

On questions of decay and existence for the viscous Camassa–Holm equations

Quelques questions de décroissance et existence pour les équations visqueuses de Camassa–Holm

Clayton Bjorland ^{*,1}, Maria E. Schonbek ²

Department of Mathematics, UC Santa Cruz, Santa Cruz, CA 95064, USA

Received 23 July 2006; received in revised form 5 July 2007; accepted 22 July 2007

Available online 17 October 2007

Abstract

We consider the viscous n -dimensional Camassa–Holm equations, with $n = 2, 3, 4$ in the whole space. We establish existence and regularity of the solutions and study the large time behavior of the solutions in several Sobolev spaces. We first show that if the data is only in L^2 then the solution decays without a rate and that this is the best that can be expected for data in L^2 . For solutions with data in $H^m \cap L^1$ we obtain decay at an algebraic rate which is optimal in the sense that it coincides with the rate of the underlying linear part.

© 2007 Elsevier Masson SAS. All rights reserved.

Résumé

On considère les équations visqueuses de Camassa–Holm dans \mathbb{R}^n , $n = 2, 3, 4$. Nous établissons l'existence et la régularité des solutions. Nous étudions le comportement asymptotique des solutions dans plusieurs espaces de Sobolev quand le temps tend vers l'infini. On montre que si la donnée est seulement dans L^2 la solution décroît vers zéro, mais la décroissance ne peut être uniforme. Pour les solutions avec donnée dans $L^1 \cap H^m$ on obtient une décroissance algébrique avec une vitesse qui est optimale dans le sens où elle coïncide avec les solutions correspondant à l'équation linéaire.

© 2007 Elsevier Masson SAS. All rights reserved.

MSC: 35B40; 35K55

Keywords: Navier–Stokes-alpha; Camassa–Holm; Existence; Regularity; Decay

* Corresponding author.

E-mail addresses: cbjorland@math.ucsc.edu (C. Bjorland), schonbek@math.ucsc.edu (M.E. Schonbek).

¹ The work of M. Schonbek was partially supported by NSF Grant DMS-0600692.

² The work of C. Bjorland was partially supported by NSF Grant OISE-0630623.

1. Introduction

The Viscous Camassa–Holm equations (VCHE) are commonly written

$$\begin{aligned} v_t + u \cdot \nabla v + v \cdot \nabla u^T + \nabla \pi &= \nu \Delta v \\ u - \alpha^2 \Delta u &= v \\ \nabla \cdot v &= 0 \end{aligned} \tag{1.1}$$

Here we adopt the notation $(v \cdot \nabla u^T)_i = \sum_j v_j \partial_i u_j$. These equations rose from work on shallow water equations [3], which led to [11,16], where the equations are derived by considering variational principles and Lagrangian averaging. In light of this derivation the equations are sometimes called the Lagrangian Averaged Navier–Stokes equations. In [9], the equations were derived as a “filtered” Navier–Stokes equation, which obeys a modified Kelvin circulation theorem along filtered velocities. In this setting they are sometimes referred to as the Navier–Stokes- α equations, where α is the parameter in the filter. Solutions to the VCHE are closely related to solutions of the famous Navier–Stokes equation (NSE), but the filter allows bounds that are currently unobtainable for the NSE, making them in some ways better suited for computational turbulence study, see [12].

In [9,10] these equations were studied in relation to turbulence theory, this treatment includes existence and uniqueness theorems on the torus in three dimensions. The two dimensional case was considered on the torus and the sphere in [14]. Global existence and uniqueness in three dimensions was proved on bounded domains with zero (non-slip) boundary conditions in [16]. These equations have also been studied in terms of large eddy simulation and turbulent pipe flow in [4–6], and [8]. In this paper we extend the current existence theorems and study the large time behavior of solutions.

This paper is organized as follows. Section two consists of notation and conventions used throughout. Section three contains preliminary discussion of the VCHE and several useful lemmas. In section four we state existence and uniqueness results for the VCHE, proofs of these statements are contained in Appendix A. In the next two sections we continue the decay program of M.E. Schonbek [20,21,23,25,27]. The main result of chapter five considers solutions of the VCHE in the whole space and we prove that the energy of a solution corresponding to data only in $L^2(\mathbb{R}^n)$ decays to zero following the arguments in [18]. We then demonstrate, by constructing counter examples, that no uniform rate of decay can exist which depends only on the initial energy. In chapter six we consider decay for solutions with initial data in $L^1 \cap L^2$. We show, using the Fourier Splitting Method, that the energy of a solution decays at the rate expected from the linear part, this is the same rate of decay as solutions to the NSE. For solutions with initial data in $H^m \cap L^1$ we calculate the decay of derivatives using again the Fourier Splitting Method with an inductive argument. In section seven we examine how solutions of the VCHE approach solutions of the NSE strongly on intervals of regularity for the NSE.

2. Notation

In this paper, L^p denotes the standard Lebesgue space with norm $\|\phi\|_p = (\int |\phi|^p)^{1/p}$. We use $\langle u, v \rangle = \int uv$ to denote the standard inner product on the Hilbert space L^2 . Compactly supported solenoidal vector fields (subsets of $\Sigma = \{\phi \in C_0^\infty(\Omega) | \nabla \phi = 0\}$) will be needed to describe incompressible solutions with zero boundary conditions. L_σ^p will denote the completion of Σ in the norm $\|\cdot\|_p$. $W^{m,p}$ will be used to denote the standard Sobolev spaces with the convention that $H^m = W^{m,2}$ (and $L^2 = H^0$). The completion of Σ under the H^m norm will be denoted by H_σ^m and $(H_\sigma^m)'$ will be the dual space. To denote the Fourier Transform of a function ϕ we will use either $\hat{\phi}$ or $\mathcal{F}(\phi)$, with $\check{\phi}$ or $\mathcal{F}^{-1}(\phi)$ the inverse transform. Throughout we will use C to denote an arbitrary constant which may change line to line, to emphasis the dependence of a constant on a number, say ν , we will write $C(\nu)$.

3. Preliminaries

The Kelvin-filtered Navier–Stokes equations (KFNSE) are given by the formula

$$\begin{aligned} \frac{\partial v}{\partial t} + u \cdot \nabla v + v \cdot \nabla u^T + \nabla \pi &= \nu \Delta v \\ \nabla \cdot v &= \nabla \cdot u = 0 \end{aligned}$$

$$v = \mathcal{O}u$$

In the above, $u = g * v$ represents a spatially filtered fluid velocity and \mathcal{O} is the inverse of this convolution. The term $u \cdot \nabla v$ is similar to “mollifying” the Navier–Stokes equations, originally done by Leray, [15], to approximate solutions. The term $v \cdot \nabla u^T = \sum v_j \nabla u_j$ allows the solution to obey a modification of the Kelvin circulation theorem where circulation is conserved around a loop moving with the filtered velocity u . In two and three dimensions, using the identity

$$u \cdot \nabla v + \sum v_j \nabla u_j = -u \times (\nabla \times v) + \nabla(v \cdot u) \tag{3.1}$$

and including the term $\nabla(v \cdot u)$ in the pressure, the KFNSE can be written as

$$\begin{aligned} \frac{\partial v}{\partial t} + \nabla \pi &= u \times (\nabla \times v) + v \Delta v \\ \nabla \cdot u &= \nabla \cdot v = 0 \\ v &= \mathcal{O}u \end{aligned}$$

The following lemma will show that the bilinear term in the Kelvin-filtered Navier–Stokes equations behaves similar to the bilinear term in the Navier–Stokes equations.

Lemma 3.1. *Let u and v be smooth divergence free functions with compact support, then*

$$\begin{aligned} \langle u \cdot \nabla v, u \rangle + \langle v \cdot \nabla u^T, u \rangle &= 0 \\ \langle u \times (\nabla \times v), u \rangle &= 0 \end{aligned}$$

Proof. The second equality is a consequence of the first, the identity (3.1), and the fact that u is divergence free. To see the first inequality we just need to rearrange the terms and then integrate by parts

$$\sum_{i,j} \int_{\mathbb{R}^n} v_j \partial_i u_j u_i dx = - \sum_{i,j} \int_{\mathbb{R}^n} u_i \partial_i v_j u_j dx \quad \square$$

Using this lemma, we can formally multiply the KFNSE by u to find

$$\left\langle \frac{\partial}{\partial t} v, u \right\rangle + v \langle \nabla v, \nabla u \rangle = 0 \tag{3.2}$$

By choosing \mathcal{O} to be the Helmholtz operator $\mathcal{O} = 1 - \alpha^2 \Delta$ we recover the Viscous Camassa–Holm equations

$$\begin{aligned} v_t + u \cdot \nabla v + v \cdot \nabla u^T + \nabla \pi &= v \Delta v \\ u - \alpha^2 \Delta u &= v \\ \nabla \cdot v &= 0 \end{aligned}$$

In the case of the VCHE, (3.2) becomes

$$\frac{1}{2} \frac{d}{dt} (\langle u, u \rangle + \alpha^2 \langle \nabla u, \nabla u \rangle) + v (\langle \nabla u, \nabla u \rangle + \alpha^2 \langle \Delta u, \Delta u \rangle) = 0 \tag{3.3}$$

This relation gives a priori estimates on u :

$$\|u(\cdot, t)\|_2^2 + \alpha^2 \|\nabla u(\cdot, t)\|_2^2 + 2v \int_0^t \|\nabla u(\cdot, t)\|_2^2 dt + 2v\alpha^2 \int_0^t \|\nabla^2 u(\cdot, t)\|_2^2 dt \leq \|u_0\|_2^2 + \alpha^2 \|\nabla u_0\|_2^2 \tag{3.4}$$

4. Existence of solutions for the VCHE

Existence and uniqueness of solutions for the VCHE on periodic domains in three dimensions was proved first in [10] using the Galerkin method. The most general existence and uniqueness theorems in three dimensions are provided in [16] which relies on a fixed point argument. The theorems in [16] assume the initial data $u_0 \in H_0^1 \cap H^s$ with $s \in [3, 5)$ and $u = Au = 0$ on the boundary, where A is the Stokes operator. Here we state extended results which cover the whole space in dimensions $2 \leq n \leq 4$, proofs are included in Appendix A. As an intermediate step, we provide a new existence proof on bounded domains in dimensions $2 \leq n \leq 4$, using the Galerkin Method, with initial data $v_0 \in L^2$, and $u = v = 0$ on the boundary. Our bounded result in three dimensions is slightly stronger than [16], by assuming $v_0 \in L^2$ we have only implied $u \in H_\sigma^2$.

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ be any open bounded subset or $\Omega = \mathbb{R}^n$, $n = 2, 3, 4$. A weak solution to the VCHE (1.1), with zero (no-slip) boundary conditions in the case of Ω bounded, is a pair of functions, u, v , such that

$$\begin{aligned} v &\in L^\infty([0, T]; L_\sigma^2(\Omega)) \cap L^2([0, T]; H_\sigma^1(\Omega)) \\ \partial_t v &\in L^2([0, T]; (H_\sigma^1)'(\Omega)) \\ u &\in L^\infty([0, T]; H_\sigma^2(\Omega)) \cap L^2([0, T]; H_\sigma^3(\Omega)) \end{aligned}$$

as well as $v(x, 0) = v_0$, and for any $\phi \in L^2([0, T]; H_\sigma^1(\Omega))$ with $\phi(T) = 0$ the following equalities are satisfied:

$$-\int_0^T \langle v, \partial_t \phi \rangle ds + \int_0^T \langle u \cdot \nabla v, \phi \rangle ds + \int_0^T \langle \phi \cdot \nabla u, v \rangle ds + v \int_0^T \langle \nabla v, \nabla \phi \rangle ds = \langle v_0, \phi(0) \rangle$$

and for a.e. $t \in [0, T]$,

$$\langle u, \phi \rangle + \alpha^2 \langle \nabla u, \nabla \phi \rangle = \langle v, \phi \rangle$$

Theorem 4.2. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary or $\Omega = \mathbb{R}^n$, $n = 2, 3, 4$. Given initial data $v_0 \in H_\sigma^M(\Omega)$, $M \geq 0$, there exists a unique weak solution to the VCHE (1.1) in the sense of Definition 4.1. This solution satisfies the estimate (3.4) as well as

$$\|\partial_t^p \nabla^m v(t)\|_2^2 + v \int_0^t \|\partial_t^p \nabla^{m+1} v(s)\|_2^2 ds \leq C(\|v_0\|_{H_\sigma^M}) \quad (4.1)$$

for all $m + 2p \leq M$.

Proof. Existence is given by Theorems A.4 and A.6 in Appendix A. The regularity statement is Theorem A.8 and uniqueness is Theorem A.9. The proofs follow from the construction of approximate solutions using the Galerkin method on bounded domains. A priori bounds are obtained through energy methods. Using a compactness lemma we are able to find a strongly convergent subsequence which allows the limit of the approximate solutions to pass through the non-linearity. To extend to unbounded domains we solve the problem in balls of radius $\{R_t\}$ (a sequence tending to infinity), and then invoke a diagonal argument. Regularity is established through an inductive argument relying on energy methods. \square

Next, we will state a corollary that describes the action of the filter and will be used many times in the following two sections.

Corollary 4.3.

$$\begin{aligned} \|\partial_t^p \nabla^m u\|_2^2 + 2\alpha^2 \|\partial_t^p \nabla^{m+1} u\|_2^2 + \alpha^4 \|\partial_t^p \nabla^{m+2} u\|_2^2 &= \|\partial_t^p \nabla^m v\|_2^2 \\ \|\partial_t^p \nabla^m u\|_n^2 + \|\partial_t^p \nabla^{m+1} u\|_n^2 &\leq C \|\partial_t^p \nabla^m v\|_2^2 \end{aligned}$$

$$\|\partial_t^p \nabla^m u(t)\|_n^2 + \nu \int_0^t \|\partial_t^p \nabla^{m+1} u(s)\|_n^2 ds \leq C(\|v_0\|_{H_0^M})$$

for all $m + 2k \leq M$, where C is a constant which depends only on α, n, m , and k (in the last bound the constant depends also on $\|v_0\|_{H_0^M}$).

