \frac{D}{r_{\Omega}})^{1/n'}$, so that the support of w_a is contained in Ω .

Our second result is a counterpart of Theorem 1.1 for inequality (1.9), and tells us that a remainder term can be added to the left-hand side of (1.9), which depends on the deviation of u from the functions given by (1.10). Such a deviation can now be controlled by an exponential estimate. Precisely, recall that, for $D \geq R_{\Omega}$, the expressions

$$\|u\|_{L^{\infty,n}(\log L)^{-1}(\Omega),D} = \left(\int_0^{|\Omega|} \frac{u^*(s)^n}{(n + \log \frac{\omega_n D^n}{s})^n} \frac{ds}{s} \right)^{1/n}$$

define a family of equivalent norms in the Lorentz–Zygmund space $L^{\infty,n}(\log L)^{-1}(\Omega)$, and set, for $C > 0$,

$$d_{C,D,Q}(u) = \inf_{a \in \mathbb{R}} \int_{\Omega} \left(\exp \left(\frac{C|u(x) - w_a(x)|^{n'}}{\|u\|_{L^{\infty,n}(\log L)^{-1}(\Omega),D}^{n'}} \right) - 1 \right) dx. \tag{1.11}$$

Then we have the following

Theorem 1.2. *Let Ω be an open bounded subset of \mathbb{R}^n , $n \geq 2$, containing 0. Let $D > R_\Omega$ and $Q > (1 + \log \frac{D}{r_\Omega})^{1/n'}$. Then a positive constant $C = C(n, R_\Omega, D, Q)$ exists such that*

$$\left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|u(x)|^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx [1 + d_{C,D,Q}(u)^{n^2}] \leq \int_\Omega |\nabla u|^n dx \tag{1.12}$$

for every $u \in W_0^{1,n}(\Omega)$.

A few comments on Theorem 1.2 are in order. The presence of the norm $\|\cdot\|_{L^{\infty,n}(\text{Log } L)^{-1}(\Omega)}$ in the definition of $d_{C,D,Q}(\cdot)$ is related to the fact that, in analogy with (1.1) and (1.4), inequality (1.9) is equivalent to

$$\omega_n^{1/n} (n-1) \|u\|_{L^{\infty,n}(\text{Log } L)^{-1}(\Omega), D} \leq \|\nabla u\|_{L^n(\Omega)} \tag{1.13}$$

for $u \in W_0^{1,n}(\Omega)$. Inequality (1.13) goes back (apart from the constant) to [11,25,27], and has recently been shown to be optimal as far as the norm on the left-hand side is concerned [17,20]. On the other hand, the norm $\|\cdot\|_{L^{\infty,n}(\text{Log } L)^{-1}(\Omega), D}$ cannot be used to measure the distance of u from the family (1.10), since $w_a \notin L^{\infty,n}(\text{Log } L)^{-1}(\Omega)$. The exponential term in (1.11) serves as a replacement for this norm, in the same spirit as $\|\cdot\|_{L^{p^*,\infty}(\mathbb{R}^n)}$ replaces $\|\cdot\|_{L^{p^*,p}(\mathbb{R}^n)}$ in (1.6), and is related to the classical embedding theorem of [29,32,34], which states that

$$\int_\Omega \left(\exp\left(\frac{C|u(x)|^{n'}}{\|\nabla u\|_{L^n(\Omega)}^{n'}}\right) - 1 \right) dx \leq 1 \tag{1.14}$$

for some positive constant $C = C(n, |\Omega|)$ and for every $u \in W_0^{1,n}(\Omega)$. Observe that (1.14) is equivalent to

$$\|u\|_{\text{Exp } L^{n'}(\Omega)} \leq C \|\nabla u\|_{L^n(\Omega)} \tag{1.15}$$

for some positive constant $C = C(n, |\Omega|)$ and for every $u \in W_0^{1,n}(\Omega)$, where

$$\|u\|_{\text{Exp } L^{n'}(\Omega)} = \inf \left\{ \lambda > 0: \int_\Omega \left(\exp\left(\frac{|u(x)|^{n'}}{\lambda^{n'}}\right) - 1 \right) dx \leq 1 \right\},$$

the Luxemburg norm in the Orlicz space associated with the Young function given by $e^{t^{n'}} - 1$ for $t \geq 0$. Recall that a Young function is a convex function from $[0, \infty)$ into $[0, \infty)$ vanishing at 0. Inequalities (1.14) and (1.15) are slightly weaker than (1.13), for

$$L^{\infty,n}(\text{Log } L)^{-1}(\Omega) \subsetneq \text{Exp } L^{n'}(\Omega) \tag{1.16}$$

(with continuous embedding). However, the remainder $d_{C,D,Q}(u)$ appearing in (1.12) is again optimal, in that the function $e^{t^{n'}} - 1$ cannot be replaced by any other Young function growing essentially faster near infinity. Indeed, $\text{Exp } L^{n'}(\Omega)$, unlike $L^{p^*}(\Omega)$, agrees with its corresponding weak space, and is the smallest rearrangement invariant space containing the family (1.10) (Proposition 3.3, Section 3).

2. The case $1 < p < n$

A quite simple proof of inequality (1.1) relies upon symmetrization. Recall that the symmetric rearrangement of a measurable function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, which decays to zero at infinity, is the function $u^\star : \mathbb{R}^n \rightarrow [0, \infty]$ obeying

$$u^\star(x) = u^*(\omega_n |x|^n) \quad \text{for } x \in \mathbb{R}^n, \tag{2.1}$$

where the decreasing rearrangement $u^* : [0, \infty) \rightarrow [0, \infty]$ is given by

$$u^*(s) = \sup \{ t > 0: |\{x \in \mathbb{R}^n: |u(x)| > t\}| > s \} \quad \text{for } s \geq 0. \tag{2.2}$$

When the domain of u is not the whole of \mathbb{R}^n , the function u^\star is defined similarly, after continuing u by 0 outside its domain.

The Hardy–Littlewood inequality [6, Chapter 2, Theorem 2.2] implies that

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \leq \int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx \tag{2.3}$$

for every u as above. On the other hand, the Pólya–Szegő principle asserts that, if u is also weakly differentiable with $|\nabla u| \in L^p(\mathbb{R}^n)$ for some $p \geq 1$, then the same properties are inherited by u^\star , and

$$\int_{\mathbb{R}^n} |\nabla u^\star|^p dx \leq \int_{\mathbb{R}^n} |\nabla u|^p dx \tag{2.4}$$

[12,26,30]. Owing to (2.3) and (2.4), inequality (1.1) is reduced to the well known one-dimensional Hardy inequality

$$\left(\frac{1}{p^\star}\right)^p \int_0^\infty \phi(s)^p s^{-p/n} ds \leq \int_0^\infty (-\phi'(s))^p s^{p/n'} ds, \tag{2.5}$$

for every non-increasing locally absolutely continuous function $\phi : (0, \infty) \rightarrow [0, \infty)$ such that $\lim_{s \rightarrow +\infty} \phi(s) = 0$ (see e.g. [6,28]).

Loosely speaking, our approach to Theorem 1.1 consists in proving the stability of the argument outlined above. To be more specific, we shall establish strengthened versions of inequalities (2.3) and (2.5), containing quantitative information on the gap between their two sides. The former of these quantitative inequalities will enable us to show that, if the difference between the right-hand side and the left-hand side of (1.1) is small, then u is close to u^\star . The latter will be used to prove that, in the same circumstance, u^\star is close to some function having the form (1.3). Inequality (1.7) will then easily follow from these two pieces of information.

The enhanced version of (2.3) is the object of our first lemma.

Lemma 2.1. *Let $n \geq 2$ and let $1 < p < n$. Then a positive constant $C = C(p, n)$ exists such that*

$$\int_{\mathbb{R}^n} \frac{u(x)^p}{|x|^p} dx + C \left(\int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx \right)^{-\frac{n+p}{n-p}} \left(\int_{\mathbb{R}^n} |u(x) - u^\star(x)|^{p^\star} dx \right)^2 \leq \int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx \tag{2.6}$$

for every nonnegative measurable function u in \mathbb{R}^n decaying to zero at infinity and making the right-hand side of (2.6) finite.

Proof. A key tool in our derivation of (2.6) is a Hardy–Littlewood inequality with a remainder term contained in [15, Theorem 1.2 and Remark 1.4]. A special case of that result tells us the following. Let B be either a ball $B_R(0)$ or \mathbb{R}^n . Let $g : B \rightarrow [0, \infty)$ be any radially strictly decreasing function, decaying to zero at infinity if $B = \mathbb{R}^n$, and such that the function $\theta : (0, |B|) \rightarrow [0, \infty)$ given by

$$\theta(s) = \operatorname{ess\,sup}_{r \in (0,s)} \frac{1}{-g^{\star'}(r)} \quad \text{for } s \in (0, |B|), \tag{2.7}$$

is finite, locally absolutely continuous, and fulfills $\lim_{s \rightarrow 0^+} \theta(s) = 0$. Let $f : B \rightarrow [0, \infty)$ be any function, decaying to zero at infinity if $B = \mathbb{R}^n$, and such that the quasi-norm

$$\|f\|_{A^q} = \left(\int_0^{|B|} f^\star(s)^q \theta'(s) ds \right)^{1/q} \tag{2.8}$$

is finite for some $q \in [1, \infty)$. Then

$$\int_B f(x)g(x) dx + \frac{1}{Cq} \|f\|_{A^q}^{-q} \|f - f^\star\|_{L^{\frac{q+1}{2}}(B)}^{q+1} \leq \int_B f^\star(x)g(x) dx, \tag{2.9}$$

for some absolute positive constant C .

An application of (2.9) with $B = \mathbb{R}^n$, $f(x) = u(x)^p$, $g(x) = |x|^{-p}$ and $q = \frac{n+p}{n-p}$ yields

$$\int_{\mathbb{R}^n} \frac{u(x)^p}{|x|^p} dx + C \|u\|_{L^{p^*, \frac{p(n+p)}{n-p}}(\mathbb{R}^n)}^{-\frac{p(n+p)}{n-p}} \|u^p - (u^\star)^p\|_{L^{\frac{2n}{n-p}}(\mathbb{R}^n)}^{\frac{2n}{n-p}} \leq \int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx \tag{2.10}$$

for some positive constant $C = C(p, n)$. Since

$$|s - r|^p \leq |s^p - r^p| \quad \text{for every } r, s \geq 0 \text{ and for every } 1 < p < \infty, \tag{2.11}$$

we have

$$\|u - u^\star\|_{L^{2p^*}(\mathbb{R}^n)}^{2p^*} \leq \|u^p - (u^\star)^p\|_{L^{\frac{2n}{n-p}}(\mathbb{R}^n)}^{\frac{2n}{n-p}}. \tag{2.12}$$

Moreover, by (1.5),

$$\|u\|_{L^{p^*, \frac{p(n+p)}{n-p}}(\mathbb{R}^n)} \leq C \|u\|_{L^{p^*, p}(\mathbb{R}^n)} = C \omega_n^{-1/n} \left(\int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx \right)^{1/p} \tag{2.13}$$

for some positive constant $C = C(p, n)$. Inequality (2.6) follows from (2.10)–(2.13). \square

The next result is concerned with a quantitative version of (2.5). In the statement, we set

$$\delta_p(\phi) = \inf_{a \geq 0} \frac{\|\phi(s) - as^{-1/p^*}\|_{L^{p^*, \infty}(0, \infty)}}{\left(\int_0^\infty \phi(s)^p s^{-p/n} ds\right)^{1/p}}, \quad 1 < p < n, \tag{2.14}$$

for any nonnegative function $\phi \in L^{p^*, p}(0, \infty)$.

Lemma 2.2. *Let $n \geq 2$ and let $1 < p < n$. Set $\gamma = \max\{p^2, 2p\}$. Then there exists a constant $C = C(p, n)$ such that*

$$\left(\frac{1}{p^*}\right)^p \left(\int_0^\infty \phi(s)^p s^{-p/n} ds\right) [1 + C \delta_p(\phi)^\gamma] \leq \int_0^\infty (-\phi'(s))^p s^{p/n'} ds \tag{2.15}$$

for every non-increasing locally absolutely continuous function $\phi : (0, \infty) \rightarrow [0, \infty)$, making the right-hand side of (2.15) finite and such that $\lim_{s \rightarrow +\infty} \phi(s) = 0$.

Proof. Since we are assuming that

$$\int_0^\infty (-\phi'(s))^p s^{p/n'} ds < \infty, \tag{2.16}$$

we have $\int_0^\infty \phi(s)^p s^{-p/n} ds < \infty$, owing to (2.5). Thus, we may suppose, without loss of generality, that

$$\int_0^\infty \phi(s)^p s^{-p/n} ds = 1. \tag{2.17}$$

Define

$$\varepsilon(\phi) = (p^*)^p \int_0^\infty (-\phi'(s))^p s^{p/n'} ds - 1. \tag{2.18}$$

Fix any $R > r > 0$. An integration by parts yields

$$\int_r^R \phi(s)^p s^{-p/n} ds = \frac{n}{n-p} (\phi(R)^p R^{1-p/n} - \phi(r)^p r^{1-p/n}) + p^* \int_r^R (-\phi'(s)) s^{1-p/n} \phi(s)^{p-1} ds. \tag{2.19}$$

Since the integral in (2.17) is convergent, the left-hand side of (2.19) has a finite limit as $r \rightarrow 0^+$ and as $R \rightarrow \infty$. The same property is enjoyed by the integral on the right-hand side of (2.19), as a consequence of Hölder’s inequality and of the convergence of the integrals in (2.16) and (2.17). Thus, $\lim_{r \rightarrow 0^+} \phi(r)r^{1-p/n}$ and $\lim_{R \rightarrow +\infty} \phi(R)^p R^{1-p/n}$ exist and are finite. Again the convergence of the integral in (2.17) entails that both limits must be 0, namely that

$$\lim_{r \rightarrow 0^+} \phi(r)^p r^{1-p/n} = \lim_{R \rightarrow +\infty} \phi(R)R^{1-p/n} = 0. \tag{2.20}$$

In conclusion, from (2.17) and (2.19) we infer that

$$1 = p^* \int_0^\infty (-\phi'(s))s^{1-p/n}\phi(s)^{p-1} ds. \tag{2.21}$$

Now, observe that a positive constant $C = C(p)$ exists such that

$$\frac{a^p}{p} + \frac{b^{p'}}{p'} - ab \geq \begin{cases} C|a - b^{\frac{1}{p-1}}|^p & \text{if } p \geq 2, \\ C \frac{|a - b^{\frac{1}{p-1}}|^2}{\max\{b^{\frac{1}{p-1}}, |a - b^{\frac{1}{p-1}}|\}^{2-p}} & \text{if } 1 < p < 2, \end{cases} \tag{2.22}$$

for $a, b > 0$. Indeed, one has

$$\frac{t^p}{p} + \frac{1}{p'} - t \geq \begin{cases} C|t - 1|^p & \text{if } p \geq 2, \\ C \min\{|t - 1|^2, |t - 1|^p\} & \text{if } 1 < p < 2, \end{cases}$$

for some positive constant $C = C(p)$ and for every $t \geq 0$, whence, (2.22) follows, on taking $t = rs^{\frac{1}{1-p}}$.

Let us distinguish the cases where $2 \leq p < n$ and $1 < p < 2$. If $p \geq 2$, by (2.22) and (2.17) we have

$$\begin{aligned} p^* \int_0^\infty -\phi'(s)s^{1-p/n}\phi(s)^{p-1} ds \\ \leq \frac{(p^*)^p}{p} \int_0^\infty (-\phi'(s))^p s^{p/n'} ds + \frac{1}{p'} - C \int_0^\infty |p^*(-\phi'(s))s^{1/n'} - \phi(s)s^{-1/n}|^p ds, \end{aligned} \tag{2.23}$$

for some positive constant $C = C(p)$. From (2.21), (2.23) and (2.18) we get that

$$\int_0^\infty |p^*(-\phi'(s))s^{1/n'} - \phi(s)s^{-1/n}|^p ds \leq C\epsilon(\phi) \tag{2.24}$$

for some positive constant $C = C(p)$. On defining the function $\psi : (0, \infty) \rightarrow [0, \infty)$ as

$$\psi(s) = s^{1/p^*}\phi(s) \quad \text{for } s > 0, \tag{2.25}$$

inequality (2.24) reads

$$\int_0^\infty |\psi'(s)|^p s^{p-1} ds \leq C\epsilon(\phi) \tag{2.26}$$

for some positive constant $C = C(p, n)$.