Proof. This is an application of the Gagliardo–Nirenberg–Sobolev inequality to the bounds in the previous theorem. Differentiating the filter relation shows

$$\partial_t^p \nabla^m u - \alpha^2 \partial_t^p \nabla^m \Delta u = \partial_t^p \nabla^m v$$

Squaring this relation then integrating by parts gives

$$\|\partial_t^p \nabla^m u\|_2^2 + 2\alpha^2 \|\partial_t^p \nabla^{m+1} u\|_2^2 + \alpha^4 \|\partial_t^p \nabla^{m+2} u\|_2^2 = \|\partial_t^p \nabla^m v\|_2^2$$

This is the first bound in the corollary. Applying the Gagliardo–Nirenberg–Sobolev inequality to $\|u\|_n$ and using the previous equality shows

$$\begin{aligned} \|\partial_t^p \nabla^m u\|_n^2 &\leq C \|\partial_t^p \nabla^m v\|_2^2 \\ \|\partial_t^p \nabla^{m+1} u\|_n^2 &\leq C \|\partial_t^p \nabla^m v\|_2^2 \end{aligned}$$

This is the second set of bounds. Combining this with the regularity bounds in the theorem give the final set of bounds. \square

5. Large time behavior of the VCHE: non-uniform decay

In bounded domains it is easy to see that the energy of a solution decays exponentially using the Poincaré inequality

$$\|u\|_2^2 \leq C(\Omega) \|\nabla u\|_2^2$$

Indeed, start with the energy estimate (3.3) and apply the Poincaré inequality to find

$$\frac{1}{2} \frac{d}{dt} (\langle u, u \rangle + \alpha^2 \langle \nabla u, \nabla u \rangle) + C(\Omega) \nu (\langle u, u \rangle + \alpha^2 \langle \nabla u, \nabla u \rangle) \leq 0$$

This differential inequality implies

$$\langle u, u \rangle + \alpha^2 \langle \nabla u, \nabla u \rangle \leq C(\|v_0\|_2) e^{-C(\Omega, \nu)t}$$

The situation in unbounded domains is more delicate. If the initial data is assumed only in L^2 then the solution decays to zero but we are unable to determine the rate without more information, the precise statements of this idea are contained in Theorems 5.2 and 5.4.

We will follow [18] to show that the solutions in the whole space, constructed in Theorem 4.2, approach zero as time becomes large. The idea is to split the solution into low and high frequency parts using a cut-off function and generalized energy inequalities to show that both the high and low frequency terms approach zero. The idea of splitting into low and high frequency was first used in [17].

Lemma 5.1. *Solutions of the VCHE constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$ satisfy the following generalized energy inequalities. Let $E \in C^1([0, \infty))$ and $\psi \in C^1([0, \infty); C^1 \cap L^2(\mathbb{R}^n))$, then*

$$\begin{aligned} E(t) \|\psi(t) * v(t)\|_2^2 &= E(s) \|\psi(s) * v(s)\|_2^2 + \int_s^t E'(\tau) \|\psi(\tau) * v(\tau)\|_2^2 d\tau \\ &\quad + 2 \int_s^t E(\tau) \langle \psi'(\tau) * v(\tau), \psi(\tau) * v(\tau) \rangle d\tau - 2 \int_s^t E(\tau) \|\nabla \psi(\tau) * v(\tau)\|_2^2 d\tau \end{aligned}$$

$$\begin{aligned}
 & - 2 \int_s^t E(\tau) \langle u \cdot \nabla v, \psi(\tau) * \psi(\tau) * v(\tau) \rangle d\tau \\
 & - 2 \int_s^t E(\tau) \langle v \cdot \nabla u^T, \psi(\tau) * \psi(\tau) * v(\tau) \rangle d\tau
 \end{aligned} \tag{5.1}$$

For $E \in C^1([0, \infty))$ and $\tilde{\psi} \in C^1(0, \infty; L^\infty(\mathbb{R}^n))$ we have

$$\begin{aligned}
 E(t) \|\tilde{\psi}(t)\hat{v}(t)\|_2^2 &= E(s) \|\tilde{\psi}(s)\hat{v}(s)\|_2^2 + \int_s^t E'(\tau) \|\tilde{\psi}(\tau)\hat{v}(\tau)\|_2^2 d\tau + 2 \int_s^t E(\tau) \langle \tilde{\psi}'(\tau)\hat{v}(\tau), \tilde{\psi}(\tau)\hat{v}(\tau) \rangle d\tau \\
 & - 2 \int_s^t E(\tau) \|\xi \tilde{\psi}(\tau)\hat{v}(\tau)\|_2^2 d\tau - 2 \int_s^t E(\tau) \langle \mathcal{F}(u \cdot \nabla v), \tilde{\psi}^2(\tau)\hat{v}(\tau) \rangle d\tau \\
 & - 2 \int_s^t E(\tau) \langle \mathcal{F}(v \cdot \nabla u^T), \tilde{\psi}^2(\tau)\hat{v}(\tau) \rangle d\tau
 \end{aligned} \tag{5.2}$$

Proof. The proof of the first inequality is accomplished by multiplying the VCHE by $E(t)\psi * \psi * v$ then integrating by parts and in time. The second inequality is obtained by first taking the Fourier Transform of the VCHE, then multiplying by $\tilde{\psi}^2\hat{v}$ and integrating. \square

Theorem 5.2. Let v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$ and $v_0 \in L^2_\sigma(\mathbb{R}^n)$, then

$$\lim_{t \rightarrow 0} \|v(t)\|_2 = 0 \tag{5.3}$$

Proof. We work in frequency space. We split the energy into low and high frequency parts

$$\|\hat{v}\|_2 \leq \|\phi\hat{v}\|_2 + \|(1 - \phi)\hat{v}\|_2 \tag{5.4}$$

with $\phi = e^{-|\xi|^2}$ will be chosen below. To estimate the low frequency part of the energy, begin with the generalized energy estimate (5.1). Temporarily fix t then choose $E = 1$ (the constant function) and

$$\psi(\tau) = \mathcal{F}^{-1}[e^{-|\xi|^2(t+1-\tau)}]$$

Note that ψ and $\mathcal{F}(\psi)$ are rapidly decreasing functions for $\tau < t + 1$. The relation $\hat{\psi}' = |\xi|^2\hat{\psi}$ shows the third and fourth terms in (5.1) add to zero. Note $\phi = e^{-|\xi|^2} = \psi(t)$ and apply the Plancherel Theorem to see

$$\|\phi\hat{v}(t)\|_2^2 \leq \|e^{|\xi|^2(t-s)}\phi\hat{v}(s)\|_2^2 + 2 \int_s^t \|\check{\phi}^2 * (u \cdot \nabla v - v \cdot \nabla u^T), e^{2\Delta(t-\tau)}v(\tau)\| d\tau \tag{5.5}$$

With Hölder inequality, Young’s inequality, and the Gagliardo–Nirenberg–Sobolev inequality we bound

$$\begin{aligned}
 \|\check{\phi}^2 * u \cdot \nabla v, e^{2\Delta(t-\tau)}v(\tau)\| &\leq \|\check{\phi}^2 * u \cdot \nabla v\|_2 \|e^{2\Delta(t-\tau)}v(\tau)\|_2 \\
 &\leq C \|\check{\phi}^2\|_{\frac{2n}{n+2}} \|u\|_{\frac{2n}{n-2}} \|\nabla v\|_2 \|v\|_2 \\
 &\leq C(\phi) \|v\|_2 \|\nabla u\|_2 \|\nabla v\|_2
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \|\check{\phi}^2 * v \cdot \nabla u^T, e^{2\Delta(t-\tau)}v(\tau)\| &\leq \|\check{\phi}^2 * v \cdot \nabla u^T\|_2 \|e^{2\Delta(t-\tau)}v(\tau)\|_2 \\
 &\leq C \|\check{\phi}^2\|_{\frac{2n}{n+2}} \|v\|_{\frac{2n}{n-2}} \|\nabla u\|_2 \|v\|_2 \\
 &\leq C(\phi) \|v\|_2 \|\nabla u\|_2 \|\nabla v\|_2
 \end{aligned}$$

Using the triangle inequality, Hölder’s inequality, and (4.1) in (5.5) yields

$$\|\phi \hat{v}(t)\|_2^2 \leq \|e^{|\xi|^2(t-s)} \phi \hat{v}(s)\|_2^2 + 2C(\phi) \|v_0\|_2 \left(\int_s^t \|\nabla u\|_2^2 d\tau \right)^{1/2} \left(\int_s^t \|\nabla v\|_2^2 d\tau \right)^{1/2}$$

As the first term on the RHS tends to zero, applying the limit $t \rightarrow \infty$ yields

$$\limsup_{t \rightarrow \infty} \|\phi \hat{v}(t)\|_2^2 \leq 2C(\phi) \|v_0\|_2 \left(\int_s^\infty \|\nabla u\|_2^2 d\tau \right)^{1/2} \left(\int_s^\infty \|\nabla v\|_2^2 d\tau \right)^{1/2}$$

The bounds (3.4) and (4.1) show $\|\nabla u\|_2^2$ and $\|\nabla v\|_2^2$ are integrable on the positive real line, letting $s \rightarrow \infty$ leaves

$$\limsup_{t \rightarrow \infty} \|\phi \hat{v}(t)\|_2^2 \rightarrow 0 \tag{5.6}$$

To estimate the high frequency start with the generalized energy inequality (5.2) and chose $\tilde{\psi} = 1 - e^{-|\xi|^2} = 1 - \phi$. Let $B_G(t) = \{\xi: |\xi| \leq G(t)\}$ where $G(t)$ will be selected later and use $\langle u \cdot \nabla v, v \rangle = 0$ to replace $\tilde{\psi}^2$ by $1 - \tilde{\psi}^2$ in the 5th term on the RHS of (5.2).

$$\begin{aligned} E(t) \|\tilde{\psi} \hat{v}(t)\|_2^2 &\leq E(s) \|\tilde{\psi} \hat{v}(s)\|_2^2 + \int_s^t E'(\tau) \int_{B_G(\tau)} |\tilde{\psi} \hat{v}(\tau)|^2 d\xi d\tau \\ &\quad + \int_s^t (E'(\tau) - 2E(\tau)G^2(\tau)) \int_{B_G^c(\tau)} |\tilde{\psi} \hat{v}(\tau)|^2 d\xi d\tau \\ &\quad + 2 \int_s^t E(\tau) \langle \mathcal{F}(u \cdot \nabla v + v \cdot \nabla u^T), (1 - \tilde{\psi}^2(\tau)) \hat{v}(\tau) \rangle d\tau \\ &\quad + 2 \int_s^t E(\tau) \langle \mathcal{F}(v \cdot \nabla u^T), \hat{v}(\tau) \rangle d\tau \end{aligned} \tag{5.7}$$

We remark both $(1 - \tilde{\psi}^2)$ and $\phi = \mathcal{F}^{-1}(1 - \tilde{\psi}^2)$ are rapidly decreasing functions. Using again Hölder’s inequality and the Plancherel theorem, then Young’s inequality and the Gagliardo–Nirenberg–Sobolev inequality allows

$$\begin{aligned} \langle \mathcal{F}(u \cdot \nabla v + v \cdot \nabla u^T), (1 - \tilde{\psi}^2(\tau)) \hat{v}(\tau) \rangle &= \langle (1 - \tilde{\psi}^2(\tau)) \mathcal{F}(u \cdot \nabla v + v \cdot \nabla u^T), \hat{v}(\tau) \rangle \\ &\leq \|\mathcal{F}^{-1}(1 - \tilde{\psi}^2(\tau)) * (u \cdot \nabla v + v \cdot \nabla u^T)\|_2 \|v\|_2 \\ &\leq C \|1 - \tilde{\psi}\|_{\frac{2n}{n+2}} (\|u\|_{\frac{2n}{n-2}} \|\nabla v\|_2 + \|v\|_{\frac{2n}{n-2}} \|\nabla u\|_2) \|v\|_2 \\ &\leq C(\phi) \|v\|_2 \|\nabla u\|_2 \|\nabla v\|_2 \end{aligned}$$

Similarly use Hölder’s inequality with the Plancherel theorem, then the Gagliardo–Nirenberg–Sobolev inequality, and Corollary 4.3 to bound

$$\begin{aligned} \langle \mathcal{F}(v \cdot \nabla u^T), \hat{v}(\tau) \rangle &\leq \|v \cdot \nabla u^T\|_2 \|v\|_2 \\ &\leq C \|v\|_{\frac{2n}{n-2}} \|\nabla u\|_n \|v\|_2 \\ &\leq C \|v\|_2 \|\nabla v\|_2^2 \end{aligned}$$

Choosing $E(t) = (1 + t)^\beta$ and $G^2(t) = \beta/2(1 + t)$ in (5.7), so that $E' - 2EG^2 = 0$, and taking $\beta > 0$ sufficiently large, leaves

$$\begin{aligned} \|(1 - \phi)\hat{v}(t)\|_2^2 &\leq \frac{(1 + s)^\beta}{(1 + t)^\beta} \|(1 - \phi)\hat{v}(s)\|_2^2 + \int_s^t \frac{\beta(1 + \tau)^{\beta-1}}{(1 + t)^\beta} \int_{B_G(\tau)} |(1 - \phi)\hat{v}(\tau)|^2 d\xi d\tau \\ &\quad + C \|v_0\|_2 \int_s^t \frac{(1 + \tau)^\beta}{(1 + t)^\beta} \|\nabla v\|_2 (\|\nabla v\|_2 + \|\nabla u\|_2) d\tau \end{aligned}$$

For $\xi \in B_G(t)$ and t sufficiently large, $\tilde{\psi} = |1 - \phi| \leq |\xi|^2$. Therefore $|1 - \phi|^2 \leq \beta^2/4(1 + t)^2$ and the second term on the RHS can be bounded as

$$\begin{aligned} \int_s^t \frac{\beta^3(1 + \tau)^{-3}}{4} \int_{A(\tau)} |\hat{v}(\tau)|^2 d\xi d\tau &\leq \int_s^t \frac{\beta^3(1 + \tau)^{-3}}{4} \|v(\tau)\|_2^2 d\tau \\ &\leq \frac{\beta^3}{4} \|v_0\|_2^2 \int_s^t (1 + \tau)^{-3} d\tau \\ &\leq \frac{\beta^3}{8} \|v_0\|_2^2 (1 + s)^{-2} \end{aligned}$$