Assume now that $1 < p < 2$. An analogous argument as above leads to

$$\int_0^\infty |\psi'(s)|^2 s^{\frac{2}{p}(p-1)} \max\{\phi(s)s^{-1/n}, |p^*(-\phi'(s))s^{1/n'} - \phi(s)s^{-1/n}|\}^{p-2} ds \leq C\epsilon(\phi) \tag{2.27}$$

for some positive constant $C = C(p, n)$. An application of Hölder’s inequality and estimate (2.27) yield

$$\int_0^\infty |\psi'(s)|^p s^{p-1} ds \leq C\epsilon(\phi)^{p/2} \left(\int_0^\infty \max\{\phi(s)s^{-1/n}, |p^*(-\phi'(s))s^{1/n'} - \phi(s)s^{-1/n}|\}^p ds \right)^{(2-p)/2} \tag{2.28}$$

for some positive constant $C = C(p, n)$. Since the maximum in (2.28) does not exceed $p^*(-\phi'(s))s^{1/n'} + \phi(s)s^{-1/n}$, via (2.17) and (2.18) we deduce that

$$\int_0^\infty |\psi'(s)|^p s^{p-1} ds \leq C\epsilon(\phi)^{p/2} \left((p^*)^p \int_0^\infty (-\phi'(s))^p s^{p/n'} ds + 1 \right)^{(2-p)/2} = C\epsilon(\phi)^{p/2} (\epsilon(\phi) + 2)^{(2-p)/2} \tag{2.29}$$

for some positive constant $C = C(p, n)$.

Assume, for a moment, that

$$\epsilon(\phi) \leq 1. \tag{2.30}$$

Then, given any $p \in [1, n)$, either inequality (2.26) or (2.29), according to whether $2 \leq p < n$ or $1 < p < 2$, tells us that

$$\int_0^\infty |\psi'(s)|^p s^{p-1} ds \leq C\epsilon(\phi)^\beta, \tag{2.31}$$

for some positive constant $C = C(p, n)$, where $\beta = \min\{1, \frac{p}{2}\}$. Notice that, in particular, (2.31) recovers the fact that $\epsilon(\phi) > 0$, since, otherwise, (2.25) yields $\phi(s) = Cs^{-1/p^*}$ for some constant C , in contrast with (2.16) and (2.17). Now, recall that $\gamma = \max\{p^2, 2p\}$, and define

$$A = \{s > 0: \psi(s) > \epsilon(\phi)^{1/\gamma}\}. \tag{2.32}$$

Since ψ is a (locally absolutely) continuous function and, by (2.20),

$$\lim_{s \rightarrow 0^+} \psi(s) = \lim_{s \rightarrow +\infty} \psi(s) = 0,$$

the set A is open and bounded. Given any $s \in A$, denote by (a_s, b_s) the connected component of A containing s . One has $0 < a_s < b_s < \infty$, and $\psi(a_s) = \psi(b_s) = \epsilon(\phi)^{1/\gamma}$. Thus, if $s \in A$, the following chain holds

$$\begin{aligned} |\psi(s) - \epsilon(\phi)^{1/\gamma}| &= \left| \int_{a_s}^s \psi'(r) dr \right| \leq \int_{a_s}^s |\psi'(r)| dr \leq \left(\int_0^\infty |\psi'(r)|^p r^{p-1} dr \right)^{1/p} \left(\int_A \frac{dr}{r} \right)^{1/p'} \\ &\leq C\epsilon(\phi)^{\beta/p} \left(\int_A \frac{dr}{r} \right)^{1/p'} \leq C\epsilon(\phi)^{\frac{\beta}{p} - \frac{p-1}{\gamma}} \left(\int_A \psi(r)^p \frac{dr}{r} \right)^{1/p'} \\ &= C\epsilon(\phi)^{1/\gamma} \left(\int_A \phi(r)^p r^{-p/n} dr \right)^{1/p'} \leq C\epsilon(\phi)^{1/\gamma}, \end{aligned} \tag{2.33}$$

where the third inequality holds owing to (2.31), the fourth inequality follows from the very definition of A , and the last inequality is due to (2.17). On the other hand, if $s \in (0, \infty) \setminus A$, then trivially

$$|\psi(s) - \epsilon(\phi)^{1/\gamma}| \leq 2\epsilon(\phi)^{1/\gamma}. \tag{2.34}$$

As a consequence of (2.33) and (2.34), a constant $C = C(p, n)$ exists such that

$$s^{1/p^*} |\phi(s) - \epsilon(\phi)^{1/\gamma} s^{-1/p^*}| \leq C\epsilon(\phi)^{1/\gamma} \quad \text{for } s > 0, \tag{2.35}$$

whence

$$\|\phi(s) - \epsilon(\phi)^{1/\gamma} s^{-1/p^*}\|_{L^{p^*, \infty}(0, \infty)} \leq C\epsilon(\phi)^{1/\gamma}.$$

Thus, under assumption (2.30), we have shown that

$$\inf_{a \geq 0} \|\phi(s) - as^{-1/p^*}\|_{L^{p^*, \infty}(0, \infty)} \leq C\epsilon(\phi)^{1/\gamma} \tag{2.36}$$

for some constant $C = C(p, n)$. Inequality (2.36) continues to hold even if (2.30) is dropped. Indeed, if $\epsilon(\phi) > 1$, then a constant $C = C(p, n)$ exists such that

$$\epsilon(\phi)^{1/\gamma} > 1 = \left(\int_0^\infty \phi(s)^p s^{-p/n} ds \right)^{1/p} \geq C \|\phi\|_{L^{p^*, \infty}(0, \infty)} \geq C \inf_{a \geq 0} \|\phi(s) - as^{-1/p^*}\|_{L^{p^*, \infty}(0, \infty)},$$

where the second inequality holds thanks to (1.5). Inequality (2.15) follows from (2.36). \square

We are now in position to accomplish the proof of Theorem 1.1.

Proof of Theorem 1.1. Assume, for the time being, that

$$u \geq 0 \tag{2.37}$$

and that

$$\int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx = 1. \tag{2.38}$$

Define

$$E(u) = \left(\frac{p}{n-p} \right)^p \int_{\mathbb{R}^n} |\nabla u|^p dx - \int_{\mathbb{R}^n} \frac{u(x)^p}{|x|^p} dx.$$

The quantity $E(u)$ can be rewritten as

$$\begin{aligned} E(u) &= \left(\frac{p}{n-p} \right)^p \left(\int_{\mathbb{R}^n} |\nabla u|^p dx - \int_{\mathbb{R}^n} |\nabla u^\star|^p dx \right) \\ &\quad + \left(\left(\frac{p}{n-p} \right)^p \int_{\mathbb{R}^n} |\nabla u^\star|^p dx - 1 \right) + \left(1 - \int_{\mathbb{R}^n} \frac{u(x)^p}{|x|^p} dx \right). \end{aligned} \tag{2.39}$$

Moreover, by (2.4), $\int_{\mathbb{R}^n} |\nabla u|^p dx - \int_{\mathbb{R}^n} |\nabla u^\star|^p dx \geq 0$. Thus, by (1.1) applied to u^\star and (2.38),

$$\begin{aligned} 0 &< \omega_n^{p/n} \left((p^*)^p \int_0^\infty (-u^*(s))^p s^{p/n'} ds - \int_0^\infty u^*(s)^p s^{-p/n} ds \right) \\ &= \left(\frac{p}{n-p} \right)^p \int_{\mathbb{R}^n} |\nabla u^\star|^p dx - 1 \leq E(u) \end{aligned} \tag{2.40}$$

and, by (2.3) and (2.38),

$$0 \leq 1 - \int_{\mathbb{R}^n} \frac{u(x)^p}{|x|^p} dx \leq E(u). \tag{2.41}$$

From (2.40), (2.38) and Lemma 2.2 we deduce that

$$\inf_{a \geq 0} \|u^*(s) - as^{-1/p^*}\|_{L^{p^*, \infty}(0, \infty)} \leq CE(u)^{1/\gamma} \tag{2.42}$$

for some constant $C = C(p, n)$, where $\gamma = \max\{p^2, 2p\}$. On the other hand, by (2.41), (2.38) and Lemma 2.1, we have

$$\left(\int_{\mathbb{R}^n} |u(x) - u^\star(x)|^{p^*} dx \right)^2 \leq CE(u) \tag{2.43}$$

for some constant $C = C(p, n)$. Inequalities (2.43) and (1.5) entail that

$$\|u - u^\star\|_{L^{p^*,\infty}(\mathbb{R}^n)} \leq C E(u)^{1/2p^*} \tag{2.44}$$

for some constant $C = C(p, n)$. Owing to (2.42) and (2.44) one has

$$\begin{aligned} \inf_{a \geq 0} \|u - v_a\|_{L^{p^*,\infty}(\mathbb{R}^n)} &\leq \|u - u^\star\|_{L^{p^*,\infty}(\mathbb{R}^n)} + \inf_{a \geq 0} \|u^\star - v_a\|_{L^{p^*,\infty}(\mathbb{R}^n)} \\ &= \|u - u^\star\|_{L^{p^*,\infty}(\mathbb{R}^n)} + \inf_{a \geq 0} \|u^\star(s) - as^{-1/p^*}\|_{L^{p^*,\infty}(0,\infty)} \\ &\leq C(E(u)^{1/2p^*} + E(u)^{1/\gamma}) \end{aligned} \tag{2.45}$$

for some constant $C = C(p, n)$. If $E(u) \leq 1$, inequality (2.45) yields

$$\inf_{a \geq 0} \|u - v_a\|_{L^{p^*,\infty}(\mathbb{R}^n)} \leq C E(u)^{1/2p^*} \tag{2.46}$$

for some constant $C = C(p, n)$, since $\gamma \leq 2p^*$. Inequality (2.46) holds also if $E(u) > 1$, since, in this case,

$$\begin{aligned} E(u)^{\frac{1}{2p^*}} > 1 &= \left(\int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx \right)^{1/p} = \omega_n^{1/n} \|u\|_{L^{p^*,p}(\mathbb{R}^n)} \\ &\geq C \|u\|_{L^{p^*,\infty}(\mathbb{R}^n)} \geq C \inf_{a \geq 0} \|u - v_a\|_{L^{p^*,\infty}(\mathbb{R}^n)} \end{aligned} \tag{2.47}$$

for some positive constant $C = C(p, n)$. Inequality (2.46) tells us that

$$\left(\frac{n-p}{p} \right)^p \int_{\mathbb{R}^n} \frac{u(x)^p}{|x|^p} dx + C \inf_{a \geq 0} \|u - v_a\|_{L^{p^*,\infty}(\mathbb{R}^n)}^{2p^*} \leq \int_{\mathbb{R}^n} |\nabla u|^p dx, \tag{2.48}$$

for some positive constant $C = C(p, n)$, under assumptions (2.37) and (2.38). Replacing u by $u(\int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx)^{-1/p}$ (a function fulfilling (2.38)) in (2.48) yields

$$\inf_{a \geq 0} \|u - v_a\|_{L^{p^*,\infty}(\mathbb{R}^n)}^{2p^*} \leq C \left(\int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx \right)^{\frac{2p^*}{p}-1} \left(\int_{\mathbb{R}^n} |\nabla u|^p dx - \left(\frac{n-p}{p} \right)^p \int_{\mathbb{R}^n} \frac{u(x)^p}{|x|^p} dx \right) \tag{2.49}$$

for some positive constant $C = C(p, n)$, and for every u fulfilling the sole additional sign assumption (2.37). Now, given any u as in the statement, define $u_+ = \frac{|u|+u}{2}$ and $u_- = \frac{|u|-u}{2}$, the positive and the negative parts of u , respectively, so that $u = u_+ - u_-$. Then

$$\begin{aligned} \inf_{a \in \mathbb{R}} \|u - v_a\|_{L^{p^*,\infty}(\mathbb{R}^n)} &= \inf_{b,c \geq 0} \|u_+ - u_- - v_{b-c}\|_{L^{p^*,\infty}(\mathbb{R}^n)} \\ &\leq \inf_{b \geq 0} \|u_+ - v_b\|_{L^{p^*,\infty}(\mathbb{R}^n)} + \inf_{c \geq 0} \|u_- - v_c\|_{L^{p^*,\infty}(\mathbb{R}^n)}. \end{aligned} \tag{2.50}$$

Owing to (2.49) applied to the nonnegative functions u_+ and u_- , one has

$$\begin{aligned} &\sum_{\pm} \inf_{a \geq 0} \|u_{\pm} - v_a\|_{L^{p^*,\infty}(\mathbb{R}^n)} \\ &\leq C \sum_{\pm} \left(\int_{\mathbb{R}^n} \frac{u_{\pm}^\star(x)^p}{|x|^p} dx \right)^{\frac{1}{p} - \frac{1}{2p^*}} \left(\int_{\mathbb{R}^n} |\nabla u_{\pm}|^p dx - \left(\frac{n-p}{p} \right)^p \int_{\mathbb{R}^n} \frac{u_{\pm}(x)^p}{|x|^p} dx \right)^{\frac{1}{2p^*}} \\ &\leq C \left(\int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx \right)^{\frac{1}{p} - \frac{1}{2p^*}} \sum_{\pm} \left(\int_{\mathbb{R}^n} |\nabla u_{\pm}|^p dx - \left(\frac{n-p}{p} \right)^p \int_{\mathbb{R}^n} \frac{u_{\pm}(x)^p}{|x|^p} dx \right)^{\frac{1}{2p^*}} \\ &\leq C \left(\int_{\mathbb{R}^n} \frac{u^\star(x)^p}{|x|^p} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \right)^{-\frac{1}{2p^*}} \sum_{\pm} \left(\int_{\mathbb{R}^n} |\nabla u_{\pm}|^p dx - \left(\frac{n-p}{p} \right)^p \int_{\mathbb{R}^n} \frac{u_{\pm}(x)^p}{|x|^p} dx \right)^{\frac{1}{2p^*}} \end{aligned}$$

$$\leq C' \|u\|_{L^{p^*,p}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \right)^{-\frac{1}{2p^*}} \left(\int_{\mathbb{R}^n} |\nabla u|^p dx - \left(\frac{n-p}{p} \right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \right)^{\frac{1}{2p^*}} \tag{2.51}$$

for some positive constants $C = C(p, n)$ and $C' = C'(p, n)$. Notice that the second inequality holds owing the fact that $\frac{1}{p} - \frac{1}{2p^*} > 0$, and the third one is a consequence of (2.3). Inequality (1.7) follows from (2.50) and (2.51). \square

We conclude this section by demonstrating the sharpness of the $L^{p^*,\infty}$ norm in the definition of $d_p(\cdot)$ in Theorem 1.1.