Letting $t \rightarrow \infty$ shows

$$\limsup_{t \rightarrow \infty} \|(1 - \phi)\hat{v}(t)\|_2^2 \leq \frac{\alpha^3}{8} \|v_0\|_2^2 (1 + s)^{-2} + C \|v_0\|_2 \left(\int_s^\infty \|\nabla v\|_2^2 d\tau + \int_s^\infty \|\nabla u\|_2^2 d\tau \right) \tag{5.8}$$

The bounds (3.4) and (4.1) again show $\|\nabla v\|_2^2$ and $\|\nabla u\|_2^2$ are integrable on the real line. Letting $s \rightarrow \infty$ proves

$$\limsup_{t \rightarrow \infty} \|(1 - \phi)\hat{v}(t)\|_2^2 = 0$$

Combining this with (5.6) and the Plancherel theorem completes this proof. \square

Corollary 5.3. *Let v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$ corresponding to $v_0 \in H_0^1(\mathbb{R}^n)$. Then*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|v(\tau)\|_2 d\tau = 0$$

Proof. Given an $\epsilon > 0$ we can choose a large s such that $\|v\|_2 \leq \epsilon$ for $\tau > s$, this follows directly from the previous theorem. Then

$$\begin{aligned} \frac{1}{t} \int_0^t \|v(\tau)\|_2 d\tau &= \frac{1}{t} \int_0^s \|v(\tau)\|_2 d\tau + \frac{1}{t} \int_s^t \int \|v(\tau)\|_2 d\tau \\ &\leq \frac{1}{t} \int_0^s \|v(\tau)\|_2 d\tau + \epsilon \frac{t - s}{t} \end{aligned} \tag{5.9}$$

Note that ϵ was chosen arbitrarily and let $t \rightarrow \infty$ to finish the proof. \square

We have shown that the energy of a solution to the VCHE will tend to zero as time becomes large, now we will provide a counter example to show that there is no uniform rate of decay based only on the initial energy of the system. This is analogous to a result proved in [24]. The idea is to take a family of initial data with a parameter ϵ that have constant L^2 norm, but norms of higher derivatives of the initial data can be taken arbitrarily small by picking ϵ

sufficiently small. It is then possible to bound the higher derivative norms of the solution arbitrarily small by taking ϵ small. Combining this with the energy relation (3.3) allows us to place a lower bound on the energy of the solution which depends on ϵ . By choosing ϵ small we can guarantee that a solution will remain away from zero for any finite amount of time.

Theorem 5.4. *Let v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$ and $v_0 \in L^2_\sigma(\mathbb{R}^n)$. There exists no function $G(t, \beta) : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following two properties:*

$$\begin{aligned} \|v\|_2 &\leq G(t, \|v_0\|_2) \\ \lim_{t \rightarrow \infty} G(t, \beta) &= 0 \quad \forall \beta \end{aligned} \tag{5.10}$$

Proof. Fix $u_0(x)$ to be any smooth function of compact support and write $u_0^\epsilon(x) = \epsilon^{n/2}u_0(\epsilon x)$. Let $v_0^\epsilon = u_0^\epsilon - \alpha^2 \Delta u_0^\epsilon$ and v^ϵ the solution of the VCHE given by Theorem 4.2 corresponding to the initial data v_0 . Note $\|u_0^\epsilon\|_2 = \|u_0\|_2$ and $\|\nabla^m u_0^\epsilon\|_2 = \epsilon^m \|\nabla^m u_0\|_2$ for all $\epsilon > 0$. Also,

$$\begin{aligned} \|v_0^\epsilon\|_2^2 &= \|u_0^\epsilon\|_2^2 + \alpha^2 \|\nabla u_0^\epsilon\|_2^2 + \alpha^4 \|\Delta u_0^\epsilon\|_2^2 \\ &= \|u_0\|_2^2 + \alpha^2 \epsilon^2 \|\nabla u_0\|_2^2 + \alpha^4 \epsilon^4 \|\Delta u_0\|_2^2 \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} \|\nabla v_0^\epsilon\|_2^2 &= \|\nabla u_0^\epsilon\|_2^2 + \alpha^2 \|\Delta u_0^\epsilon\|_2^2 + \alpha^4 \|\nabla \Delta u_0^\epsilon\|_2^2 \\ &= \epsilon^2 \|\nabla u_0\|_2^2 + \alpha^2 \epsilon^4 \|\Delta u_0\|_2^2 + \alpha^4 \epsilon^6 \|\nabla \Delta u_0\|_2^2 \end{aligned} \tag{5.12}$$

From the two previous inequalities and Corollary 4.3 we obtain a constant $C = C(\|u_0\|_{H^3})$, such that for all $\epsilon > 0$

$$\begin{aligned} \|v^\epsilon\|_2^2 &\leq C \\ \|\nabla v^\epsilon\|_2^2 &\leq C\epsilon^2 \end{aligned} \tag{5.13}$$

Multiply the VCHE (1.1) by Δv^ϵ , then integrating by parts yields

$$\frac{1}{2} \frac{d}{dt} \|\nabla v^\epsilon\|_2^2 + v \|\Delta^2 v^\epsilon\|_2^2 = \langle u^\epsilon \cdot \nabla v^\epsilon, \Delta v^\epsilon \rangle + \langle \Delta v^\epsilon \cdot \nabla u^\epsilon, v^\epsilon \rangle$$

Use the relation $\langle u^\epsilon, \nabla v^\epsilon, v^\epsilon \rangle = 0$, the Hölder inequality, Sobolev inequality, and then the Cauchy Inequality to see

$$\begin{aligned} |\langle u^\epsilon \cdot \nabla v^\epsilon, \Delta v^\epsilon \rangle| &= |(-1)\langle (\nabla u^\epsilon) \cdot \nabla v^\epsilon, \nabla v^\epsilon \rangle| \\ &\leq C \|\nabla u^\epsilon\|_n \|\nabla v^\epsilon\|_2 \|\nabla v^\epsilon\|_{\frac{2n}{n-2}} \\ &\leq \frac{\nu}{4} \|\Delta v^\epsilon\|_2^2 + C \|\nabla u^\epsilon\|_n^2 \|\nabla v^\epsilon\|_2^2 \end{aligned}$$

Similarly,

$$\begin{aligned} |\langle \Delta v^\epsilon \cdot \nabla u^\epsilon, v^\epsilon \rangle| &\leq C \|\Delta v^\epsilon\|_2 \|\nabla u^\epsilon\|_n \|v^\epsilon\|_{\frac{2n}{n-2}} \\ &\leq \frac{\nu}{4} \|\Delta v^\epsilon\|_2^2 + C \|\nabla v^\epsilon\|_2^2 \|\nabla u^\epsilon\|_n^2 \end{aligned}$$

Applied to (5.13):

$$\frac{1}{2} \frac{d}{dt} \|\nabla v^\epsilon\|_2^2 + \frac{\nu}{2} \|\Delta v^\epsilon\|_2^2 \leq C \|\nabla v^\epsilon\|_2^2 \|\nabla u^\epsilon\|_n^2 \tag{5.14}$$

By (3.4) and Corollary 4.3,

$$\int_0^\infty \|\nabla u^\epsilon\|_n^2 dt \leq \|u_0^\epsilon\|_2^2 + \|\nabla u_0^\epsilon\|_2^2 \leq \|v_0^\epsilon\|_2^2$$

This bound, combined with (5.13) and (5.14) yields

$$\|\nabla v^\epsilon\|_2^2 \leq \|\nabla v_0^\epsilon\|_2^2 e^{C\|v_0^\epsilon\|_2^2} \leq C\epsilon^2 e^{C\epsilon^2}$$

Again, apply Corollary 4.3

$$\|\nabla u^\epsilon\|_2^2 + \alpha^2 \|\Delta u^\epsilon\|_2^2 \leq \|\nabla v^\epsilon\|_2^2 \leq \|\nabla v_0^\epsilon\|_2^2 e^{C\|v_0^\epsilon\|_2^2}$$

This together with the energy estimate (3.3) implies

$$\frac{1}{2} \frac{d}{dt} (\|u^\epsilon\|_2^2 + \alpha^2 \|\nabla u^\epsilon\|_2^2) \geq -C\epsilon^2$$

or,

$$\begin{aligned} \|u^\epsilon\|_2^2 + \alpha^2 \|\nabla u^\epsilon\|_2^2 &\geq \|u_0^\epsilon\|_2^2 + \alpha^2 \|\nabla u_0^\epsilon\|_2^2 - C\epsilon^2 t \\ &= \|u_0\|_2^2 + \epsilon^2 \alpha^2 \|\nabla u_0\|_2^2 - C\epsilon^2 t \\ &\geq \|u_0\|_2^2 - C\epsilon^2 t \end{aligned} \tag{5.15}$$

From this we can deduce that there is no function $G(t, \beta, \cdot)$, continuous and approaching zero in t for each fixed β , such that $\|u\|_2 \leq G(t, \|u_0\|_2)$. If there was such a function, then at some t_0 it would satisfy the bound $G(t_0, \|u_0\|_2) \leq \|u_0\|_2/2$. By choosing ϵ sufficiently small in (5.15), i.e. $\epsilon^2 < \|u_0\|_2^2/4Ct_0$, we have found initial data with a solution which cannot satisfy this estimate. \square

6. Large time behavior of the VCHE: Algebraic decay

Although there is no uniform rate of decay for solutions with data exclusively in L^2 , we now show that there is a uniform rate of decay depending on the L^2 and L^1 norm of the initial data. Theorem 6.10 contains the most general decay result in this section.

There is a relation between the shape of the Fourier Transform of the initial data near the origin and the decay rate of a solution to a parabolic equation with this data. By requiring the initial data to be absolutely integrable (in L^1) we are in turn requiring the Fourier Transform of the initial data to be bounded. Using the Fourier Splitting Method it will also be shown that solutions in the whole space decay algebraically in H^M as $t \rightarrow \infty$ for initial data in $L^1 \cap H^M$, $M \geq 0$. The decay obtained is the same as for the linear part (the heat equation). Note that the initial conditions can be weakened to require only that $v_0 \in X$ where $X = \{v_0 | v_0(t) \leq C(1+t)^{-\beta}\}$ where $v_0(t)$ is the solution of the heat equation with initial data v_0 . The decay rate will depend on the relation between β and the number of dimensions. For similar results corresponding to the Navier–Stokes equations see [27,29], and [30].

The Fourier Splitting Method was originally applied to parabolic conservation laws in [22], and later applied to the NSE in [23]. In [24] the decay rate was made sharp in dimension $n > 2$ through a bootstrap method and logarithmic decay was shown for $n = 2$. In [31] the decay rate for $n = 2$ was made sharp through a bootstrap argument involving the Gronwall inequality. In this section we combine ideas from all of these papers in a slightly different way which allows us to prove the optimal energy decay rate in dimensions $n \geq 2$ without appealing to a bootstrap argument although we still use a bootstrap argument to obtain decay rates for higher derivatives. This same argument is also applicable to the NSE.

The first goal of this section is to obtain a decay rate for the filtered velocity u , which is accomplished by applying the Fourier Splitting Method to the natural energy relation (3.3). This decay rate is then used with an inductive argument to obtain decay rates for the unfiltered velocity v and all of its derivatives. We start by finding estimates on $\|\hat{v}\|_\infty$.

Lemma 6.1. *Let v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in L_\sigma^2 \cap L^1(\mathbb{R}^n)$. Then,*

$$|\mathcal{F}(v)| \leq C \left[1 + \left(\int_0^t \|u(s)\|_2^2 \right)^{1/2} \left(\int_0^t \|\nabla v(s)\|_2^2 \right)^{1/2} \right]$$

where the constant depends only on the initial data, the dimension of space, and the constants in the VCHE (but not α).

Proof. Use the identity

$$\sum_i \nabla(u_i v_i) = \sum_i u_i \nabla v_i + \sum_i v_i \nabla u_i \tag{6.1}$$

and write the Fourier transform of the solution $\mathcal{F}(v)$ as

$$\mathcal{F}(v) = e^{-\nu t|\xi|^2} \mathcal{F}(v_0) + \int_0^t e^{-\nu(t-s)|\xi|^2} \Psi(\xi, s) ds$$

where

$$\Psi(\xi, t) = \xi \cdot \mathcal{F}\left(\pi + \sum_i u_i v_i\right) - \mathcal{F}(u \cdot \nabla v - u \cdot \nabla v^T) \tag{6.2}$$

We would like first to bound Ψ , in that direction we have the following estimate which relies on the bound $\|\mathcal{F}(\phi)\|_\infty \leq \|\phi\|_1$ and Young’s inequality

$$|\mathcal{F}(u \cdot \nabla v - u \cdot \nabla v^T)| \leq C \|u\|_2 \|\nabla v\|_2$$

Also, taking the divergence of the VCHE (1.1) shows

$$\Delta\left(\pi + \sum_i u_i v_i\right) = \operatorname{div}(u \nabla v - u \nabla v^T)$$

Using the estimate immediately above and the Fourier transform leaves

$$\left| \xi \mathcal{F}\left(\pi + \sum_i u_i v_i\right) \right| \leq C \|u\|_2 \|\nabla v\|_2$$

Now we can bound the integrand

$$|\Psi(\xi, t)| \leq C \|u\|_2 \|\nabla v\|_2$$

Now take the supremum over ξ of (6.1) and apply the Cauchy–Schwartz Inequality:

$$|\mathcal{F}(v)| \leq |\mathcal{F}(v_0)| + C \left(\int_0^t \|u(s)\|_2^2 ds \right)^{1/2} \left(\int_0^t \|\nabla v(s)\|_2^2 ds \right)^{1/2}$$

The bound $|\mathcal{F}(v_0)|_\infty \leq \|v_0\|_1$ finishes the proof. \square

Theorem 6.2. Let v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in L^2_\sigma \cap L^1(\mathbb{R}^n)$. The solution satisfies the “energy” decay rate

$$\int_{\mathbb{R}^n} v \cdot u dx = \|u\|_2^2 + \alpha^2 \|\nabla u\|_2^2 \leq C(t + 1)^{-n/2}$$

where the constant depends only on the initial data, the dimension of space, and the constants in the VCHE (but not α).