Proposition 2.3. *Let $n \geq 2$ and let $1 < p < n$. Then $L^{p^*,\infty}(\mathbb{R}^n)$ is the smallest rearrangement invariant space containing the functions v_a given by (1.3).*

Proof. The function v_1 (and hence v_a for every $a \in \mathbb{R}$) is easily seen to belong to $L^{p^*,\infty}(\mathbb{R}^n)$. Now, let $X(\mathbb{R}^n)$ be any r.i. space such that $v_1 \in X(\mathbb{R}^n)$. We have to show that

$$L^{p^*,\infty}(\mathbb{R}^n) \subset X(\mathbb{R}^n). \tag{2.52}$$

Assume that $u \in L^{p^*,\infty}(\mathbb{R}^n)$. Then

$$u^*(s) \leq \|u\|_{L^{p^*,\infty}(\mathbb{R}^n)} s^{-\frac{1}{p^*}} = C \|u\|_{L^{p^*,\infty}(\mathbb{R}^n)} v_1^*(s) \quad \text{for } s > 0, \tag{2.53}$$

for some positive constant $C = C(p, n)$. Since we are assuming that $v_1 \in X(\mathbb{R}^n)$, inequality (2.53), via a basic property of r.i. spaces [6, Cor. 4.7, Chap. 2], entails that $u \in X(\mathbb{R}^n)$ as well. Inclusion (2.52) follows. \square

3. The case $p = n$

The outline of the proof of Theorem 1.2 is analogous to that of Theorem 1.1, and relies upon Lemmas 3.1 and 3.2 below, which replace Lemmas 2.1 and 2.2, respectively. We shall limit ourselves to establishing these new lemmas, and to sketching the derivation of Theorem 1.2 from these lemmas. The first lemma provides us with a quantitative version of the inequality

$$\int_{B_R(0)} \frac{|u(x)|^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \leq \int_{B_R(0)} \frac{u^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx, \tag{3.1}$$

which holds for every $u \in L^{\infty,n}(\log L)^{-1}(B_R(0))$, provided that $D > 0$, and follows from the Hardy–Littlewood inequality.

Lemma 3.1. *Let $n \geq 2$ and let $0 < R < D$. Then a constant $C = C(n, R, D)$ exists such that*

$$\begin{aligned} & \int_{B_R(0)} \frac{u(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \\ & + \int_{B_R(0)} \frac{u^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \left[\int_{B_R(0)} \left(\exp \left(\frac{C|u^\star - u|^{n'}}{\left(\int_{B_R(0)} \frac{u^\star(y)^n}{|y|^n (1 + \log \frac{D}{|y|})^n} dy \right)^{1/(n-1)}} \right) - 1 \right) dx \right]^{2(n-1)} \\ & \leq \int_{B_R(0)} \frac{u^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \end{aligned} \tag{3.2}$$

for every nonnegative measurable function u in $B_R(0)$ making the right-hand side of (3.2) finite.

Proof. The idea is to reduce (3.2) to a family of estimates relying upon (2.9), via an extrapolation argument. Indeed, defined $\Psi : [0, \infty) \rightarrow [0, \infty)$ as

$$\Psi(t) = e^{t^{n'}} - 1 \quad \text{for } t \geq 0, \tag{3.3}$$

we shall derive (3.2) from an application of (2.9), with $B = B_R(0)$ and $f, g : B_R(0) \rightarrow [0, \infty)$ given by

$$f(x) = u(x)^n \quad \text{for } x \in B_R(0) \tag{3.4}$$

and

$$g(x) = \frac{1}{|x|^n (1 + \log \frac{D}{|x|})^n} \quad \text{for } x \in B_R(0), \tag{3.5}$$

to each term of a power series expansion of the function Ψ .

We may clearly assume, without loss of generality, that

$$\int_{B_R(0)} \frac{u^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx = 1. \tag{3.6}$$

Set,

$$\varepsilon(u) = 1 - \int_{B_R(0)} \frac{u(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx. \tag{3.7}$$

It is easily verified that the function g given by (3.5) is radially decreasing, and that

$$g^\star(s) = \frac{n^n \omega_n}{s} \left(n + \log \frac{\omega_n D^n}{s} \right)^{-n} \quad \text{for } s \in (0, \omega_n R^n).$$

Moreover,

$$\begin{aligned} \left(-\frac{1}{g^{\star\prime}(s)} \right)' &= \frac{1}{n^n \omega_n} s \left(n + \log \frac{\omega_n D^n}{s} \right)^n \log^{-2} \left(\frac{\omega_n D^n}{s} \right) \\ &\quad \times \left[2 \log^2 \left(\frac{\omega_n D^n}{s} \right) + n \log \left(\frac{\omega_n D^n}{s} \right) + n \right] \quad \text{for } s \in (0, \omega_n R^n). \end{aligned} \tag{3.8}$$

Thus, the function $-\frac{1}{g^{\star\prime}}$ is increasing in $(0, \omega_n R^n)$, and hence agrees with the function θ defined as in (2.7). Consequently, $\lim_{s \rightarrow 0^+} \theta(s) = 0$, and there exists a constant $C = C(n, R, D)$ such that

$$\theta'(s) \leq C s \left(n + \log \frac{\omega_n D^n}{s} \right)^n \quad \text{for } s \in (0, \omega_n R^n). \tag{3.9}$$

Therefore, if $q \geq 1$ and $\|\cdot\|_{A^q}$ is defined as in (2.8), one has by (3.4) and (3.9)

$$\|f\|_{A^q}^q = \|u^n\|_{A^q}^q \leq C \int_0^{\omega_n R^n} u^\star(s)^{nq} s \left(n + \log \frac{\omega_n D^n}{s} \right)^n ds \tag{3.10}$$

for some constant $C = C(n, R, D)$. On the other hand,

$$\begin{aligned} \frac{1}{n \omega_n^{1/n}} &= \left(\int_0^{\omega_n R^n} \frac{u^\star(r)^n}{(n + \log \frac{\omega_n D^n}{r})^n r} dr \right)^{1/n} \geq u^\star(s) \left(\int_0^s \frac{dr}{(n + \log \frac{\omega_n D^n}{r})^n r} \right)^{1/n} \\ &= \frac{u^\star(s)}{(n-1)^{1/n}} \left(n + \log \frac{\omega_n D^n}{s} \right)^{-1/n'} \quad \text{for } s \in (0, \omega_n R^n), \end{aligned} \tag{3.11}$$

where the first equality holds owing to (3.6). Combining (3.10) and (3.11) yields

$$\|u^n\|_{A^q}^q \leq C \int_0^{\omega_n R^n} s \left(n + \log \frac{\omega_n D^n}{s} \right)^{q(n-1)+n} ds \tag{3.12}$$

for some constant $C = C(n, R, D)$. Given any integer $k \geq n - 1$, choose $q = \frac{2k}{n-1} - 1$. From (3.12), via a change of variable in the integral on the right-hand side, one can easily infer that

$$\|u^n\|_{\Lambda^{\frac{2k}{n-1}-1}}^{\frac{2k}{n-1}-1} \leq \frac{C}{4^k} \int_0^\infty e^{-r} r^{2k+1} dr = \frac{C}{4^k} (2k + 1)! \tag{3.13}$$

for some constant $C = C(n, R, D)$. Hence, via (2.9) and (2.11) with $p = n$, we get

$$\int_{B_R(0)} |u^\star(x) - u(x)|^{n'k} dx \leq C \frac{\sqrt{k}\sqrt{(2k+1)!}}{2^k} \varepsilon(u)^{1/2} \tag{3.14}$$

for some constant $C = C(n, R, D)$ and for $k \geq n - 1$. On the other hand, if $1 \leq k < n - 1$, Hölder’s inequality and inequality (3.14), with $k = n - 1$, yield

$$\int_{B_R(0)} |u^\star(x) - u(x)|^{n'k} dx \leq C \varepsilon(u)^{\frac{k}{2(n-1)}} \tag{3.15}$$

for some constant $C = C(n, R, D)$. Thus, under the additional assumption that

$$\varepsilon(u) \leq 1, \tag{3.16}$$

inequalities (3.14) and (3.15) tell us that

$$\int_{B_R(0)} |u^\star(x) - u(x)|^{n'k} dx \leq C \frac{\sqrt{k}\sqrt{(2k+1)!}}{2^k} \varepsilon(u)^{\frac{1}{2(n-1)}} \quad \text{for every } k \geq 1, \tag{3.17}$$

and for some constant $C = C(n, R, D)$. Owing to (3.17), given $\lambda > 0$, one has

$$\begin{aligned} \int_{B_R(0)} \Psi\left(\frac{|u^\star(x) - u(x)|}{\lambda}\right) dx &= \sum_{k=1}^\infty \frac{\lambda^{-n'k}}{k!} \int_{B_R(0)} |u^\star(x) - u(x)|^{n'k} dx \\ &\leq C \varepsilon(u)^{\frac{1}{2(n-1)}} \sum_{k=1}^\infty \lambda^{-n'k} \frac{\sqrt{k}\sqrt{(2k+1)!}}{2^k k!} \leq C' \varepsilon(u)^{\frac{1}{2(n-1)}} \sum_{k=1}^\infty \lambda^{-n'k} k^{3/4} \end{aligned} \tag{3.18}$$

for some constants $C = C(n, R, D)$ and $C' = C'(n, R, D)$. Note that we have made use of Stirling’s formula in the last inequality. Since the series on the rightmost side of (3.18) converges provided that $\lambda > 1$, and since

$$\Psi\left(\frac{s}{M}\right) \leq \frac{1}{M} \Psi(s) \quad \text{for every } s > 0 \text{ and } M \geq 1, \tag{3.19}$$

we deduce that, under assumption (3.16),

$$\int_{B_R(0)} \Psi(C|u^\star(x) - u(x)|) dx \leq \varepsilon(u)^{\frac{1}{2(n-1)}} \tag{3.20}$$

for some constant $C = C(n, R, D)$. When (3.16) is not in force, namely if $\varepsilon(u) > 1$, one has

$$\varepsilon(u)^{\frac{1}{2(n-1)}} > 1 \geq \int_{B_R(0)} \Psi\left(\frac{u(x)}{\|u\|_{\text{Exp } L^{n'}(\Omega)}}\right) dx \geq \int_{B_R(0)} \Psi(Cu(x)) dx \tag{3.21}$$

for some constant $C = C(n, R, D)$, where the second inequality is due to the very definition of the norm $\|\cdot\|_{\text{Exp } L^{n'}(B_R(0))}$, whereas the last one is a consequence of (1.16) and of (3.6). Inasmuch as (3.21) holds also with u replaced by u^\star ,

$$\int_{B_R(0)} \Psi\left(\frac{C}{2}|u^\star(x) - u(x)|\right) dx \leq \frac{1}{2} \int_{B_R(0)} \Psi(Cu^\star(x)) dx + \frac{1}{2} \int_{B_R(0)} \Psi(Cu(x)) dx < \varepsilon(u)^{\frac{1}{2(n-1)}}.$$

Thus, inequality (3.20), with a suitable constant $C = C(n, R, D)$, holds for every u as in the statement satisfying (3.6). Inequality (3.2) is a straightforward consequence of (3.20). \square

A limiting case of (2.5) tells us that, given any $K \geq 0$ and $L > 0$,

$$\left(\frac{n-1}{n}\right)^n \left(\int_0^L \phi(s)^n \left(K + \log \frac{L}{s}\right)^{-n} \frac{ds}{s}\right) \leq \int_0^L (-\phi'(s))^n s^{n-1} ds \tag{3.22}$$

for every non-increasing, locally absolutely continuous function $\phi : (0, L] \rightarrow [0, \infty)$ such that $\phi(L) = 0$. (In fact, (3.22) can be derived from (2.5), via a change of variable.)

Fixed any $L \geq l > 0$ and $K \geq 0$, set

$$\delta_{n,C}(\phi) = \inf_{a \geq 0} \int_0^L \left(\exp\left(\frac{C|\phi(s) - a[(K + \log \frac{L}{s})^{1/n'} - (K + \log \frac{l}{s})^{1/n'}] + |n'|}{(\int_0^L \phi(s)^n (K + \log \frac{L}{s})^{-n} \frac{ds}{s})^{1/(n-1)}}\right) - 1 \right) ds \tag{3.23}$$

for $C > 0$ and for any nonnegative function $\phi \in L^{\infty,n}(\text{Log } L)^{-1}(0, L)$. Then we have the following quantitative version of (3.22).

Lemma 3.2. *Let $n \geq 2$, let $L \geq l > 0$ and let $K \geq 0$. Then there exists a constant $C = C(n, l, L, K)$ such that*

$$\left(\frac{n-1}{n}\right)^n \left(\int_0^L \phi(s)^n \left(K + \log \frac{L}{s}\right)^{-n} \frac{ds}{s}\right) [1 + \delta_{n,C}(\phi)^{n^2}] \leq \int_0^L (-\phi'(s))^n s^{n-1} ds \tag{3.24}$$

for every non-increasing, locally absolutely continuous function $\phi : (0, L] \rightarrow [0, \infty)$, making the right-hand side of (3.24) finite and such that $\phi(L) = 0$.

Proof. Assume, without loss of generality, that

$$\int_0^L \phi(s)^n \left(K + \log \frac{L}{s}\right)^{-n} \frac{ds}{s} = 1, \tag{3.25}$$

and define

$$\eta(\phi) = (n')^n \int_0^L (-\phi'(s))^n s^{n-1} ds - 1. \tag{3.26}$$

An integration by parts (whose details can be justified as in the proof of Lemma 2.2) and the use of (2.22) yield

$$\begin{aligned} 1 &= \int_0^L \phi(s)^n \left(K + \log \frac{L}{s}\right)^{-n} \frac{ds}{s} = n' \int_0^L -\phi'(s) \phi(s)^{n-1} \left(K + \log \frac{L}{s}\right)^{-n+1} ds \\ &\leq \frac{(n')^n}{n} \int_0^L (-\phi'(s))^n s^{n-1} ds + \frac{1}{n'} - C \int_0^L \left| n'(-\phi'(s))s^{1/n'} - \phi(s)s^{-1/n} \left(K + \log \frac{L}{s}\right)^{-1} \right|^n ds \end{aligned} \tag{3.27}$$

for some positive constant $C = C(n)$. Hence, upon defining $\psi : (0, L] \rightarrow [0, \infty)$ as

$$\psi(s) = \phi(s) \left(K + \log \frac{L}{s}\right)^{-1/n'} \quad \text{for } s \in (0, L],$$

one gets

$$\int_0^L |\psi'(s)|^n \left(K + \log \frac{L}{s}\right)^{n-1} s^{n-1} ds \leq C \eta(\phi) \tag{3.28}$$

and for some constant $C = C(n)$. An argument analogous to that in the proof of (2.35), starting from (3.28) instead of (2.31), leads to

$$\left| \phi(s) - \eta(\phi)^{1/n^2} \left(K + \log \frac{L}{s} \right)^{1/n'} \right| \leq C \eta(\phi)^{1/n^2} \left(K + \log \frac{L}{s} \right)^{1/n'} \quad \text{for } s \in (0, L], \tag{3.29}$$

and for some constant $C = C(n)$. Thus,

$$\begin{aligned} & \left| \phi(s) - \eta(\phi)^{1/n^2} \left[\left(K + \log \frac{L}{s} \right)^{1/n'} - \left(K + \log \frac{L}{l} \right)^{1/n'} \right]_+ \right| \\ & \leq \left| \phi(s) - \eta(\phi)^{1/n^2} \left(K + \log \frac{L}{s} \right)^{1/n'} \right| \\ & \quad + \left| \eta(\phi)^{1/n^2} \left(K + \log \frac{L}{s} \right)^{1/n'} - \eta(\phi)^{1/n^2} \left[\left(K + \log \frac{L}{s} \right)^{1/n'} - \left(K + \log \frac{L}{l} \right)^{1/n'} \right] \chi_{(0,l)}(s) \right| \\ & \leq C \eta(\phi)^{1/n^2} \left(K + \log \frac{L}{s} \right)^{1/n'} + \eta(\phi)^{1/n^2} \left[\left(K + \log \frac{L}{l} \right)^{1/n'} + \left(K + \log \frac{L}{s} \right)^{1/n'} \chi_{(l,L)}(s) \right] \\ & \leq C \eta(\phi)^{1/n^2} \left(K + \log \frac{L}{s} \right)^{1/n'} + 2\eta(\phi)^{1/n^2} \left(K + \log \frac{L}{l} \right)^{1/n'} \quad \text{for } s \in (0, L], \end{aligned} \tag{3.30}$$

where C is the constant appearing in (3.29) and χ_I stands for the characteristic function of the set I . When $\eta(\phi) \leq 1$, inequality (3.30) and the convexity of Ψ entail that