Proof. The previous lemma, with the bound (4.1), yields

$$|\hat{v}|^2 \leq C \left[1 + \int_0^t \|u(s)\|_2^2 ds \right] \tag{6.3}$$

Now we begin work with the energy estimate (3.3). Using the Plancherel Theorem we rewrite it as

$$\frac{d}{dt} \int_{\mathbb{R}^n} (1 + \alpha^2 |\xi|^2) \hat{u}^2 d\xi + 2\nu \int_{\mathbb{R}^n} |\xi|^2 (1 + \alpha^2 |\xi|^2) \hat{u}^2 d\xi = 0$$

Let $B(\rho)$ be the ball of radius ρ where $\rho^2 = f'(t)/(2vf(t))$, and f is a positive, increasing function to be specified later. To simplify our equations we write $E^2 = \hat{u} \cdot \hat{v} = (1 + \alpha^2|\xi|^2)\hat{u}^2$. Then,

$$\frac{d}{dt} \int_{\mathbb{R}^n} E^2 d\xi + 2v\rho^2 \int_{B^c(\rho)} E^2 d\xi \leq 0$$

or

$$\frac{d}{dt} \int_{\mathbb{R}^n} E^2 d\xi + 2v\rho^2 \int_{\mathbb{R}^n} E^2 d\xi \leq 2v\rho^2 \int_{B(\rho)} E^2 d\xi \tag{6.4}$$

Recall the relation between u and v , that is $v = u - \alpha^2\Delta u$ which has Fourier Transform $\hat{u} = \hat{v}/(1 + \alpha^2|\xi|^2)$. Combining this with (6.3) we see

$$\begin{aligned} \|E\|_\infty^2 &= \frac{\|\hat{v} \cdot \hat{v}\|_\infty}{(1 + \alpha^2|\xi|^2)} \\ &\leq C \left[1 + \int_0^t \|u(s)\|_2^2 ds \right] \end{aligned}$$

With this bound we can estimate the integral on the right-hand side of (6.4).

$$\frac{d}{dt} \int_{\mathbb{R}^n} E^2 d\xi + 2v\rho^2 \int_{\mathbb{R}^n} E^2 d\xi \leq C\rho^{2+n} \left[1 + \int_0^t \|u(s)\|_2^2 ds \right]$$

We now have a differential inequality which can be solved using the integrating factor f to find

$$\frac{d}{dt} \left(f \int_{\mathbb{R}^n} E^2 d\xi \right) \leq C f' \left(\frac{f'}{f} \right)^{n/2} \left[1 + \int_0^t \|u(s)\|_2^2 ds \right]$$

Choose $f = (1 + t)^{n/2+1}$ so that $f'/f = (n/2 + 1)/(1 + t)$ and integrate in time from 0 to r .

$$(1 + r)^{n/2+1} \int_{\mathbb{R}^n} E^2(\xi, r) d\xi \leq \int_{\mathbb{R}^n} E^2(\xi, 0) d\xi + C \int_0^r \left(1 + \int_0^t \|u(s)\|_2^2 ds \right) dt$$

Note $\|u\|_2^2 \leq \int_{\mathbb{R}^n} E^2 d\xi$, then using the Tonelli theorem we can evaluate the integral on the RHS as

$$\int_0^r \left(1 + \int_0^t \|u(s)\|_2^2 ds \right) dt \leq r + \int_0^r (r - s) \|u(s)\|_2^2 ds$$

which leaves

$$(1 + r)^{n/2+1} \int_{\mathbb{R}^n} E^2(\xi, r) d\xi \leq C(1 + r) + C \int_0^r (r - s) \int_{\mathbb{R}^n} E^2(\xi, s) d\xi ds$$

This is of the form

$$\phi \leq C(1 + r) + C \int_0^r \phi(s)(r - s)(1 + s)^{-n/2+1} ds$$

with $\phi = (1 + r)^{n/2+1} \int_{\mathbb{R}^n} E^2(\xi, r) d\xi$. The Gronwall inequality now shows

$$(1 + r)^{n/2+1} \int_{\mathbb{R}^n} E^2(\xi, r) d\xi \leq C(1 + r) \exp \left(C \int_0^r (r - s)(1 + s)^{-n/2-1} ds \right)$$

For $n \geq 2$ the integral $\int_0^r (r-s)(1+s)^{-n/2-1} ds$ is bounded independent of r . Applying the Plancherel theorem one more time finishes the proof. \square

Next we work out of order and establish the decay rate for the homogeneous H^1 norm of v using a similar argument as the previous theorem.

Theorem 6.3. *Let v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$ corresponding to $v_0 \in H^1_\sigma \cap L^1(\mathbb{R}^n)$. The solution satisfies the decay rate*

$$\|\nabla v\|_2^2 \leq C(t+1)^{-1-n/2}$$

where the constant depends only on the initial data, the dimension of space, and the constants in the VCHE (ν, α) .

Proof. Multiply the VCHE by Δv , use the identity (6.1), after recalling that Δv is divergence free use the Hölder inequality to obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla v\|_2^2 + \nu \|\Delta v\|_2^2 \leq C \|u\|_n \|\nabla v\|_{\frac{2n}{n-2}} \|\Delta v\|_2$$

After using the Sobolev inequality, Corollary 4.3, and the previous theorem, this becomes

$$\frac{1}{2} \frac{d}{dt} \|\nabla v\|_2^2 + \nu \|\Delta v\|_2^2 \leq C(1+t)^{-n/2} \|\Delta v\|_2^2$$

We will now restrict ourselves to t large enough so that $C(1+t)^{-1} < \nu/2$, this implies

$$\frac{d}{dt} \|\nabla v\|_2^2 + \nu \|\Delta v\|_2^2 \leq 0$$

The next step is to apply the Fourier Splitting method as in the previous theorem. Let $B(\rho)$ be the ball of radius ρ where $\rho^2 = f'/(vf)$ and f is a positive increasing function to be specified later, using the Plancherel theorem:

$$\frac{d}{dt} \|\xi \hat{v}\|_2^2 + \nu \rho^2 \|\xi \hat{v}\|_2^2 \leq \nu \rho^4 \int_{B(\rho)} |\hat{v}|^2 \xi$$

Lemma 6.1 with Theorem 6.2 imply

$$|\hat{v}|^2 \leq C \left[1 + \left(\int_0^t (1+s)^{-n/2} ds \right) \left(\int_0^t \|\nabla v\|_2^2 ds \right) \right]$$

With this bound the previous line becomes

$$\frac{d}{dt} \|\xi \hat{v}\|_2^2 + \nu \rho^2 \|\xi \hat{v}\|_2^2 \leq C \nu \rho^{4+n} \left[1 + \left(\int_0^t (1+s)^{-n/2} ds \right) \left(\int_0^t \|\nabla v\|_2^2 ds \right) \right]$$

Set $f = (1+t)^{n/2+2}$ and use it as an integrating factor

$$\frac{d}{dt} ((1+t)^{n/2+2} \|\xi \hat{v}\|_2^2) \leq C \left[1 + \left(\int_0^t (1+s)^{-n/2} ds \right) \left(\int_0^t \|\nabla v\|_2^2 ds \right) \right]$$

Again, as in the previous theorem, integrate in time from 0 to r , then use the Tonelli theorem and the Plancherel theorem

$$(1+r)^{n/2+2} \|\nabla v\|_2^2 \leq C(1+r) \left(\int_0^r \left(\int_0^t (1+s)^{-n/2} ds \right) dt \right) \int_0^r \|\nabla v(s)\|_2^2 ds$$

The Gronwall inequality now shows

$$(1+r)^{n/2+2} \|\nabla v\|_2^2 \leq C(1+r)e^A$$

where

$$A = \left[\left(\int_0^r \left(\int_0^t (1+s)^{-n/2} ds \right) dt \right) \left(\int_0^r (1+t)^{-n/2-2} dt \right) \right]$$

Note A is finite, hence,

$$\|\nabla v(r)\|_2^2 \leq C(1+r)^{-n/2-1} \quad \square$$

Corollary 6.4. *Let v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in L^2_\sigma \cap L^1(\mathbb{R}^n)$. Then,*

$$\begin{aligned} |\mathcal{F}(v)| &\leq C \\ |\mathcal{F}(u)| &\leq C \end{aligned}$$

where the constant depends only on the initial data, the dimension of space, and the constants in the VCHE.

Proof. Combine Lemma 6.1 with Theorems 6.2 and 6.3. \square

Corollary 6.5. *Let v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in L^2_\sigma \cap L^1(\mathbb{R}^n)$. Then*

$$\|v\|_2^2 \leq C(t+1)^{-n/2}$$

where the constant depends only on the initial data, the dimension of space, and the constants in the VCHE.

Proof. In Theorem 6.2 we have shown that

$$\|u\|_2^2 + \alpha^2 \|\nabla u\|_2^2 \leq C(t+1)^{-n/2} \tag{6.5}$$

Differentiating the Helmholtz equation, then squaring it and integrating shows

$$\|\nabla u\|_2^2 + 2\alpha^2 \|\nabla^2 u\|_2^2 + \alpha^4 \|\nabla^3 u\|_2^2 = \|\nabla v\|_2^2$$

Combine this with Theorem 6.3,

$$\|\nabla^2 u\|_2^2 \leq C(t+1)^{-n/2-1}$$

With (6.5) we see

$$\|v\|_2^2 = \|u\|_2^2 + 2\alpha^2 \|\nabla u\|_2^2 + \alpha^4 \|\Delta u\|_2^2 \leq C(t+1)^{-n/2} \quad \square$$

We now turn our attention to a more general situation involving the Fourier Splitting Method. This next theorem will be used in the remaining decay proofs.

Theorem 6.6. *Let $\|\nabla^m w(0)\|_2 < \infty$. Given an energy inequality of the form*

$$\frac{1}{2} \frac{d}{dt} \|\nabla^m w\|_2^2 + \nu \|\nabla^{m+1} w\|_2^2 \leq C(1+t)^\gamma$$

and the bound

$$|\hat{w}(\xi, t)| \leq C(1+t)^\beta$$

which holds for $|\xi|^2 < \frac{d}{\nu(1+t)}$, we can deduce the asymptotic behavior

$$\|\nabla^m w\|_2^2 \leq C[(1+t)^{-m-n/2+2\beta} + (1+t)^{\gamma+1}]$$

Proof. We proceed directly with Fourier Splitting. Apply Plancherel’s Theorem and break up the integral on the LHS.

$$\frac{1}{2} \frac{d}{dt} \|\xi^k \hat{w}\|_2^2 + \nu \rho^2 \|\xi^k \hat{w}\|_2^2 \leq \nu \rho^{2k+2} \int_{B(\rho)} \hat{w}^2 d\xi + C(1+t)^\gamma$$

Choose, for some large d ,

$$\rho^2 = \frac{d}{\nu(1+t)}$$

Then, using the assumption for the bound on \hat{w} and performing the integration on the RHS we have

$$\frac{d}{dt} ((1+t)^d \|\xi^m \hat{w}\|_2^2) \leq C[(1+t)^{-m-1+d+2\beta-n/2} + (1+t)^{\gamma+d}]$$

Integration in time and another application of the Plancherel theorem finishes the proof. \square

For the first application of the above theorem we will compute the decay rate for all spacial derivatives for solutions of the VCHE.

Theorem 6.7. Let v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in H_\sigma^K \cap L^1(\mathbb{R}^n)$. These solutions satisfy the following decay rate for all $m \leq K$

$$\|\nabla^m v\|_2^2 \leq C(t+1)^{-m-n/2}$$

Proof. The cases $m = 0, 1$ are Theorems 6.5 and 6.3 respectively. To prove the remaining cases, we first find an inequality in a form suitable for Theorem 6.6, then using inductive arguments establish decay. Having previously established regularity of solutions, we proceed formally. Let $M \leq K$ then multiply the VCHE (1.1) by $\Delta^M v$ and integrate by parts to find

$$\frac{d}{dt} \|\nabla^M v\|_2^2 + \nu \|\nabla^{M+1} v\|_2^2 \leq I_{M,0} + J_{M,0}$$

where

$$I_{M,0} = \sum_{m=0}^M \binom{M}{m} \langle \nabla^m u \cdot \nabla^{M+1} v, \nabla^{M-m} v \rangle$$

$$J_{M,0} = \sum_{m=0}^{M-1} \binom{M-1}{m} \langle \nabla^{M+1} v \cdot \nabla^{m+1} u, \nabla^{M-m} v \rangle$$

Using the Holder inequality, the Sobolev inequality, Corollary 4.3, and the Cauchy inequality we find

$$I_{M,0} = C \sum_{m=0}^M \|\nabla^m u\|_n \|\nabla^{M-m} v\|_{\frac{2n}{n-2}} \|\nabla^{M+1} v\|_2$$

$$\leq C \|v\|_2^2 \|\nabla^{M+1} v\|_2^2 + C \|\nabla v\|_2^2 \|\nabla^M v\|_2^2 + C \sum_{m=2}^M \|\nabla^{m-1} v\|_2^2 \|\nabla^{M+1-m} v\|_2^2 + \frac{\nu}{4} \|\nabla^{M+1} v\|_2^2$$

We treat the other term in a similar way.

$$J_{M,0} \leq C \sum_{m=0}^{M-1} \|\nabla^{M+1} v\|_2 \|\nabla^{m+1} u\|_n \|\nabla^{M-m} v\|_{\frac{2n}{n-2}}$$

$$\leq C \|v\|_2^2 \|\nabla^{M+1} v\|_2^2 + C \|\nabla v\|_2^2 \|\nabla^M v\|_2^2 + C \sum_{m=2}^{M-1} \|\nabla^{m-1} v\|_2^2 \|\nabla^{M+1-m} v\|_2^2 + \frac{\nu}{4} \|\nabla^{M+1} v\|_2^2$$

Together, this leaves

$$\frac{d}{dt} \|\nabla^M v\|_2^2 + \frac{\nu}{2} \|\nabla^{M+1} v\|_2^2 \leq C \|v\|_2^2 \|\nabla^{M+1} v\|_2^2 + C \|\nabla v\|_2^2 \|\nabla^M v\|_2^2 + C \sum_{m=2}^M \|\nabla^{m-1} v\|_2^2 \|\nabla^{M+1-m} v\|_2^2 \quad (6.6)$$

The remaining part of this proof will proceed by induction where the base case is Theorems 6.3 and 6.5. We assume (inductive assumption) that the decay

$$\|\nabla^m v\|_2^2 \leq C(t + 1)^{-m-n/2}$$

holds for all $m < M$ and will show that it holds for $m = M$. With this inductive assumption (6.6) becomes

$$\frac{d}{dt} \|\nabla^M v\|_2^2 + \frac{\nu}{4} \|\nabla^{M+1} v\|_2^2 \leq C(1 + t)^{-n/2} \|\nabla^{M+1} v\|_2^2 + C(1 + t)^{-1-n/2} \|\nabla^M v\|_2^2 + C(1 + t)^{-M-n/2} \quad (6.7)$$

Consider t large enough so that $C(1 + t)^{-n/2} \leq \nu/4$. Subtracting the first term on the RHS, (6.7) becomes

$$\frac{d}{dt} \|\nabla^M v\|_2^2 + \frac{\nu}{4} \|\nabla^{M+1} v\|_2^2 \leq C(1 + t)^{-1-n/2} \|\nabla^M v\|_2^2 + C(1 + t)^{-M-n/2}$$

The next step is to apply the bound $\|\nabla^M v\|_2^2 \leq C$ (Theorem 4.2) with Theorem 6.6 to obtain the decay rate

$$\|\nabla^M v\|_2^2 \leq C(1 + t)^{-n/2}$$

Continuing with a bootstrap argument, placing this new bound into (6.7) and again using Theorem 6.6 the optimal decay rate is obtained and the proof is complete. \square

The next goal is to extend the decay results to time derivatives of the solution. To begin we will compute a frequency bound for the spacial derivatives of solutions to the VCHE. This next lemma will be used inductively with Theorem 6.6 to compute decay rates for the L^2 norm of all time derivatives.