$$\begin{aligned} & \int_0^L \Psi \left(\frac{|\phi(s) - \eta(\phi)^{1/n^2} [(K + \log \frac{L}{s})^{1/n'} - (K + \log \frac{L}{l})^{1/n'}]_+}{4C} \right) ds \\ & \leq \frac{1}{2} \int_0^L \Psi \left(\frac{\eta(\phi)^{1/n^2}}{2} \left(K + \log \frac{L}{s} \right)^{1/n'} \right) ds + \frac{1}{2} \int_0^L \Psi \left(\frac{\eta(\phi)^{1/n^2}}{C} \left(K + \log \frac{L}{l} \right)^{1/n'} \right) ds \\ & \leq \frac{\eta(\phi)^{1/n^2}}{2} \int_0^L \Psi \left(\frac{1}{2} \left(K + \log \frac{L}{s} \right)^{1/n'} \right) ds + \frac{\eta(\phi)^{1/n^2}}{2} C' \\ & = \frac{\eta(\phi)^{1/n^2}}{2} \int_0^L \left(\exp \left(\frac{K}{2^{n'}} \right) \left(\frac{L}{s} \right)^{\frac{1}{2^{n'}}} - 1 \right) ds + \frac{\eta(\phi)^{1/n^2}}{2} C' = \eta(\phi)^{1/n^2} C'' \end{aligned} \tag{3.31}$$

for some constants $C' = C'(n, l, L, K)$ and $C'' = C''(n, l, L, K)$. Notice that in the second inequality we have exploited (3.19). If $\eta(\phi) > 1$, then

$$\begin{aligned} \eta(\phi)^{1/n^2} > 1 &= \int_0^L \Psi \left(\frac{\phi(s)}{\|\phi\|_{\text{Exp } L^{n'}(0,L)}} \right) ds \geq \int_0^L \Psi(C\phi(s)) ds \\ &\geq \inf_{a \geq 0} \int_0^L \Psi \left(C \left| \phi(s) - a \left[\left(K + \log \frac{L}{s} \right)^{1/n'} - \left(K + \log \frac{L}{l} \right)^{1/n'} \right]_+ \right| \right) ds \end{aligned} \tag{3.32}$$

for some constant $C = C(n, L, K)$, where the second inequality holds owing to (1.16) and to (3.25). Estimate (3.24) follows from (3.31) and (3.32), via (3.19). \square

Proof of Theorem 1.2, sketched. Assume, for the time being, that

$$u \geq 0 \tag{3.33}$$

and

$$\int_{B_R(0)} \frac{u^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx = 1. \tag{3.34}$$

Set

$$F(u) = \left(\frac{n}{n-1}\right)^n \int_{B_R(0)} |\nabla u|^n dx - \int_{B_R(0)} \frac{u(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx,$$

where u is continued by 0 outside Ω . Similarly to (2.40) and (2.41), we get

$$0 < n^n \omega_n \left(\left(\frac{n}{n-1}\right)^n \int_0^{|\Omega|} (-u^{\star\prime}(s) s^{1/n'})^n ds - \int_0^{|\Omega|} \frac{u^\star(s)^n}{s(n + \log \frac{\omega_n D}{s})^n} ds \right) \leq F(u) \tag{3.35}$$

and

$$0 \leq 1 - \int_{B_R(0)} \frac{u(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \leq F(u). \tag{3.36}$$

From (3.35), via Lemma 3.2, (3.34) and (3.19), one can infer that

$$\inf_{a \geq 0} \int_{B_R(0)} \Psi(C|u^\star(x) - w_a(x)|) dx \leq F(u)^{1/n^2} \tag{3.37}$$

for some positive constant $C = C(n, R_\Omega, D, Q)$. Moreover, from (3.36), via Lemma 3.1 and (3.34), we get that

$$\int_{B_R(0)} \Psi(C|u(x) - u^\star(x)|) dx \leq F(u)^{(1/2)(n-1)} \tag{3.38}$$

for some positive constant $C = C(n, R_\Omega, D)$. On making use of the convexity of Ψ and distinguishing the cases where $F(u) \leq 1$ and $F(u) > 1$, one can deduce from (3.37) and (3.38) that a constant $C = C(n, R_\Omega, D, Q)$ exists such that

$$\inf_{a \geq 0} \int_{\Omega} \Psi(C|u(x) - w_a(x)|) dx \leq F(u)^{1/n^2} \tag{3.39}$$

for every $u \in W_0^{1,n}(\Omega)$ fulfilling (3.33) and (3.34). Replacing u by $u(\int_{B_R(0)} \frac{u^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx)^{-1/n}$ in (3.39) yields

$$\begin{aligned} & \int_{B_R(0)} \frac{u^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \inf_{a \geq 0} \left[\int_{B_R(0)} \Psi \left(\frac{C|u(x) - w_a(x)|}{\left(\int_{B_R(0)} \frac{u^\star(y)^n}{|y|^n (1 + \log \frac{D}{|y|})^n} dy\right)^{1/n}} \right) dx \right]^{n^2} \\ & \leq \left(\frac{n}{n-1}\right)^n \int_{B_R(0)} |\nabla u|^n dx - \int_{B_R(0)} \frac{u(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \end{aligned} \tag{3.40}$$

for every $u \in W_0^{1,n}(\Omega)$ fulfilling (3.34). The following chain holds for every $u \in W_0^{1,n}(\Omega)$:

$$\begin{aligned} & \int_{B_R(0)} \frac{|u(x)|^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \inf_{a \geq 0} \left[\int_{B_R(0)} \Psi \left(\frac{C|u(x) - w_a(x)|}{2 \left(\int_{B_R(0)} \frac{u^\star(y)^n}{|y|^n (1 + \log \frac{D}{|y|})^n} dy\right)^{1/n}} \right) dx \right]^{n^2} \\ & \leq \int_{B_R(0)} \frac{|u(x)|^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \left[\frac{1}{2} \sum_{\pm} \inf_{a \geq 0} \left(\int_{B_R(0)} \Psi \left(\frac{C|u_{\pm}(x) - w_a(x)|}{\left(\int_{B_R(0)} \frac{u^\star(y)^n}{|y|^n (1 + \log \frac{D}{|y|})^n} dy\right)^{1/n}} \right) dx \right) \right]^{n^2} \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{2} \int_{B_R(0)} \frac{u^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \sum_{\pm} \inf_{a \geq 0} \left(\int_{B_R(0)} \Psi \left(\frac{C|u_{\pm}(x) - w_a(x)|}{(\int_{B_R(0)} \frac{u^\star(y)^n}{|y|^n (1 + \log \frac{D}{|y|})^n} dy)^{1/n}} \right) dx \right)^{n^2} \\
 &= \frac{1}{2} \sum_{\pm} \inf_{a \geq 0} \left(\int_{B_R(0)} \frac{u^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \right)^{1-n} \left[\frac{\int_{B_R(0)} \Psi \left(\frac{C|u_{\pm}(x) - w_a(x)|}{(\int_{B_R(0)} \frac{u^\star(y)^n}{|y|^n (1 + \log \frac{D}{|y|})^n} dy)^{1/n}} \right) dx}{(\int_{B_R(0)} \frac{u^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx)^{-1/n}} \right]^{n^2} \\
 &\leq \frac{1}{2} \sum_{\pm} \inf_{a \geq 0} \left(\int_{B_R(0)} \frac{u_{\pm}^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \right)^{1-n} \left[\frac{\int_{B_R(0)} \Psi \left(\frac{C|u_{\pm}(x) - w_a(x)|}{(\int_{B_R(0)} \frac{u_{\pm}^\star(y)^n}{|y|^n (1 + \log \frac{D}{|y|})^n} dy)^{1/n}} \right) dx}{(\int_{B_R(0)} \frac{u_{\pm}^\star(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx)^{-1/n}} \right]^{n^2} \\
 &\leq \frac{1}{2} \sum_{\pm} \left(\left(\frac{n}{n-1} \right)^n \int_{B_R(0)} |\nabla u_{\pm}|^n dx - \int_{B_R(0)} \frac{u_{\pm}(x)^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \right) \\
 &= \frac{1}{2} \left(\left(\frac{n}{n-1} \right)^n \int_{B_R(0)} |\nabla u|^n dx - \int_{B_R(0)} \frac{|u(x)|^n}{|x|^n (1 + \log \frac{D}{|x|})^n} dx \right), \tag{3.41}
 \end{aligned}$$

where C is the constant appearing in (3.40). Note that the first inequality is a consequence of the convexity of Ψ , the second one holds owing to (3.1) and to an elementary algebraic inequality, the third one follows from the fact that $1 - n < 0$ and that the function $t \mapsto \frac{\Psi(t)}{t}$ is increasing (for Ψ is a Young function), and the last one follows from an application of (3.40) with u replaced by u_+ and u_- . Obviously, (3.41) implies (1.12). \square

We conclude with a counterpart of Proposition 2.3, demonstrating the optimality of the choice of the function $e^{n'} - 1$ in the definition of $d_{C,D,Q}(\cdot)$.