Lemma 6.8. *Let $P \geq 1$ and v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in H^1_\sigma \cap L^1(\mathbb{R}^n)$. If*

$$\|\partial_t^p \nabla^m v\|_2^2 \leq C(1 + t)^{-2p-m-n/2}$$

for all $p < P$ and $m = 0, 1$, then

$$|\partial_t^P \hat{v}(\xi)| \leq C(1 + t)^{-P}$$

for $|\xi|^2 \leq \frac{d}{\nu(1+t)}$. Here the constant depends only on the initial data, the dimension of space, and the constants in the VCHE.

Remark 6.9. Note that the conclusion for $P = 0$ is true by Corollary 6.4.

Proof. The chain rule

$$\frac{d}{dt} \int_0^t f(t, s) ds = f(t, t) + \int_0^t \frac{\partial f(t, s)}{\partial t} ds$$

applied to (6.2) shows

$$\partial_t^P \mathcal{F}(v) = (-1)^P |\xi|^{2P} e^{-t|\xi|^2} \mathcal{F}(v_0) + \sum_{p=0}^{P-1} (-|\xi|^2)^{P-1-p} \partial_t^p \Psi(\xi, t) + \int_0^t (-|\xi|^2)^P e^{-(t-s)|\xi|^2} \Psi(\xi, s) ds$$

We bound Ψ (defined by (6.2)) similar to the proof of Lemma 6.1 but using the assumptions of this lemma.

$$\partial_t^p \Psi(\xi, t) = \partial_t^p A + \partial_t^p B + \partial_t^p C$$

$$\begin{aligned}
 |\partial_t^p A| &= \left| \partial_t^p \sum_j \xi_j \mathcal{F}(u_j v) \right| \\
 &\leq \sum_{l=0}^p C |\xi| \|\partial_t^l v\|_2 \|\partial_t^{p-l} v\|_2 \\
 &\leq C(1+t)^{-p-n/2-1/2} \\
 |\partial_t^p B| &= \left| \partial_t^p \sum_j \mathcal{F}(u_j \nabla v_j^T) \right| \\
 &\leq \sum_{l=0}^p C \|\partial_t^l v\|_2 \|\partial_t^{p-l} \nabla v\|_2 \\
 &\leq C(1+t)^{-p-n/2-1/2} \\
 |\partial_t^p C| &= |\partial_t^p \xi \mathcal{F}(\pi)| \\
 &\leq |\partial_t^p A| + |\partial_t^p B| \\
 &\leq C(1+t)^{-p-n/2-1/2}
 \end{aligned}$$

The bound $|\hat{v}| \leq C$ (Corollary 6.4) and $|\xi| < d/\sqrt{v(1+t)}$ finish the proof. \square

Theorem 6.10. *Let v be the solution of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in H_\sigma^K \cap L^1(\mathbb{R}^n)$. These solutions satisfy the following decay rate for all $m + 2p \leq K$*

$$\|\partial_t^p \nabla^m v\|_2^2 \leq C(t+1)^{-2p-m-n/2}$$

where the constant depends only on the initial data, the dimension of space, and the constants in the VCHE.

Proof. This proof follows closely the proof of Theorem 6.7, we first find an inequality in a form suitable for Theorem 6.6, then using inductive arguments we establish decay. Choose P and M such that $M + 2P \leq K$, then apply ∂_t^P to the VCHE (1.1), multiply by $\partial_t^P \Delta^M v$ and integrate by parts to see

$$\frac{d}{dt} \|\partial_t^P \nabla^M v\|_2^2 + v \|\partial_t^P \nabla^{M+1} v\|_2^2 \leq I_{M,P} + J_{M,P}$$

where

$$\begin{aligned}
 I_{M,P} &= \sum_{p=0}^P \sum_{m=0}^M \binom{P}{p} \binom{M}{m} \langle \partial_t^p \nabla^m u \cdot \partial_t^p \nabla^{M+1} v, \nabla^{M-m} \partial_t^{P-p} v \rangle \\
 J_{M,P} &= \sum_{p=0}^P \sum_{m=0}^{M-1} \binom{P}{p} \binom{M-1}{m} \langle \partial_t^p \nabla^{M+1} v \cdot \nabla \partial_t^p \nabla^m u, \partial_t^{P-p} \nabla^{M-m} v \rangle
 \end{aligned}$$

or, in the case $M = 0$,

$$J_{0,P} = \sum_{p=0}^P \binom{P}{p} \langle \partial_t^p v \cdot \nabla \partial_t^p u, \partial_t^{P-p} v \rangle$$

Use the Holder inequality, the Sobolev inequality, Corollary 4.3, and the Cauchy inequality we find, for $M > 0$,

$$\begin{aligned}
 I_{M,P} &= C \sum_{p=0}^P \sum_{m=0}^M \|\partial_t^p \nabla^m u\|_n \|\partial_t^{P-p} \nabla^{M-m} v\|_{\frac{2n}{n-2}} \|\partial_t^p \nabla^{M+1} v\|_2 \\
 &\leq C \sum_{p=0}^P \|\partial_t^p v\|_2^2 \|\partial_t^{P-p} \nabla^{M+1} v\|_2^2 + C \sum_{p=0}^P \|\partial_t^p \nabla v\|_2^2 \|\partial_t^{P-p} \nabla^M v\|_2^2
 \end{aligned}$$

$$+ C \sum_{p=0}^P \sum_{m=2}^M \|\partial_t^p \nabla^{m-1} v\|_2^2 \|\partial_t^{P-p} \nabla^{M+1-m} v\|_2^2 + \frac{\nu}{4} \|\partial_t^P \nabla^{M+1} v\|_2^2$$

Similarly for the second term if $M > 0$,

$$\begin{aligned} J_{M,P} &\leq C \sum_{p=0}^P \sum_{m=0}^{M-1} \|\partial_t^p \nabla^{M+1} v\|_2 \|\partial_t^p \nabla^{m+1} u\|_n \|\partial_t^{P-p} \nabla^{M-m} v\|_{\frac{2n}{n-2}} \\ &\leq C \sum_{p=0}^P \|\partial_t^p v\|_2^2 \|\partial_t^{P-p} \nabla^{M+1} v\|_2^2 + C \sum_{p=0}^P \|\partial_t^p \nabla v\|_2^2 \|\partial_t^{P-p} \nabla^M v\|_2^2 \\ &\quad + C \sum_{p=0}^P \sum_{m=2}^{M-1} \|\partial_t^p \nabla^{m-1} v\|_2^2 \|\partial_t^{P-p} \nabla^{M+1-m} v\|_2^2 + \frac{\nu}{4} \|\partial_t^P \nabla^{M+1} v\|_2^2 \end{aligned}$$

In the case $M = 0$ the estimate is

$$I_{0,P} + J_{0,P} \leq C \sum_{p=0}^P \|\partial_t^p v\|_2^2 \|\partial_t^{P-p} \nabla v\|_2^2 + \frac{\nu}{4} \|\partial_t^P \nabla v\|_2^2$$

We have shown in the case $M > 0$

$$\begin{aligned} \frac{d}{dt} \|\partial_t^P \nabla^M v\|_2^2 + \frac{\nu}{2} \|\partial_t^P \nabla^{M+1} v\|_2^2 &\leq C \sum_{p=0}^P \|\partial_t^p v\|_2^2 \|\partial_t^{P-p} \nabla^{M+1} v\|_2^2 + C \sum_{p=0}^P \|\partial_t^p \nabla v\|_2^2 \|\partial_t^{P-p} \nabla^M v\|_2^2 \\ &\quad + C \sum_{p=0}^P \sum_{m=2}^M \|\partial_t^p \nabla^{m-1} v\|_2^2 \|\partial_t^{P-p} \nabla^{M+1-m} v\|_2^2 \end{aligned} \tag{6.8}$$

and in the case $M = 0$,

$$\frac{d}{dt} \|\partial_t^P v\|_2^2 + \frac{\nu}{2} \|\partial_t^P \nabla v\|_2^2 \leq C \sum_{p=0}^P \|\partial_t^p v\|_2^2 \|\partial_t^{P-p} \nabla v\|_2^2 \tag{6.9}$$

We now begin the inductive part of our argument where the base case is Theorem 6.7. Pick $P \leq K/2$ and assume (inductive assumption) the decay

$$\|\partial_t^p \nabla^m v\|_2^2 \leq C(t+1)^{-2p-m-n/2} \tag{6.10}$$

holds for all $p < P$ and m such that $2p + m \leq K$. The inductive claim is that the decay holds for $p = P$ with m such that $2P + m \leq K$. To prove the inductive claim it will be shown first that the decay rate holds for $p = P$ and $m = 0$ using (6.9). Then, using (6.8) it will be shown that the decay rate holds for the remaining values of m using another inductive argument.

To establish the decay for $p = P$ and $m = 0$, apply the inductive assumption to (6.9) to find

$$\frac{d}{dt} \|\partial_t^P v\|_2^2 + \frac{\nu}{2} \|\partial_t^P \nabla v\|_2^2 \leq C(1+t)^{-n/2} \|\partial_t^P \nabla v\|_2^2 + C(1+t)^{-2P-1-n}$$

Take t large enough so that $C(1+t)^{-n/2} \leq \nu/4$ and move the first term on the RHS to the left side

$$\frac{d}{dt} \|\partial_t^P v\|_2^2 + \frac{\nu}{2} \|\partial_t^P \nabla v\|_2^2 \leq C(1+t)^{-2P-1-n}$$

Now, an application of Theorem 6.6 with Lemma 6.8 establishes the decay (6.10) for $p = P$ and $m = 0$. This is the base case for the next inductive argument. Assume (inductive assumption) the decay (6.10) holds for $m \leq M + 1$ when $p < P$, and $m < M$ when $p = P$, we will show that this implies the decay holds for $m = M$ and $p = P$. Proving this inductive claim will finish the proof. Begin by applying the inductive assumption to (6.8).

$$\frac{d}{dt} \|\partial_t^P \nabla^M v\|_2^2 + \frac{\nu}{2} \|\partial_t^P \nabla^{M+1} v\|_2^2 \leq C(1+t)^{-n/2} \|\partial_t^P \nabla^{M+1} v\|_2^2 + C(1+t)^{-n/2-1} \|\partial_t^P \nabla^M v\|_2^2 + C(1+t)^{-2p-M-n}$$

Take t large so that $C(1+t)^{-n/2} \leq \nu/4$ and move the first term on the RHS to the LHS. Then apply Theorem 6.6 with Lemma 6.8 to establish the decay rate

$$\|\partial_t^P \nabla^m v\|_2^2 \leq C(t+1)^{-n/2} \tag{6.11}$$

Another bootstrap argument gives the optimal decay and finishes the proof. \square

7. Convergence of the VCHE to the NSE in the whole space

To understand how solutions of the VCHE approach solutions of the NSE as the filter constant α approaches zero we must first understand how a solution u of the Helmholtz equation

$$u - \alpha^2 \Delta u = v \tag{7.1}$$

approaches v when taking α to zero. To begin we state a theorem concerning the Helmholtz equation in all of space, the theorem is standard elliptic theory and no proof is given. This theorem can be proved using elliptic estimates and interpolation or if one multiplies the Helmholtz equation by e^{τ/α^2} and divides by α^2 it can be thought of as the heat equation and the bounds follow from estimates on the heat kernel.

Theorem 7.1. *Given $v \in L^p(\mathbb{R}^n)$, $p \in (1, \infty)$, there exists a $u \in W^{1,p}(\mathbb{R}^n)$ that is a weak solution to the Helmholtz equation $u - \alpha^2 \Delta u = v$. Moreover, this function satisfies*

$$\begin{aligned} \|u\|_p &\leq \|v\|_p \\ \|u\|_q &\leq \frac{C(n, p, q)}{\alpha^{1+\gamma}} \|v\|_p \quad \text{for } \gamma = \frac{n}{2} \left(\frac{1}{p} - \frac{1}{q} \right) < 1 \\ \|\nabla u\|_q &\leq \frac{C(n, p, q)}{\alpha^{3/2+\gamma}} \|v\|_p \quad \text{for } \gamma = \frac{n}{2} \left(\frac{1}{p} - \frac{1}{q} \right) < \frac{1}{2} \end{aligned}$$

If $n(2/p - 1) < 1$ then the solution is unique.

Proof. Standard elliptic theory. \square

A solution u of the Helmholtz equation corresponding to v will approach v weakly as the filter parameter tends to zero. Indeed, fix $v \in L^p(\mathbb{R}^n)$ and let $\{\alpha_i\}$ be a sequence tending to zero. By the above theorem, for each α_i there is a weak solution $u_{\alpha_i} \in W^{1,p}(\mathbb{R}^n)$ of the Helmholtz equation such that

$$\int_{\mathbb{R}^n} u_{\alpha_i} \cdot \phi \, dx + \alpha_i^2 \int_{\mathbb{R}^n} \nabla u_{\alpha_i} \cdot \nabla \phi \, dx = \int_{\mathbb{R}^n} v \cdot \phi \, dx$$

The functions u_{α_i} are bounded in $L^p(\mathbb{R}^n)$ independent of α_i , so there exists a (possible) subsequence α_{i_j} with a weak limit in $L^p(\mathbb{R}^n)$. Also, for $1/p + 1/q = 1$

$$\alpha_i^2 \int_{\mathbb{R}^n} \nabla u \cdot \nabla \phi \, dx \leq \alpha_i^2 \|\nabla u\|_p \|\nabla \phi\|_q \leq C(n) \alpha_i^{1/2} \|v\|_p \|\nabla \phi\|_q$$

which approaches zero as $\alpha_i \rightarrow 0$. This proves that $u_{\alpha_i} \rightharpoonup v$ in $L^p(\mathbb{R}^n)$. We can do better than this if v is sufficiently differentiable.

Theorem 7.2. *Let $v \in W^{1,p}(\mathbb{R}^n)$ and let u be the corresponding solution to the Helmholtz equation (7.1). Then*

$$\|u - v\|_q \leq C(n, p, q) \alpha^{1/2-\gamma} \|\nabla v\|_p \quad \text{for } \gamma = \frac{n}{2} \left(\frac{1}{p} - \frac{1}{q} \right) < \frac{1}{2}$$

If α is a sequence tending to zero and u_α are solutions the Helmholtz equation, then $u_\alpha \rightarrow v$ strongly in $L^q(\mathbb{R}^n)$ for $1/p - 1/q < 1/n$.

Proof. If u and v satisfy the Helmholtz equation, then

$$\|u - v\|_q \leq \alpha^2 \|\Delta u\|_q \tag{7.2}$$

The Helmholtz equation is linear, so the derivatives of the functions obey the relation $\nabla u - \alpha^2 \Delta \nabla u = \nabla v$. Applying Theorem 7.1 to this PDE with the restriction on γ allows the bound

$$\|\Delta u\|_q \leq \frac{C(n, p, q)}{\alpha^{3/2+\gamma}} \|\nabla v\|_p$$

Together with (7.2),

$$\|u - v\|_q \leq C(n, p, q) \alpha^{1/2-\gamma} \|\nabla v\|_p$$

The second statement is an immediate consequence of this. \square

In [9,10], the authors show how the solutions of the VCHE approach a solution of the NSE weakly when the parameter in the filter tends to zero ($\alpha \rightarrow 0$). We will show how solutions to the VCHE approach solutions to the NSE strongly as $\alpha \rightarrow 0$ when the solution to the NSE is known to be regular. The proof requires estimates on the solution of the VCHE which are independent of α , but in regions of time where the NSE is known to be regular by some functional analytic arguments, the passive bound on the filter make this assumption reasonable.

For example, solutions of the Navier–Stokes equation obey the Prodi Inequality [19]

$$\frac{d}{dt} \|\nabla u\|_2^2 \leq C \|\nabla u\|_2^{2n}$$

This can be used to prove existence of a strong solution in some time interval $[0, T]$ or regular solutions for all time if the initial data is small. The Prodi inequality is proved through energy estimates, using the passive bound for the filter in Theorem 7.1 and following the same energy arguments allows the same bound for solutions of the VCHE. This bound will be independent of α , so we can apply the following theorem to conclude that in some closed interval $[0, T]$ the solution of the VCHE approaches a solution to the NSE strongly.

Theorem 7.3. *Let $\{\alpha_i\}$ be a sequence of filter coefficients tending to zero and v_{α_i} the solutions of the VCHE (1.1) constructed in Theorem 4.2 with $\Omega = \mathbb{R}^n$ corresponding to $w_0 \in H_o^1(\mathbb{R}^n)$. Let w be the solution the NSE with initial conditions w_0 . In any time interval $[0, T]$ where a solution to the NSE is known to be regular, if there exists a bound*

$$\sup_{\alpha_i} \sup_{t \in [0, T]} (\|v_{\alpha_i}\|_l + \|\nabla v_{\alpha_i}\|_l) < C$$

which is independent of α , then v_α approaches w strongly in $L^\infty([0, T], L^q(\mathbb{R}^n))$ as $\alpha \rightarrow 0$, where $q = \frac{2l}{l-2}$.

Proof. We begin with a mild form of the solutions to both problems. We are working in a time domain with known regularity so these are the unique solutions. If \mathbb{P} is the Leray projector onto the divergence free subspace of L^2 and Φ is the heat kernel, then

$$w(t) = \Phi(t) * w_0 - \int_0^t \Phi(t-s) * \mathbb{P}[w \cdot \nabla w](s) ds$$

$$v(t) = \Phi(t) * w_0 - \int_0^t \Phi(t-s) * \mathbb{P}\left[u \cdot \nabla v + \sum u_j \nabla v_j\right](s) ds$$

By adding and subtracting cross terms we see

$$w(t) - v(t) = - \int_0^t \Phi(t-s) * \mathbb{P}\left[(w-u) \cdot \nabla w + u \cdot \nabla(w-v)\right] + \mathbb{P}\left[u_j \nabla(v_j - w_j) + (u_j - w_j) \nabla w_j\right](s) ds$$

The first term in the integrand is bounded using Young’s inequality and the definition of the projector

$$\begin{aligned} \|\Phi(t-s) * \mathbb{P}[(w-u) \cdot \nabla w](s)\|_q &\leq \|\Phi(t-s)\|_p \|(w-u) \cdot \nabla w\|_2 \\ &\leq \|\Phi(t-s)\|_p \|w-u\|_q \|\nabla w\|_l \end{aligned}$$

where $1/q + 1 = 1/p + 1/2$ and $1/2 = 1/q + 1/l$. Using Theorem 7.2 with $\gamma = (1/2 - 1/q)n/2 < 1/2$ we obtain

$$\|\Phi(t-s) * \mathbb{P}[(w-u) \cdot \nabla w](s)\|_q \leq \|\Phi(t-s)\|_p \|\nabla w\|_l (\|w-v\|_q + C\alpha^{1/2-\gamma} \|\nabla v\|_2)$$

The fourth term can be bounded in essentially the same way. We approach the second term in a slightly different way, by first passing the derivative to the heat kernel. These functions are smooth functions of the whole space so the projector will commute with the derivative.

$$\|\Phi(t-s) * \mathbb{P}[u \cdot \nabla(w-v)]\|_q = \|\nabla \Phi(t-s) * \mathbb{P}[u \cdot (w-v)]\|_q$$

Then by using Young’s inequality and the definition of the projector

$$\|\Phi(t-s) * \mathbb{P}[u \cdot \nabla(w-v)]\|_q \leq \|\nabla \Phi(t-s)\|_p \|u\|_l \|w-v\|_q$$

To bound the third term, start with the product rule and again pass the derivative the heat kernel

$$\|\Phi(t-s) * \mathbb{P}[u_j \nabla(v_j - w_j)]\|_q = \|\nabla \Phi(t-s) * \mathbb{P}[u_j(w_j - v_j)]\|_q + \|\Phi(t-s) * \mathbb{P}[\nabla u_j(w_j - v_j)]\|_q$$

Then, using Young’s inequality and again the definition of the projector

$$\|\Phi(t-s) * \mathbb{P}[u_j \nabla(v_j - w_j)]\|_q \leq (\|\nabla \Phi(t-s)\|_p \|u_j\|_l + \|\Phi(t-s)\|_p \|\nabla u_j\|_l) \|w_j - v_j\|_q$$

Putting all of these bounds together and estimating the heat kernel yields

$$\|v-w\|_q \leq A\alpha^{1/2-\gamma} + B \int_0^t \frac{1}{(t-s)^\delta} \|v-w\|_q(s) ds$$

$$A = C \int_0^t \|\nabla v\|_2 ds$$

$$B = C \sup_{s \in [0, T]} (\|\nabla w\|_l + \|u\|_l + \|\nabla u\|_l)$$

Here, $\delta = 1/2 + (1 - 1/p)n/2 < 1$ by the assumption $l > n$. Application of the Gronwall inequality finishes the proof. For example, a modified Gronwall inequality [26] now shows

$$\|v-w\|_q \leq A\alpha^{1/2-\gamma} \Upsilon(B\Gamma(1-\delta)t^\delta)$$

$$\Upsilon(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n(1-\delta) + 1)}$$

See also [1]. Letting $\alpha \rightarrow 0$ we see that $v \rightarrow w$ strongly in $L^q(\mathbb{R}^3)$. \square

Acknowledgement

The authors would like to thank an anonymous referee for helpful advice and comments.

Appendix A

Here we construct a weak solution to the VCHE. Due to the close relation between the VCHE and the Navier–Stokes equation, our proof is similar to known existence proofs for the NSE. See, for example, [1,2,7,13,15,28]. First, we construct solutions on any bounded Ω with smooth boundary using the Galerkin method, this is where the Stokes operator is known to be compact thanks to the Poincaré Inequality. Special care is taken to use inequalities which do not depend on the size of Ω so we can use these solutions to prove existence of a weak solution in unbounded

domains. The only step that requires Ω bounded is in the compact inclusion used to obtain the strong convergence necessary to pass limits through the non-linear term. This problem is overcome by working in the compact support of the test functions.

To begin we recall a standard but useful elliptic estimate.

Remark A.1. Let $2 \leq n \leq 4$ and $\Omega \subset \mathbb{R}^n$ be an open set with smooth boundary. If $u \in H^2_\sigma$, and $v \in L^2_\sigma$ satisfy the Helmholtz equation

$$u - \alpha^2 \Delta u = v$$

on Ω , then

$$\|u\|_n \leq C \|v\|_2 \tag{A.1}$$

$$\|\nabla u\|_n \leq C \|v\|_2 \tag{A.2}$$

$$\|u\|_2^2 + 2\alpha^2 \|\nabla u\|_2^2 + \alpha^4 \|\Delta u\|_2^2 = \|v\|_2^2 \tag{A.3}$$

where the constants C depend only on α and n .

The stationary Stokes equation

$$\Delta u + \nabla p = v$$

$$u|_{\partial\Omega} = 0$$

is known to have a solution $u \in H^1_\sigma(\Omega)$ for each $v \in L^2_\sigma(\Omega)$ when Ω is an open bounded set. Solving this PDE defines an operator $L^2_\sigma(\Omega) \rightarrow H^1_\sigma(\Omega)$. Composing this with the compact inclusion $H^1_\sigma(\Omega) \rightarrow L^2_\sigma(\Omega)$ gives a compact and self-adjoint operator $L^2_\sigma(\Omega) \rightarrow L^2_\sigma(\Omega)$, which we call the *Stokes operator*.

Lemma A.2. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. There exists an orthonormal basis of $L^2_\sigma(\Omega)$, $\{\omega_j\}_{j=1}^\infty$, where each ω_j is an eigenfunction of the Stokes Operator on Ω . The associated eigenvalues are all positive real numbers and the eigenvectors are smooth and approach zero on the boundary. Let $H_m = \text{span}\{\omega_1, \dots, \omega_m\}$ and let P_m be the orthogonal projection $P_m : L^2_\sigma(\Omega) \rightarrow H_m$. Given $v_0 \in C^\infty_0(\Omega)$, for each m there is an approximate solution

$$v_m = \sum_{j=1}^m g_{jm}(t) \omega_j$$

and

$$u_m = \sum_{j=1}^m \frac{g_{jm}(t)}{1 + \alpha^2 \lambda_j} \omega_j$$

where $g_{jm} \in C^1([0, T_m])$ for some time T_m . These approximate solutions satisfy the following relations:

$$\begin{aligned} \langle \partial_t v_m, \omega_i \rangle + \langle u_m \cdot \nabla v_m, \omega_i \rangle - \langle \omega_i \cdot \nabla v_m, u_m \rangle &= v \langle \Delta v_m, \omega_i \rangle \\ v_m(0) &= P_m v_0 \end{aligned} \tag{A.4}$$

Proof. Owing to spectral theory the Stokes operator (self-adjoint, compact) has a countable number of positive eigenvalues λ_i , and associated smooth, divergence-free eigenfunctions ω_i which form a basis for $L^2_\sigma(\Omega)$. These functions satisfy the relation

$$-\Delta \omega_i = \lambda_i \omega_i$$

To determine the scalars g_{im} we construct a system of m ODE's.

$$\frac{dg_{im}}{dt} + v \lambda_i g_{im} + \sum_{j,k=1}^m \frac{g_{jm} g_{km}}{1 + \alpha^2 \lambda_k} (\langle \omega_k \cdot \nabla \omega_j, \omega_i \rangle - \langle \omega_i \cdot \nabla \omega_j, \omega_k \rangle) = 0$$

Local existence of solutions to ODE's give existence of solutions g_{im} , which are defined for some time interval $[0, T_m]$. \square

The bounds in the next lemma will prove that T_m can be bounded independent of m , and in fact $T_m = \infty$ for all m .