Proposition 3.3. *Let Ω be an open bounded subset of \mathbb{R}^n , $n \geq 2$, having finite measure and containing 0. Then the Orlicz space $\text{Exp } L^{n'}(\Omega)$ is the smallest rearrangement invariant space on Ω containing the functions w_a given in (1.10) (with $D \geq R_\Omega$ and $Q > (1 + \log(D/r_\Omega))^{1/n'}$). Hence, in particular, $\int_\Omega \Phi(\lambda w_a(x)) dx = \infty$ for every $\lambda > 0$ and for every Young function $\Phi(t)$ growing essentially faster than $e^{n'} - 1$ near infinity.*

Recall that a Young function Φ_1 is said to increase essentially faster than another Young function Φ_2 near infinity if $\lim_{t \rightarrow +\infty} \frac{\Phi_2(\lambda t)}{\Phi_1(t)} = 0$ for every $\lambda > 0$.

Proof of Proposition 3.3. The fact that $w_a \in \text{Exp } L^{n'}(\Omega)$ is easily verified. Now, let $X(\Omega)$ be any r.i. space containing w_1 . We have to show that

$$\text{Exp } L^{n'}(\Omega) \subset X(\Omega). \tag{3.42}$$

Let u be any function from $\text{Exp } L^{n'}(\Omega)$. Then

$$1 \geq \int_\Omega \Psi \left(\frac{|u(x)|}{\|u\|_{\text{Exp } L^{n'}(\Omega)}} \right) dx = \int_0^{|\Omega|} \Psi \left(\frac{u^*(r)}{\|u\|_{\text{Exp } L^{n'}(\Omega)}} \right) dr \geq s \Psi \left(\frac{u^*(s)}{\|u\|_{\text{Exp } L^{n'}(\Omega)}} \right)$$

for $s \in (0, |\Omega|)$. Hence,

$$u^*(s) \leq \|u\|_{\text{Exp } L^{n'}(\Omega)} \left(\log \left(1 + \frac{1}{s} \right) \right)^{1/n'} \leq C \|u\|_{\text{Exp } L^{n'}(\Omega)} w_1^*(s) \quad \text{for } s \in (0, |\Omega|),$$

and for some positive constant $C = C(n, D, Q)$. Since $w_1 \in X(\Omega)$, by [6, Cor. 4.7, Chap. 2] $u \in X(\Omega)$ as well, and (3.42) follows.

The last assertion in the statement is a consequence of (3.42) and of a basic property of Orlicz spaces [1, Theorem 8.12]. \square

References

- [1] R.A. Adams, *Sobolev Spaces*, Academic Press, Orlando, 1975.
- [2] Adimurthi, N. Chaudhuri, M. Ramaswamy, An improved Hardy–Sobolev inequality and its applications, *Proc. Amer. Math. Soc.* 130 (2002) 489–505.
- [3] Adimurthi, M.J. Esteban, An improved Hardy–Sobolev inequality in $W^{1,p}$ and its applications to Schrödinger operators, *NoDEA Nonlin. Differential Equation Appl.* 12 (2005) 243–263.
- [4] G. Barbatis, S. Filippas, A. Tertikas, A unified approach to improved L^p Hardy inequalities with best constants, *Trans. Amer. Math. Soc.* 356 (2004) 2169–2196.
- [5] T. Bartsch, T. Weth, M. Willem, A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator, *Calc. Var. Part. Diff. Eq.* 18 (2003) 57–75.
- [6] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.
- [7] G. Bianchi, H. Egnell, A note on the Sobolev inequality, *J. Funct. Anal.* 100 (1991) 18–24.
- [8] H. Brezis, E. Lieb, Sobolev inequalities with remainder terms, *J. Funct. Anal.* 62 (1985) 73–86.
- [9] H. Brezis, M. Marcus, I. Shafir, Extremal functions for Hardy’s inequality with weights, *J. Funct. Anal.* 171 (2000) 177–191.
- [10] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* 36 (1983) 437–477.
- [11] H. Brezis, S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, *Comm. Part. Diff. Eq.* 5 (1980) 773–789.
- [12] J.E. Brothers, W.P. Ziemer, Minimal rearrangements of Sobolev functions, *J. Reine Angew. Math.* 384 (1988) 419–431.
- [13] A. Cianchi, A quantitative Sobolev inequality in BV , *J. Funct. Anal.* 237 (2006) 466–481.
- [14] A. Cianchi, Sharp Morrey–Sobolev inequalities and the distance from extremals, *Trans. Amer. Math. Soc.*, in press.
- [15] A. Cianchi, A. Ferone, A strengthened version of the Hardy–Littlewood inequality, preprint.
- [16] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, The sharp Sobolev inequality in quantitative form, preprint.
- [17] M. Cwikel, E. Pustylnik, Sobolev type embeddings in the limiting case, *J. Fourier Anal. Appl.* 4 (1998) 433–446.
- [18] A. Detalla, T. Horiuchi, H. Ando, Missing terms in Hardy–Sobolev inequalities and its applications, *Far East J. Math. Sci.* 14 (2004) 333–359.
- [19] J. Dolbeault, M.J. Esteban, M. Loss, L. Vega, An analytical proof of Hardy-like inequalities related to the Dirac operator, *J. Funct. Anal.* 216 (2004) 1–21.
- [20] D.E. Edmunds, R.A. Kerman, L. Pick, Optimal Sobolev imbeddings involving rearrangement invariant quasi-norms, *J. Funct. Anal.* 170 (2000) 307–355.
- [21] N. Fusco, F. Maggi, A. Pratelli, The sharp quantitative Sobolev inequality in BV , preprint.
- [22] S. Filippas, V.G. Maz’ya, A. Tertikas, On a question of Brezis and Marcus, *Calc. Var. Part. Diff. Eq.* 25 (2006) 491–501.
- [23] F. Gazzola, H.C. Grunau, E. Mitidieri, Hardy inequalities with optimal constants and remainder terms, *Trans. Amer. Math. Soc.* 356 (2004) 2149–2168.
- [24] F. Gazzola, H.C. Grunau, M. Squassina, Existence and non-existence results for critical growth biharmonic elliptic equations, *Calc. Var. Part. Diff. Eq.* 18 (2003) 117–143.
- [25] K. Hansson, Imbedding theorems of Sobolev type in potential theory, *Math. Scand.* 45 (1979) 77–102.
- [26] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Math., vol. 1150, Springer-Verlag, Berlin, 1985.
- [27] V.G. Maz’ya, *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.
- [28] B. Opic, A. Kufner, *Hardy-type Inequalities*, Longman Scientific Technical, Harlow, 1990.
- [29] S.I. Pohozaev, On the imbedding Sobolev theorem for $pl = n$, in: *Doklady Conference, Section Math., Moscow Power Inst.*, 1965, pp. 158–170 (Russian).
- [30] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* 110 (1976) 353–372.
- [31] J. Tidblom, A Hardy inequality in the half-space, *J. Funct. Anal.* 221 (2005) 482–495.
- [32] N.S. Trudinger, On imbeddings into Orlicz spaces and some applications, *J. Math. Mech.* 17 (1967) 473–483.
- [33] J.L. Vazquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, *J. Funct. Anal.* 173 (2000) 103–153.
- [34] V.I. Yudovich, Some estimates connected with integral operators and with solutions of elliptic equations, *Soviet Math. Dokl.* 2 (1961) 746–749.