Lemma A.3. *For $2 \leq n \leq 4$, the approximate solutions constructed in Lemma A.2 have the following bounds, which do not depend on T, Ω or m .*

$$\begin{aligned} \|v_m\|_{L^\infty([0, T]; L^2_\sigma(\Omega))} + \|\nabla v_m\|_{L^2([0, T]; L^2_\sigma(\Omega))} &\leq C(n, \alpha, \nu, \|v_0\|_2) \\ \|\partial_t v_m\|_{L^2([0, T]; (H^1_\sigma)'(\Omega))} &\leq C(n, \alpha, \nu, \|v_0\|_2) \end{aligned}$$

Proof. Similar to formal multiplication of the VCHE (1.1) by u , multiply (A.4) by $\frac{1}{1+\alpha^2\lambda_i} g_{im}$, sum, then apply Lemma 3.1 to see

$$\|u_m\|_2^2 + \alpha^2 \|\nabla u_m\|_2^2 + 2\nu \int_0^T \|\nabla u_m\|_2^2 dt + 2\alpha^2 \nu \int_0^T \|\Delta u_m\|_2^2 dt = \|u_0\|_2^2 + \alpha^2 \|\nabla u_0\|_2^2 \tag{A.5}$$

With the Poincaré inequality and (A.3) this becomes first bound in the theorem. Using (A.1) we deduce

$$\|u_m\|_n^2 + \int_0^\infty \|\nabla u_m\|_n^2 dt < C(n, \alpha, \nu, \|u_0\|_2, \|\nabla u_0\|_2) \tag{A.6}$$

To bound the derivative start with (A.4). Any $\phi \in H^1_\sigma$ can be written as a sum of the ω_i so each approximate solution satisfies

$$\langle \partial_t v_m, \phi \rangle + \langle u_m \cdot \nabla v_m, \phi \rangle - \langle \phi \cdot \nabla v_m, u_m \rangle = \nu \langle \Delta v_m, \phi \rangle$$

After integration by parts and applying the Hölder inequality with the Gagliardo–Nirenberg–Sobolev inequality we find

$$|\langle \partial_t v_m, \phi \rangle| \leq C \|u_m\|_n \|\nabla v_m\|_2 \|\nabla \phi\|_2 + C \|\nabla v_m\|_2 \|\nabla \phi\|_2$$

As ϕ was chosen arbitrarily we conclude

$$\|\partial_t v_m\|_{(H^1_\sigma)'} \leq C(\|u_m\|_n \|\nabla v_m\|_2 + \|\nabla v_m\|_2)$$

This, together with (A.5) and (A.6), proves the second bound in the theorem. \square

Theorem A.4. *Let $\Omega \in \mathbb{R}^n$, $2 \leq n \leq 4$ be a bounded set with smooth boundary and $v_0 \in C^\infty_0(\Omega)$. Then, there exists a weak solution to the VCHE (1.1) in the sense of Definition (4.1).*

Proof. Thanks to Lemmas A.2 and A.3 we only need to prove the convergence of the approximate solutions. Lemma A.3 shows how the sequence v_m remains bounded, so using a possible subsequence and the Banach–Alaoglu theorem there exists a function

$$\begin{aligned} v &\in L^\infty([0, T]; L^2_\sigma(\Omega)) \cap L^2([0, T]; H^1_\sigma(\Omega)) \\ \partial_t v &\in L^2([0, T]; (H^1_\sigma)'(\Omega)) \end{aligned}$$

such that

$$v_m \rightharpoonup v \quad \text{in } L^\infty([0, T]; L^2_\sigma(\Omega)) \text{ weak } * \tag{A.7}$$

$$v_m \rightharpoonup v \quad \text{in } L^2([0, T]; H^1_\sigma(\Omega)) \text{ weakly} \tag{A.8}$$

We will now show that v is a weak solution to the VCHE (1.1).

By the construction of our approximate solutions and integration by parts, we know for any basis vector $\omega_j \in L^2_\sigma(\Omega)$ and any smooth scalar function of time $\phi_j(t)$ such that $\phi_j(T) = 0$,

$$\int_0^T \langle v_m, \phi'_j \omega_j \rangle ds + \int_0^T \langle u \cdot \nabla v, \phi \omega_j \rangle ds + \int_0^T \langle \phi \omega_j \cdot \nabla u, v \rangle ds + \int_0^T \langle \nabla v, \nabla \phi \omega_j \rangle ds = \langle v_m(0), \phi(0) \omega_j \rangle$$

The convergence (A.7) and (A.8) implies

$$\begin{aligned} \int_0^t \langle v_m, \phi'_j \omega_j \rangle ds &\rightarrow \int_0^t \langle v, \phi'_j \omega_j \rangle ds \\ \int_0^t \langle \nabla v_m, \phi_j \nabla \omega_j \rangle ds &\rightarrow \int_0^t \langle \nabla v, \phi_j \nabla \omega_j \rangle ds \end{aligned}$$

Also,

$$\langle v_m(0), \phi_j(0) \omega_j \rangle = \langle P_m(v_0), \phi_j(0) \omega_j \rangle \rightarrow \langle v_0, \phi_j(0) \omega_j \rangle \tag{A.9}$$

Passing through the non-linear terms will require strong convergence, so we use the fact that the bounds in Lemma A.3 imply (see [7], Lemma 8.2) the existence of a possible subsequence v_m such that

$$v_m \rightarrow v \text{ in } L^2([0, T]; L^2(\Omega)) \text{ strongly} \tag{A.10}$$

Theorem 7.1 give the existence of a function u which satisfies

$$u - \alpha^2 \Delta u = v$$

Similar to (A.3),

$$\|u_m - u\|_2^2 + \alpha^2 \|\nabla(u_m - u)\|_2^2 + \alpha^4 \|\nabla^2(u_m - u)\|_2^2 = \|v_m - v\|_2^2$$

In particular, applying the Gagliardo–Nirenberg–Sobolev Inequality shows $\|u_m - u\|_n^2 \leq C \|v_m - v\|_2^2$. This, with the strong convergence (A.10), shows how u_m approaches u strongly.

We can now prove the convergence of the non-linear terms

$$\begin{aligned} \int_0^T \langle u_m \cdot \nabla v_m, \phi_j \omega_j \rangle ds &\rightarrow \int_0^T \langle u \cdot \nabla v, \phi_j \omega_j \rangle ds \\ \int_0^t \langle \phi_j \omega_j \cdot \nabla v_m, u_m \rangle ds &\rightarrow \int_0^t \langle \phi_j \omega_j \cdot \nabla v, u \rangle ds \end{aligned}$$

Indeed, adding and subtracting the cross terms, then using the Hölder Inequality, the Gagliardo–Nirenberg–Sobolev Inequality

$$|\langle u_m \cdot \nabla v_m, \phi \omega_j \rangle - \langle u \cdot \nabla v, \phi_j \omega_j \rangle| \leq A_1 + B_1$$

$$\begin{aligned} A_1 &= | \langle (u_m - u) \cdot \nabla v_m, \phi_j \omega_j \rangle | \\ &\leq \|u_m - u\|_n \|\nabla v_m\|_2 \|\phi_j \omega_j\|_{\frac{2n}{n-2}} \\ &\leq \|v_m - v\|_2 \|\nabla v_m\|_2 \|\phi_j \nabla \omega_j\|_2 \end{aligned}$$

Due to the strong convergence (A.10) the bound in Lemma A.3, and the Hölder inequality, we see $\int_0^T A_1 ds \rightarrow 0$. Similarly,

$$\begin{aligned} B_1 &= | \langle u \cdot \nabla(v_m - v), \phi_j \omega_j \rangle | \\ &= | \langle u \cdot \nabla \phi_j \omega_j, (v_m - v) \rangle | \\ &\leq \|u\|_n \|\phi_j \nabla \omega_j\|_{\frac{2n}{n-2}} \|v_m - v\|_2 \\ &\leq C \|v\|_2 \|\phi_j \nabla \omega_j\|_{\frac{2n}{n-2}} \|v_m - v\|_2 \end{aligned}$$

Again, owing to (A.10), Lemma A.3, and the Hölder inequality, $\int_0^T B_1 ds \rightarrow 0$. Putting this together,

$$\int_0^T \langle u_m \cdot \nabla v_m, \phi_j \omega_j \rangle ds \rightarrow \int_0^T \langle u \cdot \nabla v, \phi_j \omega_j \rangle ds$$

The remaining non-linear term is handled in a similar way

$$|\langle \phi_j \omega_j \cdot \nabla v_m, u_m \rangle - \langle \phi_j \omega_j \cdot \nabla v, u \rangle| \leq A_2 + B_2$$

$$\begin{aligned} A_2 &= |\langle \phi_j \omega_j \cdot \nabla v_m, (u_m - u) \rangle| \\ &\leq \|\phi_j \omega_j\|_{\frac{2n}{n-2}} \|\nabla v_m\|_2 \|u_m - u\|_n \\ &\leq C \|\phi_j \nabla \omega_j\|_2 \|\nabla v_m\|_2 \|v_m - v\|_2 \end{aligned}$$

$$\begin{aligned} B_2 &= |\langle \phi_j \omega_j \cdot \nabla (v_m - v), u \rangle| \\ &= |\langle \phi_j \omega_j \cdot \nabla u, v_m - v \rangle| \\ &\leq \|\phi_j \omega_j\|_{\frac{2n}{n-2}} \|v_m - v\|_2 \|\nabla u\|_n \\ &\leq C \|\phi_j \nabla \omega_j\|_2 \|v_m - v\|_2 \|v\|_2 \end{aligned}$$

Applying (A.10) with Lemma A.3 and the Hölder inequality shows

$$\int_0^T \langle \phi_j \omega_j \cdot \nabla v_m, u_m \rangle ds \rightarrow \int_0^T \langle \phi_j \omega_j \cdot \nabla v, u \rangle ds$$

Since the ω_j are dense in L^2_σ and ϕ_j is an arbitrary smooth function the proof is complete. \square

Corollary A.5. *The conclusions of Theorem A.4 hold with the relaxed hypothesis $v_0 \in L^2_\sigma(\Omega)$.*

Proof. Note that all of the bounds attained in Lemma A.3 and used in the proof of the previous theorem depend only on the L^2 norm of the initial data. Let $v_0^i \in C_0^\infty(\Omega)$ be a sequence of functions approaching v_0 strongly in H_0^1 such that

$$\|v_0^i\|_{H_0^1} \leq \|v_0\|_{H_0^1}$$

Such a sequence can be constructed using standard mollifiers and cutoff functions. Considering each v_0^i as initial data, Theorem A.4 and its corollary give the existence of a weak solution v^i in the sense of Definition 4.1. Applying (A.3), we see that these weak solutions satisfy the bounds

$$\|v^i\|_{L^\infty([0,T];L^2_\sigma(\Omega))} + \|\nabla v^i\|_{L^2([0,T];L^2_\sigma(\Omega))} \leq C(n, \alpha, v, \|v_0\|_2)$$

$$\|\partial_t v^i\|_{L^2([0,T];(H_0^1)'(\Omega))} \leq C(n, \alpha, v, \|v_0\|_2)$$

and for each $\phi \in H_0^1$ the relation

$$\int_0^T \langle v^i, \partial_t \phi \rangle ds + \int_0^T \langle u^i \cdot \nabla v^i, \phi \rangle ds + \int_0^T \langle \phi \cdot \nabla u^i, v^i \rangle ds + \int_0^T \langle \nabla v^i, \nabla \phi \rangle ds = \langle v_0, \phi \rangle \tag{A.11}$$

As before, using the Banach–Alaoglu Theorem and extracting a possible subsequence implies that there exists a function

$$v \in L^\infty([0, T]; L^2_\sigma(\Omega)) \cap L^2([0, T]; H_0^1(\Omega))$$

$$\partial_t v \in L^2([0, T]; (H_0^1)'(\Omega))$$

such that

$$v^i \rightharpoonup v \quad \text{in } L^\infty([0, T]; L^2_\sigma(\Omega)) \text{ weak } *$$

$$v^i \rightharpoonup v \quad \text{in } L^2([0, T]; H^1_\sigma(\Omega)) \text{ weakly}$$

Passing the limits through (A.11) follows by the same steps as in the proof of the previous theorem. \square

Theorem A.6. *Let $v_0 \in L^2_\sigma(\mathbb{R}^n)$. Then, there exists a weak solution in the sense of Definition 4.1, with initial data v_0 in the whole space \mathbb{R}^n , $2 \leq n \leq 4$.*

Proof. Let R_i be a sequence tending to infinity and χ_{R_i} a smooth cutoff function which is equal to 1 inside the ball of radius $R_i - \epsilon$ and zero on the boundary of the ball with radius R_i . Corollary A.5 now gives existence of a weak solution v^{R_i} on the ball of radius R_i with initial conditions $v_0 \chi_{R_i}$. Extend v^{R_i} to all of \mathbb{R}^n by setting it equal to zero outside the ball of radius R_i . All of the bounds in Lemma A.3 were found independent of the size of Ω , so here they hold independent of R_i . Using the Banach–Alaoglu Theorem we have the existence of a function

$$v \in L^\infty([0, T]; L^2_\sigma(\mathbb{R}^n)) \cap L^2([0, T]; H^1_\sigma(\mathbb{R}^n))$$

$$\partial_t v \in L^2([0, T]; (H^1_\sigma)'(\mathbb{R}^n))$$

such that

$$v^{R_i} \rightharpoonup v \quad \text{in } L^\infty([0, T]; L^2_\sigma(\mathbb{R}^n)) \text{ weak } * \tag{A.12}$$

$$v^{R_i} \rightharpoonup v \quad \text{in } L^2([0, T]; H^1_\sigma(\mathbb{R}^n)) \text{ weakly} \tag{A.13}$$

There exists an orthogonal basis $\{\phi_i\}$ for $L^2([0, T]; (\mathbb{R}^n))$ where each function in the basis is smooth and has compact support in space. For R_i larger then the support of ϕ , Theorem A.4 with it’s corollary show

$$\int_0^T \langle v^{R_i}, \partial_t \phi \rangle ds + \int_0^T \langle u^{R_i} \cdot \nabla v^{R_i}, \phi \rangle ds + \int_0^T \langle \phi \cdot \nabla u^{R_i}, v^{R_i} \rangle ds + \int_0^T \langle \nabla v^{R_i}, \nabla \phi \rangle ds = \langle v_0, \phi \rangle$$

The limit $m \rightarrow \infty$ can be passed through the linear terms just as before. In the (compact) support of each basis function ϕ_j , we have the strong convergence to pass the limit through the non-linear terms. A diagonal argument shows this convergence holds as $R_i \rightarrow \infty$. \square

In the above existence theorems, the pressure term can be found by either taking the divergence of the VCHE and solving the Poisson equation, or using a famous result of de Rham. See, for example, [28].

Theorem A.7. *The solutions to the VCHE constructed in Theorems A.4 and A.6, with initial data in $v_0 \in H^K_\sigma$, satisfy the bound*

$$\|\nabla^M v\|_2^2 + \int_0^t \|\nabla^{M+1} v\|_2^2 \leq C(n, \alpha, \nu, \|v\|_{H^K_0}) \tag{A.14}$$

for all $M \leq K$.

Proof. We will do the calculations formally and note that these bounds can be applied to the approximate solutions constructed in Theorem A.2, this proof proceeds by induction. The inductive assumption is that the following bound holds for all $m < M$.

$$\|\nabla^m v\|_2^2 + \int_0^T \|\nabla^{m+1} v\|_2^2 dt \leq C$$

The base case ($m = 0$) is true by Lemma A.3, we will now show that it holds for $m = M$. The bound A.1 with the inductive assumption implies

$$\|\nabla^m u\|_n^2 + \int_0^T \|\nabla^{m+1} u\|_n^2 dt \leq C$$

Multiply the VCHE (1.1) by $\Delta^M v$ and integrate by parts to find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla^M v\|_2^2 + \nu \|\nabla^{M+1} v\|_2^2 &\leq I_M + J_M \\ I_M &= \sum_{m=0}^M \binom{M}{m} \langle \nabla^m u \cdot \nabla \nabla^{M-m} v, \nabla^M v \rangle \\ J_M &= \sum_{m=0}^M \binom{M}{m} \langle \nabla^M v \cdot \nabla \nabla^m u, \nabla^{M-m} v \rangle \end{aligned} \tag{A.15}$$

The two integrals on the RHS are estimated essentially the same way. The key difference is that in the first one we use the relation $\langle u \cdot \nabla v, v \rangle = 0$ while in the second we can place an extra derivative on u .

With application of $\langle u, \nabla v, v \rangle = 0$ the first bound becomes

$$I_M = \sum_{m=1}^M \binom{M}{m} \langle \nabla^m u \cdot \nabla \nabla^{M-m} v, \nabla^M v \rangle$$

Hölder’s inequality, the Sobolev inequality, and Cauchy’s inequality show

$$\begin{aligned} I_M &\leq C \sum_{m=1}^M \|\nabla^m u\|_n \|\nabla^{M+1-m} v\|_2 \|\nabla^M v\|_{\frac{2n}{n-2}} \\ &\leq C \sum_{m=1}^M \|\nabla^m u\|_n^2 \|\nabla^{M+1-m} v\|_2^2 + \frac{\nu}{4} \|\nabla^{M+1} v\|_2^2 \end{aligned}$$

Similarly for the second term

$$\begin{aligned} J_M &\leq C \sum_{m=0}^M \|\nabla^M v\|_{\frac{2n}{n-2}} \|\nabla^{m+1} u\|_n \|\nabla^{M-m} v\|_2 \\ &\leq \frac{\nu}{4} \|\nabla^{M+1} v\|_2^2 + C \sum_{m=0}^M \|\nabla^{m+1} u\|_n^2 \|\nabla^{M-m} v\|_2^2 \end{aligned}$$

Equation (A.15) becomes

$$\frac{d}{dt} \|\nabla^M v\|_2^2 \leq C \sum_{m=0}^M \|\nabla^{m+1} u\|_n^2 \|\nabla^{M-m} v\|_2^2$$

The Gronwall inequality with application of the inductive assumption finish the proof. \square

Theorem A.8. *The solution to the VCHE constructed in Theorems A.4 and A.6, with initial data in $v_0 \in H_\sigma^K$, satisfies the bounds*

$$\|\partial_t^p \nabla^m v\|_2^2 + \int_0^t \|\partial_t^p \nabla^{m+1} v\|_2^2 \leq C(n, \alpha, \nu, \|v\|_{H_0^K}) \tag{A.16}$$

for all $M + 2P \leq K$.

Proof. To prove this, we will bound the time derivatives of the solution in terms of the space derivatives, then use the previous theorem to establish regularity. We will do the calculations formally and note that these bounds can be applied to the approximate solutions constructed in Theorem A.2.

Apply $\partial_t^P \nabla^M$ to the solution of the VCHE, from this we have the inequality

$$\|\partial_t^{P+1} \nabla^M v\|_2^2 \leq C (\|\partial_t^P \nabla^{M+2} v\|_2^2 + \|\partial_t^P \nabla^M (u \cdot \nabla v)\|_2^2 + \|\partial_t^P \nabla^M (v \cdot \nabla u^T)\|_2^2)$$

Using the Gagliardo–Nirenberg–Sobolev inequality and (A.1) we can bound the first term on the right-hand side as

$$\begin{aligned} \|\partial_t^P \nabla^M (u \cdot \nabla v)\|_2^2 &= \sum_{p=0}^P \sum_{m=0}^M \binom{P}{p} \binom{M}{m} \|\partial_t^p \nabla^m u\|_n^2 \|\partial_t^{P-p} \nabla^{M+1-m} v\|_{\frac{2n}{n-2}}^2 \\ &\leq C \sum_{p=0}^P \sum_{m=0}^M \|\partial_t^p \nabla^m v\|_2^2 \|\partial_t^{P-p} \nabla^{M+2-m} v\|_2^2 \end{aligned}$$

Similarly for the second term,

$$\begin{aligned} \|\partial_t^P \nabla^M (v \cdot \nabla u^T)\|_2^2 &= \sum_{p=0}^P \sum_{m=0}^M \binom{P}{p} \binom{M}{m} \|\partial_t^p \nabla^{m+1} u\|_n^2 \|\partial_t^{P-p} \nabla^{M-m} v\|_{\frac{2n}{n-2}}^2 \\ &\leq C \sum_{p=0}^P \sum_{m=0}^M \|\partial_t^p \nabla^{m+1} v\|_2^2 \|\partial_t^{P-p} \nabla^{M+1-m} v\|_2^2 \end{aligned}$$

Putting this together we can deduce

$$\|\partial_t^{P+1} \nabla^M v\|_2^2 \leq C \|\partial_t^P v\|_{H_0^{M+2}}^2$$

This implies, for all M, P , such that $M + 2P \leq K$,

$$\|\partial_t^P \nabla^M v\|_2^2 \leq C \|v\|_{H_0^K}^2$$

Appealing to Theorem A.7 finishes the proof. \square

The previous theorem demonstrates how the norms $\|v\|_{H^m}$ and $\|v\|_{H^{m+1}}$ can be bounded in terms of $\|v_0\|_{H^m}$. Since the PDE is parabolic we can expect regularity from interior estimates but the bounds will not depend explicitly on the initial conditions.

Theorem A.9. *The solution to the VCHE constructed in Theorems A.4 and A.6 is unique.*

Proof. Let v and w be two solutions to the VCHE (1.1) with the same initial conditions. Let u and ω be the corresponding “filtered” velocities. The difference solves the PDE

$$(v - w)_t - \nu \Delta (v - w) + \nabla p + u \cdot \nabla v - \omega \cdot \nabla w + v \cdot \nabla u^T - w \cdot \nabla \omega^T = 0$$

with zero initial conditions. Multiplying this relation by $v - w$ and integrating by parts leaves

$$\frac{1}{2} \frac{d}{dt} \|v - w\|_2^2 + \nu \|\nabla (v - w)\|_2^2 = \langle (u - \omega) \cdot \nabla w, (v - w) \rangle + \langle (v - w) \cdot \nabla (u - \omega), v \rangle + \langle (v - w) \cdot \nabla \omega, v - w \rangle$$

Using Hölder’s inequality, the Gagliardo–Nirenberg–Sobolev inequality, Cauchy’s inequality, and (A.6), estimate the RHS

$$\begin{aligned} \langle (u - \omega) \cdot \nabla w, (v - w) \rangle &\leq \|u - \omega\|_n \|\nabla w\|_2 \|v - w\|_{\frac{2n}{n-2}} \\ &\leq C \|v - w\|_2^2 \|\nabla w\|_2^2 + \frac{\nu}{4} \|\nabla (v - w)\|_2^2 \\ \langle (v - w) \cdot \nabla (u - \omega), v \rangle &\leq \|v - w\|_2 \|\nabla (u - \omega)\|_n \|v\|_{\frac{2n}{n-2}} \end{aligned}$$

$$\begin{aligned} &\leq C \|v - w\|_2^2 \|\nabla v\|_2^2 + \frac{\nu}{8} \|\nabla(v - w)\|_2^2 \\ \langle (v - w) \cdot \nabla \omega, v - w \rangle &\leq \|v - w\|_2 \|\nabla \omega\|_n \|v - w\|_{\frac{2n}{n-2}} \\ &\leq C \|v - w\|_2^2 \|w\|_2^2 + \frac{\nu}{8} \|\nabla(v - w)\|_2^2 \end{aligned}$$

After using the bounds in Lemma A.3 we have

$$\frac{1}{2} \frac{d}{dt} \|v - w\|_2^2 + \frac{\nu}{2} \|\nabla(v - w)\|_2^2 \leq C \|v - w\|_2^2$$

By assumption $\|v_0 - w_0\|_2 = 0$, so $\|v - w\|_2 = 0$ for all $t \in [0, T]$. \square

References

- [1] M. Ben-Artzi, Global solutions of two-dimensional Navier–Stokes and Euler equations, *Arch. Rational Mech. Anal.* 128 (4) (1994) 329–358.
- [2] L. Caffarelli, R. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of the Navier–Stokes equations, *Comm. Pure Appl. Math.* 35 (6) (1982) 771–831.
- [3] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* 71 (11) (1993) 1661–1664.
- [4] S. Chen, C. Foias, D.D. Holm, E. Olson, E.S. Titi, S. Wynne, Camassa–Holm equations as a closure model for turbulent channel and pipe flow, *Phys. Rev. Lett.* 81 (24) (1998) 5338–5341.
- [5] S. Chen, C. Foias, D.D. Holm, E. Olson, E.S. Titi, S. Wynne, The Camassa–Holm equations and turbulence, in: *Predictability: Quantifying Uncertainty in Models of Complex Phenomena*, Los Alamos, NM, 1998, *Phys. D* 133 (1–4) (1999) 49–65.
- [6] S. Chen, C. Foias, D.D. Holm, E. Olson, E.S. Titi, S. Wynne, A connection between the Camassa–Holm equations and turbulent flows in channels and pipes, in: *The International Conference on Turbulence*, Los Alamos, NM, 1998, *Phys. Fluids* 11 (8) (1999) 2343–2353.
- [7] P. Constantin, C. Foias, *Navier–Stokes Equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.
- [8] J.A. Domaradzki, D.D. Holm, Navier–Stokes-alpha model: Les equations with nonlinear dispersion, 2001.
- [9] C. Foias, D.D. Holm, E.S. Titi, The Navier–Stokes-alpha model of fluid turbulence, in: *Advances in Nonlinear Mathematics and Science*, *Phys. D* 152/153 (2001) 505–519.
- [10] C. Foias, D.D. Holm, E.S. Titi, The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory, *J. Dynam. Differential Equations* 14 (1) (2002) 1–35.
- [11] D.D. Holm, J.E. Marsden, T.S. Ratiu, The Euler–Poincaré equations and semidirect products with applications to continuum theories, *Adv. Math.* 137 (1) (1998) 1–81.
- [12] D.D. Holm, E.S. Titi, Computational models of turbulence: The lans- α model and the role of global analysis, *SIAM News* 38 (2005).
- [13] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, *Math. Nachr.* 4 (1951) 213–231.
- [14] A.A. Ilyin, E.S. Titi, Attractors for the two-dimensional Navier–Stokes- α model: an α -dependence study, *J. Dynam. Differential Equations* 15 (4) (2003) 751–778.
- [15] J. Leray, Essai sur le mouvement d’un liquide visqueux emplissant l’espace, *Acta Math.* 63 (1934) 193–248.
- [16] J.E. Marsden, S. Shkoller, Global well-posedness for the Lagrangian averaged Navier–Stokes (LANS- α) equations on bounded domains, in: *Topological Methods in the Physical Sciences*, London, 2000, *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.* 359 (1784) (2001) 1449–1468.
- [17] K. Masuda, Weak solutions of Navier–Stokes equations, *Tohoku Math. J.* (2) 36 (4) (1984) 623–646.
- [18] T. Ogawa, S.V. Rajopadhye, M.E. Schonbek, Energy decay for a weak solution of the Navier–Stokes equation with slowly varying external forces, *J. Funct. Anal.* 144 (2) (1997) 325–358.
- [19] G. Prodi, Teoremi di tipo locale per il sistema di Navier–Stokes e stabilità delle soluzioni stazionarie, *Rend. Sem. Mat. Univ. Padova* 32 (1962) 374–397.
- [20] M. Schonbek, The Fourier splitting method, in: *Advances in Geometric Analysis and Continuum Mechanics*, Stanford, CA, 1993, *Internat. Press*, Cambridge, MA, 1995, pp. 269–274.
- [21] M.E. Schonbek, Decay of solutions to parabolic conservation laws, *Comm. Partial Differential Equations* 5 (5) (1980) 449–473.
- [22] M.E. Schonbek, Sharp rate of decay of solutions to 2-dimensional Navier–Stokes equations, *Comm. Partial Differential Equations* 7 (1) (1980) 449–473.
- [23] M.E. Schonbek, L^2 decay for weak solutions of the Navier–Stokes equations, *Arch. Rational Mech. Anal.* 88 (3) (1985) 209–222.
- [24] M.E. Schonbek, Large time behaviour of solutions to the Navier–Stokes equations, *Comm. Partial Differential Equations* 11 (7) (1986) 733–763.
- [25] M.E. Schonbek, Large time behaviour of solutions to the Navier–Stokes equations in H^m spaces, *Comm. Partial Differential Equations* 20 (1–2) (1995) 103–117.
- [26] M.E. Schonbek, T.P. Schonbek, Moments and lower bounds in the far-field of solutions to quasi-geostrophic flows, *Discrete Contin. Dyn. Syst.* 13 (5) (2005) 1277–1304.
- [27] M.E. Schonbek, M. Wiegner, On the decay of higher-order norms of the solutions of Navier–Stokes equations, *Proc. Roy. Soc. Edinburgh Sect. A* 126 (3) (1996) 677–685.
- [28] R. Temam, *Navier–Stokes Equations. Theory and Numerical Analysis*, AMS Chelsea Publishing, Providence, RI, 2001, Reprint of the 1984 edition.

- [29] M. Wiegner, Decay results for weak solutions of the Navier–Stokes equations on \mathbf{R}^n , *J. London Math. Soc.* (2) 35 (2) (1987) 303–313.
- [30] M. Wiegner, Higher order estimates in further dimensions for the solutions of Navier–Stokes equations, in: *Evolution Equations*, Warsaw, 2001, in: *Banach Center Publ.*, vol. 60, Polish Acad. Sci., Warsaw, 2003, pp. 81–84.
- [31] L.H. Zhang, Sharp rate of decay of solutions to 2-dimensional Navier–Stokes equations, *Comm. Partial Differential Equations* 20 (1–2) (1995) 119–127.